# Safe-by-Repair: A Convex Optimization Approach for Repairing Unsafe Two-Level Lattice Neural Network Controllers

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Abstract—In this paper, we consider the problem of repairing a data-trained Rectified Linear Unit (ReLU) Neural Network (NN) controller for a discrete-time, input-affine system. That is we assume that such a NN controller is available, and we seek to repair unsafe closed-loop behavior at one known "counterexample" state while simultaneously preserving a notion of safe closed-loop behavior on a separate, verified set of states. To this end, we further assume that the NN controller has a Two-Level Lattice (TLL) architecture, and exhibit an algorithm that can systematically and efficiently repair such an network. Facilitated by this choice, our approach uses the unique semantics of the TLL architecture to divide the repair problem into two significantly decoupled sub-problems, one of which is concerned with repairing the un-safe counterexample - and hence is essentially of local scope - and the other of which ensures that the repairs are realized in the output of the network - and hence is essentially of global scope. We then show that one set of sufficient conditions for solving each these sub-problems can be cast as a convex feasibility problem, and this allows us to formulate the TLL repair problem as two separate, but significantly decoupled, convex optimization problems. Finally, we evaluate our algorithm on a TLL controller on a simple dynamical model of a four-wheel-car.

#### I. INTRODUCTION

The proliferation of Neural Networks (NNs) as safety-critical controllers has made obtaining provably correct NN controllers vitally important. However, most current techniques for doing so involve a repeatedly training and verifying a NN until adequate safety properties have been achieved. Such methods are not only inherently computationally expensive (because training and verification of NNs are), their convergence properties can be extremely poor. For example, when verifying multiple safety properties, such methods can cycle back and forth between safety properties, with each subsequent retraining achieving one safety property by undoing another one.

An alternative approach obtains safety-critical NN controllers by *repairing* an existing NN controller. Specifically, it is assumed that an already-trained NN controller is available that performs in a *mostly* correct fashion, albeit with some specific, known instances of incorrect behavior. But rather than using retraining techniques, repair entails *systematically* altering the parameters of the original controller *in a limited way*, so as to retain the original safe behavior while simultaneously correcting the unsafe behavior. The objective of repair is to exploit as much as possible the safety that was learned during the training of the original NN parameters, rather than allowing re-training to *unlearn* safe behavior.

Despite these advantages, the NN repair problem is challenging because it has two main objectives, both of which are at odds with each other. In particular, repairing an unsafe behavior requires altering the NN's response in a *local* region of the state space, but changing even a few neurons generally affects the *global* response of the NN – which could undo the initial safety guarantee supplied with the network. This tension is especially relevant for general deep NNs, and repairs realized on neurons in their latter layers. This is especially the case for repairing controllers, where the relationship between specific neurons and their importance to the overall safety properties is difficult to discern. As a result, there has been limited success in studying NN controller repair, especially for nonlinear systems.

In this paper, we exhibit an explicit algorithm that can repair a NN controller for a *discrete-time, input-affine nonlinear* system. The cornerstone of our approach is to consider NN controllers of a specific architecture: in particular, the recently proposed Two-Level Lattice (TLL) NN architecture [1]. The TLL architecture has unique neuronal semantics, and those semantics greatly facilitate finding a balance between the local and global trade-offs inherent in NN repair. In particular, by assuming a TLL architecture, we can separate the problem of controller repair into two *significantly decoupled* problems, one consisting of essentially only local considerations and one consisting of essentially only global ones.

Related Work: Repairing (or patching) NNs can be traced to the late 2000s. An early result on patching connected transfer learning and concept drift with patching [2]; another result established fundamental requirements to apply classifier patching on NNs by using inner layers to learn a patch for concept drift in an image classifier network [3]. Another approach based on a Satisfiability Modulo Theory (SMT) formulation of the repair problem was proposed by [4] where they changed the parameters of a classifier network to comply with a safety specification, i.e. where the designer knows exactly the subset of the input space to be classified. This prior work nonetheless is heuristic-based and so not guaranteed to produced desired results, which was noticed by [5] who cast the problem of patching (minimal repair) as a verification problem for NNs (including Deep ones). However, this work focused on a restricted version of the problem in which the changes in weights are limited to a single layer. Finally, [6] proposed a verification-based approach for repairing DNNs but not restricted to modifying the output; instead, proposed to identify and modify the most relevant neurons that causes the safety violation using gradient guidance.

#### **II. PRELIMINARIES**

#### A. Notation

We will denote the real numbers by  $\mathbb{R}$ . For an  $(n \times m)$  matrix (or vector), A, we will use the notation  $[\![A]\!]_{i,j}$  to

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denote the element in the  $i^{th}$  row and  $j^{th}$  column of A. Analogously, the notation  $[\![A]\!]_{i,\cdot}$  will denote the  $i^{\text{th}}$  row of A, and  $[\![A]\!]_{\cdot,j}$  will denote the  $j^{\text{th}}$  column of A; when A is a vector instead of a matrix, both notations will return a scalar corresponding to the corresponding element in the vector. Let  $\mathbf{0}_{n,m}$  be an  $(n \times m)$  matrix of zeros. We will use bold parenthesis  $(\cdot)$  to delineate the arguments to a function that returns a function. We use the functions First and Last to return the first and last elements of an ordered list (or a vector in  $\mathbb{R}^n$ ). The function Concat concatenates two ordered lists, or two vectors in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  along their (common) nontrivial dimension to get a third vector in  $\mathbb{R}^{n+m}$ . Finally,  $B(x; \delta)$ denotes an open Euclidean ball centered at x with radius  $\delta$ . The norm  $\left\|\cdot\right\|$  will refer to the Euclidean norm.

#### B. Dynamical Model

In this paper, we will consider the general case of a discrete-time input-affine nonlinear system  $\Sigma$  specified by:

$$\Sigma : \{ x_{i+1} = f(x_i) + g(x_i)u_i$$
 (1)

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the input. In addition,  $f: \mathbb{R}^n \to \mathbb{R}^n$  and  $g: \mathbb{R}^n \to \mathbb{R}^n$  are continuous and smooth functions of x.

**Definition 1** (Closed-loop Trajectory). Let  $u : \mathbb{R}^n \to \mathbb{R}^m$ . Then a closed-loop trajectory of the system (1) under u, starting from state  $x_0$ , will be denoted by the sequence  $\{\zeta_i^{x_0}(u)\}_{i=0}^{\infty}. \text{ That is } \zeta_{i+1}^{x_0}(u) = f(\zeta_i^{x_0}(u)) + g(\zeta_i^{x_0}(u)) \cdot u(\zeta_i^{x_0}(u)) \text{ and } \zeta_0^{x_0}(u) = x_0.$ 

Definition 2 (Workspace). We will assume that trajectories of (1) are confined to a connected, compact workspace,  $X_{WS}$ with non-empty interior, of size  $ext(X_{ws}) \triangleq \sup_{x \in X_{ws}} ||x||$ .

#### C. Neural Networks

We will exclusively consider Rectified Linear Unit Neural Networks (ReLU NNs). A K-layer ReLU NN is specified by composing K layer functions, each of which may be either linear and nonlinear. A nonlinear layer with i inputs and  $\mathfrak{o}$  outputs is specified by a ( $\mathfrak{o} \times \mathfrak{i}$ ) real-valued matrix of weights, W, and a  $(o \times 1)$  real-valued matrix of biases, b as follows:  $L_{\theta}$  :  $z \mapsto \max\{Wz + b, 0\}$  with the max function taken element-wise, and  $\theta \triangleq (W, b)$ . A *linear* layer is the same as a nonlinear layer, only it omits the nonlinearity  $\max\{\cdot, 0\}$ ; such a layer will be indicated with a superscript lin, e.g.  $L_{\theta}^{\text{lin}}$ . Thus, a K-layer ReLU NN function as above is specified by K layer functions  $\{L_{\theta^{(i)}} : i = 1, \dots, K\}$  that are *composable*: i.e. they satisfy  $i_i = o_{i-1} : i = 2, ..., K$ . We will annotate a ReLU function MV by a list of its parameters  $\Theta \triangleq (\theta^{|1}, \ldots, \theta^{|K})^1$ .

