Practical stability and attractors of systems with bounded perturbations

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***Abstract*— The classical Lyapunov analysis of stable fixed points is extended to perturbed dynamical systems that may not have any fixed point due to perturbations. Practical stability is meant here to assess the convergence of such systems. This is achieved by investigating a parametric optimization problem encoding some worst-case Lie derivative. Key properties of this parametric optimization problem are formulated. The proposed framework is finally applied to a class of perturbed linear systems tracking a highly nonlinear reference.**

I. INTRODUCTION

Lyapunov's method is one of the most widely used tools to assess the stability of dynamical systems [10], [12]. Broadly applied for both linear and non-linear systems, it can be generalized to study the stability of a larger class of dynamical systems, such as uncertain and non-continuous systems [7]. Lyapunov's theory is intrinsically local in its formulation and, while it allows for the analytical assessment of global stability in general, evaluating the exact Region of Attraction (ROA) of an equilibrium is a hard problem, even when the dynamics are known. The ROA's estimation problem has been extensively studied in the literature [2] and we can find several well known approaches to the problem. Classically, we can estimate the ROA by evaluating the invariant sub-level sets of a given fixed Lyapunov function. In the case of polynomial systems, the evaluation is performed solving optimization problems with linear matrix constraints, exploiting the sum-of-squares relaxation in polynomial optimization [5], [9], [4]. It's possible to generalize this approach to allow for parametric Lyapunov functions [14], [4]. These techniques can be generalized to compute the robust ROA of perturbed systems under bounded disturbances [8], [11].

All these methods require that there exists a stable and robust (i.e. perturbation-independent) equilibrium for the system. This is a major drawback, since in a large class of real-world problems this hypothesis is not satisfied. An important example of this is the problem of tracking a moving target with unknown dynamics, which is a common problem that cannot, a priori, allow for a robust equilibrium point. In these problems it could still be possible however to discuss the *practical stability* of the system, i.e. to assess whether the system, while technically unstable in theory, can be considered to be stable in practice. Consider, for instance, the case where the tracking system quickly reaches a small neighborhood of the target while never actually converging to it. Motivated by this, we want to relax the requirement of a robust equilibrium by using instead the concept of *attractor*. An attractor is a set of states which is positively invariant and attracting, i.e. such that the system converges to it (possibly in infinite time) when its initial state is close enough to the attractor. In the tracking system example discussed above, the attractor would be the small neighborhood of the target; this characterization, however, holds in more general settings.

The main contribution of this paper is to characterize the practical stability of perturbed systems with bounded perturbations by exploiting a given candidate Lyapunov function. The proposed definition of *value function* associated to the system and candidate Lyapunov function allows characterizing the system's attractors, hence extending the classical Lyapunov analysis of attracting fixed points. Our formulation does not require any underlying structure for the dynamics of the system, relaxing the classical hypothesis of polynomial dynamics and allowing for the study of a large class of non-linear, perturbed systems.

The paper is organized as follows. In Section II we generalize the classical ROA's estimation problem to the identification of an attractor and the estimation of its ROA, and we show that this problem can be framed as a parametric optimization problem, using the candidate Lyapunov function's level sets as parameter. In Section III we study the continuity of the parametric optimization problem's solution, showing that, under common assumptions, we can expect it to be well-behaved. Finally, in Section IV we apply our results to discuss the practical stability of a class of perturbed linear systems.

II. PRACTICAL STABILITY OF PERTURBED SYSTEMS

We consider a perturbed system $\dot{x} = f(x, w)$, where $w \in \mathcal{W}$ represents a time-dependent, unknown perturbation, with f continuous[1] with respect to both x and w and $\mathcal{W}$ a compact set. We also define a *candidate Lyapunov function*, that we denote with $V(x)$, to be a continuously differentiable function satisfying:

$$V(x) \geq 0 \quad \text{and} \quad V(x_0) = 0 \tag{1}$$

for some x_0. For practical purposes, one can think of V to be a standard Lyapunov function (see, e.g., [12]) for the nominal system $\dot{x} = f(x, 0)$. If $f(x, 0)$ is a feedback system,

[1] Lipschitz continuity with respect to x would enforce uniqueness of trajectories, but is not required for Lyapunov-based stability analysis.

we can then see our method as a characterization of the perturbation's effects on the nominal controller. Still, this is not a strict requirement for our analysis and the weaker hypotheses (1) suffice.

