

Robust Safe Control Synthesis with Disturbance Observer-Based Control Barrier Functions

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Abstract—In a complex real-time operating environment, external disturbances and uncertainties adversely affect the safety, stability, and performance of dynamical systems. This paper presents a robust stabilizing safety-critical controller synthesis framework with control Lyapunov functions (CLFs) and control barrier functions (CBFs) in the presence of disturbance. A high-gain input observer method is adapted to estimate the time-varying unmodelled dynamics of the CBF with an error bound using the first-order time derivative of the CBF. This approach leads to an easily tunable low-order disturbance estimator structure with a design parameter as it utilizes only the CBF constraint. The estimated unknown input and associated error bound are used to ensure robust safety and exponential stability by formulating a CLF-CBF quadratic program. The proposed method is applicable to both relative degree one and higher relative degree CBF constraints. The efficacy of the proposed approach is demonstrated using numerical simulations of an adaptive cruise control system and a Segway platform with an external disturbance.

I. INTRODUCTION

Real-time safety is a necessity in many control applications, for instance, autonomous vehicles, robotic systems, and spacecraft. Therefore, provable safety-critical control of dynamical systems has drawn increasing attention in recent years. Control barrier functions (CBFs) are a tool to handle safety constraints in the form of forward invariance of a set [1]. CBFs can be unified with stability and performance requirements, encoded by the time derivative of a control Lyapunov function (CLF), in an online quadratic program (CLF-CBF-QP) to ensure safety and control objectives simultaneously [2]. This optimization framework has been widely applied to multi-agent systems [3], autonomous driving [4], and wheeled robots [5] due to its computational efficiency. Although these applications guarantee optimization constraints for high-fidelity dynamical models, real-time control systems generally include unavoidable uncertainties and disturbances that might cause performance degradation, and in some cases, even lead to unsafe operations [6], [7].

To address the robustness mentioned above issue of CLF-CBF-QP, the input-to-state safe CBF (ISSf-CBF) technique, which provides a robust stabilizing safe controller via a larger forward invariant set, has been introduced in [8]. The infinity norm of the bounded disturbance is used to synthesize the controller directly without a model of the unknown input. More recently, tunable ISSf-CBF (TISSf-CBF) has been proposed to reduce the conservatism of the ISSf-CBF method due to the worst-case disturbance assumption [9]. In [10]

and [11] robust CBF approaches have been investigated for uncertain systems to guarantee safety.

Disturbance observer theory, a well-studied robust control tool, has been used with the ISSf-CBF method and worst-case disturbance bound to attenuate external disturbances for safety-critical control of an autonomous surface vehicle [12]. In [13], robust safety constraints have been enforced by an adaptive pointwise unmodeled dynamic estimation law. This work considers relative-degree one systems in which the first time derivative of the CBF depends on the control signal. However, this restrictive assumption is violated in several robotic systems, as most safety constraints have relative degree greater than one.

In this study, a high-gain disturbance observer-based robust CLF-CBF-QP is formulated to guarantee the exponential stability and safety of a disturbed nonlinear system in the presence of time-varying unknown inputs. This disturbance observer scheme integrates the first-order time derivative of CBF and CLF with an input observer approach to estimate the unmodelled system dynamics within an exponential error bound. Since this method uses only the first-order CBF or CLF constraint, it presents a simple disturbance estimation framework with only one design parameter that needs to be tuned. We then formulate a CLF-CBF-QP containing estimated disturbance and error bound-based constraints that provide a robust, safe stabilizing control input. Moreover, the proposed safe control method is appropriate for high relative degree CBF constraints. Finally, we demonstrate the applicability of this method using adaptive cruise control (ACC) and Segway platform simulation examples.

The rest of this paper is organized as follows. The preliminaries are introduced in Section II. Section III provides the disturbance observer-based robust CLF-CBF-QP scheme. Simulation results are presented in Section IV, Section V concludes the paper.

II. PRELIMINARIES

Notation: The notation used in this study is fairly standard. \mathbb{R} , \mathbb{R}^+ , \mathbb{R}_0^+ represent the set of real, positive real and non-negative real numbers, respectively. The Euclidean norm of a matrix is denoted by $\|\cdot\|$, and $\|\cdot\|_\infty$ represents the infinity norm. A continuous function $\alpha : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ belongs to class- \mathcal{K}_∞ ($\alpha \in \mathcal{K}_\infty$) if it is strictly increasing, $\alpha(0) = 0$, $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$, and a continuous function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ belongs to extended class- \mathcal{K}_∞ ($\alpha \in \mathcal{K}_{\infty,e}$) if it is strictly monotonically increasing, $\alpha(0) = 0$, $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$, $\alpha(r) \rightarrow -\infty$ as $r \rightarrow -\infty$. For a given set $\mathcal{C} \subset \mathbb{R}^n$, $\partial\mathcal{C}$ and $\text{Int}(\mathcal{C})$ denote its boundary and interior, respectively.

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We consider a nominal nonlinear control affine and disturbed nonlinear control affine systems given by

$$\dot{x} = f(x) + g(x)u, \quad (1)$$

$$\dot{x} = f(x) + g(x)u + g(x)d(t), \quad (2)$$

where $x \in X \subseteq \mathbb{R}^n$ is the state, $u \in U \subseteq \mathbb{R}^m$ is the control input, $d \in D \subseteq \mathbb{R}^m$ is the bounded disturbance as $\sup_{t \geq 0} \|d(t)\| \leq d_b \in \mathbb{R}_0^+$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are locally Lipschitz.

