Optimal convergence rates of totally asynchronous optimization

Xuyang Wu, Sindri Magnússon, Hamid Reza Feyzmahdavian, and Mikael Johansson

Abstract—Asynchronous optimization algorithms are at the core of modern machine learning and resource allocation systems. However, most convergence results consider bounded information delays and several important algorithms lack guarantees when they operate under total asynchrony. In this paper, we derive explicit convergence rates for the proximal incremental aggregated gradient (PIAG) and the asynchronous block-coordinate descent (Async-BCD) methods under a specific model of total asynchrony, and show that the derived rates are order-optimal. The convergence bounds provide an insightful understanding of how the growth rate of the delays deteriorates the convergence times of the algorithms. Our theoretical findings are demonstrated by a numerical example.

I. INTRODUCTION

Distributed and parallel algorithms are powerful tools for solving large-scale problems. These algorithms coordinate multiple computing nodes to solve the overall problem. The coordination can be synchronous, meaning that each node needs to wait for all other nodes to successfully conclude their computations and communications before proceeding to the next iteration. This is clearly inefficient: the slowest node dictates the convergence speed, systems become sensitive to single node failures, and the implementation overhead for synchronization can be large. Therefore, asynchronous algorithms that need no synchronization are often preferred [1]–[3]. However, compared to synchronous algorithms, asynchronous algorithms are more difficult to analyze, and their convergence properties are not as well understood.

Early efforts on establishing convergence properties of asynchronous algorithms were made in the 1980s by Bertsekas and Tsitsiklis, e.g., [4]–[6]. They considered two models for asynchrony, partial asynchrony ("bounded delays") and total asynchrony ("unbounded delays"), and analyzed convergence for certain classes of algorithms under these two models. However, these algorithm classes do not cover many modern optimization algorithms, such AsySPA [2], PIAG [7], Async-BCD [8], Arock [9], and Asynchronous SGD [10].

In the last decade, the convergence of many modern algorithms is established under partial asynchrony, e.g., AsySPA [2], PIAG [7], [11], [12], Async-BCD [8], [13], [14], ARock [9], [15], and Asynchronous SGD [10]. However, only a few algorithms are shown to work under total asynchrony [16]–[18]. In particular, [16], [17] study Asynchronous SGD and [18] focuses on a delay-tolerant averaged proximal gradient algorithm. None of these papers cover total asynchrony for PIAG and Async-BCD, which are the focus of our work. Moreover, different from us, the works [16]–[18] do not characterize how the unbounded delay affects convergence rates or explores the existence of an optimal rate.

This paper studies two asynchronous optimization algorithms, PIAG and Async-BCD, under total asynchrony. None of these algorithms has been proven to converge under total asynchrony before. We make the following contributions:

- We derive explicit convergence rates for PIAG and Async-BCD under a model of total asynchrony.
- We prove that the derived convergence rates for the two methods are optimal in terms of order.
- We use the convergence bounds to provide insight and understanding of how the growth rate of delays slows down the convergence of PIAG and Async-BCD.

Notation and Preliminaries

We use \mathbb{N} and \mathbb{N}_0 to denote the set of natural numbers and the set of natural numbers including zero, respectively. We let $[m] = \{1, \ldots, m\}$ for any $m \in \mathbb{N}$ and represent $x \in \mathbb{R}^d$ as $x = (x^{(1)}, \ldots, x^{(m)})$, where each $x^{(i)} \in \mathbb{R}^{d^{(i)}}$ and $\sum_{i=1}^n d^{(i)} = d$. We define the proximal operator of a function $r : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ as

$$\operatorname{prox}_{r}(x) = \operatorname*{argmin}_{y \in \mathbb{R}^{d}} r(y) + \frac{1}{2} \|y - x\|^{2}.$$

We say a differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is L-smooth if

$$\|\nabla f(x) - \nabla f(x+h)\| \le L \|h\|, \ \forall x, h \in \mathbb{R}^d.$$