Given f and V, we propose to define the function $m(c)$ that associates to a target value c of the candidate Lyapunov function the worst-case Lie derivative on the corresponding level set:

$$m(c) = \sup_{x \in \mathcal{V}_c} \Phi(x) \tag{2a}$$

$$\Phi(x) = \sup_{w \in \mathcal{W}} \nabla V(x)^T f(x, w), \tag{2b}$$

where $\mathcal{V}_c = \{x \in \mathcal{X} : V(x) = c\}$ is the level set with value c. We call this function the *value function*[2] associated to f and V, or simply the value function when the associated system and candidate Lyapunov function are clear from the context. Sufficient conditions for the two supremum to be maximum are investigated in the next section. The obvious usefulness of the value function is summarized in the following two properties, valid under the typical assumption that the level sets of the candidate Lyapunov function involved in these properties are compact:

1) If $m(c) < 0$ inside the interval $(\underline{c}, \overline{c}]$, for some $0 < \underline{c} < \overline{c}$, then the sublevel set $\mathcal{V}_{\leq \underline{c}}$ is an attractor and the sublevel set $\mathcal{V}_{\leq \overline{c}}$ is inside its region of attraction, i.e. $\mathcal{V}_{\leq \overline{c}}$ is an *estimated region of attraction* for the attractor.
2) If $m(c) < 0$ inside $(\underline{c}, +\infty)$ for some $0 < \underline{c}$, then the sublevel set $\mathcal{V}_{\underline{c}}$ is a global attractor.

As discussed in the introduction, for a *practically stable* system we typically expect that $m(c) > 0$ inside $(0, c^*)$ and $m(c) < 0$ inside $(c^*, +\infty)$: provided that V is proper, i.e., has bounded level sets, all trajectories will converge inside the positively invariant set $\mathcal{V}_{c^*}$. The usefulness of the value function defined here is of course balanced by the difficulty of computing it. Its formal evaluation is carried out for a class of perturbed linear system in Section IV.

In the general case, the quest of finding an interval where the value function is negative provides a strong temptation: computing its roots is of course a simpler and appealing approach. The roots of the value function can be characterized by using the Karush–Kuhn–Tucker (KKT) conditions applied to the optimization problem[3] (2):

$$\nabla^2 V(x) f(x, w) + \left(f_x(x, w)^T + \lambda I\right) \nabla V(x) = 0 \tag{3a}$$
$$\left(f_w(x, w)^T + \mu I\right) \nabla g(w) = 0 \tag{3b}$$
$$V(x) - c = 0 \tag{3c}$$
$$\mu g(w) = 0 \tag{3d}$$
$$\nabla V(x)^T f(x, w) = 0, \tag{3e}$$

[2]The theoretical machinery, i.e., Berge's maximum theorem and the hemicontinuity of set valued functions, used throughout the paper comes mainly from economics, the field in which the name *value function* has been defined.

[3]By substituting the definition of $\Phi(x)$ into (2a), we can see $m(c)$ as the solution of a single optimization problem in x and w.

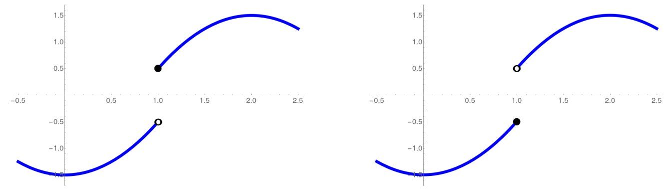

Fig. 1. Example of upper (on the left) and lower (on the right) semicontinuous functions, where the solid black point denotes $h(1)$. In practice, the only difference is whether $h(1)$ is at the top or bottom portion of the discontinuity.

where $\nabla^2 V$ denotes the Hessian of V, f_x, f_w the Jacobians of f w.r.t. x and w, $g(w)$ represents the constraints on w (i.e. $\mathcal{W} = \{w : g(w) \leq 0\}$) and λ, $\mu \in \mathbb{R}$ are the KKT multipliers. In addition to the usual KKT conditions, we consider c as a variable and we add the constraint $\dot{V}(x, w) = 0$ in order to find a solution compatible with $m(c) = 0$. However, this simple intuition hides a trap: if the value function is not continuous, we might incur in a change of sign without passing through zero, rendering this approach useless. Thus, before we succumb to temptation, It is crucial to characterize the properties of f, V that guarantee the value function's continuity.

