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Differentially Private Distributed Mismatch Tracking Algorithm for Constraint-Coupled Resource Allocation Problems

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Abstract-This paper considers privacy-concerned distributed constraint-coupled resource allocation problems over an undirected network, where each agent holds a private cost function and obtains the solution via only local communication. With privacy concerns, we mask the exchanged information with independent Laplace noise against a potential attacker with potential access to all network communications. We propose a differentially private distributed mismatch tracking algorithm (diff-DMAC) to achieve cost-optimal distribution of resources while preserving privacy. Adopting constant stepsizes, the linear convergence property of diff-DMAC in mean square is established under the standard assumptions of Lipschitz gradients and strong convexity. Moreover, it is theoretically proven that the proposed algorithm is ϵ -differentially private. And we also show the trade-off between convergence accuracy and privacy level. Finally, a numerical example is provided for verification.

I. INTRODUCTION

Distributed resource allocation problem (DRAP) has received much attention due to its wide applications in various domains, including smart grids [1], wireless networks [2] and robot networks [3]. The goal of DRAP is to achieve the cost-optimal distribution of limited resources among users to meet their demands, local constraints, and possibly certain coupled global constraints, see [4] for example. Conventional centralized approaches are subject to performance limitations of the central entity, such as a single point of failure, limited scalability. Moreover, it may raise privacy concerns when the central agent is not reliable enough. Alternatively, distributed approaches which operate with only local information have better robustness and scalability, especially for large-scale systems [5]–[7].

To implement algorithms in a distributed manner, the information exchange between agents in the network is unavoidable which can raise concerns about privacy disclosure. For example, in the supply-demand optimization of the power grid, attackers can infer users' life patterns from certain published information that is computed based on the demand from users [8]. While the aforementioned works consider an ambitious suite of topics under various constraints imposed by real-world applications, privacy is an aspect generally absent in their treatments.

Recently, privacy preservation becomes an increasingly critical issue for distributed systems. Encryption-based algorithms are proposed for distributed optimization problems [9], [10] and proved to be effective. However, the encryption technique leads to a high computational complexity which seems undesirable in large-scale networks. To preserve the privacy of the agents' costs, an asynchronous heterogeneous-stepsize algorithm is proposed in [11], but the method assumes that the infomation of communication topology is unavailable to attackers.

Another thread to address distributed optimization problems is to adopt differential privacy technique which is robust to arbitrary auxiliary information exposed to attackers, including communication topology information, thus well suited for multi-agents scenarios. Motivated by the privacy concerns in EV charging networks, the work in [8] presents a differentially private distributed algorithm to solve distributed optimization problem, but an extra central agent is needed. In [12], the authors design a completely distributed algorithm that guarantees differential privacy by perturbing the states with Laplace noise. It requires the stepsize to be decaying to guarantee the convergence, resulting in a low convergence rate. Adopting the gradient-tracking scheme, a differentially private distributed algorithm is proposed in [13] which has a linear convergence rate with constant stepsizes. However, the proposed algorithm in [13] is only applicable to unconstrained problems. For solving the economic dispatch problem where both global and local constraints are considered, a privacy-preserving distributed optimization algorithm is proposed in [14] for quadratic cost functions and its effctiveness in preserving privacy is guaranteed through qualitative analysis. The extension to general DRAPs is performed in [15] for strongly convex and Lipschitz smooth cost functions. Additionally, quantitative analysis of privacy is provided. However, the work only considers scalar cost functions and the coupling within constraints is ignored.

In this paper, a differentially private distributed mismatch tracking algorithm (diff-DMAC) is proposed to solve constraint-coupled DRAPs with individual constraints and a global linear coupling constraint while preserving the privacy of the local cost functions. To meet the global coupledconstraint, a mismatch tracking step is introduced to obtain the supply-demand mismatch in the network. And the tracked global mismatch is then implemented for error compensation. With a constant stepsize, diff-DMAC has a provable linear convergence rate for strongly convex and Lipschitz smooth

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cost functions. We further present a rigorous analysis of the algorithm's differential privacy and thus provide a stronger privacy guarantee. Finally, its convergence properties and effectiveness in privacy-preserving are numerically validated.

The rest of the paper is organized as follows. Section II formulates the constraint-coupled DRAP. The proposed algorithm is developed in Section III. And the theoretical analyses about its convergence and privacy are given in Section IV and V, respectively. Then the algorithm is numerically tested in Section VI. Finally, Section VII concludes the paper.

Notation: Vectors default to columns if not otherwise specified. Bold letter $\mathbf{x} \in \mathbb{R}^{n \times p}$ is defined as $\mathbf{x} = [x_1^{\mathsf{T}}, \cdots, x_n^{\mathsf{T}}]^{\mathsf{T}}$. The Kronecker product is denoted by \otimes . Let $\mathbf{1}_n$ be the *n*-dimension vector with all one entries. For vectors, we use $\|\cdot\|$ to denote the 2-norm. And for matrices, $\|\cdot\|$ denotes the spectral norm. $\underline{\lambda}(X)$ denotes the minimum eigenvalue of matrix X. We use $blkdiag(X_1, \cdots, X_n)$ to refer to the block-diagonal matrix with X_1, \cdots, X_n as blocks. For a random variable $x \in \mathbb{R}$, $Lap(\theta)$ denotes the Laplace distribution with pdf $f = (x|\theta) = \frac{1}{2\theta}e^{-\frac{|x|}{\theta}}$, where $\theta > 0$. If $x \sim Lap(\theta)$, we have $\mathbb{E}[|x|] = \theta$, $\mathbb{E}[x^2] = 2\theta^2$. $\mathbb{P}(\cdot)$ is used to denote the probability. $\mathbb{B}(S)$ denotes the set of Borel subsets of topological space S.

II. PROBLEM FORMULATION

We consider the following general constraint-coupled distributed resource allocation problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^{n_p}} f(\boldsymbol{x}) = \sum_{i=1}^n f_i(x_i)$$

s.t. $\sum_{i=1}^n A_i x_i = \sum_{i=1}^n d_i, \ x_i \in \mathcal{X}_i, \forall i,$ (1)

where $f_i : \mathbb{R}^p \to \mathbb{R}$ is the agent *i*'s private cost function, $x_i \in \mathbb{R}^p$ is the decision vector of agent *i* and d_i denotes the local resource demand. \mathcal{X}_i is a local convex and closed set which encodes local constraints of agent *i*. $A_i \in \mathbb{R}^{m \times p}$ $(p \ge m)$ is the coupling matrix with full row rank and $A_i^{\mathsf{T}}A_i$ is invertible, i.e., $\lambda(A_i^{\mathsf{T}}A_i) > 0$.

Many practical problems take the form of problem (1), e.g., [1], [16]. One typical example is the economic dispatch of multi-microgrid (multi-MG) systems as shown in Fig. 1, where both microturbines and non-dispatchable distributed generators are involved to provide energy to meet the loads, and the outputs of non-dispatchable distributed generators can fluctuate significantly [16]. When the supply-demand balance in the MG cannot be maintained, the coordination among MGs is needed and both within-MG, between-MG optimization problems should be considered to improve systems' reliability [17].