A. Stability and Control Lyapunov Functions

CLFs allow the formulation of optimization-based stabilizing controllers, and exponential stability requirements can be reduced to finding a CLF for system (1). Therefore, CLFs are useful to represent closed-loop control objectives in a CLF-CBF-QP, for instance, reaching a target set [1], [14].

Definition 1 (Exponential stability): The equilibrium point, $x = 0$, of the nonlinear system (1) is *exponentially stable* if there are constants $\beta_1, \beta_2, \beta_3 \in \mathbb{R}^+$ such that $\|x(0)\| \leq \beta_1 \implies \|x(t)\| \leq \beta_2 e^{-\beta_3 t} \|x(0)\| \forall t \geq 0$.

Definition 2 (Control Lyapunov function [15], [16]):

For the nominal system (1), a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ is an *exponentially stabilizing control Lyapunov function*, if there exists constants $c, \zeta_1, \zeta_2, \lambda \in \mathbb{R}^+$ and control signal $u \in U$ such that $\forall x \in X : \zeta_1 \|x\|^c \leq V(x) \leq \zeta_2 \|x\|^c$,

$$\inf_{u \in U} \dot{V}(x, u) \triangleq \underbrace{\frac{\partial V}{\partial x} f(x)}_{L_f V(x)} + \underbrace{\frac{\partial V}{\partial x} g(x) u}_{L_g V(x)} \leq -\lambda V(x), \quad (3)$$

where $L_f V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, $L_g V(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are the Lie derivatives of $V(x)$ with respect to $f(x)$, $g(x)$, respectively.

In real-time applications, external disturbances such as external load and friction may deteriorate the stability or safety of dynamical systems. In such cases, the definition of a CLF in (3) can be extended to input-to-state stabilizing CLF (ISS-CLF) according to the disturbance input $d(t)$ in (2).

Definition 3 (Input-to-state stabilizing CLF [15]): For the disturbed system (2), a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ is an *exponential input-to-state stabilizing control Lyapunov function (ISS-CLF)*, if there exists $c, \zeta_1, \zeta_2, \lambda \in \mathbb{R}^+$, $\iota \in \mathcal{K}_\infty$, $u \in U$ such that $\forall x \in X, \forall d \in D : \zeta_1 \|x\|^c \leq V(x) \leq \zeta_2 \|x\|^c$,

$$\inf_{u \in U} L_f V(x) + L_g V(x)(u + d) \leq -\lambda V(x) + \iota(\|d\|_\infty). \quad (4)$$

Given a $V(x)$ and $\lambda \in \mathbb{R}^+$ for (2), we define the set of exponentially stabilizing controllers for $\forall x \in X, \forall d \in D$ as

$$K_{\text{CLF}}(t, x, d) \triangleq \left\{ u \in U \mid \dot{V}(x, u, d) \leq -\lambda V(x) \right\}, \quad (5)$$

which states that robust exponential stability can be achieved by synthesizing a control input that applies the CLF condition (3) to the disturbed system (2).

B. Safety and Control Barrier Functions

Control barrier functions are a useful tool for rendering the set $\mathcal{C} \subset \mathbb{R}^n$ as forward invariant throughout its state-space. We note that set \mathcal{C} is forward invariant if, for every initial condition $x(0) \in \mathcal{C}$, the solution of (1) satisfies $x(t) \in \mathcal{C} \forall t \geq 0$.

We consider a set $\mathcal{C} \subset \mathbb{R}^n$ defined as a 0-superlevel set of a continuously differentiable function $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathcal{C} \triangleq \{x \in \mathbb{R}^n : h(x) \geq 0\}, \quad (6)$$

$$\partial \mathcal{C} \triangleq \{x \in \mathbb{R}^n : h(x) = 0\}, \quad (7)$$

$$\text{Int}(\mathcal{C}) \triangleq \{x \in \mathbb{R}^n : h(x) > 0\}. \quad (8)$$

The nominal closed-loop system (1) is safe on the set \mathcal{C} if \mathcal{C} is forward invariant [1]. CBFs can be utilized to design safe controllers for system (1) with respect to set \mathcal{C} .

Definition 4 (Control barrier function [1]): Let $\mathcal{C} \subset \mathbb{R}^n$ be the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by (6)-(8). Then, $h(x)$ is a CBF for system (1) on \mathcal{C} if there exists $\alpha \in \mathcal{K}_{\infty, e}$ such that $\forall x \in \mathcal{C}$:

$$\sup_{u \in U} \dot{h}(x, u) \triangleq \underbrace{\frac{\partial h}{\partial x} f(x)}_{L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x) u}_{L_g h(x)} \geq -\alpha(h(x)). \quad (9)$$

Given a $h(x)$, $\alpha \in \mathcal{K}_{\infty, e}$ for system (2), we define the set of robust safe controllers for $\forall x \in X, \forall d \in D$ as

$$K_{\text{CBF}}(t, x, d) \triangleq \left\{ u \in U \mid \dot{h}(x, u, d) \geq -\alpha(h(x)) \right\}. \quad (10)$$