III. Continuity of the value function

As seen in Section II, the continuity of the value function plays an important role in choosing the strategy to evaluate it. We want to study the continuity of $m(c)$ by investigating its *upper semicontinuity* and *lower semicontinuity* separately. Intuitively, the continuity property can be "split" into these two sub-properties, and we find that a function is continuous when both hold at the same time. For a scalar function (like $m(c)$), the following definition applies [1].

Definition 1: Let $h : \mathcal{H} \subset \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then:

1) The function h is called *upper semicontinuous* at $\bar{c}$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that:
$$|c - \bar{c}| < \delta \implies h(c) \leq h(\bar{c}) + \epsilon \tag{4}$$
2) The function h is called *lower semicontinuous* at $\bar{c}$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that:
$$|c - \bar{c}| < \delta \implies h(c) \geq h(\bar{c}) - \epsilon \tag{5}$$

A function h which is upper (lower) semicontinuous on its whole domain is simply called upper (lower) semicontinuous. If it is both upper and lower semicontinuous, then it is a *continuous* function.
An example of the difference between upper and lower semicontinuity can be seen in Figure 1. While, technically, in this definition we ask the codomain of h to be the extended reals $\mathbb{R} \cup \{-\infty, +\infty\}$, in practice, for $m(c)$, we can consider to have the set of real numbers $\mathbb{R}$ as codomain. Moreover, due to the positive definiteness of V, its domain is $\mathbb{R}_+$.

We are ready to discuss our results. First of all, we show that, under typical assumptions on f and V, the function $\Phi(x)$ is continuous.

Proposition 1: Let $V : \mathcal{X} \to \mathbb{R}$ be a function of class $\mathcal{C}^1$, and let $f : \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$ be continuous in x and w, where

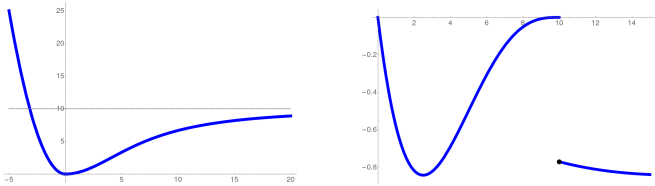

Fig. 2. On the left, plot of $V(x)$ defined in (7). The dotted line represents the horizontal asymptote for $x \to +\infty$. On the right, the corresponding value function, where the solid black point denotes the value of m at the discontinuity.

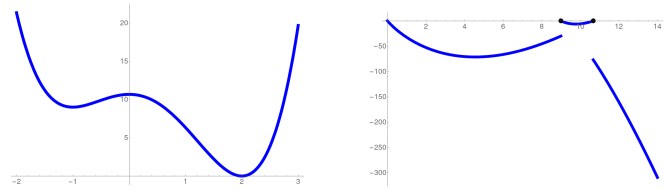

Fig. 3. On the left, plot of $V(x)$ defined in (9). On the right, the corresponding value function, where the solid black points denote the values of m at the discontinuities.

$\mathcal{W}$ is a compact set. Then, $\Phi(x)$ is a continuous function and the maximum is attained, i.e. $\Phi(x)$ can be written as a $\max$ instead of $\sup$.

Proof: Being both $\nabla V(x)$ and $f(x, w)$ continuous, the product $\nabla^T V(x) f(x, w)$ is continuous as well. Thus, by considering $\mathcal{W}$ as a constant (and so continuous) set-valued function with nonempty compact values, we can directly apply Berge's maximum theorem [1, Th. 17.31], which proves that $\Phi(x)$ is continuous. ■

While promising, the continuity of $\Phi(x)$ is not sufficient to guarantee the continuity of the value function. Consider, for instance, the unperturbed, scalar system defined as:

$$\dot{x} = f(x) = \begin{cases} -\frac{10x}{(0.1x^2+5)^2} & \text{if } x > 0 \\ -2x & \text{if } x \leq 0 \end{cases}, \tag{6}$$

and let:

$$V(x) = \begin{cases} \frac{x^2}{0.1x^2+5} & \text{if } x > 0 \\ x^2 & \text{if } x \leq 0 \end{cases} \tag{7}$$

be its candidate Lyapunov function. The plot of V is shown in Figure 2.