Assumption 1: Each cost function $f_i : \mathbb{R}^p \to \mathbb{R}$ is strongly convex and Lipschitz smooth, i.e.,

$$\|\nabla f_i(x) - \nabla f_i(x')\| \le L_i \|x - x'\|$$
 (2)

and

$$(x - x')^{\mathsf{T}} (\nabla f_i(x) - \nabla f_i(x')) \ge \varphi_i ||x - x'||^2$$
 (3)

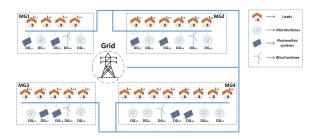


Fig. 1. Multi-microgrid systems with distributed generators.

 $\forall x, x' \in \mathbb{R}^p$, where $\varphi_i > 0$ and $L_i < \infty$ are the strong convexity and Lipschitz constants, respectively. Define $\underline{\varphi} = \min_i \{\varphi_i\}$ and $\overline{L} = \max_i \{L_i\}$.

The communication network over which agents exchange information can be represented by an undirected graph $\mathcal{G} =$ $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \dots, n\}$ is the set of agents and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ denotes the set of edges, accompanied with a nonnegative weighted matrix \mathcal{W} . For any $i, j \in \mathcal{N}$ in the network, $w_{ij} > 0$ denotes agent j can exchange information with agent i. The collection of all individual agents that agent i can communicate with is defined as its neighbors set \mathcal{N}_i .

Assumption 2: The graph \mathcal{G} is undirected and connected and the weight matrix $\mathcal{W} \in \mathbb{R}^{n \times n}$ is doubly stochastic, i.e., $\mathcal{W}\mathbf{1}_n = \mathbf{1}_n$ and $\mathbf{1}_n^{\mathsf{T}} \mathcal{W} = \mathbf{1}_n^{\mathsf{T}}$.

Assumptions 1-2 are standard when solving related problems.

In this problem, agents want to achieve the optimum while preserving the privacy of local cost functions which can be commercially sensitive in practice. To preserve privacy of functions, we add noise ζ to mask the exchanged information $\mathbf{x}(k)$ at each time instant as $\mathbf{z}(k) = \mathbf{x}(k) + \zeta(k)$. Let \mathcal{F} be the function set $\{f_i\}_{i=1}^n$, the update of \mathbf{x} can be expressed in impact form

$$\mathbf{x}(k+1) = g(\mathbf{x}(k), \boldsymbol{\zeta}(k), \mathcal{W}, \mathcal{F}),$$

where $g(\cdot)$ denotes the update rule. Therefore, exchanged information $\mathbf{z} = {\mathbf{z}(k)}_{k=0}^{\infty}$ can be determined if the initialization $\mathbf{x}(0)$, the added noises $\boldsymbol{\zeta} = {\{\boldsymbol{\zeta}(k)\}_{k=0}^{\infty}}$, the communication topology \mathcal{W} and the function set \mathcal{F} are known. To provide theoretical guarantees on the privacy for their cost functions, the concept of differential privacy is introduced, see [15].

Definition 1: Given $\delta > 0$ and two function sets $\mathcal{F}^{(1)} = \{f_i^{(1)}\}_{i=1}^n$ and $\mathcal{F}^{(2)} = \{f_i^{(2)}\}_{i=1}^n$, the two subsets are δ -adjacent if there exists $\delta' \in (-\delta, \delta)$ and some i_0 such that

$$f_i^{(1)} = f_i^{(2)}, \forall i \neq i_0; \nabla f_{i_0}^{(1)}(x) = \nabla f_{i_0}^{(2)}(x + \delta'), \forall x \in \mathcal{X}_{i_0}^{(1)},$$

where $\mathcal{X}_{i_0}^{(l)}$ is the domain of $f_{i_0}^{(l)}, l = 1, 2$ and they satisfy that $\mathcal{X}_{i_0}^{(2)} = \{x + \delta' | x \in \mathcal{X}_{i_0}^{(1)}\}.$ Definition 2: Given $\delta, \epsilon > 0$, for any two δ -adjacent function

Definition 2: Given $\delta, \epsilon > 0$, for any two δ -adjacent function sets $\mathcal{F}^{(1)}, \mathcal{F}^{(2)}$ and any observation $\mathcal{O} \subset \mathbb{B}((\mathbb{R}^{nm})^{\mathbb{N}})$, the process keeps ϵ -differentially private if

$$\mathbb{P}\{\boldsymbol{\zeta} \in \Omega | Z_{T^{(1)}}(\boldsymbol{\zeta}) \in \mathcal{O}\} \le e^{\epsilon} \mathbb{P}\{\boldsymbol{\zeta} \in \Omega | Z_{T^{(2)}}(\boldsymbol{\zeta}) \in \mathcal{O}\},\$$

where we define $T^{(l)} = \{x(0), \mathcal{W}, \mathcal{F}^{(l)}\}, Z_{T^{(1)}}(\boldsymbol{\zeta}) = \mathbf{z}$ and the observation \mathcal{O} encodes all the information collected by the potential attacker.

The attacker is assumed to know all auxiliary information, including the exchanged information between agents, communication topology, etc., except the parameter δ' . Intuitively, differential privacy guarantees that the two function sets can not be distinguished through the released information and thus avoids the leakage of the optimal operating point.

III. ALGORITHM DEVELOPMENT

In this section, a distributed algorithm is designed to solve constraint-coupled DRAPs with privacy concerns. The following Proposition is presented to give the optimal conditions for constraint-coupled DRAP.

Proposition 1: $x^{\star} = [x_1^{\star \intercal}, \cdots, x_n^{\star \intercal}] \in \mathbb{R}^{n \times p}$ is the optimal solution of (1) if it satisfies

i) $\sum_{i=1}^{n} A_i x_i^{\star} = \sum_{i=1}^{n} d_i$, ii) there exists $\mu^{\star} \in \mathbb{R}^m$ such that

$$x = \arg\min_{\mathbf{z}\in\mathcal{X}} \{f(\mathbf{z}) - (\mathbf{1}_n \otimes \mu^*)^\mathsf{T} \mathbf{A} \mathbf{z}\},\tag{4}$$

where \mathcal{X} is defined as the Cartesian product $\mathcal{X} = \mathcal{X}_1 \times \cdots \times$ \mathcal{X}_n and the matrix $\mathbf{A} = blkdiag(A_1, \cdots, A_n)$.