Definition 5 (Exponential CBF (ECBF) [17]): Let $\mathcal{C} \subset \mathbb{R}^n$ be the 0-superlevel set of an r -times continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $L_g L_f^{r-1} h(x) \neq 0$ and $L_g L_f^2 h(x) = L_g L_f^3 h(x) = \dots = L_g L_f^{r-2} h(x) = 0 \forall x \in \mathcal{C}$. Then, $h(x)$ is an ECBF for system (1) on \mathcal{C} if there exists a row vector $K_\alpha \in \mathbb{R}^r$ such that $\forall x \in \mathcal{C}$:

$$\sup_{u \in U} h^r(x, u) \triangleq L_f^r h(x) + L_g L_f^{r-1} h(x) u \geq -K_\alpha \eta_b(x) \quad (11)$$

where $\eta_b(x) = [h(x) \ \dot{h}(x) \ \ddot{h}(x) \ \dots \ h^{r-1}(x)]^T$.

Definition 6 (Input-to-state safe CBF [8]): Let $\mathcal{C} \subset \mathbb{R}^n$ be the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$. Then, $h(x)$ is an ISSf-CBF for disturbed system (2) on \mathcal{C} if there exists $\alpha \in \mathcal{K}_{\infty, e}$, $\iota \in \mathcal{K}_\infty$ such that $\forall x \in \mathcal{C}$:

$$\sup_{u \in U} L_f h(x) + L_g h(x) u \geq -\alpha(h(x)) - \iota(\|d\|_\infty). \quad (12)$$

When valid CLFs and CBFs are given for system (1), the stability and safety constraints can be enforced to compute pointwise safe control inputs via following CLF-CBF-QP:

$$\begin{aligned} u^*(x) = \arg \min_{u \in U, \delta \in \mathbb{R}} \quad & \|u - k(x)\|^2 + p\delta^2 \\ \text{s.t.} \quad & \dot{h}(x, u) \geq -\alpha(h(x)) \\ & \dot{V}(x, u) \leq -\lambda V(x) + \delta \end{aligned}$$

where $k(x)$ is the nominal feedback controller, $\delta \in \mathbb{R}$ is a relaxation variable that is penalized by a constant $p \in \mathbb{R}^+$.

Similarly, one can combine ISS-CLF (4) and ISSf-CBF (12) constraints to synthesize pointwise safe control inputs for the disturbed system (2) via the following ISSf-CBF-QP [8], [9]:

$$\begin{aligned} u^*(x) = & \arg \min_{u \in U, \delta \in \mathbb{R}} \|u - k(x)\|^2 + p\delta^2 \\ \text{s.t.} \quad & \dot{h}(x, u) \geq -\alpha(h(x)) + \epsilon \|L_g h(x)\|^2 \\ & \dot{V}(x, u) \leq -\lambda V(x) + \delta \end{aligned}$$

where $\epsilon \in \mathbb{R}^+$ is a user-defined constant. Safe controller synthesizing using ISSf-CBF-QP is a way to handle unmodeled system dynamics. However, this QP conservatively ensures the safety requirements for the disturbed system (2) due to the worst-case disturbance input assumption.

C. High-Gain Input Disturbance Observer

In order to define robust, exponentially stabilizing, safe controllers using (5) and (10) we need to measure the time-varying disturbance $d(t) \in D$ that is not directly available in real-time applications. Using a disturbance observer framework, our objective is to replace $d(t)$ with an estimated disturbance term $\hat{d}(t)$ and the upper bound of the associated estimation error. Note that the dependence on time t will be omitted for simplicity throughout the rest of the paper, and it will be used only if necessary.

Specifically, we consider a high-gain input disturbance observer that is proposed in [18]. Let us define a first-order dynamical system

$$\dot{z}_d = v_d + w_d, \quad (13)$$

where $z_d \in \mathbb{R}$ and $v_d \in \mathbb{R}$ are known or measured variables, and $w_d \in \mathbb{R}$ is the unknown time-varying unmodelled dynamics or disturbance input of the system that needs to be estimated. Define estimated disturbance $\hat{w}_d \in \mathbb{R}$ as

$$\hat{w}_d = k_d z_d - \varepsilon_d, \quad (14)$$

where $k_d \in \mathbb{R}^+$ is the disturbance observer gain to be tuned, and $\varepsilon_d \in \mathbb{R}$ is an auxiliary variable satisfying

$$\dot{\varepsilon}_d = -k_d \varepsilon_d + k_d v_d + k_d^2 z_d. \quad (15)$$

Then, the error dynamics of this disturbance estimation method, $e_d = (w_d - \hat{w}_d) \in \mathbb{R}$, is obtained as

$$e_d = w_d + \varepsilon_d - k_d z_d. \quad (16)$$

Definition 7 (Estimation error quantified observer): A disturbance observer is called an *estimation error quantified observer* for system (13) if it generates a disturbance estimation \hat{w}_d with an error bound $\|e_d\|$ such that $\forall t \geq 0$,

$$\|e_d\| \leq M_d(t, w_d, \hat{w}_d), \quad (17)$$

where $M_d(t, w_d, \hat{w}_d) : \mathbb{R}_0^+ \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}_0^+$. If we assume that \dot{w}_d is bounded as $\sup_{t \geq 0} \|\dot{w}_d\| \leq b_1$, then the high-gain disturbance observer model given in (13)-(16) is an estimation error quantified disturbance observer with the following error bound [18]:

$$\|e_d\| \leq \sqrt{e_d(0)^2 e^{-k_d t} + b_1^2 / k_d^2}. \quad (18)$$

III. DISTURBANCE OBSERVER-BASED SAFETY-CRITICAL CONTROL

In this section, we address the issue of having disturbances by proposing a new disturbance observer-based robust safety-critical framework. Specifically, we adapt the high-gain disturbance observer scheme to estimate the time-varying effect of disturbance on the time derivative of CBF with the associated error bound. Next, we use the estimated part of the CBF constraint and error bound to construct a robust safety constraint.