We call f is \hat{L} -block-wise smooth with respect to a partition $x = (x^{(1)}, \ldots, x^{(m)})$ if for all $i, j \in [m]$, and $h^{(j)} \in \mathbb{R}^{d^{(j)}}$,

$$\|\nabla_i f(x + U^{(j)} h^{(j)}) - \nabla_i f(x)\| \le \hat{L} \|h^{(j)}\|.$$
(1)

Here, $\nabla_j f(\cdot)$ is the partial gradient of f with respect to the jth block and $U^{(j)} : \mathbb{R}^{d^{(j)}} \to \mathbb{R}^d$ maps $h^{(j)} \in \mathbb{R}^{d^{(j)}}$ into a d-dimensional vector where the jth block is $h^{(j)}$ and other blocks are zero. For an L-smooth function f and a convex function $r : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we say P(x) = f(x) + r(x) satisfies the proximal PL condition [19] for $\sigma > 0$ if

$$\sigma(P(x) - P^{\star}) \le -L\hat{P}(x), \ \forall x \in \operatorname{dom}(P),$$
(2)

where $P^{\star} = \min_{x \in \mathbb{R}^d} P(x)$ and

$$\hat{P}(x) = \min_{y \in \mathbb{R}^d} \{ \langle \nabla f(x), y - x \rangle + \frac{L}{2} \| y - x \|^2 + r(y) - r(x) \}.$$

X. Wu and M. Johansson are with the Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden. Email: {xuyangw,mikaelj}@kth.se.

S. Magnússon is with the Department of Computer and System Science, Stockholm University, SE-164 07 Stockholm, Sweden. Email: sindri.magnusson@dsv.su.se.

H. R. Feyzmahdavian is with ABB Cooperate Research, SE-721 78 Västerås, Sweden. Email: hamid.feyzmahdavian@se.abb.com

This work was supported in part by the funding from Digital Futures and in part by the Swedish Research Council (Vetenskapsrådet) under grant 2020-03607.

II. PROBLEM STATEMENT

We focus on optimization problems on the form

$$\min_{x \in \mathbb{R}^d} P(x) = f(x) + r(x), \tag{3}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is smooth and possibly non-convex, and $r: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex but possibly nondifferentiable. Such a composite structure is common in, for example, machine learning where f is a loss and r is a regularizer. The function r can also represent the indicator function of a convex set.

We consider the proximal incremental aggregated gradient (PIAG) algorithm [7] and the asynchronous block-coordinate descent (Async-BCD) method [8] to solve (3).

A. PIAG

The algorithm solves problem (3) for f on the form

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} f^{(i)}(x)$$

using the following update

$$g_k = \frac{1}{n} \sum_{i=1}^n \nabla f^{(i)}(x_{k-\tau_k^{(i)}}), \qquad (4)$$

$$x_{k+1} = \operatorname{prox}_{\gamma_k r}(x_k - \gamma_k g_k), \tag{5}$$

where k is the iteration index, $\tau_k^{(i)} \in [0,k]$ is the delay of the gradient $\nabla f^{(i)}$ at iteration k, and $\gamma_k \ge 0$ is the step-size. The update (4)–(5) is often implemented in the parameter server architecture [20], where each worker i computes $\nabla f^{(i)}(x_{k-\tau^{(i)}})$ and the master aggregates all the most recent local gradients to form (4) and updates the iterate using (5). The implementation of PIAG is detailed in Algorithm 1.