Proof: i) is one of the feasible conditions of (1) which must be satisfied. Problem (4) can be decomposed into nsubproblems $x_i^{\star} = \arg \min_{z \in \mathcal{X}_i} \{ f_i(z) - \mu^{\star \mathsf{T}} A_i z \}, \forall i =$ $1, \cdots, n$. It follows that

$$f_i(x_i^{\star}) - \mu^{\star \mathsf{T}} A_i x_i^{\star} \leq f_i(x_i) - \mu^{\star \mathsf{T}} A_i x_i, \ \forall x_i \in \mathcal{X}_i.$$

Summing the above inequality from i = 1 to n yields

$$\sum_{i=1}^{n} f_i(x_i^{\star}) \le \sum_{i=1}^{n} f_i(x_i) - \mu^{\star \mathsf{T}} (\sum_{i=1}^{n} A_i x_i - \sum_{i=1}^{n} A_i x_i^{\star}).$$

If \mathbf{x}^{\star} satisfies condition i), $\sum_{i=1}^{n} A_i x_i - \sum_{i=1}^{n} A_i x_i^{\star} = 0$ holds for any feasible solution x of (1). Thus, we have $f(\mathbf{x}^{\star}) < f(\mathbf{x})$ which completes the proof.

In the following, we will use Proposition 1 to develop a distributed algorithm for solving the DRAP (1). Based on the condition ii) of Proposition 1, the x-update is designed and the consensus protocol [18] is adopted to drive μ_i in each agent to same value

$$\boldsymbol{\mu}(k+1) = \mathbf{W}\boldsymbol{\mu}(k),\tag{5}$$

$$\mathbf{x}(k+1) = \arg\min_{\mathbf{z}\in\mathcal{X}} \{f(\mathbf{z}) - \boldsymbol{\mu}^{\mathsf{T}}(k+1)\mathbf{A}\mathbf{z}\}, \qquad (6)$$

where we define $\mathbf{W} = \mathcal{W} \otimes I_m$.

Inspired by the gradient-tracking scheme [19], a tracking step is introduced here to track the global supply-demand mismatch as follows

$$\mathbf{y}(k+1) = \mathbf{W}\mathbf{y}(k) + \mathbf{A}\mathbf{x}(k+1) - \mathbf{A}\mathbf{x}(k).$$
(7)

Under Assumption 2, if the initialization condition $(\mathbf{1}_n \otimes$ I_m)^{T}($\mathbf{Ax}(0) - \mathbf{d}$) = ($\mathbf{1}_n \otimes I_m$)^{T} $\mathbf{y}(0)$ holds, iteration (7) can infer that

$$\sum_{i=1}^{n} y_i(k) = \sum_{i=1}^{n} A_i x_i(k) - \sum_{i=1}^{n} d_i, \ \forall k > 0.$$
 (8)

To guarantee the condition i) of Proposition 1 holds, the tracked mismatch is used as a compensation term for μ update (5). Thus, the proposed algorithm is obtained

$$\boldsymbol{\mu}(k+1) = \mathbf{W}\boldsymbol{\mu}(k) - \alpha \mathbf{y}(k), \tag{9}$$

$$\mathbf{x}(k+1) = \arg\min_{\mathbf{z}\in\mathcal{X}} \{f(\mathbf{z}) - \boldsymbol{\mu}^{\mathsf{T}}(k+1)\mathbf{A}\mathbf{z}\},\qquad(10)$$

$$\mathbf{y}(k+1) = \mathbf{W}\mathbf{y}(k) + \mathbf{A}\mathbf{x}(k+1) - \mathbf{A}\mathbf{x}(k).$$
(11)

The following Lemma shows the fixed point of the proposed algorithm is consistent with the optimal solution of the problem when Assumption 2 holds.

Lemma 1: Under Assumption 2, any fixed point of the proposed algorithm satisfies the optimal conditions of constraintcoupled DRAP.

Proof: It is not difficult to check that any fixed point $(\boldsymbol{\mu}_{F}, \mathbf{x}_{F}, \mathbf{y}_{F})$ of the proposed algorithm satisfies

$$(I_{nm} - \mathbf{W})\boldsymbol{\mu}_F = -\alpha \mathbf{y}_F,\tag{12}$$

$$\mathbf{x}_F = \arg\min_{\mathbf{z}\in\mathcal{X}} \{f(\mathbf{z}) - \boldsymbol{\mu}_F^{\mathsf{T}} \mathbf{A} \mathbf{z}\},\tag{13}$$

$$(I_{nm} - \mathbf{W})\mathbf{y}_F = 0. \tag{14}$$

Under Assumption 2, from (12) and (14), we have

$$(\mathbf{1}_n \otimes I_m)^{\mathsf{T}} \mathbf{y}_F = 0, \ \mathbf{y}_F = \mathbf{1}_n \otimes y_F.$$
 (15)

Thus, $\mathbf{y}_F = 0$ holds and combining it with (12) yields

$$\boldsymbol{\mu}_F = \mathbf{1}_n \otimes \boldsymbol{\mu}_F. \tag{16}$$

Moreover, with proper initialization, from (8) we have

$$(\mathbf{1}_n \otimes I_m)^{\mathsf{T}} (\mathbf{A} \mathbf{x}_F - \mathbf{d}) = (\mathbf{1}_n \otimes I_m)^{\mathsf{T}} \mathbf{y}_F = 0.$$
 (17)

where (13), (16) and (17) correspond to the optimal conditions in Proposition 1 which completes the proof.

With privacy concerns, agents exchange noise-masked information against potential attackers to prevent information disclosure. In [20], Laplace noise is shown to satisfy the necessary and sufficient condition of differential privacy. Furthermore, due to the noise accumulation in the tracking process as $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} A_i x_i - \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} \sum_{t=0}^{n-1} \zeta_i(t)$, the added noise ζ_i needs to decay to ensure the convergence of y_i . Likewise, η_i also needs to decay for the convergence of μ_i and y_i . Thus, the added noises are set to $\eta_i(k) \sim Lap(\theta_{ik}^{\eta}), \zeta_i(k) \sim Lap(\theta_{ik}^{\zeta})$ where $\theta_{ik}^{\eta} = d_{\eta_i} q_i^{\ k}, \theta_{ik}^{\zeta} = d_{\zeta_i} q_i^{\ k}, q_i \in (0, 1)$. That is, two diminishing Laplace noises are added to the communication process of μ_i , y_i , respectively. Both of them are shown to be necessary in Section V.

Based on the mismatch tracking scheme and the information-masked protocol, a differentially private distributed mismatch tracking algorithm (diff-DMAC) is proposed which is shown in Algorithm 1 in details.

IV. CONVERGENCE ANALYSIS FOR DIFF-DMAC

In this section, we will establish the linear convergence rate of the proposed algorithm in mean square. And the upper and lower bounds of the convergence accuracy is also provided. We set $d_i = 0$, $\forall i$ in the following analysis for simplicity while nonzero scenarios can be analyzed similarly.