The CLF-CBF-QP for disturbed system (2) can be formulated using the linear constraints of the exponentially stabilizing and safe controller sets in (5) and (10) as

$$\begin{aligned} u^*(x) = & \arg \min_{u \in U, \delta \in \mathbb{R}} \|u - k(x)\|^2 + p\delta^2 \\ \text{s.t.} \quad & L_f h(x) + L_g h(x)u + L_g h(x)d \geq -\alpha(h(x)) \\ & L_f V(x) + L_g V(x)u + L_g V(x)d \leq -\lambda V(x) + \delta \end{aligned} \quad (19)$$

where $L_g h(x)d$, $L_g V(x)d$ are unknowns since they depend on immeasurable disturbance d . The objective function is set to modify desired feedback control input minimally.

If we consider $h : \mathbb{R}^n \rightarrow \mathbb{R}$ to be a CBF for system (2) on set \mathcal{C} , the time derivative of $h(x)$ is given by

$$\dot{h}(x, u, d) = \underbrace{L_f h(x) + L_g h(x)u}_{a(x, u)} + \underbrace{L_g h(x)d}_{b(x, d)}, \quad (20)$$

where $a(x, u) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is the known part of $\dot{h}(x, u, d)$, and $b(x, d) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ needs to be estimated. The effect of the disturbance on the $\dot{h}(x, u, d)$ is obvious. Furthermore, the first-order system dynamics of (20) is in the form of an observation problem in standard format (13). Therefore, the high-gain input disturbance observer can be defined for estimation of $b(x, d)$, i.e., $\hat{b}(x, u, d)$, as

$$\hat{b}(x, u, d) = k_b h(x, u, d) - \varepsilon_b, \quad (21)$$

$$\dot{\varepsilon}_b = -k_b \varepsilon_b + k_b a(x, u) + k_b^2 h(x, u, d), \quad (22)$$

$$e_b = b(x, d) - \hat{b}(x, u, d) = b(x, d) + \varepsilon_b - k_b h(x, u, d), \quad (23)$$

where $k_b \in \mathbb{R}^+$ is the disturbance observer gain, $e_b \in \mathbb{R}$ is the estimation error dynamic, and $\varepsilon_b \in \mathbb{R}$ is the auxiliary variable.

Assumption 1: There exists a constant $b_h \in \mathbb{R}_0^+$ such that $\sup_{t \geq 0} \|\dot{b}(x, d)\| \leq b_h$.

Assumption 2: The CBF and CLF have relative degree one, i.e., $L_g h(x) \neq 0$, $L_g V(x) \neq 0 \forall x \in X$ in (19).

Note that, firstly, we consider the relative degree one systems using Assumption 2, then we extend our results to the higher relative degree systems via an ECBF.

Theorem 1: The estimation error dynamics of the disturbance observer given in (23), under Assumption 1 and Assumption 2, convergences to a set defined by

$$\|e_b(t, x, u, d)\| \leq \sqrt{(e_b(0)^2 - b_h^2 / k_b^2) e^{-k_b t} + b_h^2 / k_b^2}. \quad (24)$$

Proof: In order to prove the convergence of the proposed disturbance observer to its steady-state value and derive an estimation error bound, we choose a Lyapunov function $V_d : \mathbb{R} \rightarrow \mathbb{R}_0^+$ as follows by omitting the dependence on (t, x, u, d, \hat{d}) for simplicity:

$$V_d = \frac{1}{2}(e_b)^2 = \frac{1}{2}(b - \hat{b})^2. \quad (25)$$

Then, time derivative of V_d is given by

$$\dot{V}_d = (b - \hat{b})(\dot{b} - \dot{\hat{b}}) = \frac{1}{2} \frac{d(b - \hat{b})^2}{dt}. \quad (26)$$

Substituting (21) and its time derivative into (26) yields

$$\dot{V}_d = (b + \epsilon_b - k_b h)(\dot{b} - k_b \dot{h} + \dot{\epsilon}_b), \quad (27)$$

and, substituting (20) and (22) into (27), we have

$$\begin{aligned} \dot{V}_d &= (b + \epsilon_b - k_b h)(\dot{b} - k_b b - k_b \epsilon_b + k_b^2 h) \\ &= (b - \hat{b})(\dot{b} - k_b(b - \hat{b})) \\ &= -k_b(b - \hat{b})^2 + (b - \hat{b})\dot{b}. \end{aligned} \quad (28)$$

Noting that,

$$-k_b(b - \hat{b})^2 + (b - \hat{b})\dot{b} \leq -k_b(b - \hat{b})^2 + \|b - \hat{b}\|b_h, \quad (29)$$

we obtain

$$\frac{1}{2} \frac{d(b - \hat{b})^2}{dt} \leq -k_b(b - \hat{b})^2 + \|b - \hat{b}\|b_h. \quad (30)$$