The number of layers and the dimensions of the matrices  $\theta^{|i|} = (W^{|i|}, b^{|i|})$  specify the *architecture* of the ReLU NN. Therefore, we will denote the architecture of the ReLU NN  $\mathscr{M}_{\Theta}$  by  $\operatorname{Arch}(\Theta) \triangleq ((n, \mathfrak{o}_1), (\mathfrak{i}_2, \mathfrak{o}_2), \dots, (\mathfrak{i}_K, m)).$ 

# D. Special NN Operations

Definition 3 (Sequential (Functional) Composition). Let  $\mathcal{M}_{\Theta_1}$  and  $\mathcal{M}_{\Theta_2}$  be two NNs where  $Last(Arch(\Theta_1)) = (\mathfrak{i}, \mathfrak{c})$  and  $First(Arch(\Theta_2)) = (\mathfrak{c}, \mathfrak{o})$ . Then the functional *composition* of  $\mathcal{M}_{\Theta_1}$  and  $\mathcal{M}_{\Theta_2}$ , i.e.  $\mathcal{M}_{\Theta_1} \circ \mathcal{M}_{\Theta_2}$ , is a

well defined NN, and can be represented by the parameter *list*  $\Theta_1 \circ \Theta_2 \triangleq \texttt{Concat}(\Theta_1, \Theta_2).$ 

**Definition 4.** Let  $\mathcal{M}_{\Theta_1}$  and  $\mathcal{M}_{\Theta_2}$  be two K-layer NNs with parameter lists:  $\Theta_i = ((W_i^{|1}, b_i^{|1}), \dots, (W_i^{|K}, b_i^{|K})), i =$ 1,2. Then the **parallel composition** of  $\mathcal{M}_{\Theta_1}$  and  $\mathcal{M}_{\Theta_2}$  is a NN given by the parameter list

$$\Theta_1 \parallel \Theta_2 \triangleq \left( \left( \begin{bmatrix} W_1^{|1|} \\ W_2^{|1|} \end{bmatrix}, \begin{bmatrix} b_1^{|1|} \\ b_2^{|1|} \end{bmatrix} \right), ..., \left( \begin{bmatrix} W_1^{|K|} \\ W_2^{|K|} \end{bmatrix}, \begin{bmatrix} b_1^{|K|} \\ b_2^{|K|} \end{bmatrix} \right) \right).$$
(2)

That is  $\Theta_1 \| \Theta_2$  accepts an input of the same size as (both)  $\Theta_1$ and  $\Theta_2$ , but has as many outputs as  $\Theta_1$  and  $\Theta_2$  combined.

Definition 5 (n-element min/max NNs). An n-element min **network** is denoted by the parameter list  $\Theta_{\min_n}$ .  $\mathcal{M}(\Theta_{\min_n})$  :  $\mathbb{R}^n \to \mathbb{R}$  such that  $\mathcal{M}(\Theta_{\min_n})(x)$  is the the minimum from among the components of x (i.e. minimum according to the usual order relation < on  $\mathbb{R}$ ). An *n*-element max network is denoted by  $\Theta_{\max_n}$ , and functions analogously. These networks are described in [1].

#### E. Two-Level-Lattice (TLL) Neural Networks

In this paper, we will be especially concerned with ReLU NNs that have the Two-Level Lattice (TLL) architecture, as introduced with the AReN algorithm in [1]. Thus we define a TLL NN as follows.

Definition 6 (TLL NN [1, Theorem 2]). A NN that maps  $\mathbb{R}^n \to \mathbb{R}$  is said to be **TLL NN of size** (N, M) if the size of its parameter list  $\Xi_{N,M}$  can be characterized entirely by integers N and M as follows.

$$\Xi_{N,M} \triangleq \Theta_{\max M} \circ ((\Theta_{\min N} \circ \Theta_{S_1}) \parallel \dots \parallel (\Theta_{\min N} \circ \Theta_{S_M})) \circ \Theta_{\ell}$$
(3)

where

- $\Theta_{\ell} \triangleq ((W_{\ell}, b_{\ell}));$

• each  $\Theta_{S_j}$  has the form  $\Theta_{S_j} = (S_j, \mathbf{0}_{M,1})$ ; and •  $S_j = [\llbracket I_N \rrbracket_{\iota_1}^T, \ldots \llbracket I_N \rrbracket_{\iota_N}^T]^T$  for some sequence  $\iota_k \in \{1, \ldots, N\}$ , where  $I_N$  is the  $(N \times N)$  identity matrix. The matrices  $\Theta_{\ell}$  will be referred to as the **linear function** matrices of  $\Xi_{N,M}$ . The matrices  $\{S_j | j = 1, \dots, M\}$  will be referred to as the selector matrices of  $\Xi_{N,M}$ . Each set  $s_i \triangleq \{k \in \{1, \dots, N\} | \exists \iota \in \{1, \dots, N\} . [S_i]_{\iota,k} = 1\}$  is said

to be the selector set of  $S_j$ . A multi-output TLL NN with range space  $\mathbb{R}^m$  is defined using *m* equally sized scalar TLL NNs. That is we denote such a network by  $\Xi_{N,M}^{(m)}$ , with each output component denoted by  $\Xi_{N,M}^i$ ,  $i = 1, \ldots, m$ .

#### **III. PROBLEM FORMULATION**

The main problem we consider in this paper is one of TLL NN repair. In brief, we take as a starting point a TLL NN controller that is "mostly" correct in the sense that is provably safe under a specific set of circumstances (states); here we assume that safety entails avoiding a particular, fixed subset of the state space. However, we further suppose that this TLL NN controller induces some additional, *unsafe* behavior of (1) that is explicitly observed, such as from a more expansive application of a model checker; of course this unsafe behavior necessarily occurs in states not covered by the original safety guarantee. The repair problem, then, is to "repair" the given TLL controller so that this additional unsafe behavior is made safe, while

<sup>&</sup>lt;sup>1</sup>That is  $\Theta$  is not the concatenation of the  $\theta^{(i)}$  into a single large matrix, so it preserves information about the sizes of the constituent  $\theta^{(i)}$ .

simultaneously preserving the original safety guarantees associated with the network.

The basis for the problem in this paper is thus a TLL NN controller that has been designed (or trained) to control (1) in a safe way. In particular, we use the following definition to fix our notion of "unsafe" behavior for (1).

**Definition 7** (Unsafe Operation of (1)). Let  $G_u$  be an  $(K_u \times$ n) real-valued matrix, and let  $h_u$  be an  $(K_u \times 1)$  real vector, which together define a set of **unsafe states**  $X_{\text{unsafe}} \triangleq \{x \in$  $\mathbb{R}^n | G_u x \ge h_u \}.$ 

Then, we mean that a TLL NN controller is safe with respect to (1) and  $X_{\text{unsafe}}$  in the following sense.

**Definition 8** (Safe TLL NN Controller). Let  $X_{safe} \subset \mathbb{R}^n$  be a set of states such that  $X_{\text{safe}} \cap X_{\text{unsafe}} = \emptyset$ . Then a TLL NN controller  $\mathfrak{u} \triangleq \mathscr{M}(\Xi_{N,M}^{(m)}) : \mathbb{R}^n \to \mathbb{R}^m$  is safe for (1) on horizon T (with respect to  $X_{safe}$  and  $X_{unsafe}$ ) if:

$$\forall x_0 \in X_{\text{safe}}, i \in \{1, \dots, T\}. \left(\zeta_i^{x_0}(\mathscr{M}(\Xi_{N,M}^{(m)})) \notin X_{\text{unsafe}}\right).$$
(4)

That is  $\mathcal{M}(\Xi_{N,M}^{(m)})$  is safe (w.r.t.  $X_{safe}$ ) if all of its length-T trajectories starting in  $X_{safe}$  avoid the unsafe states  $X_{unsafe}$ .

The design of safe controllers in the sense of Definition 8 has been considered in a number of contexts; see e.g. [7]. Often this design procedure involves training the NN using data collected from an expert, and verifying the result using one of many available NN verifiers [7].

However, as noted above, we further suppose that a given TLL NN which is safe in the sense of Definition 8 nevertheless has some unsafe behavior for states that lie outside  $X_{safe}$ . In particular, we suppose that a model checker (for example) provides to us a *counterexample* (or witness) to unsafe operation of (1).

Definition 9 (Counterexample to Safe Operation of (1)). Let  $X_{\text{safe}} \subset \mathbb{R}^n$ , and let  $\mathfrak{u} \triangleq \mathscr{M}(\Xi_{N,M}^{(m)})$  be a TLL controller that is safe for (1) on horizon T w.r.t  $X_{\text{safe}}$  and  $X_{\text{unsafe}}$ . A counter example to the safe operation of (1) is a state  $x_{c.e.} \notin X_{safe}$  such that

$$f(x_{\textit{c.e.}}) + g(x_{\textit{c.e.}}) \cdot \mathfrak{u}(x_{\textit{c.e.}}) = \zeta_1^{x_{\textit{c.e.}}}(\mathfrak{u}) \in X_{\textit{unsafe}}.$$
 (5)

That is starting (1) in  $x_{c.e.}$  results in an unsafe state in the next time step.

We can now state the main problem of this paper.

Problem 1. Let dynamics (1) be given, and assume its trajectories are confined to compact subset of states,  $X_{ws}$ (see Definition 2). Also, let  $X_{\text{unsafe}} \subset X_{\text{ws}}$  be a specified set of unsafe states for (1), as in Definition 7. Furthermore, let  $\mathfrak{u} = \mathscr{M}(\Xi_{N,M}^{(m)})$  be a TLL NN controller for (1) that is safe on horizon T with respect to a set of states  $X_{\mathsf{safe}} \subset X_{\mathsf{ws}}$ (see Definition 8), and let  $x_{c.e.}$  be a counterexample to safety in the sense of Definition 9.

Then the TLL repair problem is to obtain a new TLL controller  $\mathbf{\tilde{u}} = \mathcal{M}(\mathbf{\tilde{\Xi}}_{N,M}^{(m)})$  with the following properties:

- (i)  $\bar{u}$  is also safe on horizon T with respect to  $X_{safe}$ ;
- (ii) the trajectory  $\zeta_1^{x_{c.e.}}(\bar{\mathfrak{u}})$  is safe i.e. the counterexample
- (ii) the instance of the second seco

(iv) the selector matrices of  $\overline{\Xi}_{N,M}^{(m)}$  and  $\Xi_{N,M}^{(m)}$  are identical - *i.e.*  $\overline{S}_k = S_k$  for  $k = 1, \dots, M$ ; and

(v) 
$$\|\overline{W}_{\ell} - W_{\ell}\|_2 + \|\overline{b}_{\ell} - b_{\ell}\|_2$$
 is minimized.