Algorithm 1 : diff-DMAC Algorithm

- 1: Initialization: $x_i(0)$ and $\mu_i(0)$ are arbitrarily assigned and $y_i(0) = A_i x_i(0) - d_i$.
- 2: for k = 0, 1, ... do
- 3: Adds noise $\eta_i(k), \zeta_i(k)$ to get $z_{\mu_i}(k)$ and $z_{y_i}(k)$ as $z_{\mu_j}(k) = \mu_j(k) + \eta_j(k), z_{y_j}(k) = y_j(k) + \zeta_j(k).$
- 4: Broadcasts $z_{\mu_i}(k)$ and $z_{y_i}(k)$ to $j \in \mathcal{N}_i$.
- 5: Receives $z_{\mu_j}(k)$ and $z_{y_j}(k)$ from $j \in \mathcal{N}_i$
- 6: Updates $\mu_i(k+1)$ through $\mu_i(k+1) = \sum_{j=1}^n w_{ij} z_{\mu_j}(k) - \alpha y_i(k)$ 7: Updates $x_i(k+1)$ through $x_i(k+1) = \arg \min_{z \in \mathcal{X}_i} \{f_i(z) - \mu_i^{\mathsf{T}}(k+1)A_iz\}$ 8: Updates $y_i(k+1)$ through $y_i(k+1) = \sum_{j=1}^n w_{ij} z_{y_j}(k) + A_i x_i(k+1) - A_i x_i(k).$
- 9: end for

Let $\nabla f(x) = [\nabla f_1(x_1)^{\mathsf{T}}, \cdots, \nabla f_n(x_n)^{\mathsf{T}}]^{\mathsf{T}}$ and $\bar{\mathbf{W}} = \frac{\mathbf{1}_n^{\mathsf{T}} \mathbf{1}_n}{n} \otimes I_m$, $\mathbf{\check{W}} = I_{nm} - \bar{\mathbf{W}}$, $\mathbf{\check{W}} = \mathbf{W}\mathbf{\check{W}}$. From (7), (9), we have that

$$\bar{\boldsymbol{\mu}}(k+1) = \bar{\boldsymbol{\mu}}(k) + \bar{\boldsymbol{\eta}}(k) - \alpha \bar{\boldsymbol{y}}(k), \qquad (18)$$

$$\bar{\boldsymbol{y}}(k+1) = \bar{\boldsymbol{y}}(k) + \bar{\boldsymbol{\zeta}}(k) + \bar{\mathbf{W}}\mathbf{A}(\mathbf{x}(k+1) - \mathbf{x}(k)), \quad (19)$$

$$\check{\boldsymbol{\mu}}(k+1) = \tilde{\mathbf{W}}(\check{\boldsymbol{\mu}}(k) + \check{\boldsymbol{\eta}}(k)) - \alpha \check{\boldsymbol{y}}(k),$$
(20)

$$\check{\boldsymbol{y}}(k+1) = \tilde{\mathbf{W}}(\check{\boldsymbol{y}}(k) + \check{\boldsymbol{\zeta}}(k)) + \check{\mathbf{W}}\mathbf{A}(\mathbf{x}(\mathbf{k}+1) - \mathbf{x}(\mathbf{k})),$$
(21)

where $\bar{\mu} = \bar{\mathbf{W}}\mu$, $\check{\mu} = \check{\mathbf{W}}\mu$ and the doubly stochasticity of the weight matrix is used here. Under Assumption 2, as a consequence of the Perron–Frobenius theorem, we have that $\|\mathcal{W} - \frac{\mathbf{1}^{\intercal}\mathbf{1}}{n}\| < 1$, and thus

$$\bar{\lambda} = \|\tilde{\mathbf{W}}\| = \|(\mathcal{W} - \frac{\mathbf{1}^{\intercal}\mathbf{1}}{n}) \otimes I_m\| < 1.$$
(22)

Lemma 2: For positive a_1 , a_2 , $a_3 > 0$, if $a_1a_2a_3 > 1$ holds, there exist γ_1 , γ_2 , $\gamma_3 > 0$ such that

$$\gamma_1 < a_1 \gamma_2, \ \gamma_2 < a_2 \gamma_3, \ \gamma_3 < a_3 \gamma_1.$$
 (23)

The proof is omitted. Note that for any $\beta > 0$, $\gamma'_i = \beta \gamma_i$, i = 1, 2, 3, also satisfy the condition (23). Define $C = \{1 + (\frac{\|\mathbf{A}\|^2 \alpha^2}{\varphi^2} - \frac{2\alpha}{\overline{L}}) \cdot \underline{\lambda} (A^{\mathsf{T}} A)\}^{\frac{1}{2}}$. Based on the above results, the following Theorem is proved by induction argument.

Theorem 1: Given a constant $r \in (r_{LB}, \infty)$, where r_{LB} is defined as $r_{LB} = \max\{q_{\zeta_i}, q_{\eta_i}, C, \bar{\lambda}\}$. Suppose Assumptions 1,2 hold, α satisfies

$$\alpha < \frac{\underline{\varphi}^2}{2\|\mathbf{A}\|^2 \overline{L}} \text{ and } \frac{(r-C)\underline{\varphi}}{\alpha\|\mathbf{A}\|} (\frac{(r-\bar{\lambda})^2 \underline{\varphi}}{2\alpha\|\mathbf{A}\|} - 1) > 1.$$
 (24)

then the sequences generated by diff-DMAC satisfy

$$\mathbb{E}[\|\check{\boldsymbol{\mu}}(k)\|] \le \gamma_1 r^k, \tag{25}$$

$$\mathbb{E}[\|\check{\boldsymbol{y}}(k)\|] \le \gamma_2 r^k, \tag{26}$$

$$\mathbb{E}[\|\boldsymbol{x}(k) - \boldsymbol{x}^{\infty}\|] \le \gamma_3 r^k, \tag{27}$$

$$\mathbb{E}[\|\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty}\|] \le \gamma_4 r^k, \tag{28}$$

for some constant $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$, where \mathbf{x}^{∞} and $\boldsymbol{\mu}^{\infty}$ satisfy

$$\bar{\mathbf{W}}\mathbf{A}\mathbf{x}^{\infty} = -\sum_{k=1}^{\infty} \bar{\boldsymbol{\zeta}}(k), \ \bar{\mathbf{W}}\boldsymbol{\mu}^{\infty} = \boldsymbol{\mu}^{\infty}, \tag{29}$$

$$x_i^{\infty} = \arg\min_{z \in \mathcal{X}_i} \{\nabla f_i(z) - \mu_i^{\infty \mathsf{T}} A_i z\}, \forall i.$$
(30)

Proof: Combining (18) and (29), we have that

$$(k+1) - \bar{\boldsymbol{\mu}}^{\infty} = \bar{\boldsymbol{\mu}}(k) - \bar{\boldsymbol{\mu}}^{\infty} - \alpha \mathbf{W} \mathbf{A}(\mathbf{x}(k) - \mathbf{x}^{\infty}) + \alpha \sum_{t=k}^{\infty} \bar{\boldsymbol{\zeta}}(t) + \bar{\boldsymbol{\eta}}(k).$$
(31)

Under Assumption 1, the function

 $\bar{\mu}$

$$f_i^*(A_i^\mathsf{T}\mu_i) = \arg\max_{z\in\mathcal{X}_i} \{\mu_i^\mathsf{T}A_i z - f_i(z)\}$$

is differentiable and f_i^* satisfies $\frac{1}{L_i}$ -strongly convex and $\frac{1}{\varphi_i}$ -Lipschitz smooth [21]. Furthermore, it follows (10) and the Proposition B.25 in [22] that

$$x_i(k) = \nabla f_i^*(A_i^{\mathsf{T}} \mu_i(k)), \forall i.$$
(32)