Therefore, we need to define an upper bound for $\|b - \hat{b}\|b_h$. Now, consider the following inequality

$$(k_b\|b - \hat{b}\| - b_h)^2 = k_b^2\|b - \hat{b}\|^2 - 2\|b - \hat{b}\|k_b b_h + b_h^2 \geq 0, \quad (31)$$

which results

$$\|b - \hat{b}\|b_h \leq \frac{k_b\|b - \hat{b}\|^2}{2} + \frac{b_h^2}{2k_b}. \quad (32)$$

Substituting this upper bound into (30), we obtain

$$2\dot{V}_d = \frac{d(b - \hat{b})^2}{dt} \leq -k_b(b - \hat{b})^2 + \frac{b_h^2}{k_b}. \quad (33)$$

Integration of (33) yields the following inequality

$$\|b - \hat{b}\| \leq \sqrt{e^{b_\alpha} e^{-k_b t} + b_h^2/k_b^2}, \quad (34)$$

where $b_\alpha \in \mathbb{R}$. Finally, solving (34) for e^{b_α} with initial conditions leads to

$$e^{b_\alpha} \geq (b(0) - \hat{b}(0))^2 - b_h^2/k_b^2 \quad (35)$$

for $t > 0$; therefore, we can replace e^{b_α} with a constant using (35) as $e^{b_\alpha} = (b(0) - \hat{b}(0))^2 - b_h^2/k_b^2$ that leads to

$$\|b - \hat{b}\| \leq \underbrace{\sqrt{((b(0) - \hat{b}(0))^2 - b_h^2/k_b^2)e^{-k_b t} + b_h^2/k_b^2}}_{M_b(t, x, u, d, \hat{d})}, \quad (36)$$

which is the statement of the theorem. \blacksquare

Remark 1: Since $(b(0) - \hat{b}(0))^2 > (b(0) - \hat{b}(0))^2 - b_h^2/k_b^2$, Theorem 1 provides a less conservative upper bound with a

convergence guarantee for the estimation error dynamics than (18), derived in [18].

The proposed high-gain disturbance observer is an estimation error quantified disturbance observer for the time derivative of $h(x)$. Moreover, (36) implies that $\hat{b}(x, u, d) \rightarrow b(x, d)$ if $k_b \rightarrow \infty$, and if $(L_g h)^{-1}$ exists, one can easily compute the estimated disturbance as

$$\hat{d} = (L_g h)^{-1} \hat{b}(x, d). \quad (37)$$

Hence, if there exists a constant $b_d \in \mathbb{R}_0^+$ such that $\sup_{t \geq 0} \|\hat{d}\| \leq b_d$, we can derive a bound for d_e ,

$$\hat{d} = d - d_e, \quad (38)$$

as $\|d_e\| \leq M_d(t, x, u, d, \hat{d}) := \|(L_g h)^{-1}\|M_b(t, x, u, d, \hat{d})$. Then, plugging in for d in the constraints of optimization problem (19) yields

$$L_f h(x) + L_g h(x)(u + \hat{d} + d_e) \geq -\alpha(h(x)), \quad (39)$$

$$L_f V(x) + L_g V(x)(u + \hat{d} + d_e) \leq -\lambda V(x). \quad (40)$$

Lemma 1: Consider the disturbed system (2) with an estimated error quantified disturbance observer that provides \hat{d} with an error bound $\|d_e\| \leq M_d(t, x, u, d, \hat{d})$. Suppose that a safe set $\mathcal{C} \subset \mathbb{R}^n$ and a 0-superlevel set of the continuously differentiable function $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, and $\alpha \in \mathcal{K}_{\infty, e}$ are given for the nominal system (1). If a control signal $u \in U$ satisfies

$$L_f h(x) + L_g h(x)u + \underbrace{L_g h(x)\hat{d}}_{\hat{b}} - \underbrace{\|L_g h(x)\|M_d}_{M_b} \geq -\alpha(h(x)), \quad (41)$$

then the robust CBF constraint in (19) is also guaranteed.

Proof: Our objective is to show that $L_g h(x)d$ in (19) is an upper bound for $L_g h(x)\hat{d} - \|L_g h(x)\|M_d \forall t \geq 0$. We have

$$\begin{aligned} L_g h(x)d &= L_g h(x)(\hat{d} + d_e) \\ &= L_g h(x)(\hat{d} + d - \hat{d}) \\ &\geq L_g h(x)(\hat{d}) - \|L_g h(x)\|(\|d - \hat{d}\|) \\ &\geq \underbrace{L_g h(x)(\hat{d})}_{\hat{b}} - \underbrace{\|L_g h(x)\|M_d(t, d, \hat{d})}_{M_b}, \end{aligned} \quad (42)$$

which means that (41) \implies (10). \blacksquare

Remark 2: Lemma 1 provides a sufficient robust safety condition via a modified CBF, however, if $\hat{b} \rightarrow b$, (41) becomes a necessary and sufficient robust safety constraint.