In particular, iii, iv) and v) justify the designation of this problem as one of "repair". That is the repair problem is to fix the counterexample while keeping the network as close as possible to the original network under consideration. Note: the formulation of Problem 1 only allows repair by means of altering the *linear layers* of  $\Xi_{N,M}^{(m)}$ ; c.f. (*iii*) and (*iv*).

#### **IV. FRAMEWORK**

The TLL NN repair problem described in Problem 1 is challenging because it has two main objectives, which are at odds with each other. In particular, repairing a counterexample requires altering the NN's response in a local region of the state space, but changing even a few neurons generally affects the global response of the NN - which could undo the initial safety guarantee supplied with the network. This tension is especially relevant for general deep NNs, and repairs realized on neurons in their latter layers. It is for this reason that we posed Problem 1 in terms of TLL NNs: our approach will be to use the unique semantics of TLL NNs to balance the trade-offs between local NN alteration to repair the defective controller and global NN alteration to ensure that the repaired controller activates at the counterexample. Moreover, locally repairing the defective controller at  $x_{c.e.}$  entails a further trade off between two competing objectives of its own: actually repairing the counterexample – Problem 1(ii) – without causing a violation of the original safety guarantee for  $X_{safe}$  – i.e. Problem 1(i). Likewise, global alteration of the TLL to ensure correct activation of our repairs will entail its own trade-off: the alterations necessary to achieve the correct activation will also have to be made without sacrificing the safety guarantee for  $X_{safe}$  – i.e. Problem 1(*i*).

We devote the remainder of this section to two crucial subsections, one for each side of this local/global dichotomy. Our goal in these two subsections is to describe constraints on a TLL controller that are sufficient to ensure that it accomplishes the repair described in Problem 1. Thus, the results in this section should be seen as optimization constraints around which we can build our algorithm to solve Problem 1. The algorithmic details and formalism are presented in Section V.

#### A. Local TLL Repair

We first consider in isolation the problem of repairing the TLL controller in the vicinity of the counterexample  $x_{c.e.}$ , but under the assumption that the altered controller will remain the active there. The problem of actually guaranteeing that this is the case will be considered in the subsequent section. Thus, we proceed with the repair by establishing constraints on the alterations of those parameters in the TLL controller associated with the affine controller instantiated at and around the state  $x_{c.e.}$ . To be consistent with the literature, we will refer to any individual affine function instantiated by a NN as one of its *local linear functions*.

**Definition 10** (Local Linear Function). Let  $f : \mathbb{R}^n \to \mathbb{R}$ be CPWA. Then a local linear function of f is a linear function  $\ell : \mathbb{R}^n \to \mathbb{R}$  if there exists an open set  $\mathfrak{O}$  such that  $\ell(x) = \mathsf{f}(x)$  for all  $x \in \mathfrak{O}$ .

The unique semantics of TLL NNs makes them especially well suited to this local repair task because in a TLL NN, its local linear functions appear directly as neuronal parameters. In particular, all of the local linear functions of a TLL NN are described *directly* by parameters in its linear layer; i.e.  $\Theta_{\ell} = (W_{\ell}, b_{\ell})$  for scalar TLL NNs or  $\Theta_{\ell}^{\kappa} = (W_{\ell}^{\kappa}, b_{\ell}^{\kappa})$ for the  $\kappa^{\text{th}}$  output of a multi-output TLL (see Definition 6). This follows as a corollary of the following relatively straightforward proposition, borrowed from [8]:

**Proposition 1** ([8, Proposition 3]). Let  $\Xi_{N,M}$  be a scalar *TLL NN with linear function matrices*  $\Theta_{\ell} = (W_{\ell}, b_{\ell})$ . Then every local linear function of  $\mathscr{M}(\Xi_{N,M})$  is exactly equal to  $\ell_i : x \mapsto [W_{\ell}x + b_{\ell}]_{i,\cdot}$  for some  $i \in \{1, \ldots, N\}$ .

Similarly, let  $\Xi_{N,M}^{(m)}$  be a multi-output TLL, and let  $\ell$  be any local linear function of  $\mathscr{M}(\Xi_{N,M}^{(m)})$ . Then for each  $\kappa \in \{1, \ldots, m\}$ , the  $\kappa^{th}$  component of  $\ell$  satisfies  $\llbracket \ell \rrbracket_{\kappa, \cdot} = x \mapsto \llbracket W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \rrbracket_{i_{\kappa}, \cdot}$  for some  $i_{\kappa} \in \{1, \ldots, N\}$ .

**Corollary 1.** Let  $\Xi_{N,M}^{(m)}$  be a TLL over domain  $\mathbb{R}^n$ , and let  $x_{c.e.} \in \mathbb{R}^n$ . Then there exist m integers  $act_k \in \{1, \ldots, N\}$  for  $k = 1, \ldots, m$  and a closed, connected set with non-empty interior,  $R_a \subset \mathbb{R}^n$  such that

• 
$$x_{c.e.} \in R_a$$
; and  
•  $\llbracket \mathscr{M}(\Xi_{N,M}^{(m)}) \rrbracket_k = x \mapsto \llbracket W_\ell^k x + b^k \rrbracket_{act_k}$  on the set  $R_a$ .

Corollary 1 is actually a strong statement: it indicates that in a TLL, each local linear function is described directly by *its own* linear-function-layer parameters and those parameters describe *only* that local linear function.

Thus, as a consequence of Corollary 1, "repairing" the problematic local controller (local linear function) of the TLL controller in Problem 1 involves the following steps:

- identify which of the local linear functions is realized by the TLL controller at x<sub>c.e.</sub> – i.e. identifying the indices of the active local linear function at x<sub>c.e.</sub> viz. indices act<sub>κ</sub> ∈ {1,..., N} for each output κ as in Corollary 1;
- 2) <u>establish constraints</u> on the parameters of that local linear function so as to ensure repair of the counterexample; i.e. altering the elements of the rows  $[\![W_{\ell}^{\kappa}]\!]_{act_{\kappa},\cdot}$  and  $[\![b_{\ell}^{\kappa}]\!]_{act_{\kappa}}$  for each output  $\kappa$  such that the resulting linear controller repairs the counterexample as in Problem 1(*ii*); and
- 3) <u>establish constraints</u> to ensure the repaired parameters do not induce a violation of the safety constraint for the guaranteed set of safe states,  $X_{safe}$ , as in Problem 1(*i*).

## We consider these three steps in sequence as follows.

1) Identifying the Active Controller at  $x_{c.e.}$ : From Corollary 1, all of the possible linear controllers that a TLL controller realizes are exposed directly in the parameters of its linear layer matrices,  $\Theta_{\ell}^{\kappa}$ . Crucially for the repair problem, once the active controller at  $x_{c.e.}$  has been identified, the TLL parameters responsible for that controller immediately evident. This is the starting point for our repair process.

Since a TLL consists of two levels of lattice operations, it is straightforward to identify which of these affine functions is in fact active at  $x_{c.e.}$ ; for a given output,  $\kappa$ , this is can be done by evaluating  $W_{\ell}^{\kappa}x_{c.e.} + b_{\ell}^{\kappa}$  and comparing the components thereof according to the selector sets associated with the TLL controller. That is the index of the active controller for output  $\kappa$ , denoted by  $act_{\kappa}$ , is determined by the following two expressions:

$$\mu_k^{\kappa} \triangleq \arg\min_{i \in S_k^{\kappa}} \llbracket W_{\ell}^{\kappa} x_{\text{c.e.}} + b_{\ell}^{\kappa} \rrbracket_i$$
(6)

$$\operatorname{act}_{\kappa} \triangleq \arg \max_{j \in \{\mu_{\kappa}^{\kappa} | k=1,\dots,M\}} \llbracket W_{\ell}^{\kappa} x_{\mathsf{c.e.}} + b_{\ell}^{\kappa} \rrbracket_{j}$$
(7)

These expressions mirror the computations that define a TLL network, as described in Definition 6; the only difference is that max and min are replaced by arg max and arg min, respectively, so as to retrieve the index of interest instead of the network's output.

2) Repairing the Affine Controller at  $x_{c.e.}$ : Given the result of Corollary 1, the parameters of the network that result in a problematic controller at  $x_{c.e.}$  are readily apparent. Moreover, since these parameters are obviously in the linear layer of the original TLL, they are alterable under the requirement in Problem 1 that only linear-layer parameters are permitted to be used for repair. Thus, in the current context, local repair entails simply correcting the elements of the matrices  $[\![W^k_\ell]\!]_{act_k}$  and  $[\![b^k_\ell]\!]_{act_k}$ . It is thus clear that a "repaired" controller should satisfy

$$f(x_{\text{c.e.}}) + g(x_{\text{c.e.}}) \begin{bmatrix} \llbracket W_{\ell}^{1} x_{\text{c.e.}} + b_{\ell}^{1} \rrbracket_{\operatorname{act}_{1}} \\ \vdots \\ \llbracket W_{\ell}^{m} x_{\operatorname{c.e.}} + b_{\ell}^{m} \rrbracket_{\operatorname{act}_{m}} \end{bmatrix} \notin X_{\operatorname{unsafe}}.$$
(8)

Then (8) represents a *linear constraint* in the local controller to be repaired, and this constraint imposes the repair property in Problem 1(ii). That is provided that the repaired controller described by  $\{act_{\kappa}\}$  remains active at the counterexample; as noted, we consider this problem in the global stasis condition subsequently.