It is easy to verify the continuity and continuous differentiability of f and V, respectively. Moreover, we can see that $f(x) = -V'(x)$, meaning that the Lie derivative is $\dot{V}(x) = -(V'(x))^2$, which is negative for all $x \in \mathbb{R}\backslash\{0\}$. Even with a scalar and globally asymptotically stable system, we can see, in Figure 2, that the value function is not upper semicontinuous: this is due to a sudden change of the worst-case Lie derivative at the level of the horizontal asymptote.

The lower semicontinuity of the value function is not guaranteed either. Consider, for instance, the system:

$$\dot{x} = f(x) = -4x(x^2 - x - 2), \tag{8}$$

with candidate Lyapunov function:

$$V(x) = x^4 - \frac{4}{3}x^3 - 4x^2 + \frac{32}{3}. \tag{9}$$

Similarly to the previous example, we have a system of the form $f(x) = -V'(x)$. As we can see in Figure 3, in this case as well, the value function is discontinuous, in particular it is not lower semicontinuous: this is due to sudden changes of the worst-case Lie derivative at the levels of the local minimizer and local maximizer.

Motivated by this, we want, in the following theorem, to independently study the upper and lower semicontinuity of the value function. The theorem's proof relies on the same foundations of the already introduced Berge's maximum theorem: as we will see, the upper and lower semicontinuity of $m(c)$ depend on the continuity of $\Phi(x)$ and the hemicontinuity[4] of $\mathcal{V}_c$, seen as a set-valued function $\mathcal{V}_c : \mathbb{R}_+ \rightrightarrows \mathcal{X}$. We have the following.

Theorem 1: Let V, f satisfy the same hypotheses of Prop. 1. Then, the following holds true:

1) If V is radially unbounded, then $m(c)$ is upper semicontinuous and the maximum is attained, i.e. $m(c)$ can be written as a $\max$ instead of $\sup$.
2) If, for all $x \in \mathcal{V}_{\bar{c}}$, $\nabla V(x) \neq 0$, then $m(c)$ is lower semicontinuous at $\bar{c}$.

Remark 1: Case 1 rules out situations illustrated by (6)–(7), while Case 2 rules out situations illustrated by (8)–(9).

Proof: Due to Proposition 1, $\Phi(x)$ is continuous in the theorem's hypotheses. Thus, we focus in this proof on the upper and lower hemicontinuity of $\mathcal{V}_c$.

We start with the first statement. Being V continuous, then the preimage of a closed set through V is closed as well, meaning that $\mathcal{V}_c$ is closed. Moreover, the radial unboundness of V implies that $\mathcal{V}_c$ is bounded, for all $c \in \mathbb{R}_+$. Thus, by [3, Cor. 21], $\mathcal{V}_c$ is upper hemicontinuous, which means that, by [1, Lemma 17.30], $m(c)$ is a $\max$ and it is upper semicontinuous.

The second statement's proof is proved in a similar fashion, albeit more involved. Firstly, we show that $\mathcal{V}_c$ is *inner hemicontinuous* at $\bar{c}$, which is a property equivalent to the lower hemicontinuity [3, Prop. 23]. By its definition, we have that $\mathcal{V}_c$ is inner hemicontinuous at $\bar{c}$ if, for all $\bar{x} \in \mathcal{V}_{\bar{c}}$, it holds true that:

$$\forall \{c_k\} \to \bar{c}, \exists x_k \in \mathcal{V}_{c_k} : x_k \to \bar{x} \tag{10}$$

We prove the inner hemicontinuity of $\mathcal{V}_c$ by contradiction. Suppose that $\mathcal{V}_c$ is not inner hemicontinuous at $\bar{c}$. That means that there exists $\bar{x} \in \mathcal{V}_{\bar{c}}$ such that:

$$\exists \{c_k\} \to \bar{c} : \forall x_k \in \mathcal{V}_{c_k}, x_k \not\to \bar{x}. \tag{11}$$

Consider the restriction of V, centered in $\bar{x}$ and along the direction of its gradient evaluated at $\bar{x}$, i.e. the scalar function $\bar{V}(h) = V(\bar{x} + \nabla h V(\bar{x}))$. Notice that, by the continuity of

[4]Hemicontinuity generalized the concept of continuity for set-valued functions. Intuitively, a set-valued function is not lower hemicontinuous if we have a sudden "explosion" of its value (e.g. a new connected component appearing), and similarly it is not upper hemicontinuous if we have a sudden "implosion". For a formal introduction see, e.g., [1].