From (30) and (32), we can obtain that

$$\begin{aligned} \|\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty} - \alpha \mathbf{W} \mathbf{A}(\mathbf{x}(k) - \mathbf{x}^{\infty}) \| \\ &= \|\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty} - \alpha \bar{\mathbf{W}} \mathbf{A}(\nabla f^{*}(\mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}(k)) - \nabla f^{*}(\mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}^{\infty})) \| \\ &\leq \left\{ \|\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty} \|^{2} + \alpha^{2} \|\nabla f^{*}(\mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}(k)) - \nabla f^{*}(\mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}^{\infty}) \| \right. \\ &\left. - 2\alpha(\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty})^{\mathsf{T}} \mathbf{A}(\nabla f^{*}(\mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}(k)) - \nabla f^{*}(\mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}^{\infty})) \right\}^{\frac{1}{2}} \\ &+ \alpha \|\nabla f^{*}(\mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}(k)) - \nabla f^{*}(\mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}(k)) \| \\ &\leq C \|\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty}\| + \frac{\|\mathbf{A}\|^{2}\alpha}{\underline{\varphi}} \|\check{\boldsymbol{\mu}}(k)\|, \end{aligned}$$
(33)

where C < 1 under (24) and the last inequality follows from the strongly convexity and Lipschitz smoothness of f^* .

We will prove (25)-(28) by induction. When $\kappa = 0$, it is not difficult to find $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ such that (25)-(28) hold. We assume that for all $\kappa \leq k$, (25)-(28) hold. Considering $\kappa = k + 1$, we obtain from (31) and (33) that

$$\mathbb{E}[\|\bar{\boldsymbol{\mu}}(k+1) - \boldsymbol{\mu}^{\infty}\|] \\
\leq C\mathbb{E}[\|\bar{\boldsymbol{\mu}}(k) - \boldsymbol{\mu}^{\infty}\|] + \frac{\|\mathbf{A}\|\alpha}{\underline{\varphi}}\mathbb{E}[\|\check{\boldsymbol{\mu}}(k)\|] \\
+ \mathbb{E}[\|\alpha\sum_{t=k}^{\infty}\bar{\boldsymbol{\zeta}}(t)\|] - \mathbb{E}[\|\bar{\boldsymbol{\eta}}(k)\|] \\
\leq C\gamma_4 r^k + \frac{\|\mathbf{A}\|\alpha}{\bar{\varphi}}\gamma_1 r^k + nm\bar{q}^k\bar{d}_{\eta} + \frac{\alpha nm\bar{q}^k\bar{d}_{\zeta}}{1 - \bar{q}}.$$
(34)

Combining (20)-(22) gives

$$\mathbb{E}[\|\check{\boldsymbol{\mu}}(k+1)\|] \leq \bar{\lambda}\mathbb{E}[\|\check{\boldsymbol{\mu}}(k)\|] + \alpha\mathbb{E}[\|\check{\boldsymbol{y}}(k)\|] + \bar{\lambda}\mathbb{E}[\|\check{\boldsymbol{\eta}}(k)\|] \\ \leq \bar{\lambda}\gamma_1 r^k + \alpha\gamma_2 r^k + \bar{\lambda}nm\bar{q}^k\bar{d}_{\eta}, \qquad (35)$$

$$\mathbb{E}[\|\check{\boldsymbol{y}}(k+1)\|] \leq \bar{\lambda}\mathbb{E}[\|\check{\boldsymbol{y}}(k)\|] + \|\mathbf{A}\|\mathbb{E}[\|\boldsymbol{x}(k) - \boldsymbol{x}^{\infty}\|] \\ + \|\mathbf{A}\|\mathbb{E}[\|\boldsymbol{x}(k+1) - \boldsymbol{x}^{\infty}\|] + \bar{\lambda}\mathbb{E}[\|\check{\boldsymbol{\zeta}}(k)\|] \\ \leq \bar{\lambda}\gamma_{2}r^{k} + \|\mathbf{A}\|\gamma_{3}r^{k} + \bar{\lambda}nm\bar{q}^{k}\bar{d}_{\zeta} \\ + \|\mathbf{A}\|\mathbb{E}[\|\boldsymbol{x}(k+1) - \boldsymbol{x}^{\infty}\|].$$
(36)

Under Assumption 1, (32) implies that

$$\mathbb{E}[\|\boldsymbol{x}(k+1) - \boldsymbol{x}^{\infty}\|] = \mathbb{E}[\|\nabla f^{*}(\mathbf{A}\boldsymbol{\mu}(k+1)) - \nabla f^{*}(\mathbf{A}\boldsymbol{\mu}^{\infty})\|] \\ \leq \frac{\|\mathbf{A}\|}{\underline{\varphi}} \mathbb{E}[\|\check{\boldsymbol{\mu}}(k+1)\|] + \frac{\|\mathbf{A}\|}{\underline{\varphi}} \mathbb{E}[\|\bar{\boldsymbol{\mu}}(k+1) - \boldsymbol{\mu}^{\infty}\|],$$
(37)

where the last inequality is due to the Lipschitz smoothness of f^* .

Note that constant $\overline{\lambda}, C \in (0, 1)$, due to Lemma 2, there exist $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ such that

$$C\gamma_4 + \frac{\|\mathbf{A}\|_{\alpha}}{\underline{\varphi}}\gamma_1 < \gamma_4 r, \tag{38}$$

$$\bar{\lambda}\gamma_1 + \alpha\gamma_2 < \gamma_1 r, \tag{39}$$

$$\lambda \gamma_2 + 2 \|\mathbf{A}\| \gamma_3 < \gamma_2 r, \tag{40}$$

$$\frac{\|\mathbf{A}\|}{\varphi}(\gamma_1 + \gamma_4) < \gamma_3. \tag{41}$$

Basd on equations (38)-(41), taking sufficiently large $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ yields

$$C\gamma_4 r^k + \frac{\|\mathbf{A}\|_{\alpha}}{\bar{\varphi}} \gamma_1 r^k + nm\bar{q}^k \bar{d}_{\eta} + \frac{\alpha nm\bar{q}^k \bar{d}_{\zeta}}{1-\bar{q}} \leq \gamma_4 r^{k+1},$$

$$\bar{\lambda}\gamma_1 r^k + \alpha\gamma_2 r^k + \bar{\lambda}nm\bar{q}^k \bar{d}_{\eta} \leq \gamma_1 r^{k+1},$$

$$\bar{\lambda}\gamma_2 r^k + 2\|\mathbf{A}\|\gamma_3 r^k \leq \gamma_2 r^{k+1},$$

$$\frac{\|\mathbf{A}\|}{\underline{\varphi}} (\gamma_1 + \gamma_4) r^{k+1} \leq \gamma_3 r^{k+1},$$

where we use the definition of r_{LB} and substituting these equations into (34)-(37) completes the proof.