3) Preserving the Initial Safety Condition with the Repaired Controller: One unique aspect of the TLL NN architecture is that affine functions defined in its linear layer can be reused across regions of its input space. In particular, the controller associated with the parameters we repaired in the previous step – i.e. the indices  $\{act_{\kappa}\}$  of the linear layer matrices – may likewise be activated in or around  $X_{safe}$ . The fact that we altered these controller parameters thus means that trajectories emanating from  $X_{safe}$  may be affected in turn by our repair efforts: that is the repairs we made to the controller to address Problem 1(i) may simultaneously alter the TLL in a way that **undoes** the requirement in Problem 1(i) – i.e. the initial safety guarantee on  $X_{safe}$  and  $\mathcal{M}(\Xi_{N,M}^{(m)})$ . Thus, local repair of the problematic controller must account for this safety property, too.

We accomplish this by bounding the reach set of (1) for initial conditions in  $X_{safe}$ , and for this we employ the usual strategy of bounding the relevant Lipschitz constants. Naturally, since the TLL controller is a CPWA controller operated in closed loop, these bounds will also incorporate the size of the TLL controller parameters  $\|[W_{\ell}^{\kappa}]_{i}\|$  and  $\|[b_{\ell}^{\kappa}]_{i}\|$  for  $\kappa \in \{1, \ldots, m\}$  and  $i \in \{1, \ldots, N\}$ .

In general, however, we have the following proposition.

**Proposition 2.** Consider system dynamics (1), and suppose that the state x is confined to known compact workspace,  $X_{WS}$  (see Definition 2). Also, let T be the integer time horizon from Definition 8. Finally, assume that a closed-loop CPWA  $\Psi : \mathbb{R}^n \to \mathbb{R}^m$  is applied to (1), and that  $\Psi$  has local linear functions  $\mathcal{L}_{\Psi} = \{x \mapsto w_k x + b_k | k = 1, ..., N\}.$  Moreover, define the function  $\beta$  as

$$\beta(\|w\|, \|b\|) \triangleq \sup_{x_0 \in X_{safe}} \left( \|f(x_0) - x_0\| + \|g(x_0)\| \cdot \|w\| \cdot ext(X_{ws}) + \|g(x_0)\| \cdot \|b\| \right)$$
(9)

and in turn define

$$\beta_{max}(\Psi) \triangleq \beta\Big(\max_{w \in \{w_k | k=1,...,N\}} \|w\|, \max_{b \in \{b_k | k=1,...,N\}} \|b\|\Big).$$
(10)

Finally, define the function L as in (11), and in turn define

$$L_{max}(\Psi) \triangleq L\Big(\max_{w \in \{w_k | k=1,...,N\}} \|w\|, \max_{b \in \{b_k | k=1,...,N\}} \|b\|\Big).$$
(12)
Then for all  $x_0 \in X_{\text{safe}}$ ,  $i \in \{1,...,T\}$ , we have:

Then for all 
$$x_0 \in X_{\text{safe}}$$
,  $i \in \{1, \ldots, T\}$ , we have:

$$\|\zeta_T^{x_0}(\Psi) - x_0\| \le \beta_{max}(\Psi) \cdot \sum_{k=0}^T L_{max}(\Psi)^k.$$
(13)

The proof of Proposition 2 is in Appendix VII of [9].

Proposition 2 bounds the size of the reach set for (1) in terms of an arbitrary CPWA controller,  $\Psi$ , when the system is started from  $X_{safe}$ . This proposition is naturally applied in order to find bounds for safety with respect to the unsafe region  $X_{\text{unsafe}}$  as follows.

**Proposition 3.** Let T,  $X_{\text{ws}}$ ,  $\Psi$  and  $\mathcal{L}_{\Psi}$  be as in Proposition 2, and let  $\beta_{max}$  and  $L_{max}$  be two constants s.t. for all  $\delta x \in \mathbb{R}^n$ 

$$\begin{aligned} \|\delta x\| &\leq \beta_{\max} \cdot \sum_{k=0}^{T} L_{\max}{}^{k} \\ &\implies \forall x_0 \in X_{\text{safe}} \cdot \left(x_0 + \delta x \notin X_{\text{unsafe}}\right) \end{aligned} (14)$$

If  $\beta_{max}(\Psi) \leq \beta_{max}$  and  $L_{max}(\Psi) \leq L_{max}$ , then trajectories of (1) under closed loop controller  $\Psi$  are safe in the sense that

$$\forall x_0 \in X_{\text{safe}} \forall i \in \{1, \dots, T\} \, . \, \zeta_t^{x_0}(\Psi) \notin X_{\text{unsafe}}.$$
(15)

The proof of is a more or less straightforward application of Proposition 2, and so can be found in Appendix VII of [9].

In particular, Proposition 3 states that if we find constants  $\beta_{\text{max}}$  and  $L_{\text{max}}$  that satisfy (14), then we have a way to bound the parameters of any CPWA controller (via  $\beta$  and L) so that that controller is safe in closed loop. This translates to conditions that our repaired controller must satisfy in order to preserve the safety property required in Problem 1(i).

Formally, this entails particularizing Proposition 2 and 3 to the TLL controllers associated with the repair problem.

Corollary 2. Again consider system (1) confined to workspace  $X_{WS}$  as before. Also, let  $\beta_{max}$  and  $L_{max}$  be such that they satisfy the assumptions of Proposition 3, viz. (14).

Now, let  $\Xi_{N,M}^{(m)}$  be the TLL controller as given in Problem 1, and let  $\Theta_{\ell}^{\kappa} = (W_{\ell}^{\kappa}, b_{\ell}^{\kappa})$  be its linear layer matrices for

outputs  $\kappa = 1, \ldots, m$  as usual. For this controller, define the following two quantities:

$$\Omega_W \triangleq \max_{w \in \bigcup_{\kappa=1}^m \{ [W_\ell^{\kappa}]_j | j=1,\dots,N \}} \|w\|$$
(16)

$$\Omega_b \triangleq \max_{b \in \cup_{\kappa=1}^m \{ \llbracket b_\ell^\kappa \rrbracket_j | j=1,\dots,N \}} \|b\|$$
(17)

so that  $\beta_{max}(\Xi_{N,M}^{(m)}) = \beta(\Omega_W, \Omega_b)$  and  $L_{max}(\Xi_{N,M}^{(m)}) = L(\Omega_W, \Omega_b)$ . Finally, let indices  $\{act_\kappa\}_{\kappa=1}^m$  specify the active local linear functions of  $\Xi_{N,M}^{(m)}$  that are to be repaired, as described in Subsection IV-A.1 and IV-A.2. Let  $\overline{w}_{act_{\kappa}}^{\kappa}$ and  $\tilde{b}_{act_{\kappa}}^{\kappa}$  be any repaired values of  $[\![W_{\ell}^{\kappa}]\!]_{act_{\kappa}}$ , and  $[\![b_{\ell}^{\kappa}]\!]_{act_{\kappa}}$ , respectively.

If the following four conditions are satisfied

$$\beta(\|\bar{w}_{act_{\kappa}}^{\kappa}\|, \|\bar{b}_{act_{\kappa}}^{\kappa}\|) \le \beta_{max}$$
(18)

$$\beta_{max}(\Xi_{N,M}^{(m)}) \le \beta_{max} \tag{19}$$

$$L(\|\bar{w}_{act_{\kappa}}^{\kappa}\|, \|\bar{b}_{act_{\kappa}}^{\kappa}\|) \le L_{max}$$

$$\tag{20}$$

$$L_{max}(\Xi_{N,M}^{(m)}) \le L_{max} \tag{21}$$

then the following hold for all  $x_0 \in X_{safe}$ :

$$\|\zeta_T^{x_0}(\bar{\Xi}_{N,M}^{(m)}) - x_0\| \le \beta_{max} \cdot \sum_{k=0}^T L_{max}^k$$
(22)

and hence

$$\forall i \in \{1, \dots, T\} . \zeta_i^{x_0}(\bar{\Xi}_{N,M}^{(m)}) \notin X_{\text{unsafe}}.$$
 (23)

The proof of Corollary 2 is in Appendix VII of [9].

The conclusion (22) of Corollary 2 should be interpreted as follows: the bound on the reach set of the repaired controller,  $\bar{\Xi}_{N,M}^{(m)}$ , is no worse than the bound on the reach set of the original TLL controller given in Problem 1. Hence, by the assumptions borrowed from Proposition 3, conclusion (23) of Corollary 2 indicates that the repaired controller  $\bar{\Xi}_{N,M}^{(m)}$ remains safe in the sense of Problem 1(i) – i.e. closed-loop trajectories emanating from  $X_{safe}$  remain safe on horizon T.

For the subsequent development of our algorithm, (18) and (20) will play the crucial role of ensuring that the repaired controller respects the guarantee of Problem 1(i).