V, $\bar{V}$ is continuous. Moreover, there exists an open interval $H \subset \mathbb{R}$, with $0 \in H$, such that $\bar{V}$ is strictly increasing over H. This is due to the fact that the derivative $D\bar{V}(h)$ evaluated at zero is:

$$D\bar{V}(h)|_{h=0} = \|\nabla V(\bar{x})\|^2 > 0, \tag{12}$$

and, by the continuity of ∇V, there exists a neighbourhood of the origin (i.e. an open interval H containing zero) where (12) holds true, meaning that $\bar{V}$ is strictly increasing on H. Thus, $\bar{V}$ admits a continuous inverse $\bar{V}^{-1}$ on H. Its continuity means that:

$$\forall\{c_k\} \to \bar{c}, \;\; \bar{V}^{-1}(c_k) \to \bar{V}^{-1}(\bar{c}) = 0 \tag{13}$$

Let $h_k = \bar{V}^{-1}(c_k)$. By defining the sequence $\{x_k\}_{k\in\mathbb{N}}$ as $x_k = \bar{x} + h_k \nabla V(\bar{x})$, it follows from Eq. (13) that:

$$\forall\{c_k\} \to \bar{c}, \;\; x_k \to \bar{x} \tag{14}$$

which is in contradiction with Eq. 11, proving the inner (and, thus, lower) hemicontinuity of $\mathcal{V}_c$ at $\bar{c}$. By [1, Lemma 17.29], the lower hemicontinuity of $\mathcal{V}_c$ at $\bar{c}$ implies that $m(c)$ is lower semicontinuous at $\bar{c}$, concluding the proof. ■

IV. Application to linear tracking systems with bounded velocity target

A. Upper bound of the value function for linear systems with bounded uncertain right hand side

We consider the linear system

$$\dot{z} = Az + w, \tag{15}$$

where A is a *stable* matrix (i.e. all eigenvalues have negative real part) and the uncertainty w is bounded inside $\mathcal{W} = \{w \in \mathbb{R}^n : \|w\| \le M\}$. We consider a quadratic Lyapunov function $V(z) = z^T P z$ for the nominal system with P symmetric positive-definite (SPD). The Lie derivative for the system with nominal $w = 0$ is given by $z^T(PA + A^T P)z$, which is supposed to be negative-definite. Such a matrix P is usually obtained by choosing an arbitrary SPD matrix Q, typically $Q = I$, and solving the Lyapunov equation

$$PA + A^T P + Q = 0. \tag{16}$$

The following theorem provides an upper bound for the value function associated to the system and the Lyapunov function defined above.

Theorem 2: Let P, Q be positive-definite matrices that Lyapunov's equation (16) holds. Then, $m(c) \le \overline{m}(c)$ with

$$\overline{m}(c) = \sigma_1 c + 2M\sqrt{\lambda_1}\sqrt{c}, \tag{17}$$

where σ_1 is the greatest eigenvalue of $\left(-P^{-1}Q\right)$ and λ_1 is the greatest eigenvalue of P. Furthermore, $\sigma_1 < 0$.

Proof: With these hypotheses, $m(c)$ can be written as:

$$m(c) = \max_{z\in\mathcal{V}_c} \Phi(z). \tag{18}$$

Here we have

$$\Phi(z) = \max_{w\in\mathcal{W}} \left(-z^T Q z + 2z^T P w\right) \tag{19}$$

$$= -z^T Q z + 2\max_{w\in\mathcal{W}} z^T P w. \tag{20}$$

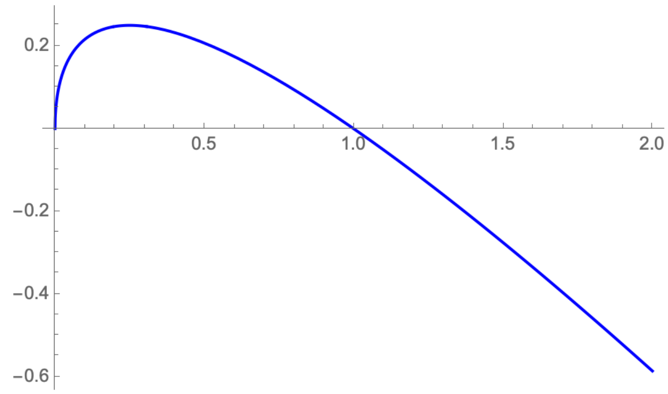

Fig. 4. Graph of $-c + \sqrt{c}$.