B. Async-BCD

Suppose that the non-smooth function r in problem (3) is separable, i.e., for a partition $x = (x^{(1)}, \ldots, x^{(m)})$ with $x^{(i)} \in \mathbb{R}^{d^{(i)}}$ and $\sum_{i=1}^{m} d^{(i)} = d$, it holds that $r(x) = \sum_{i=1}^{m} r^{(i)}(x^{(i)}) \quad \forall x \in \mathbb{R}^d$. When the dimension d of x is large, one attractive method for solving (3) is the blockcoordinate descent (BCD) method: at each $k \in \mathbb{N}_0$, randomly choose $j \in [m]$ and execute the update

$$x_{k+1}^{(j)} = \operatorname{prox}_{\gamma_k r^{(j)}} (x_k^{(j)} - \gamma_k \nabla_j f(x_k)),$$

where $\nabla_i f(\cdot)$ is the partial gradient of f with respect to the *j*th block $x^{(j)}$ and $\gamma_k \ge 0$ is the step-size. Async-BCD implements BCD using multiple processors in a shared memory setting [9]. The decision vector is stored in shared memory and at each iteration k, one worker $i_k \in [n]$ updates

$$x_{k+1}^{(j)} = \operatorname{prox}_{\gamma_k r^{(j)}} (x_k^{(j)} - \gamma_k \nabla_j f(x_{k-\tau_k})).$$
(6)

Here, $x_{k-\tau_k}$ is the decision vector that worker i_k has read from shared memory and based its partial gradient computation on. The delay τ_k measures the number of updates that other processors have performed between the read and write operations of worker i_k . The block index j is drawn by i_k uniformly at random at time $k - \tau_k$. Algorithm 2 details the implementation of Async-BCD.

Algorithm 1 PIAG [7], [11]

1: **Input**: initial iterate x_0 , number of iteration $k_{\max} \in \mathbb{N}$.

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2: Initialization:
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- 3: The master sets $k \leftarrow 0$, $g^{(i)} \leftarrow \nabla f^{(i)}(x_0) \ \forall i \in [n]$, and $g_0 \leftarrow \frac{1}{n} \sum_{i=1}^n \nabla f^{(i)}(x_0).$
- 4: while $k \leq k_{\text{max}}$: each worker $i \in [n]$ asynchronously and continuously do
- receive x_k from the master. 5:
- compute $\nabla f^{(i)}(x_k)$. 6:
- send $\nabla f^{(i)}(x_k)$ to the master. 7:
- 8: end while
- while $k \leq k_{\text{max}}$: the master **do** 9:
- Wait until a set \mathcal{R} of workers return. 10:
- for all $w \in \mathcal{R}$ do 11:
- update $g^{(w)} \leftarrow \nabla f^{(w)}(x_l)$. 12:
- end for 13:
- set $g_k \leftarrow \frac{1}{n} \sum_{i=1}^n g^{(i)}$. 14:
- determine the step-size γ_k . 15:
- update $x_{k+1} \leftarrow \operatorname{prox}_{\gamma_k r}(x_k \gamma_k g_k)$. 16:
- 17: set $k \leftarrow k+1$.
- for all $w \in \mathcal{R}$ do 18:
- 19: send x_k to worker w.
- end for 20
- 21: end while

Algorithm 2 Async-BCD

- 1: Setup: initial iterate x_0 , number of iteration $k_{\max} \in \mathbb{N}$.
- 2: while $k \leq k_{\text{max}}$: each worker $i \in [n]$ asynchronously and continuously do
- sample $j \in [m]$ uniformly at random. 3:
- 4: compute $\nabla_i f(\hat{x}_k)$ based on \hat{x}_k read at time $k - \tau_k$.
- 5:
- determine the step-size γ_k . compute $x_{k+1}^{(j)}$ by (6). 6:
- write on the shared memory. 7:
- 8: set $k \leftarrow k+1$.
- read x_k from the shared memory. 9:

10: end while

III. MAIN RESULT

In this section, we will derive convergence results for PIAG and Async-BCD under a totally asynchronous delay model. In our setting, the totally asynchronous model of Bertsekas and Tsitsiklis [21] would allow the delays to grow unbounded, as long as no processor ceases to update and

$$\lim_{k \to +\infty} k - \tau_k = +\infty, \tag{7}$$

where $\tau_k = \max_{i \in [n]} \tau_k^{(i)}$ for PIAG. Convergence *rate* results under this model are unlikely since it does not impose any strict bound on how quickly the delays can grow. We thus focus on a particular model of asynchrony that satisfies (7).