#### B. Global TLL Alteration for Repaired Controller Activation

In the context of local repair, we identified the local linear function instantiated by the TLL controller, and repaired the parameters associated with that particular function - i.e. the repairs were affected on a particular, indexed row of  $W_{\ell}^{\kappa}$ and  $b_{\ell}^{\kappa}$ . We then proceeded under the assumption that the affine function at that index would remain active in the output of the TLL network at the counterexample, even after altering its parameters. Unfortunately, this is not case in a TLL network per se, since the value of each local linear function at a point interacts with the selector matrices (see Definition 6) to determine whether it is active or not. In other words, changing the parameters of a particular indexed local linear function in a TLL will change its output value at any

$$L(\|w\|, \|b\|) \triangleq L_f + L_g \cdot \sup_{x_0 \in X_{\mathsf{safe}}} \|w\| \cdot \|x_0\| + \sup_{x_0 \in X_{\mathsf{safe}}} \|w\| \cdot \|g(x_0)\| + L_g \cdot \|b\|$$
(11)

given point (in general), and hence also the region on which said indexed local linear function is *active*. Analogous to the local alteration consider before, we thus need to **devise** global constraints sufficient to enforce the activation of the repaired controller at  $x_{c.e.}$ .

This observation is manifest in the computation structure that defines a TLL NN: a particular affine function is active in the output of the TLL if and only if it is active in the output of one of the min networks (see Definition 6), and the output of that same min network exceeds the output of all others, thereby being active at the output of the final max network (again, see Definition 6). Thus, ensuring that a particular, indexed local linear function is active at the output of a TLL entails ensuring that that function

- (a) appears at the output of one of the min networks; and
- (b) appears at the output of the max network, by exceeding the outputs of all the *other* min networks.

Notably, this sequence also suggests a mechanism for meeting the task at hand: ensuring that the repaired controller remains active at the counter example.

Formally, we have the following proposition.

**Proposition 4.** Let  $\Xi_{N,M}^{(m)}$  be a TLL NN over  $\mathbb{R}^n$  with output-component linear function matrices  $\Theta_{\ell}^{\kappa} = (W_{\ell}^{\kappa}, b_{\ell}^{\kappa})$  as usual, and let  $x_{c.e.} \in \mathbb{R}^n$ .

Then the index  $act_{\kappa} \in \{1, \ldots, N\}$  denote the local linear function that is active at  $x_{c.e.}$  for output  $\kappa$ , as described in Corollary 1, if and only if there exists index  $sel_{\kappa} \in \{1, \ldots, M\}$  such that

(i) for all 
$$i \in S_{sel_{\kappa}}^{\kappa}$$
 and any  $x \in R_a$ ,

$$\llbracket W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \rrbracket_{act_{\kappa}, \cdot} \leq \llbracket W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \rrbracket_{i, \cdot}$$
(24)

*i.e. the active local linear function "survives" the* min *network associated with selector set*  $S_{sel_r}^{\kappa}$ ; and

(ii) for all  $j \in \{1, ..., M\} \setminus \{sel_{\kappa}\}$  there exists an index  $\iota_{j}^{\kappa} \in \{1, ..., N\}$  s.t. for all  $x \in R_{a}$ 

$$\llbracket W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \rrbracket_{\iota_{i}^{\kappa},\cdot} \leq \llbracket W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \rrbracket_{act_{\kappa},\cdot}$$
(25)

i.e. the active local linear function "survives" the max network of output  $\kappa$  by exceeding the output of all of the other min networks.

This proposition follows calculations similar to those mentioned before; the proof is in Appendix VII of [9].

The "only if" portion of Proposition 4 thus directly suggests constraints to impose such that the desired local linear function  $\operatorname{act}_{\kappa}$  is active on its respective output. In particular, among the **non-active local linear functions** at  $x_{c.e.}$ , at least one must be altered from each of the selector sets  $s_j : j \in \{1, \ldots, M\} \setminus \{\operatorname{sel}_{\kappa}\}$ . The fact that these alterations must be made to local linear functions which are not active at the counterexample warrants the description of this procedure as "global alteration".

Finally, however, we note that altering these un-repaired local linear functions – i.e. those not indexed by  $act_{\kappa}$  – may create the same issue described in Section IV-A.3. Thus, for any of these global alterations additional safety constraints like (18) and (20) must be imposed on the altered parameters.

#### V. MAIN ALGORITHM

Problem 1 permits the alteration of linear-layer parameters in the original TLL controller to perform repair. In Section IV, we developed *constraints* on these parameters to perform

- first, local alteration to ensure repair of the defective controller at  $x_{c.e.}$ ; and
- subsequently, global alteration to ensure that the repaired local controller is activated at and around  $x_{c.e.}$ .

The derivations of both sets of constraints implies that they are merely sufficient conditions for their respective purposes, so there is no guarantee that any subset of them are jointly feasible. Moreover, as a "repair" problem, any repairs conducted must involve minimal alteration – Problem 1(v).

Thus, the core of our algorithm is to employ a convex solver to find the minimally altered TLL parameters that also satisfy the local and global constraints we have outlined for successful repair with respect to the other aspects of Problem 1. The fact that the local repair constraints are prerequisite to the global activation constraints means that we will employ a convex solver on two optimization problems *in sequence*: first, to determine the feasibility of local repair and effectuate that repair in a minimal way; and then subsequently to determine the feasibility of activating said repaired controller as required and effectuating that activation in a minimal way.

#### A. Optimization Problem for Local Alteration (Repair)

Local alteration for repair starts by identifying the active controller at the counterexample, as denoted by the index  $\operatorname{act}_{\kappa}$  for each output of the controller,  $\kappa$ . The local controller for each output is thus the starting point for repair in our algorithm, as described in the prequel. From this knowledge, an explicit constraint sufficient to repair the local controller at  $x_{c.e.}$  is specified directly by the dynamics: see (8).

Our formulation of a safety constraint for the locally repaired controller requires additional input, though. In particular, we need to identify constants  $\beta_{\text{max}}$  and  $L_{\text{max}}$  such that the non-local controllers satisfy (19) and (21). Then Corollary 2 implies that (18) and (20) are constraints that ensure the repaired controller satisfies Problem 1(*i*). For this we take the naive approach of setting  $\beta_{\text{max}} = \beta(\Xi_{N,M}^{(m)})$ , and then solving for the smallest  $L_{\text{max}}$  that ensures safety for that particular  $\beta_{\text{max}}$ . In particular, we set

$$L_{\max} = \inf\{L' > 0 \mid \beta_{\max} \cdot \sum_{k=0}^{T} {L'}^k = \inf_{\substack{x_s \in X_{\text{safe}} \\ x_u \in X_{\text{unsafe}}}} \|x_s - x_u\|\}.$$
(26)

Given this information the local repair optimization problem can be formulated for a multi-output TLL as:

$$\begin{aligned} \mathbf{Local} &: \min_{\overline{w}_{\mathsf{act}_{\kappa}}^{\kappa}, \overline{b}_{\mathsf{act}_{\kappa}}^{\kappa}}} \sum_{\kappa=1}^{m} \| [\![W_{\ell}^{\kappa}]\!]_{\mathsf{act}_{\kappa}} - \overline{w}_{\mathsf{act}_{\kappa}}^{\kappa} \| + \| [\![b_{\ell}^{\kappa}]\!]_{\mathsf{act}_{\kappa}} - \overline{b}_{\mathsf{act}_{\kappa}}^{\kappa} \| \\ &\text{s.t. } f(x_{\mathsf{c.e.}}) + g(x_{\mathsf{c.e.}}) \begin{bmatrix} \overline{w}_{\mathsf{act}_{1}}^{1} x_{\mathsf{c.e.}} + \overline{b}_{\mathsf{act}_{1}}^{1} \\ \vdots \\ \overline{w}_{\mathsf{act}_{m}}^{m} x_{\mathsf{c.e.}} + \overline{b}_{\mathsf{act}_{m}}^{m} \end{bmatrix} \notin X_{\mathsf{unsafe}} \\ &\forall \kappa = 1, \dots, m \quad L(\| \overline{w}_{\mathsf{act}_{\kappa}}^{\kappa} \|, \| \overline{b}_{\mathsf{act}_{\kappa}}^{\kappa} \|) \leq L_{\mathsf{max}} \\ &\forall \kappa = 1, \dots, m \quad \beta(\| \overline{w}_{\mathsf{act}_{\kappa}}^{\kappa} \|, \| \overline{b}_{\mathsf{act}_{\kappa}}^{\kappa} \|) \leq \beta_{\mathsf{max}} \\ &\forall \kappa = 1, \dots, m \quad L(\Xi_{NM}^{m}) \leq L_{\mathsf{max}} \end{aligned}$$

Note: the final collection of constraints on  $L(\Xi_{N,M}^{(m)})$  is necessary to ensure that (21) is satisfied and Corollary 2 is applicable (equation (19) is satisfied by definition of  $\beta_{\text{max}}$ ).

#### B. Optimization Problem for Global Alteration (Activation)

If the optimization problem **Local** is feasible, then the local controller at  $x_{c.e.}$  can successfully be repaired, and the global activation of said controller can be considered. Since we are starting with a local linear function we want to be active at and around  $x_{c.e.}$ , we can retain the definition of act<sub> $\kappa$ </sub> from the initialization of **Local**. Moreover, since Problem 1 preserves the selector matrices of the original TLL controller, we will define the selector indices, sel<sub> $\kappa$ </sub>, in terms of the activation pattern of the *original*, defective local linear controller (although this is not required by the repair choices we have made: other choices are possible).