Using Cauchy-Schwarz inequality and the bound on the norm of w, we obtain $z^T P w \le \|Pz\|\,\|w\| \le M\|Pz\|$, with the equality holding when w is parallel to Pz. Since $\mathcal{W}$ is a ball of radius M, such a w parallel to Pz with norm M exists and we have $\Phi(z) = -z^T Q z + 2M\|Pz\|$. This leads us to:

$$m(c) = \max_{z\in\mathcal{V}_c} \left[-z^T Q z + 2M\|Pz\|\right] \tag{21a}$$

$$\le \left(\max_{z\in\mathcal{V}_c} -z^T Q z\right) + 2M\left(\max_{z\in\mathcal{V}_c} \|Pz\|\right), \tag{21b}$$

where the optimization problem is split into two independent subproblems that can be solved independently.

For the second subproblem, we solve $\max_{z\in\mathcal{V}_c} \|Pz\|^2$. Lagrange's first-order conditions yield $P^2 z - \mu P z = 0$, which must hold for some $\mu \in \mathbb{R}$. Being P invertible, this is equivalent to solving $Pz = \mu z$, leading to

$$\max_{z^T P z = c} \|Pz\|^2 = \max_{\substack{z^T P z = c \\ \mu\in\Lambda(P)}} \mu z^T P z = \lambda_1 c, \tag{22}$$

where $\Lambda(P)$ is the spectrum of P and $\lambda_1 = \max\{\Lambda(P)\}$ is the greatest eigenvalue of P. The second subproblem maximum is therefore $\sqrt{\lambda_1 c}$.

For the first subproblem, Lagrange's first-order conditions of $\max_{z\in\mathcal{V}_c} -z^T Q z$ lead to $-Qz - \mu P z = 0$ for some $\mu \in \mathbb{R}$. We notice that the values of μ that satisfy this equation are the generalized eigenvalues [13] of the symmetric matrix pencil $(-Q, P)$, whose set we denote with $\Lambda(-Q, P)$. Being P positive-definite, we know that we have n real generalized eigenvalues (see [13, Th. 15.3.3]), that we denote, w.l.o.g., with $\sigma_1 \ge \cdots \ge \sigma_n$. Thus, by substitution, we have that:

$$\max_{z^T P z = c} -z^T Q z = \max_{\substack{z^T P z = c \\ \mu\in\Lambda(-Q,P)}} \mu z^T P z = \sigma_1 c, \tag{23}$$

which allows us to write:

$$m(c) \le \sigma_1 c + 2M\sqrt{\lambda_1}\sqrt{c}. \tag{24}$$

We conclude the proof by noticing that, being P invertible, the generalized eigenvalues of the pencil $(-Q, P)$ are equal to the eigenvalues of $-P^{-1}Q$, and that, being $-z^T Q z < 0$ for all $z \ne 0$, we have that $\sigma_1 c < 0$ for $c \ne 0$, proving that σ_1 is indeed negative. ■

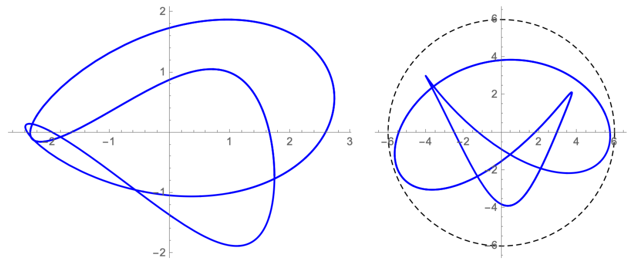

Fig. 5. Left: $r(t)$; right: $r'(t)$ with a circle of radius 6.