Assumption 1: For some $a \in (0, 1)$, $b \in [0, 1]$, and $c \ge 0$,

$$\tau_k \leq \min(k, ak^b + c), \ \forall k \in \mathbb{N}_0.$$

Clearly, Assumption 1 guarantees (7). Moreover, by varying $b \in [0, 1]$ we can move seamlessly between several interesting and important models of asynchrony. In particular,

- b = 0 yields bounded delays: $\tau_k \leq \min(k, a + c)$.
- b = 1/2 is sublinear growth: $\tau_k \le \min(k, a\sqrt{k} + c)$.
- b = 1 is a linear delay bound: $\tau_k \leq \min(k, ak + c)$.

The linearly growing delay bound is the largest polynomial growth we can have because when b > 1, the total asynchrony condition (7) no longer holds. Our analysis may be extended to other delay models, e.g., those in [15] and [22].

A. PIAG

Let us first present the convergence rate guarantees for PIAG under the delays characterized by Assumption 1. For convenient notation, we introduce

$$\phi(k) = \begin{cases} k^{1-b}, & b \in [0,1), \\ \ln k, & b = 1, \end{cases} \quad \forall k \in \mathbb{N}_0,$$

where b is the delay bound parameter in Assumption 1.

Theorem 1: Suppose that each $f^{(i)}$ is L_i -smooth, r is convex and closed, $P^* := \min_x P(x) > -\infty$, and Assumption 1 holds. Define $L = \sqrt{(1/n) \sum_{i=1}^n L_i^2}$. Let $\{x_k\}$ be generated by the PIAG algorithm with

$$\gamma_k = \frac{h}{L(a(\frac{k+c}{1-a})^b + c + 1)}, \ \forall k \in \mathbb{N}_0,$$
(8)

where $h \in (0, 1)$. Then,

(i) There exist $\xi_k \in \partial r(x_k) \ \forall k \in \mathbb{N}_0$ such that

$$\min_{t \le k} \|\nabla f(x_t) + \xi_t\|^2 = O(1/\phi(k)).$$

(ii) If each $f^{(i)}$ is convex, then

$$P(x_k) - P^* = O(1/\phi(k)).$$

(iii) If P satisfies the proximal PL-condition (2), then

$$P(x_k) - P^* = O(\lambda^{\phi(k)})$$

for some $\lambda \in (0, 1)$.

Proof: See Appendix A.

Table I extracts the relationship between the delay bound, admissible step-size, and convergence rates in Theorem 1.

delay bound	step-size	rate	rate
		(non-convex, convex)	(proximal PL)
$O(k^b), b < 1$	$O(k^{-b})$	$O(1/k^{1-b})$	$O(\lambda^{k^{1-b}})$
$O(k^b), b = 1$	O(1/k)	$O(1/\ln k)$	O(1/k)

TABLE I: asynchrony, step-size, and convergence rate.

Note that when b = 0, which corresponds to bounded delays, the convergence rates in Table I match those of PIAG under partial asynchrony [7], [11], [12], [23] and those of gradient descent [19], [24], i.e., O(1/k) for non-convex and convex objective functions and linear convergence if the proximal PL-condition holds. The table quantifies how large delays limit the admissible step-sizes and deteriorate the convergence rates, which agrees with intuition.

B. Async-BCD

Based on the block-wise smoothness assumption (1), the following theorem establishes convergence rates for Async-BCD in solving problem (3).