Thus, in order to formulate an optimization problem for global alteration, we need to define constraints compatible with Proposition 4 based on the activation/selector indices described above. Part (i) of the conditions in Proposition 4 is unambiguous at this point: it says that the desired active local linear function,  $\operatorname{act}_{\kappa}$ , must have the minimum output from among those functions selected by selector set  $s_{\operatorname{sel}_{\kappa}}$ . Part (ii) of the conditions in Proposition 4 is ambiguous however: we only need to specify *one* local linear function from each of the *other* min groups to be "forced" lower than the desired active local linear function. In the face of this ambiguity, we select these functions using indices  $t_i^{\kappa} : j \in \{1, \ldots, M\} \setminus \{\operatorname{act}_{\kappa}\}$  that are defined as follows:

$$\iota_{j}^{\kappa} \triangleq \arg\min_{i \in s_{j}^{\kappa}} \llbracket W_{\ell}^{\kappa} x_{\text{c.e.}} + b_{\ell}^{\kappa} \rrbracket_{i}.$$
(27)

That is we form our global alteration constraint out of the non-active controllers which are have the *lowest outputs among their respective min groups*. We reason that these local linear functions will in some sense require the least alteration in order to satisfy Part (*ii*) of Proposition 4, which requires their outputs to be less than the local linear function that we have just repaired.

Thus, we can formulate the global alteration optimization problem as follows:

$$\begin{split} \mathbf{Global}: & \min_{\overline{w}_{\ell}^{\kappa}, b_{\ell}^{\kappa}} \sum_{\kappa=1}^{m} \| W_{\ell}^{\kappa} - \overline{W}_{\ell} \| + \| b_{\ell}^{\kappa} - \bar{b}_{\ell} \| \\ \text{s.t.} & \forall \kappa = \{1, \dots, m\} & . [\![W_{\ell}^{\kappa}]\!]_{\operatorname{act}_{\kappa}, \cdot} = \overline{w}_{\operatorname{act}_{\kappa}}^{\kappa} \\ & \forall \kappa = \{1, \dots, m\} & . [\![b_{\ell}^{\kappa}]\!]_{\operatorname{act}_{\kappa}, \cdot} = \bar{b}_{\operatorname{act}_{\kappa}}^{\kappa} \\ & \forall \kappa = \{1, \dots, m\} \quad \forall i \in s_{\operatorname{sel}_{\kappa}} . \ \overline{w}_{\operatorname{act}_{\kappa}}^{\kappa} x_{\operatorname{c.e.}} + \bar{b}_{\operatorname{act}_{\kappa}}^{\kappa} \\ & \leq [\![W_{\ell}^{\kappa} x_{\operatorname{c.e.}} + b_{\ell}^{\kappa}]\!]_{i} \\ & \forall \kappa = \{1, \dots, m\} \\ & \forall j \in \{1, \dots, M\} \backslash \{\operatorname{sel}_{\kappa}\} . [\![W_{\ell}^{\kappa} x_{\operatorname{c.e.}} + b_{\ell}^{\kappa}]\!]_{i_{j}^{\kappa}} \\ & \leq \overline{w}_{\operatorname{act}_{\kappa}}^{\kappa} x_{\operatorname{c.e.}} + \bar{b}_{\operatorname{act}_{\kappa}}^{\kappa} \end{split}$$

where of course  $\bar{w}_{act_{\kappa}}^{\kappa}$  and  $\bar{b}_{act_{\kappa}}^{\kappa}$  are the repaired local controller parameters obtained from the optimal solution of **Local**. Note that the first two sets of equality constraints merely ensure that **Global** does not alter these parameters.

#### C. Main Algorithm

A pseudo-code description of our main algorithm is shown in Algorithm 1, as repairTLL. It collects all of the initializations from Section IV, Subsection V-A and Subsection V-B. Only the functions FindActCntrl and FindActSlctr encapsulate procedures defined in this paper; their implementation is nevertheless adequately

input : 
$$f, g$$
 system dynamics (1)  
 $X_{ws}$  workspace set  
 $\Xi_{N,M}^{(m)}$  TLL controller to repair  
 $T$  safety time horizon  
 $X_{safe}$  set of safe states under  $\Xi_{N,M}^{(m)}$   
 $x_{c.e.}$  counterexample state  
output:  $\Xi_{N,M}^{(m)}$  repaired TLL controller

#### 1 function

repairTLL  $(f, g, X_{ws}, \Xi_{NM}^{(m)}, T, X_{safe}, x_{c.e.})$  $gMaxSafe \leftarrow \sup_{x_0 \in X_{safe}} ||g(x_0)||$ 2 beta (w,b) :=  $\sup_{x_0 \in X_{safe}} ||g(x_0)||$ + gMaxSafe \* w \* ext( $X_{ws}$ ) + gMaxSafe \* b L (w,b) :=  $L_f + L_g$  \* w \*  $\sup_{x_0 \in X_{safe}} ||x_0||$ + w \* gMaxSafe +  $L_g$  \* b 3 4 5 6  $\Omega_W \leftarrow \max_{w \in \bigcup_{\kappa=1}^m \{ \llbracket W_\ell^\kappa \rrbracket_j | j=1,\dots,N \}} \|w\|$ 7  $\Omega_b \leftarrow \max_{b \in \bigcup_{\kappa=1}^m \{ [\![b_\ell^\kappa]\!]_j \mid j=1,\dots,N \}} \|b\|$ 8 betaMax  $\leftarrow$  beta ( $\Omega_W$ ,  $\Omega_b$ ) 9 dSafe  $\leftarrow \inf_{\substack{x_s \in X_{\text{safe}} \\ x_u \in X_{\text{unsafe}}}} \|x_s - x_u\|$ Lmax  $\leftarrow \inf \{L' | \text{ betaMax } * \sum_{k=0}^T L'^k = \text{dSafe} \}$ 10 11  $\begin{aligned} & \{ \operatorname{act}_{\kappa} \}_{\kappa=1}^{m} \leftarrow \operatorname{FindActCntrl} (\Xi_{N,M}^{(m)}, x_{c.e.}) \\ & \{ \operatorname{sel}_{\kappa} \}_{\kappa=1}^{m} \leftarrow \operatorname{FindActSlctr} (\Xi_{N,M}^{(m)}, x_{c.e.}) \end{aligned}$ 12 13 Initialize (*Local*,  $\{f, g, \Xi_{N,M}^{(m)}, x_{c.e.}, L, Lmax, \}$ 14 beta, betaMax,  $\{act_{\kappa}\}_{\kappa=1}^{m}, X_{unsafe}\}$ )  $sol \leftarrow Solve(Local)$ 15 if not sol.feasible() then 16 return False 17 else 18  $\{(\mathbf{w}^{\kappa},\mathbf{b}^{\kappa})\}_{\kappa=1}^{m} \leftarrow ext{ sol.optimalValue ()}$ 19 end 20 21 for  $\kappa$  in  $1, \ldots, m$  do for j in  $\{1,\ldots,M\}\backslash\{\mathit{sel}_\kappa\}$  do 22  $\| \iota_i^{\kappa} \leftarrow \arg\min_{i \in s_i} \| \llbracket W_{\ell}^{\kappa} x_{\mathsf{c.e.}} + b_{\ell}^{\kappa} \rrbracket_i \|$ 23 end 24 25 end  $\begin{array}{l} \text{Initialize} (\textit{Global}, \{f, g, \Xi_{N,M}^{(m)}, x_{\textit{c.e.}}, \texttt{L}, \texttt{Lmax}, \\ \texttt{beta}, \texttt{beta} \texttt{Max}, \{act_{\kappa}\}_{\kappa=1}^{m}, \{\iota_{j}^{\kappa}\}_{\kappa, j}, \{(\texttt{w}^{\kappa}, \texttt{b}^{\kappa})\}\}) \end{array}$ 26 sol ← Solve (Global) 27 28 if not sol.feasible() then return False 29 30 else  $| \{ (\mathbf{W}^{\kappa}, \mathbf{B}^{\kappa}) \}_{\kappa=1}^{m} \leftarrow ext{ sol.optimalValue ()}$ 31 32 **return**  $\Xi_{N,M}^{(m)}$ .setLinLayer ({( $W^{\kappa}, B^{\kappa}$ )}\_{\kappa=1}) 33 34 end

Algorithm 1: repairTLL.

described in Subsection IV-A.1 and Proposition 4, respectively. The correctness of repairTLL follows from the results in those sections.

#### VI. NUMERICAL EXAMPLES

We illustrate the results in this paper on a four-wheel car described by the following model:

$$x(t+1) = \begin{bmatrix} x_1(t) + V\cos(x_3(t)) \cdot t_s \\ x_2(t) + V\sin(x_3(t)) \cdot t_s \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t_s \end{bmatrix} v(t) \quad (28)$$

where the state  $x(t) = [p_x(t) \ p_y(t) \ \Psi(t)]^T$  for vehicle position  $(p_x \ p_y)$  and yaw angle  $\Psi$ , the and control input v is the vehicle yaw rate. The parameters are the translational speed of the vehicle, V (meters/sec); and the sampling period,  $t_s$  (sec). For the purposes of our experiments, we consider a compact workspace  $X_{ws} = [-3,3] \times [-4,4] \times [-\pi,\pi]$ ; a safe set of states  $X_{safe} = [-0.25, 0.25] \times [-\pi,\pi]$ ;  $[-0.75, -0.25] \times [-\frac{\pi}{8}, \frac{\pi}{8}]$ , which was verified using NNV [7] over 100 iterations; and an unsafe region  $X_{\text{unsafe}}$  specified by  $[0 \ 1 \ 0] \cdot x > 3$ . Furthermore, we consider model parameters: V = 0.3 m/s and  $t_s = 0.01$  seconds.