Remark 3: The proposed method is also applicable to high relative degree CBF constraints. To show this, let consider a higher relative degree disturbed system (2) with an ECBF $h(x)$ defined in Definition 5, where $L_g h(x) = 0 \forall x \in X$. The r^{th} -order time derivative of $h(x)$ is given by

$$\underbrace{h^r(x, u, d)}_{(h^{r-1}(x, u, d))} = \underbrace{L_f^r h(x) + L_g L_f^{r-1} h(x)u}_{a_e(x, u)} + \underbrace{L_g L_f^{r-1} h(x)d}_{b_e(x, d)}, \quad (43)$$

which is in the form of the first-order dynamical system (13); therefore, we can adapt the proposed input disturbance observer scheme to estimate the unknown dynamics $b_e(x, d)$. Again, if there exists a constant $b_E \in \mathbb{R}_0^+$ such that $\sup_{t \geq 0} \|\hat{b}_e(x, d)\| \leq b_E$, then a high-gain disturbance observer can be proposed to estimate $b_e(x, d)$ with an error bound $\|b_e - \hat{b}_e\| \leq M_{b_e}(t, x, u, d, \hat{d})$. Finally, the robust ECBF constraint is defined as

$$L_f^r h(x) + L_g L_f^{r-1} h(x) u + \underbrace{L_g L_f^{r-1} h(x) \hat{d}}_{\hat{b}_e(x, d)} - M_{b_e} \geq -K_\alpha \eta_b(x), \quad (44)$$

where $K_\alpha, \eta_b(x)$ are given in (11).

Note that proposed disturbance observer framework estimates the effect of the disturbance on the time derivative of $h(x)$. Therefore, Lemma 1 provides a sufficient robust CBF condition even if $(L_g h)^{-1}$ is not exactly known. However, without the estimated disturbance \hat{d} and the associated error bound, the robust CLF constraint cannot be defined. To address this issue, consider the first-order time derivative of $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\dot{V}(x, u, d) = \underbrace{L_f V(x) + L_g V(x) u}_{a_V(x, u)} + \underbrace{L_g V(x) \hat{d}}_{b_V(x, d)}, \quad (45)$$

where $b_V(x, d)$ needs to be estimated. If there exists a constant $b_h \in \mathbb{R}_0^+$ such that $\sup_{t \geq 0} \|\hat{b}_V(x, d)\| \leq b_L$, then we can design a disturbance observer to estimate $b_V(x, d)$ with an error bound $\|b_V - \hat{b}_V\| \leq M_{b_V}(t, x, u, d, \hat{d})$, $\|d_e\| = \|d - \hat{d}\| \leq M_{d_V}(t, x, u, d, \hat{d}) = \|(L_g V)^{-1} M_{b_V}(t, x, u, d, \hat{d})\|$, which leads to following lemma to provide a sufficient robust CLF condition for system (2).

Lemma 2: Consider the disturbed system (2) with an estimated error quantified observer that provides \hat{d} with an error bound $\|d_e\| \leq M_{d_V}(t, x, u, d, \hat{d})$. Suppose that a continuously differentiable exponentially stabilizing CLF function $V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$, and $\lambda \in \mathbb{R}^+$ is given for nominal system (1). If a control signal $u \in U$ satisfies

$$L_f V(x) + L_g V(x) u + \underbrace{L_g V(x) \hat{d}}_{\hat{b}_V} + \underbrace{\|L_g V(x)\| M_{d_V}}_{M_{b_V}} \leq -\lambda V(x), \quad (46)$$

then the robust CLF constraint in (19) is also guaranteed.

Proof: Our objective is to show that $L_g V(x) d$ in (19) is a lower bound of $L_g V(x) \hat{d} + \|L_g V(x)\| M_d \forall t \geq 0$. We also have

$$\begin{aligned} L_g V(x) d &= L_g V(x) (\hat{d} + d_e) \\ &= L_g V(x) (\hat{d} + d - \hat{d}) \\ &\leq L_g V(x) (\hat{d}) + \|L_g V(x)\| (\|d - \hat{d}\|) \\ &\leq \underbrace{L_g V(x) (\hat{d})}_{\hat{b}_V} + \underbrace{\|L_g V(x)\| M_d(t, d, \hat{d})}_{M_{b_V}}, \end{aligned} \quad (47)$$

which means that (46) \implies (5). \blacksquare

By Lemma 1 and Lemma 2, the pointwise safe controller for disturbed nonlinear system (2) is obtained by the follow-

ing robust CLF-CBF-QP:

$$\begin{aligned} u^*(x) &= \arg \min_{u \in U, \delta \in \mathbb{R}} \|u - k(x)\|^2 + p\delta^2 \\ &\text{s.t.} \\ &L_f h(x) + L_g h(x) u + \hat{b} - M_b \geq -\alpha(h(x)) \\ &L_f V(x) + L_g V(x) u + \hat{b}_V + M_{b_V} \leq -\lambda V(x) + \delta \end{aligned} \quad (48)$$

Note that, for higher relative degree systems, the CBF constraint in (48) needs to be replaced with the robust ECBF constraint given in (44).