If M is not zero then $\overline{m}(c)$ starts with an infinite derivate at $c = 0$, has a single maximum at $c = \lambda_1(\frac{M}{\sigma_1})^2$ and $\lim_{c\to\infty} \overline{m}(c) = -\infty$. Its typical graph is shown in Figure 4. The function $\overline{m}(c)$ has a unique positive root

$$c^* = \frac{4M^2\lambda_1}{\sigma_1^2}, \tag{25}$$

and is negative for greater values of c. The positive definite quadratic Lyapunov function being radially unbounded, the sublevel set $\mathcal{V}_{\leq c^*}$ is proved to be a global attractor of the uncertain system (15). In the typical case where $Q = I$ we have $\sigma_1 = \frac{-1}{\lambda_1}$ and the root's expression simplifies to

$$c^* = 4M^2\lambda_1^3. \tag{26}$$

B. Linear tracking systems with bounded velocity target

We consider a simple linear system with known input matrix gain A of the form:

$$\dot{x} = Au, \tag{27}$$

where $x \in \mathbb{R}^n$ is the system's state, $u \in \mathbb{R}^n$ its input and $A \in \mathbb{R}^{n\times n}$ is stable. The target reference signal $r(t) \in \mathbb{R}^n$ is measured at all times, and we assume the knowledge of an upper bound M on $\|\dot{r}(t)\|$. The proportional control $u(t) = -k(x(t) - r(t))$, $k > 0$, leads to the following closed loop system:

$$\dot{x}(t) = kA\big(x(t) - r(t)\big). \tag{28}$$

The dynamic of the tracking error $z(t) = x(t) - r(t)$ is then

$$\dot{z}(t) = kAz(t) - \dot{r}(t). \tag{29}$$

We use Theorem 2 to find a globally attracting sublevel set $\mathcal{V}_{\leq c^*}$. In order to compare the sublevel sets for different gains, we consider the Lyapunov function $V(z) = z^TPz$ with $PA + A^TP + I = 0$ independently of the gain k. Therefore, the matrix Q that satisfies the Lyapunov equation (16) is $Q = kI$. Since Q is proportional to I, the sublevel value (25) now simplifies to

$$c^* = \frac{4M^2\lambda_1^3}{k^2}. \tag{30}$$

C. Numerical application

We consider the following highly oscillating target,

$$r(t) = \begin{pmatrix} \cos(2t) + 2\sin(2t) + 0.5\sin(3t) \\ \sin(t) - \cos(3t) \end{pmatrix}, \tag{31}$$

TABLE I

Values of c^* obtained using Theorem 2 for different gain values.

k	5	10	50	100
c^*	3.73	0.932	0.0373	0.00931

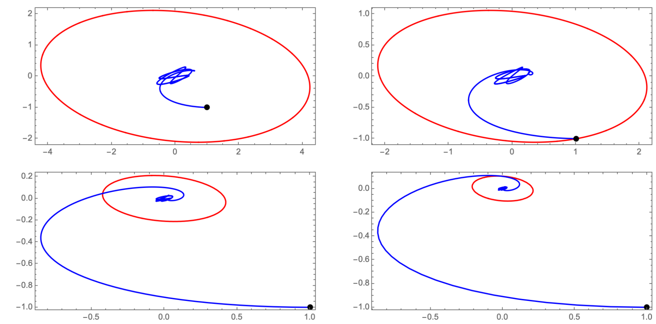

Fig. 6. Tracking error of the proportional linear tracking system for the same initial condition, whose initial error is denoted with a black point, together with the globally attracting ellipsoids, for different values of the gain $k \in \{5, 10, 50, 100\}$, with western reading direction.

whose velocity is bounded by $M = 6$, see Figure 5. The matrix A is chosen to be non symmetric and P follows solving $PA + A^TP + I = 0$:

$$A = \begin{pmatrix} -2 & 5 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \frac{3}{14} & \frac{1}{14} \\ \frac{1}{14} & \frac{6}{7} \end{pmatrix}. \tag{32}$$

The largest eigenvalue of P is $\lambda_1 = \frac{1}{28}(15 + \sqrt{85}) \approx 0.86$. The globally attractive sublevel values obtained using Theorem 2 for different values of the gain k are given in Table I. The trajectories of the system and the globally attracting sublevel sets obtained by Theorem 2 are shown in Figure 6, while in Figure 7 we can see the value of the Lyapunov function with respect to time.