Theorem 2: Suppose that f is \hat{L} -block-wise smooth with respect to the partition $x = (x^{(1)}, \ldots, x^{(m)})$, each $r^{(i)}$ is convex and closed, $P^* := \min_x P(x) > -\infty$, and that Assumption 1 holds. Let $\{x_k\}$ be generated by the Async-BCD algorithm with

$$\gamma_k = \frac{h}{\hat{L}(a(\frac{k+c}{1-a})^b + c + 1)}, \ \forall k \in \mathbb{N}_0, \tag{9}$$

where $h \in (0, 1)$. Then,

$$\min_{t \le k} E[\|\tilde{\nabla}P(x_t)\|^2] = O(1/\phi(k)),$$

where $\tilde{\nabla}P(x_t) = \hat{L}(\operatorname{prox}_{\frac{1}{\hat{L}}r}(x_t - \frac{1}{\hat{L}}\nabla f(x_t)) - x_t).$ *Proof:* See Appendix B.

In Theorem 2, $\tilde{\nabla}P(x) = \mathbf{0}$ if and only if $\mathbf{0} \in \partial P(x)$, i.e., x is a stationary point of problem (3). When b = 0, our convergence rate is of the same order compared to Async-BCD under partial asynchrony [13], [14]. The relationship between delay bound, step-size, and convergence rate of Async-BCD is summarized in Table I for non-convex objective functions. Again, a larger delay requires smaller step-sizes and leads to a slower convergence.

C. Optimal convergence rate

The next result establishes that the convergence rates in the preceding theorems are optimal under Assumption 1, and not a consequence of the particular step-size policies.

Theorem 3: The convergence rates for PIAG and Async-BCD derived in Theorems 1–2 are order-optimal.

1) *Proof of Theorem 3:* We prove our claim by constructing an objective function and a delay sequence that satisfy the assumptions of the preceding theorems, and are such that the proposed rates are optimal.

Let $r \equiv 0$ and let f be L-smooth for some L > 0. Then, both PIAG and Async-BCD reduce to

$$x_{k+1} = x_k - \gamma_k \nabla f(x_{k-\tau_k}). \tag{10}$$

Now, consider the delay sequence

$$\tau_k = k - T_t, \quad \text{if } k \in [T_t, T_{t+1}),$$
 (11)

where $\{T_t\}$ is defined by $T_0 = 0$ and

$$T_{t+1} = \max\{\kappa \in \mathbb{N}_0 : \kappa - (a\kappa^b + c) \le T_t\} + 1 \quad (12)$$

for some $a \in (0, 1)$, $b \in [0, 1]$, and $c \ge 0$. In this way, for any $k \in [T_t, T_{t+1})$, it holds that $k - (ak^b + c) \le T_t$. By (11), $\tau_k \le ak^b + c$ and $\tau_k \le k$, so $\{\tau_k\}$ satisfies Assumption 1.

By substituting (11) into (10), we obtain

$$x_{k+1} = x_{T_t} - (\sum_{\ell=T_t}^k \gamma_\ell) \nabla f(x_{T_t}), \quad \forall k \in [T_t, T_{t+1}).$$

This implies that x_k , $k \in \mathbb{N}$ is obtained by performing $\max\{t \in \mathbb{N}_0 : T_t \leq k - 1\} + 1$ steps of gradient descent starting from x_0 . Moreover, we prove in Appendix C that

$$\max\{t \in \mathbb{N}_0: \ T_t \le k - 1\} + 1 = O(\phi(k)).$$
(13)

The result now follows by observing that after $O(\phi(k))$ steps of gradient descent on a general *L*-smooth function or a general *L*-smooth and proximal PL function, we cannot obtain better order of convergence [19], [25] than those in Theorems 1–2.

IV. NUMERICAL EXPERIMENTS

We demonstrate the theoretical results in Theorems 1-2 and evaluate the practical performance of the two methods under total asynchrony in simulations. We consider a binary classification problem on the training data set of RCV1 [26] using the regularized logistic regression model:

$$f(x) = \frac{1}{N} \sum_{i=1}^{N} \left(\log(1 + e^{-p_i(q_i^T x)}) + \frac{\lambda_2}{2} \|x\|^2 \right),$$

$$r(x) = \lambda_1 \|x\|_1,$$

where p_i is the feature of the *i*th sample, q_i is the corresponding label, and N is the number of samples. We use $(\lambda_1, \lambda_2) = (10^{-5}, 10^{-4})$ in all simulations.