Finally, if $(L_g h)^{-1}$ exists, we do not need to use another disturbance observer for the robust CLF constraint since \hat{d} is obtained via (37). In this case, in order to reject the disturbance the objective function of the robust CLF-CBF-QP in (48) can be modified as $\|u - (k(x) - \hat{d})\|^2$. Furthermore, with a known \hat{d} , the constraints in (48) are defined as following robust CLF-CBF-QP:

$$\begin{aligned} u^*(x) &= \arg \min_{u \in U, \delta \in \mathbb{R}} \|u - k(x)\|^2 + p\delta^2 \\ &\text{s.t.} \\ &L_f h(x) + L_g h(x) (u + \hat{d}) - \|L_g h(x)\| M_d \geq -\alpha(h(x)) \\ &L_f V(x) + L_g V(x) (u + \hat{d}) + \|L_g V(x)\| M_d \leq -\lambda V(x) + \delta \end{aligned} \quad (49)$$

IV. SIMULATION RESULTS

A. Adaptive Cruise Control Example

In this subsection, we apply the proposed disturbance observer based robust, safe controller design methodology to an adaptive cruise control example [6], [13], in which our safety objective is to maintain the safe following distance while cruising at a constant speed. The system dynamics are in the form (2):

$$\underbrace{\begin{bmatrix} \dot{v}_l \\ \dot{v}_f \\ \dot{D} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} a_l \\ -F_r/m \\ v_l - v_d \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix}}_{g(x)} d, \quad (50)$$

where v_l [m/s] and v_f [m/s] are the velocity of the lead car and the following car, respectively, and D [m] is the distance between the lead and following cars, $F_r = f_0 + f_1 v_f + f_2 (v_f)^2$ [N] is the aerodynamic drag, m [kg] is the mass of the following car, a_l is the acceleration of the lead car, d is the external disturbance. The safety constraint requires the following car to keep a safe distance from the lead car as $D \geq v_f \tau_d$, where τ_d is the desired time headway. The CBF $h(x) = D - v_f \tau_d$ captures this. The closed-loop control objective requires cruise at a constant speed that is encoded into the QP via CLF $V(x) = (v_f - v_d)^2$. The parameters of the ACC problem simulation are given in Table 1. Since the ACC system is in the form of a single-input system and $(L_g h)^{-1} \neq 0$, we use the robust CLF-CBF-QP given in (49). The objective of robust CLF-CBF-QP is set to be $0.5u^2/m^2 + 0.5p\delta^2$. The control signal is constrained as $-0.4mg \leq u \leq 0.4mg$. For the sake of completeness, we compare the proposed disturbance observer based method

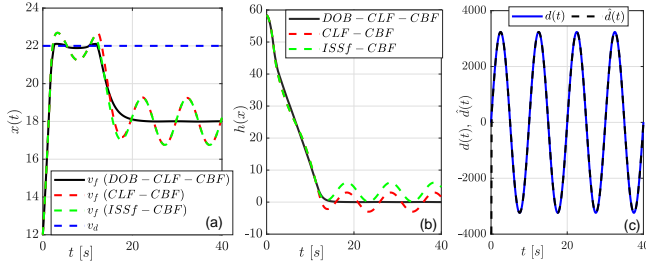


Fig. 1: Numerical simulation of the ACC system. (a) Velocity of the lead car and desired velocity. (b) Control barrier function $h(x)$. (c) Estimated and actual disturbances. Simulations are performed with disturbance observer (DOB)-based robust CLF-CBF-QP (proposed method) (black), nominal CLF-CBF-QP (red), and ISSf-CBF-QP (green).

TABLE I: Parameters in simulation for ACC example.

$m = 1650$ [kg]	$f_0 = 0.1$ [N]	$p = 100$
$g = 9.81$ [m/s ²]	$f_1 = 5$ [Ns/m]	$\lambda = 5$
$a_l = 0$ [m/s ²]	$f_2 = 0.25$ [Ns ² /m]	$\alpha(h(x)) = h$
$v_d = 22$ [m/s]	$x(0) = [18 \ 12 \ 80]^T$	$\tau_d = 1.8$ [s]
$b_h = 2.22$	$k_b = 100$	$d = 0.2g\sin(20\pi t)$

and ISSf-CBF-QP, where we consider $\epsilon = 2620000$, and CLF-CBF-QP.

Fig. 1 shows the evaluation of v_f , $h(x)$, and disturbance estimation accomplishment of the disturbance observer, respectively. As can be seen from Fig. 1-(b), the applied disturbance input causes safety violation without the disturbance estimator, i.e., CLF-CBF-QP violates the safety requirement. Although the ISSf-CBF provides a safety guarantee, it causes performance deterioration due to the conservative structure of the method, as observed from Fig. 1-(a) and Fig. 1-(b). The safety and control performance of the system is maintained using the proposed disturbance observer based robust safety control algorithm, as shown in Fig. 1-(a) and Fig. 1-(b). Furthermore, it is observed from Fig. 1-(c) that the proposed disturbance observer can effectively estimate actual disturbance.

B. Segway Platform Example

The proposed method can also be applied to uncertain systems with high relative-degree safety constraints. This section demonstrates the proposed algorithm on a Segway platform model on an inclined surface, which causes unmatched disturbance and uncertainties, with a CBF with relative degree two.