All experiments were executed on an Intel Core i5 2.5-GHz processor with 8 GB of memory. We collected 1850 data points of state-action pairs from a PI Controller used to steer the car over  $X_{ws}$  while avoiding  $X_{unsafe}$ . Then, to exhibit a NN controller with a counterexample, a TLL NN with N = 50 and M = 10 was trained from a corrupted version of this data-set: we manually changed the control on 25 data points close to  $X_{\text{unsafe}}$  so that the car would steer into it. We simulated the trajectories of the car using this TLL NN controller for different  $x_0$  and identified  $x_{c.e.} =$ [0 2.999 0.2] as a valid counterexample for safety after two time steps. Finally, to repair this faulty NN, we found all the required bounds for both system dynamics and NN parameters and a horizon of T = 7. We found the required safety constraints  $\beta_{\text{max}} = 0.0865$  and  $L_{\text{max}} = 1.4243$ . Then, from  $x_{c.e.}$  we obtained the controller  $K = [K_w \ K_b]$  where  $K_w = [-0.1442, -0.5424, -0.425]$  and  $K_b = [2.223]$ .

Next, we ran our algorithm to repair the counterexample using CVX (convex solver). The result of the first optimization problem, Local, was the linear controller:  $\bar{K}_w = [-0.0027 - 0.0487 - 0.0105]$  and  $\bar{K}_b = [-9.7845];$ this optimization required a total execution time of 1.89 sec. The result of the second optimization problem, Global successfully activated the repaired controller, and had an optimal cost of 8.97; this optimization required a total execution time of 6.53 sec. We also compare the original TLL Norms ||W|| = 6.54 and ||b|| = 5.6876 with the repaired:  $||\overline{W}|| = 11.029$  and  $||\overline{b}|| = 5.687$ .

Finally, we simulated the motion of the car using the repaired TLL NN controller for 50 steps. Shown in Fig. 1 are the state trajectories of both original faulty TLL controller and repaired TLL Controller starting from the  $x_{c.e.}$  In the latter the TLL controller met the safety specifications.

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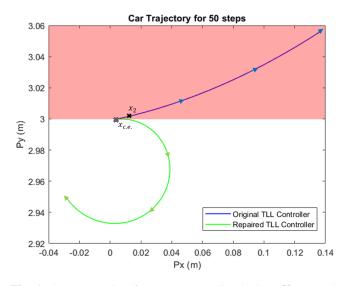


Fig. 1: System starting from  $x_{c.e.}$  goes directly into  $X_{unsafe}$  and Repaired  $x_{c.e.}$  produces a safe trajectory. Red area is  $X_{unsafe}$ , Red Cross is  $x_{c.e.}$  and Black Cross shows state after 2 steps.

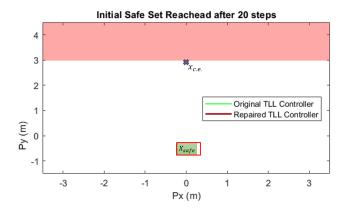


Fig. 2: Initial Safe set before and after repair for 20 steps. Red area is  $X_{\text{unsafe}}$ ; Red Cross is  $x_{\text{c.e.}}$ ;  $X_{\text{ws}} = [-3, 3] \times [-4, 4]$ 

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## VII. APPENDIX

# A. Proof of Corollary 1

*Proof.* It is straightforward to see that every point x in the domain of  $\mathscr{M}(\Xi_{N,M}^{(m)})$  belongs to the closure of some open set  $\mathfrak{D}$ , on which  $\mathscr{M}(\Xi_{N,M}^{(m)})$  is affine (i.e. equal to one of its local linear functions). For if this weren't the case, then there would be an open subset of the domain of  $\mathscr{M}(\Xi_{N,M}^{(m)})$ , where it wasn't affine, thus contradicting the CPWA property of a ReLU NN.

Thus, let  $\mathfrak{D}_{x_{c.e.}}$  be such an open set that includes  $x_{c.e.}$  in its closure, and let  $\ell : \mathbb{R}^n \to \mathbb{R}^m$  be the local linear function of  $\mathscr{M}(\Xi_{N,M}^{(m)})$  on  $\mathfrak{D}_{x_{c.e.}}$ . We can further assume that  $\mathfrak{D}_{x_{c.e.}}$  is connected without loss of generality, so set  $R_a = \overline{\mathfrak{D}}_{x_{c.e.}}$ .

By Proposition 1, there exists indices  $\{act_{\kappa}\}_{\kappa=1}^{m}$  such that

$$\llbracket \ell \rrbracket_{\operatorname{act}_{\kappa}} = x \mapsto \llbracket W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \rrbracket_{\operatorname{act}_{\kappa}}.$$
(29)

But by the definition of  $\ell$  and the above, we also have that

$$\forall x \in \mathfrak{D}_{\mathsf{c.e.}} \, \left[ W_{\ell}^{\kappa} x + b_{\ell}^{\kappa} \right]_{\mathsf{act}_{\kappa}} = \left[ \mathscr{M}(\Xi_{N,M}^{(m)})(x) \right]_{\mathsf{act}_{\kappa}}. \tag{30}$$

Thus, the conclusion of the corollary holds on  $\mathfrak{D}_{x_{\text{c.e.}}}$ ; it holds on  $R_a = \overline{D}_{x_{\text{c.e.}}}$  by continuity of  $\mathscr{M}(\Xi_{N,M}^{(m)})$ .

# B. Proof of Proposition 2

**Lemma 1.** Let  $F : x \mapsto g(x) \cdot u(x)$  for Lipschitz continuous functions  $g : \mathbb{R}^n \to \mathbb{R}^{(n \cdot m)}$  (with output an  $(n \times m)$ real-valued matrix) and  $u : \mathbb{R}^n \to \mathbb{R}^m$  with Lipschitz constants  $L_q$  and  $L_u$ , respectively.

Then on compact subset  $X \subset \mathbb{R}^n$ , F is Lipschitz continuous with Lipschitz constant  $L_F = L_g \cdot \sup_{x \in X} ||u(x)|| + L_u \cdot \sup_{x \in X} ||g(x)||.$ 

*Proof.* This follows by straightforward manipulations as follows. Let  $x, x' \in X$  and note that:

$$\begin{aligned} \|g(x)u(x) - g(x')u(x')\| \\ &= \|g(x)u(x) + (-g(x') + g(x'))u(x) - g(x')u(x')\| \\ &= \|(g(x) - g(x'))u(x) + g(x')(u(x) - u(x'))\| \\ &\leq \|g(x) - g(x')\| \cdot \|u(x)\| + \|u(x) - u(x')\| \cdot \|g(x')\| \\ &\leq (L_g \cdot \|u(x)\| + L_u \cdot \|g(x')\|) \cdot \|x - x'\| \\ &\leq (L_g \cdot \sup_{x \in X} \|u(x)\| + L_u \cdot \sup_{x' \in X} \|g(x')\|) \cdot \|x - x'\|. \end{aligned}$$

*Proof.* (Proposition 2) We will expand and bound the quantity on the left-hand side of the conclusion, (13).

$$\begin{aligned} \|\zeta_T^{x_0}(\Psi) - x_0\| \\ &= \|\zeta_T^{x_0}(\Psi) - \zeta_{T-1}^{x_0}(\Psi) + \zeta_{T-1}^{x_0}(\Psi) - x_0\| \\ &\leq \|\zeta_T^{x_0}(\Psi) - \zeta_{T-1}^{x_0}(\Psi)\| + \|\zeta_{T-1}^{x_0}(\Psi) - x_0\| \end{aligned} (31)$$

We then bound the first term as follows:

$$\begin{aligned} &\|\zeta_{T}^{x_{0}}(\Psi) - \zeta_{T-1}^{x_{0}}(\Psi)\| \\ &\leq \|f(\zeta_{T-1}^{x_{0}}(\Psi)) - f(\zeta_{T-2}^{x_{0}}(\Psi))\| \\ &+ \left\|g(\zeta_{T-1}^{x_{0}}(\Psi)) \cdot \left[w(\zeta_{T-1}^{x_{0}}(\Psi)) \cdot \zeta_{T-1}^{x_{0}}(\Psi) + b(\zeta_{T-1}^{x_{0}}(\Psi))\right] \right\| \\ &- g(\zeta_{T-2}^{x_{0}}(\Psi)) \cdot \left[w(\zeta_{T-2}^{x_{0}}(\Psi)) \cdot \zeta_{T-2}^{x_{0}}(\Psi) + b(\zeta_{T-2}^{x_{0}}(\Psi))\right] \right\| \end{aligned}$$
(32)

where the functions  $w : \mathbb{R}^n \to \mathbb{R}^n$  and  $b : \mathbb{R}^n \to \mathbb{R}$  return a (unique) choice of the linear (weights) and affine (bias) of the local linear function of  $\Psi$  that is active at their argument.