A. PIAG

We split the N samples into n = 10 batches and assign each batch to a single worker. We consider the following delay model: $\tau_0^{(i)} = 0$ for all $i \in [n]$. For all $k \in \mathbb{N}$ and $i \in [n]$, if $\tau_{k-1}^{(i)} \leq \min(k, ak^b + c) - 1$, then $\tau_k^{(i)} = \tau_{k-1}^{(i)} + 1$; Otherwise, $\tau_k^{(i)}$ is randomly drawn from $[\min(k, \lfloor ak^b + c \rfloor)]$. We use a = 0.1 and c = 0, and consider b = 0.2, 0.6 and 1 to evaluate the effect of delays. Note that the constructed delay sequence satisfies Assumption 1.

We plot the objective error $P(x_k) - P^*$ generated by PIAG in Fig 1(a), and the theoretical bound $O(1/\phi(k))$ in Theorem 1 for the convex objective functions in Fig 1(b), where the exact value of the bound is obtained by substituting (16) into (15). Although the rate $O(\lambda^{\phi(k)})$ in Theorem 1 also holds since the objective function satisfies the proximal PL-condition, it is quite slow because the parameter λ_2 is small. Observe from Fig 1(a) that PIAG tends to converge for all three b's, and the convergence speed deteriorates as b increases. These demonstrate Theorem 1. Through comparison between Fig 1(a)-1(b) that are with different y-scale, although the theoretical bound is much larger than the practical objective error for all three values of b, their decreasing speed are similar.

B. Async-BCD

We use n = 8 processors and split the decision vector x evenly into m = 14 blocks. We set $\tau_0 = 0$. For all $k \in \mathbb{N}$, $\tau_k = \tau_{k-1} + 1$ if $\tau_{k-1} + 1 \leq \min(k, \lfloor ak^b + c \rfloor)$ and is randomly drawn from $[\min(k, \lfloor ak^b + c \rfloor)]$ otherwise. Like above, we set a = 0.1 and c = 0, and consider b = 0.2, 0.6, and 1. The resulting delay sequence satisfies Assumption 1.



Fig. 1: Convergence of PIAG

Fig 2 plots the convergence of objective error $P(x_k) - P^*$ generated by Async-BCD. We observe that for small value 0.2 of b, the convergence to optimum is clear and fast and for larger values 0.6 and 1, the convergence to optimum is hard to observe which is normal because the delays corresponding to b = 0.6 and b = 1 increase too fast and coordinate-type methods often converge slowly in terms of iteration number.

V. CONCLUSION

We have derived explicit convergence rates of PIAG and Async-BCD under a model of computation that allows for a broad range of totally asynchronous behaviours. The convergence rates are optimal in terms of the order of iteration index k and reflect how asynchrony affects the convergence times of the algorithms. The theoretical results were validated in simulations. We believe that the proposed techniques apply also to other asynchronous optimization algorithms, but leave such studies for future work.



Fig. 2: Convergence of Async-BCD

APPENDIX

A. Proof of Theorem 1

The proof uses [23, Theorem 2].