Based on Theorem 1, we prove the linear convergence property of diff-DMAC and quantify its convergence accuracy.

Theorem 2: Suppose the conditions in Theorem 1 are satisfied. If α further satisfies

$$\alpha < \frac{\Phi[-(1-C) + \sqrt{(1-C)^2 + 2(1-C)(1-\bar{\lambda})^2}]}{2\|\mathbf{A}\|},$$
(42)

the sequence $\{\mathbf{x}(\mathbf{k})\}_{k\geq 0}$ generated by diff-DMAC will converge linearly to the neighbor of the optimum \mathbf{x}^* in mean square and we have

$$\frac{1}{n^2 \|\mathbf{A}\|^2} N_{\zeta} \le \mathbb{E}\{\|\boldsymbol{x}^{\infty} - \boldsymbol{x}^{\star}\|^2\} \le \frac{\overline{L}^2}{n\underline{\varphi}^2[\underline{\lambda}(\mathbf{A}^{\intercal}\mathbf{A})]^2} N_{\zeta},$$
(43)

where N_{ζ} is defined as $N_{\zeta} = \sum_{i=1}^{n} \frac{2md_{\zeta i}^2}{1-q_{\zeta i}^2}$.

Proof: Note that $\overline{\lambda}, C \in (0, 1)$, it is not difficult to see that with α chosen to satisfy (42), we can always find a constant $r \in (r_{LB}, 1)$ such that equations (25)-(28) hold. Thus, the convergence of diff-DMAC is proved by Theorem 1.

Due to (29) and $\overline{\mathbf{W}}\mathbf{A}\mathbf{x}^{\star} = 0$, we have

$$\mathbb{E}[\|\bar{\mathbf{W}}\mathbf{A}(\mathbf{x}^{\infty} - \mathbf{x}^{\star})\|^{2}] = \mathbb{E}[\|\sum_{t=0}^{\infty} \bar{\boldsymbol{\zeta}}(t)\|^{2}] = \frac{1}{n^{2}} \sum_{i=1}^{n} \frac{2md_{\zeta_{i}}^{2}}{1 - q_{\zeta_{i}}^{2}}.$$
(44)

Since $\overline{\mathbf{W}} = \frac{\mathbf{1}^{\mathsf{T}} \mathbf{1}}{n} \otimes I_m$, it is easy to obtain that

$$\mathbb{E}[\|\bar{\mathbf{W}}\mathbf{A}(\mathbf{x}^{\infty}-\mathbf{x}^{\star})\|^{2}] \leq \|\mathbf{A}\|^{2}\mathbb{E}[\|\mathbf{x}^{\infty}-\mathbf{x}^{\star}\|^{2}].$$
(45)

The optimality condition of the x-update (10) implies that

$$\begin{aligned} \nabla f_i(x_i^{\infty}) - A_i^{\mathsf{T}} \mu_i^{\infty})^{\mathsf{T}}(x_i^{\star} - x_i^{\infty}) &\geq 0, \\ (\nabla f_i(x_i^{\star}) - A_i^{\mathsf{T}} \mu_i^{\star})^{\mathsf{T}}(x_i^{\infty} - x_i^{\star}) &\geq 0, \forall i. \end{aligned}$$
(46)

Based on (46) and (47), we obtain

$$\begin{aligned} &(\mu_i^{\infty} - \mu_i^{\star})^{\mathsf{T}} A_i (x_i^{\infty} - x_i^{\star}) \\ &\geq (\nabla f_i (\mathsf{T} x_i^{\infty}) - \nabla f_i (\mathsf{T} x_i^{\star}))^{\mathsf{T}} (x_i^{\infty} - x_i^{\star}) \\ &\geq \frac{\underline{\Phi} \|A_i\|}{\overline{L}} \|\mu_i^{\infty} - \mu_i^{\star}\| \|x_i^{\infty} - x_i^{\star}\|, \forall i. \end{aligned}$$
(48)

The facts that $\bar{\mathbf{W}}\mu^{\infty} = \mu^{\infty}$ and $\bar{\mathbf{W}}\mu^{\star} = \mu^{\star}$ imply

$$\mu_i^{\infty} - \mu_i^{\star} = \frac{1}{n} \sum_{i=1}^n \mu_i^{\infty} - \frac{1}{n} \sum_{i=1}^n \mu_i^{\star}, \ \forall i \in \mathcal{V},$$
(49)
$$(\mathbf{x}^{\infty} - \mathbf{x}^{\star}) = (\boldsymbol{\mu}^{\infty} - \boldsymbol{\mu}^{\star})^{\mathsf{T}} \bar{\mathbf{W}} \mathbf{A} (\mathbf{x}^{\infty} - \mathbf{x}^{\star})$$

$$(\boldsymbol{\mu}^{\infty} - \boldsymbol{\mu}^{\star})^{\mathsf{T}} \mathbf{A} (\mathbf{x}^{\infty} - \mathbf{x}^{\star}) = (\boldsymbol{\mu}^{\infty} - \boldsymbol{\mu}^{\star})^{\mathsf{T}} \mathbf{W} \mathbf{A} (\mathbf{x}^{\infty} - \mathbf{x}^{\star})$$

$$\leq \|\boldsymbol{\mu}^{\infty} - \boldsymbol{\mu}^{\star}\| \| \bar{\mathbf{W}} \mathbf{A} (\mathbf{x}^{\infty} - \mathbf{x}^{\star}) \|,$$

(50)

Then combining (48)-(50) yields

$$\mathbb{E}[\|\mathbf{x}^{\infty} - \mathbf{x}^{\star}\|^{2}] \leq \frac{\bar{L}^{2}}{n\underline{\varphi}^{2}[\underline{\lambda}(\mathbf{A}^{\intercal}\mathbf{A})]^{2}} \mathbb{E}[\|\bar{\mathbf{W}}\mathbf{A}(\mathbf{x}^{\infty} - \mathbf{x}^{\star})\|^{2}].$$
(51)

Substituting (44) into (45), (51) completes the proof. \blacksquare

V. DIFFERENTIAL PRIVACY

In this section, we present the privacy preserving performance of the proposed algorithm by theoretical analysis.