Now, we collect the  $w(\cdot)$  and  $b(\cdot)$  terms in right-hand side of (32). That is:

$$\begin{split} \|\zeta_{T}^{x_{0}}(\Psi) - \zeta_{T-1}^{x_{0}}(\Psi)\| \\ &\leq \|f(\zeta_{T-1}^{x_{0}}(\Psi)) - f(\zeta_{T-2}^{x_{0}}(\Psi))\| \\ &+ \left\|g(\zeta_{T-1}^{x_{0}}(\Psi))w(\zeta_{T-1}^{x_{0}}(\Psi))\zeta_{T-1}^{x_{0}}(\Psi) \\ &- g(\zeta_{T-2}^{x_{0}}(\Psi)w(\zeta_{T-2}^{x_{0}}(\Psi))\zeta_{T-2}^{x_{0}}(\Psi)\| \\ &+ \left\|g(\zeta_{T-1}^{x_{0}}(\Psi))b(\zeta_{T-1}^{x_{0}}(\Psi)) - g(\zeta_{T-2}^{x_{0}}(\Psi)b(\zeta_{T-2}^{x_{0}}(\Psi))\right\| \end{split}$$

The first term in the above can be directly bounded using the Lipschitz constant of f. Also, since there are only finitely many local linear function of  $\Psi$ ,  $b(\cdot)$  takes one of finitely many values across the entire state space, and we may bound the associated term using this observation. Finally, we can Lemma 1 to the second term, noting that the linear function defined by  $w(\cdot)$  has Lipschitz constant  $||w(\cdot)||$  and there are only finitely many possible values for this quantity (one for each local linear function). This yields the following bound:

$$\begin{aligned} \|\zeta_{T}^{x_{0}}(\Psi) - \zeta_{T-1}^{x_{0}}(\Psi)\| \\ &\leq L_{f} \cdot \|\zeta_{T-1}^{x_{0}}(\Psi) - \zeta_{T-2}^{x_{0}}(\Psi)\| \\ &+ \left(L_{g} \cdot \sup_{x \in X_{ws}} \|w(x) \cdot x\| + \max_{k} \|w_{k}\| \sup_{x \in X_{ws}} \|g(x)\|\right) \\ &\quad \cdot \|\zeta_{T-1}^{x_{0}}(\Psi) - \zeta_{T-2}^{x_{0}}(\Psi)\| \\ &+ \max_{k} \|b_{k}\| \cdot L_{g} \cdot \|\zeta_{T-1}^{x_{0}}(\Psi) - \zeta_{T-2}^{x_{0}}(\Psi)\| \end{aligned}$$

If we simplify, then we see that we have

$$\begin{aligned} \|\zeta_T^{x_0}(\Psi) - \zeta_{T-1}^{x_0}(\Psi)\| \\ &\leq L_{\max}(\Psi) \cdot \|\zeta_{T-1}^{x_0}(\Psi) - \zeta_{T-2}^{x_0}(\Psi)\| \end{aligned} (33)$$

with  $L_{\max}(\Psi)$  as defined in the statement of the Proposition. Now, we expand the final term of (31) as

$$\begin{aligned} \|\zeta_{T-1}^{x_0}(\Psi) - x_0\| \\ &\leq \|\zeta_{T-1}^{x_0}(\Psi) - \zeta_{T-2}^{x_0}(\Psi)\| + \|\zeta_{T-2}^{x_0}(\Psi) - x_0\| \end{aligned} (34)$$

so that (31) can be rewritten as:

$$\begin{aligned} \|\zeta_T^{x_0}(\Psi) - x_0\| &\leq (L_{\max}(\Psi) + 1) \cdot \|\zeta_{T-1}^{x_0}(\Psi) - \zeta_{T-2}^{x_0}(\Psi)\| \\ &+ \|\zeta_{T-2}^{x_0}(\Psi) - x_0\|. \end{aligned}$$
(35)

But now we can proceed inductively, applying the bound (33) mutatis mutandis to the expression  $\|\zeta_{T-1}^{x_0}(\Psi) - \zeta_{T-2}^{x_0}(\Psi)\|$  in (35). This induction can proceed until the factor to be expanded using (33) has the form  $\|\zeta_{T-(T-1)}^{x_0}(\Psi) - \zeta_{T-(T)}^{x_0}(\Psi)\|$ , which will yield the bound:

$$\|\zeta_T^{x_0}(\Psi) - x_0\| \le \|\zeta_1^{x_0}(\Psi) - x_0\| \cdot \sum_{k=0}^T L_{\max}(\Psi)^k.$$
 (36)

Thus it remains to bound the quantity  $\|\zeta_1^{x_0}(\Psi) - x_0\|$ . We proceed to do this in a relatively straightforward way:

$$\begin{aligned} \|\zeta_{1}^{x_{0}}(\Psi) - x_{0}\| \\ &= \|f(x_{0}) + g(x_{0}) \left[w(x_{0})x_{0} + b(x_{0})\right] - x_{0}\| \\ &\leq \|f(x_{0}) - x_{0}\| + \|g(x_{0})\| \cdot \|w(x_{0})\| \cdot \|x_{0}\| \\ &+ \|g(x_{0})\| \cdot \|b(x_{0})\|. \end{aligned}$$
(37)

Finally, since we're interested in bounding the original quantity,  $\|\zeta_T^{x_0}(\Psi) - x_0\|$ , over all  $x_0 \in X_{\text{safe}}$ , we can upper-bound the above by taking a supremum over all  $x_0 \in X_{\text{safe}}$ . Thus,

$$\sup_{x \in X_{\text{safe}}} \|\zeta_T^{x_0}(\Psi) - x_0\| \le \sup_{x \in X_{\text{safe}}} \|\zeta_1^{x_0}(\Psi) - x_0\| \cdot \sum_{k=0}^T L_{\max}(\Psi)^k \quad (38)$$

where the sup on the right-hand side does not interact with the summation, since  $L_{\max}(\Psi)$  is constant with respect to  $x_0$ . The final conclusion is obtained by observing that that

$$\sup_{x \in X_{\mathsf{safe}}} \|\zeta_1^{x_0}(\Psi) - x_0\| \le \beta_{\max}(\Psi) \tag{39}$$

with  $\beta_{\max}(\Psi)$  as defined in the statement of the proposition.

# C. Proof of Proposition 3

*Proof.* This is more or less a straightforward application of Proposition 2.

Indeed, by Proposition 2 and the assumption of this proposition, we conclude that

$$\begin{aligned} \|\zeta_T^{x_0}(\Psi) - x_0\| &\leq \beta_{\max}(\Psi) \cdot \sum_{k=0}^T L_{\max}(\Psi)^k \\ &\leq \beta_{\max} \cdot \sum_{k=0}^T L_{\max}^k. \end{aligned}$$

Hence,  $\delta x = \zeta_{T_0}^{x_0}(\Psi) - x_0$  triggers the implication in (14), and we conclude that

$$\forall x_0 \in X_{\text{safe}} \ . \ x_0 + \delta x = \zeta_T^{x_0}(\Psi) \notin X_{\text{unsafe}}$$
(40)

as required.

# D. Proof of Corollary 2

*Proof.* Corollary 2 is simply a particularization of Proposition 3 to the repaired TLL network,  $\overline{\Xi}_{N,M}^{(m)}$ . It is only necessary to note that we have separate conditions to ensure that conclusion of Proposition 3 applied both to the *original* TLL network (i.e. (19) and (21)), as well as the repaired TLL parameters (i.e. (18) and (20)).

## E. Proof of Proposition 4

*Proof.* The "if" portion of this proof is suggested by the computations in Section IV-A.1, so we focus on the "only if" portion.

Thus, let  $\{\operatorname{act}_{\kappa}\}_{\kappa=1}^{m} \in \{1,\ldots,N\}^{m}$  be a set of indices, and assume that there exists an index  $\{\operatorname{sel}_{\kappa}\}_{\kappa=1}^{m} \in \{1,\ldots,M\}^{m}$  for which the "only if" assumptions of the proposition are satisfied. We will show that the local linear function with indices  $\{\operatorname{act}_{\kappa}\}_{\kappa=1}^{m}$  is in fact active on  $R_{a}$ . This will follow more or less directly by simply carrying

This will follow more or less directly by simply carrying out the computations of the TLL NN on  $R_a$ . In particular, by condition (*i*), we have that  $\mathscr{M}(\Theta_{\min_N} \circ \Theta_{S_{\mathrm{sel}\kappa}}^{\kappa} \circ \Theta_{\ell}^{\kappa}) = x \mapsto$  $[\![W_{\ell}^{\kappa}x + b_{\ell}^{\kappa}]\!]_{\mathrm{act}\kappa}$  for all  $x \in R_a$ ,  $\kappa = 1, \ldots, m$ . Then, by condition (*ii*) we have that for all  $j \in \{1, \ldots, M\} \setminus \{\mathrm{sel}_{\kappa}\}$ and  $x \in R_a$ 

$$\mathscr{M}(\Theta_{\min_{N}} \circ \Theta_{S_{\mathrm{sel}_{\kappa}}^{\kappa}} \circ \Theta_{\ell}^{\kappa})(x) \leq \mathscr{M}(\Theta_{\min_{N}} \circ \Theta_{S_{\mathrm{sel}_{\kappa}}^{\kappa}} \circ \Theta_{\ell}^{\kappa})(x).$$
(41)

The conclusion thus follows immediately from (41) and the fact that the min groups for the input to the final layer of the output's TLL,  $\Theta_{\max_M}$ .