We consider a Segway model given by Fig. 2-(a) with the position and pitch angle states $[p \ \theta]^T \in \mathbb{R} \times [0, 2\pi]$ on an unknown surface inclined by $\phi \in [0, 2\pi]$. For simplicity, let assume that $\dot{\phi} = 0$, $\ddot{\phi} = 0$. We have the following dynamics:

$$\begin{bmatrix} m_0 & mL\cos(\psi) \\ mL\cos(\psi) & J_0 \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -mgL\sin(\psi) \end{bmatrix} + \begin{bmatrix} b_t/R & -b_t - mL\dot{\theta}\sin(\psi) \\ -b_t & b_t R \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} K_m/R \\ -K_m \end{bmatrix} u, \quad (51)$$

where $\psi = \theta + \phi$. For more details on the descriptions and values of the parameters, see [19]. Note that the terms including inclination angle ϕ can be viewed as the uncertainty and disturbance. Choosing the state vector $x = [p \ \dot{p} \ \phi \ \dot{\phi}]^T$, we can get the system dynamics in the form (2) with uncertainties as

$$\dot{x} = f(x) + g(x)u + \underbrace{\Delta f(x, \phi) + \Delta g(x, \phi)u + \tau_d(x, \phi)}_{d(x, u, \phi)}, \quad (52)$$

where, the effect of inclination uncertainty is lumped into a single unmatched disturbance vector $d(x, u, \phi)$. The unmodelled dynamics of the system, which is described in [5], are also included in $\Delta f(x, \phi)$, $\Delta g(x, \phi)$. We choose a barrier function $h(x) = \pi/10 - \theta^2$, which has relative degree two, to encode constraints on the pitch angle. A Linear Quadratic Regulator (LQR) nominal controller is designed using the linearized model of the system to track the desired path. To estimate the effect of $d(x, u, \phi)$ on the time derivative of CBF $\dot{h}(x, u, d)$, the proposed disturbance estimation framework can be adapted as

$$\dot{h}(x, u, d) = \underbrace{L_f h(x) + L_g h(x)u}_{a(x, u)} + \underbrace{\frac{\partial h}{\partial x} d(x, u, \phi)}_{b_d(x, u, \phi)}. \quad (53)$$

Finally, we can modify the robust CBF constraint in (48) as

$$L_f h(x) + L_g h(x)u + \hat{b}_d(x, u, \phi) - M_{b_d} \geq -\alpha(h(x)), \quad (54)$$

where M_{b_d} is the estimation error bound of $\hat{b}_d(x, u, \phi)$. We use only the robust CBF constraint for this example.

Fig. 2-(c) and Fig. 2-(d) show numerical simulation results where the Segway moves from $[0 \ 0 \ 0.138 \ 0]^T$ to $[1 \ 0 \ 0.138 \ 0]^T$ on a surface inclined by $\phi = 20^\circ$ in its state-space. The planar Segway platform stays within the safe set with a disturbance observer-based approach while travelling on an inclined surface. On the other hand, without disturbance observer, the Segway shows unsafe behaviour. Fig. 2-(b) shows that the proposed disturbance estimation approach appropriately estimate the actual effects of the disturbance and uncertainty on the time derivative of the CBF with the defined error bound.

C. Conclusions and Future Work

In this paper, we present a disturbance observer-based robust safe controller synthesis method in the presence of disturbance or uncertainty. We first introduce a high-gain observer method to estimate the unmodelled dynamics of the CBF using only the safety constraint. Then, the estimated disturbance and associated error bound are utilized to construct a new robust CLF-CBF-QP. We show the effectiveness of the method on the numerical simulations of an adaptive cruise control system and Segway with an external disturbance. Our future work includes an extension of the proposed method to robust time-varying CBF approaches to consider the Signal Temporal Logic specifications.

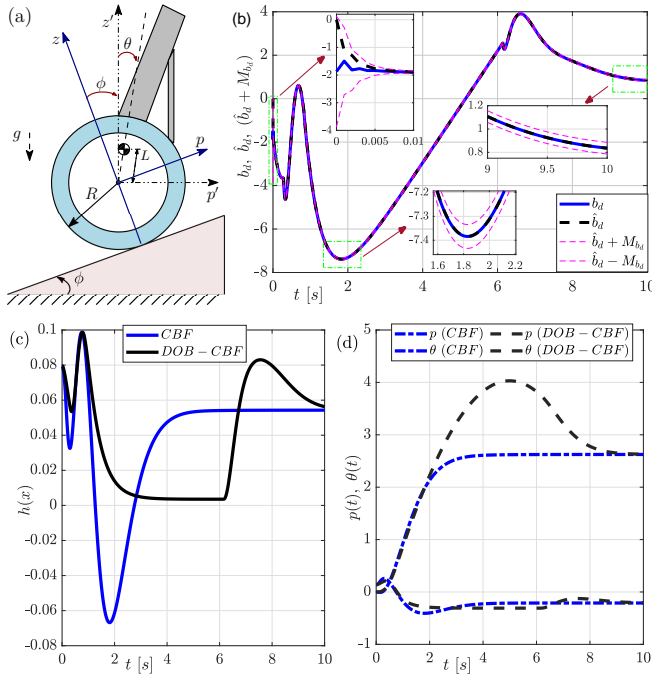


Fig. 2: Segway example. (a) Planar Segway model on an inclined surface. (b) Estimated and actual effects of the disturbance and uncertainty on the time derivative of the control barrier function. (c) Control barrier function $h(x)$. (d) Trajectories of p and θ . Simulations are performed with disturbance observer (DOB)-based robust CBF-QP (black) and nominal CBF-QP (blue).

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