Theorem 4 ([23]): Under the conditions in Theorem 1, if for some $h \in (0, 1)$,

$$\sum_{t=k-\tau_k}^k \gamma_t \le \frac{h}{L},\tag{14}$$

then

(1) There exist $\xi_k \in \partial r(x_k) \ \forall k \in \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} \gamma_{k-1} \|\nabla f(x_k) + \xi_k\|^2 \le \frac{2(h^2 - h + 1)(P(x_0) - P^{\star})}{1 - h}.$$

(2) If each $f^{(i)}$ is convex, then

$$P(x_k) - P^* \le \frac{P(x_0) - P^* + \frac{1}{2a_0} \|x_0 - x^*\|^2}{1 + \frac{1}{a_0} \sum_{t=0}^{k-1} \gamma_t}, \quad (15)$$

where $a_0 = \frac{h(h+1)}{L(1-h)}$. (3) If *P* satisfies the proximal PL-condition (2), then

$$P(x_k) - P(x^{\star}) \le e^{-\frac{3\beta\sigma(1-\bar{h})}{4(\bar{h}^2 - \bar{h} + 1)}\sum_{t=0}^{k-1}\gamma_t} (P(x_0) - P^{\star}),$$

where $\tilde{h} = \frac{1+h}{2}$ and $\beta = \min(1, \frac{1-h}{2h}\frac{L}{\sigma})$. To prove Theorem 1 using Theorem 4, we first show that

 $\{\gamma_k\}$ in (8) satisfies (14). Because $\tau_k \leq ak^b + c \leq ak + c$, for any $t \in [k - \tau_k, k]$, we have $t \ge k - \tau_k \ge (1 - a)k - c$, so that

$$\gamma_t = \frac{h}{L(a(\frac{t+c}{1-a})^b + c + 1)} \le \frac{h}{L(ak^b + c + 1)}$$

Using the above equation and $\tau_k \leq ak^b + c$, we have

$$\sum_{t=k-\tau_k}^k \gamma_t \le \frac{(\tau_k+1)h}{L(ak^b+c+1)} \le \frac{h}{L},$$

i.e., (14) holds.

Next, we show that $1/\sum_{t=0}^k \gamma_t = O(1/\phi(k)).$ For any $t \in \mathbb{N},$

$$\gamma_t = \frac{h}{L(a(\frac{t+c}{1-a})^b + c + 1)}$$

= $\frac{h(t+c)^{-b}}{L(a(1-a)^{-b} + (c+1)(t+c)^{-b})}$
\ge $\frac{h(t+c)^{-b}}{L(a(1-a)^{-b} + (c+1)^{1-b})}.$

In addition,
$$\gamma_0 = \frac{h}{L(a(\frac{c}{1-a})^b + c + 1)}$$
. Then, for any $k \in \mathbb{N}$,

$$\sum_{t=0}^{k-1} \gamma_t \ge \gamma_0 + \sum_{t=1}^{k-1} \gamma_t$$

$$\ge \gamma_0 + \int_1^k \frac{h(s+c)^{-b}}{L(a(1-a)^{-b} + (c+1)^{1-b})} ds$$

$$= \frac{h}{L(a(\frac{c}{1-a})^b + c + 1)}$$

$$+ \begin{cases} \frac{h((k+c)^{1-b} - (1+c)^{1-b})}{L(a(1-a)^{-b} + (c+1)^{1-b})(1-b)}, & b \in [0,1), \\ \frac{h\ln(\frac{k+c}{1+c})}{L(a/(1-a) + 1)}, & b = 1, \end{cases}$$
(16)

which indicates $1/\sum_{t=0}^{k-1} \gamma_t = O(1/\phi(k))$. Hence, the results in Theorem 1 for convex and proximal PL functions hold.

Also note that the result in Theorem 4 for non-convex objective functions implies

$$(\sum_{t=0}^{k-1} \gamma_t) \min_{t \le k} \|\nabla f(x_t) + \xi_t\|^2$$

$$\leq \sum_{t=1}^k \gamma_{t-1} \|\nabla f(x_t) + \xi_t\|^2$$

$$\leq \sum_{t=1}^\infty \gamma_{t-1} \|\nabla f(x_t) + \xi_t\|^2$$

$$\leq \frac{2(h^2 - h + 1)(P(x_0) - P^*)}{1 - h}.$$
(17)

By substituting (16) into (17), we obtain the result in Theorem 1 for non-convex objective functions.