Theorem 3: Suppose the conditions in Theorem 2 hold, if q_{i_0} satisfies

$$q_{i_0} \in \left(\frac{\alpha \|A_{i_0}\|^2 + \|A_{i_0}\|\sqrt{\alpha^2 \|A_{i_0}\|^2 + 4\alpha\varphi_{i_0}}}{2\varphi_{i_0}}, 1\right), \quad (52)$$

the proposed diff-DMAC preservses $\epsilon_{i_0}\text{-differential privacy}$ for the cost function of agent i_0 , where

$$\epsilon_{i_0} = \left(\frac{1}{\alpha d_{\zeta_{i_0}}} + \frac{1}{d_{\eta_{i_0}}}\right) \frac{\alpha \varphi_{i_0} \delta \|A_{i_0}\|}{\varphi_{i_0} q_{i_0}^2 - \alpha q_{i_0} - \alpha}.$$
 (53)

Proof: Two function sets have the same initialization and all auxiliary variables are known to attackers as mentioned in Section II. Therefore, exchanged variables must be same, i.e., $\boldsymbol{z}_{\boldsymbol{\mu}}^{(1)}(k) = \boldsymbol{z}_{\boldsymbol{\mu}}^{(2)}(k)$ and $\boldsymbol{z}_{\boldsymbol{y}}^{(1)}(k) = \boldsymbol{z}_{\boldsymbol{y}}^{(2)}(k)$, otherwise, the two function sets will be easily distinguished. Therefore, for any $i \neq i_0$, due to the realation $f_i^{(1)} = f_i^{(2)}$, the added noise should satisfy

$$\zeta_i^{(1)}(k) = \zeta_i^{(2)}(k), \ \eta_i^{(1)}(k) = \eta_i^{(2)}(k), \forall k, \forall i \neq i_0.$$

And for $i = i_0$, since $f_i^{(1)} \neq f_i^{(2)}$ the noise should satisfy

$$\Delta \eta_{i_0}(k) = -\Delta \mu_{i_0}(k), \ \Delta \zeta_{i_0}(k) = -\Delta y_{i_0}(k),$$
 (54)

where $\Delta \zeta_{i_0}(k) = \zeta_{i_0}^{(1)}(k) - \zeta_{i_0}^{(2)}(k)$, $\Delta \eta_{i_0}(k) = \eta_{i_0}^{(1)}(k) - \eta_{i_0}^{(2)}(k)$, $\Delta \mu_{i_0}(k) = \mu_{i_0}^{(1)}(k) - \mu_{i_0}^{(2)}(k)$ and $\Delta y_{i_0}(k) = y_{i_0}^{(1)}(k) - y_{i_0}^{(2)}(k)$. Furthermore, it follows that

$$\Delta \mu_{i_0}(k+1) = -\alpha \Delta y_{i_0}(k), \ \Delta y_{i_0}(k+1) = \Delta A_{i_0} g_{i_0}(k),$$
(55)

where $\Delta g_{i_0}(k) = \Delta x_{i_0}(k+1) - \Delta x_{i_0}(k)$ and $\Delta x_{i_0}(k) = x_{i_0}^{(1)}(k) - x_{i_0}^{(2)}(k)$. Following the analysis in [13], we need to obtain the upper bound of $\frac{f_{\zeta\eta}(\boldsymbol{\zeta}^{(1)}, \boldsymbol{\eta}^{(1)})}{f_{\zeta\eta}(\mathcal{B}(\boldsymbol{\zeta}^{(1)}, \boldsymbol{\eta}^{(1)}))}$ to quantify the privacy level ϵ_{i_0} ,

$$\frac{\mathbb{P}\{\boldsymbol{\zeta},\boldsymbol{\eta}\in\Omega|Z_{T^{(1)}}(\boldsymbol{\zeta},\boldsymbol{\eta})\in\mathcal{O}\}}{\mathbb{P}\{\boldsymbol{\zeta},\boldsymbol{\eta}\in\Omega|Z_{T^{(2)}}(\boldsymbol{\zeta},\boldsymbol{\eta})\in\mathcal{O}\}} \leq \sup_{(\boldsymbol{\zeta},\boldsymbol{\eta})}\frac{f_{\boldsymbol{\zeta}\boldsymbol{\eta}}(\boldsymbol{\zeta}^{(1)},\boldsymbol{\eta}^{(1)})}{f_{\boldsymbol{\zeta}\boldsymbol{\eta}}(\mathcal{B}(\boldsymbol{\zeta}^{(1)},\boldsymbol{\eta}^{(1)}))},$$
(56)

where $Z_{T^{(l)}}(\boldsymbol{\zeta}^{(l)}\boldsymbol{\eta}^{(l)}) = \{\boldsymbol{z}_{\boldsymbol{\mu}}^{(l)}(k), \boldsymbol{z}_{\boldsymbol{y}}^{(l)}(k)\}_{k\geq 0}, l = 1, 2,$ belongs to attacker's observation.

For $l \in \{1, 2\}$, the x-update can be reritten as

$$x_{i_0}^{(l)}(k+1) = \arg\min_{z \in \mathcal{X}_{i_0}^{(l)}} \{f_{i_0}^{(l)}(z) - \mu_{i_0}^{(l)^{\mathsf{T}}}(k+1)A_{i_0}z\}$$

From Definition 1, we have $x_{i_0}^{(1)}, x_{i_0}^{(2)} - \delta' \in \mathcal{X}_{i_0}^{(1)}$ and $x_{i_0}^{(2)}, x_{i_0}^{(1)} + \delta' \in \mathcal{X}_{i_0}^{(1)}$. Similar to the analysis of Theorem 2, the optimal condition of the written x-update yields

$$\begin{aligned} &(\mu_{i_0}^{(1)}(k) - \mu_{i_0}^{(2)}(k))^{\mathsf{T}} A_{i_0}(x_{i_0}^{(1)}(k) + \delta' - x_{i_0}^{(2)}(k)) \\ &\geq (\nabla f_{i_0}^{(1)}(x_{i_0}^{(1)}(k)) - \nabla f_{i_0}^{(2)}(x_{i_0}^{(2)}(k)))^{\mathsf{T}}(x_{i_0}^{(1)}(k) + \delta' - x_{i_0}^{(2)}(k)) \\ &\geq \frac{\varphi_{i_0}}{\|A_{i_0}\|^2} \|A_{i_0}(x_{i_0}^{(1)}(k) + \delta' - x_{i_0}^{(2)}(k))\|^2. \end{aligned}$$
(57)

The last inequality uses the relation $\nabla f_{i_0}^{(1)}(x_{i_0}^{(1)}(k)) = \nabla f_{i_0}^{(2)}(x_{i_0}^{(1)}(k) + \delta')$. Then applying Cauchy–Schwarz inequality to the LHS of (57) yields

$$\|A_{i_0}(x_{i_0}^{(1)}(k) + \delta' - x_{i_0}^{(2)}(k))\| \le \frac{\|A_{i_0}\|^2}{\varphi_{i_0}} \|\Delta \mu_{i_0}(k)\|.$$
(58)