B. Proof of Theorem 2

The proof uses [23, Theorem 3].

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Theorem 5 ([23]): Under the conditions in Theorem 2, if for some $h \in (0, 1)$,

$$\sum_{=k-\tau_k}^k \gamma_t \le \frac{h}{\hat{L}}, \quad \forall k \in \mathbb{N}_0,$$

then

$$\sum_{k=0}^{\infty} \gamma_k E[\|\tilde{\nabla}P(x_k)\|^2] \le \frac{4m(P(x_0) - P^{\star})}{1 - h}.$$

Using similar derivation of (17) in Appendix A, we have

$$\min_{t \le k} E[\|\tilde{\nabla}P(x_t)\|^2] \le \frac{4m(P(x_0) - P^*)}{(1-h)\sum_{t=0}^{k-1} \gamma_t}.$$

Moreover, similar to the derivation of (16), $1/\sum_{t=0}^{k} \gamma_t =$ $O(1/\phi(k))$. Then, we obtain the result.

C. Proof of (13)

By (12), for any $\kappa > T_{t+1}-1$, we have $\kappa - (a\kappa^b + c) > T_t$. Therefore,

$$T_{t+1} \ge T_t + aT_{t+1}^b + c$$

$$\ge T_t + aT_t^b + c,$$
(18)

where the second step uses $T_{t+1} \ge T_t$ derived from the first step.

Case 1: $b \in [0,1)$. The proof uses induction. Let $\eta = a(1-b)2^{-\frac{b}{1-b}}$. Suppose that $T_t \ge (\eta t)^{\frac{1}{1-b}}$ for some $t \in \mathbb{N}_0$, which holds naturally at t = 0, 1. Then, by (18),

$$T_{t+1} - (\eta(t+1))^{\frac{1}{1-b}}$$

$$\geq (\eta t)^{\frac{1}{1-b}} + a(\eta t)^{\frac{b}{1-b}} + c - (\eta(t+1))^{\frac{1}{1-b}}$$

$$= (\eta t)^{\frac{1}{1-b}} \underbrace{\left(1 + \frac{a}{\eta t} - (1 + \frac{1}{t})^{\frac{1}{1-b}}\right)}_{v(t)} + c.$$
(19)

Note that $v'(t) = -\frac{1}{t^2} \left(\frac{a}{\eta} - \frac{1}{1-b} \left(1 + \frac{1}{t}\right)^{\frac{b}{1-b}}\right)$, which satisfies $v'(t) \leq -\frac{1}{t^2} \left(\frac{a}{\eta} - \frac{1}{1-b} 2^{\frac{b}{1-b}}\right) = 0$ when $t \geq 1$. Hence, v(t) is monotonically decreasing on $[1, +\infty)$ and $v(t) \geq \lim_{\ell \to +\infty} v(\ell) = 0$ for all $t \geq 1$, which, together with (19), gives $T_{t+1} \geq (\eta(t+1))^{\frac{1}{1-b}}$. Hence, $t \leq \frac{(T_t)^{1-b}}{\eta}$ for all $t \in \mathbb{N}_0$ and $\max\{t \in \mathbb{N}_0 : T_t \leq k-1\} \leq \frac{(k-1)^{1-b}}{\eta}$, i.e., $\max\{t \in \mathbb{N}_0 : T_t \leq k-1\} + 1 = O(k^{1-b})$.

Case 2: b = 1. By (18),

$$T_{t+1} \ge (1+a)T_t \ge (1+a)^t T_1 \ge (1+a)^t$$

so that $t \leq \ln \frac{T_t}{1+a} + 1$. Hence, $\max\{t \in \mathbb{N}_0 : T_t \leq k-1\} \leq \ln \frac{k-1}{1+a} + 1$, i.e., $\max\{t \in \mathbb{N}_0 : T_t \leq k-1\} + 1 = O(\ln k)$.

Concluding the two cases, we complete the proof.

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