From (54), (55) and the relation (58), we have

$$\begin{split} \|\Delta\eta_{i_0}(k+1)\| &= \alpha \, \|A_{i_0}\Delta g_{i_0}(k-1)\|_2 \\ &= \alpha \Big\| A_{i_0}(x_{i_0}^{(1)}(k) - x_{i_0}^{(2)}(k) - \delta) \\ &- A_{i_0} \left(x_{i_0}^{(1)}(k-1) - x_{i_0}^{(2)}(k-1) - \delta \right) \Big\| \\ &\leq \frac{\alpha \|A_{i_0}\|^2}{\varphi_{i_0}} \left(\|\Delta\eta_{i_0}(k)\|_2 + \|\Delta\eta_{i_0}(k-1)\| \right). \end{split}$$

$$(59)$$

The two roots of $\varphi_{i_0}x^2 - \alpha \|A_{i_0}\|^2 x - \alpha \|A_{i_0}\|^2 = 0$, denoted as τ_1, τ_2 are different and all lie in (-1, 1). Therefore, from (59), we have

$$\|\Delta \eta_{i_0}(k)\|_2 \le \frac{\alpha \delta \|A_{i_0}\|}{\tau_1 - \tau_2} (\tau_1^{k-1} - \tau_2^{k-1}), \qquad (60)$$

since $\|\Delta \eta_{i_0}(1)\| = 0$ and $\|\Delta \eta_{i_0}(2)\| < \alpha \delta \|A_{i_0}\|$ which can be derived from (54) and (55). These two equations also give that $\alpha |\Delta \zeta_{i_0}(k)| = |\Delta \eta_{i_0}(k+1)|, \forall k$. Thus,

$$\frac{f_{\zeta\eta}(\boldsymbol{\zeta}^{(1)},\boldsymbol{\eta}^{(1)})}{f_{\zeta\eta}\left(\mathcal{B}(\boldsymbol{\zeta}^{(1)},\boldsymbol{\eta}^{(1)})\right)} = \prod_{k=1}^{\infty} \frac{f_L(\zeta_{i_0}^{(1)}(k),\theta_{i_k}^{\zeta})f_L(\eta_{i_0}^{(1)}(k),\theta_{i_k}^{\eta})}{f_L(\zeta_{i_0}^{(2)}(k),\theta_{i_k}^{\zeta})f_L(\eta_{i_0}^{(2)}(k),\theta_{i_k}^{\eta})} \\
\leq \prod_{k=1}^{\infty} e^{\frac{|\Delta\zeta_{i_0}(k)|}{d_{\zeta_{i_0}}q_{i_0}^k}} \prod_{k=2}^{\infty} e^{\frac{|\Delta\eta_{i_0}(k)|}{d_{\eta_{i_0}}q_{i_0}^k}}. \\
\leq e^{\left(\frac{1}{\alpha d_{\zeta_{i_0}}} + \frac{1}{d_{\eta_{i_0}}}\right) \frac{\alpha \varphi_{i_0}\delta \|A_{i_0}\|}{\varphi_{i_0}q_{i_0}^2 - \alpha q_{i_0} - \alpha}}} = e^{\epsilon_{i_0}}, \tag{61}$$

where the last inequality follows the the sum formula of geometric series. And the inequality $\varphi_{i_0}q_{i_0}^2 - \alpha \|A_{i_0}\|^2 q_{i_0}$ – $\alpha \|A_{i_0}\|^2 > 0$ holds under (52).

Combining (56) and (61) yields the result corresponding to the definition of differential privacy of the cost function in Definition 2. Thus, the proof completes.

Note that the privacy level ϵ in Theorem 3 is related to both noise ζ and η . Both of them are necessary, since if at least one of them are set to zero, there does not exist a finite number $\epsilon > 0$, i.e., the differential privacy cannot be preserved.

Since the convergence accuracy is only related to the noise $\boldsymbol{\zeta}$, the optimal value of ϵ_i can be obtained by setting $d_{\eta_i} \rightarrow$ ∞ and it gives

$$\epsilon_i^{\star} = \frac{\varphi_i \delta \|A_i\|}{d_{\zeta i}(\varphi_i q_i^2 - \alpha q_i - \alpha)}, \forall i \in \{1, \dots, n\}.$$
(62)

Combining (43) with (62), we can derive that increasing added noise leads to a high level privacy but a low accuracy which shows the trade-off between accuracy and privacy.

VI. NUMERICAL EXPERIMENTS

In this section, we conduct numerical experiments to verify our theoretical analysis. The proposed algorithm is tested on a multi-MG system with 14 microgrids [16]. Regarding the communication network, we generate an undirected graph by adding random links to a ring network. We consider the following problem with quadratic cost functions $f_i(x_i) =$ $u_i x_i^2 + v_i x_i + w_i, \forall i$

$$\min_{\boldsymbol{x} \in \mathbb{R}^{n \times 1}} f(\boldsymbol{x}) = \sum_{i=1}^{n} f_i(x_i)$$

s.t. $\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} d_i, \ x_i^{\min} \le x_i \le x_i^{\max}, \forall i.$ (63)

where coefficients a_i are randomly chosen and the parameters of the generators are adopted from [1].

Set total load demand $d_{total} = 231MW$, the exchanged information is masked by independent Laplace noise with $q = 0.98, d_{\eta} = d_{\zeta} = 1$ and we apply diff-DMAC to problem (63). Fig. 2 (a) shows how the mean square error (MSE) $\mathbb{E}[||\mathbf{x}(k) - \mathbf{x}^{\star}||^2]$ changes with iteration time k, where the expected errors are approximated by averaging over 100 simulation results. The results validate the linear convergence rate of the proposed algorithm. Fix q = 0.98, the relation between error and d_{ζ} is shown in Fig. 2 (b). We find that as d_{ζ} becomes larger, the MSE of x increases, which means larger noise brings lower accuracy. Moreover, the experimental result is strictly between the lower bound and upper bound given in Theorem 2.

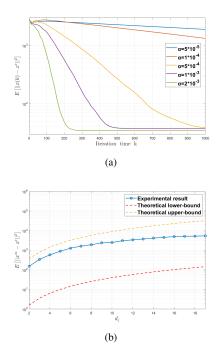


Fig. 2. (a) MSE versus Iteration time k with different stepsizes; (b) Relation between MSE and $d_{\zeta}.$

As for the differential privacy of diff-DMAC, Theorem 3 is numerically tested with different d_{ζ} as shown in Fig. 3(a). The experimental result is upper bounded by the theoretical result, i.e. $\epsilon_e \leq \epsilon$, which verifies Theorem 3. Combining Fig. 3(a) with Fig. 3(b) shows that a bigger and slower decaying noise leads to a better privacy level. However, large noise also influences the convergence accuracy and Fig. 4 validates the aforementioned trade-off between privacy and accuracy.

VII. CONCLUSION

In this paper, a differentially private distributed mismatch tracking algorithm has been proposed to solve constraintcoupled resource allocation problems with privacy concerns. Its linear convergence property has been established for strongly convex and Lipschitz-smooth cost functions. Then, the differential privacy of the proposed algorithm is theoretically proved and we also characterize the trade-off between accuracy and privacy. The theoretical results have been examined by numerical experiments.

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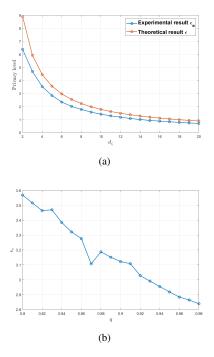


Fig. 3. (a) Privacy level versus d_{ζ} ; (b) ϵ_e versus decay coefficient q.

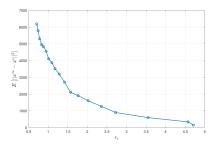


Fig. 4. Trade-off between convergence accuracy and privacy level.

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