D. Application of Theorem 1

The positive definite quadratic Lyapunov function satisfies the hypothesis of Theorem 1, which proves that the exact value function $m(c)$ is continuous. We can therefore infer

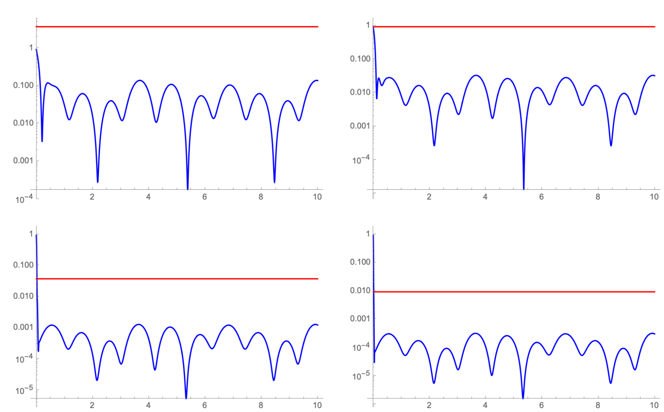

Fig. 7. Lyapunov function's value along the trajectories of the proportional linear tracking system (in blue) together with the globally attracting level set's value (in orange), for different values of the gain $k \in \{5, 10, 50, 100\}$, with western reading direction.

TABLE II

Values of c for the extremal values leading to a zero Lie derivative, with the corresponding values of the value function.

k	5	10	50	100
c_1	0.0507	0.0127	0.000507	0.000127
c_2	3.73	0.932	0.0373	0.00931
$m(c_1)$	2.21	1.10	0.221	0.110
$m(c_2)$	0	0	0	0

the sign of $m(c)$ by computing its roots. The system (3) that characterizes its roots is now

$$-Qx + Pw + \lambda Px = 0 \tag{33a}$$

$$Px + \mu w = 0 \tag{33b}$$

$$x^T Px = c \tag{33c}$$

$$\mu(w^T w - M^2) = 0 \tag{33d}$$

$$-x^T Qx + 2x^T Pw = 0. \tag{33e}$$

It is polynomial of degree three[5] and can be solved using formal computations. For each gain value, we find four solutions showing two distinct values of c, given in Table II. The solutions with c_2 correspond exactly to the solutions previously computed, while $c_1 < c_2$. We therefore expect the latter to be local maximizers. This is confirmed by evaluating $m(c_1)$ and $m(c_2)$ as follows: for these values of c, we solve the system (33) with the last equation removed. This is a square system of equations[6] that encodes Lagrange first order conditions for the Lie derivative, its solutions therefore include the global maximizer. The global maximizers obtained this way are shown in Table II and confirm that c_2 is indeed the unique positive root of $m(c)$.

Surprisingly, both $m(c)$ computed here and $\overline{m}(c)$ computed in the previous section have the same unique positive root (compare Table I and Table II), while the upper bound (21) should entail some overestimation. This coincidence is left for future investigations.

V. Conclusion

In this paper, we introduced the concept of *value function* associated to a perturbed dynamical system and a candidate Lyapunov function. We have shown that by analyzing this function it is possible to identify and study the system's *attractors*, even in cases where the system lacks robust equilibria and when its vector field is nonlinear. In applications, this translates in the ability to assess the *practical stability* of a system, providing quantitative information on the trajectories' proximity to the unperturbed equilibrium and allowing for the evaluation of a region of attraction's estimate. On the downside, the value function is difficult to compute, requiring ad-hoc, non-trivial theoretical results even in the case of linear systems with a simple perturbation structure.

[5]The only degree three monomial in the system is the complementarity constraint (33d), which is actually a simple alternative. The system can therefore be solved as two degree two systems.

[6]With respect to the square system (33), the variable c is now fixed, and one equation has been removed, hence leading to a new square system.

However, Theorem 1 opens the way to using root finding algorithms for locating the roots of $m(c)$, deducing intervals of negative values and identifying the corresponding attractors. The use of numerical solvers and computer-assisted proof methods to perform this evaluation will be the subject of future studies. In particular, the use of interval analysis [8], which do not require any particular structure for the system's vector field, to find the value function's root in a validated way seems promising and is currently under investigation.

Even though we specialize the value function's definition for perturbed systems, it can clearly be generalized to allow for a generic differential inclusion. Thus, it appears logical to try to extend this approach to a larger class of systems, naturally defined as differential inclusions (for instance, systems with discontinuous vector fields). Moreover, by using the notion of *set-valued Lie derivative*, it would be possible to relax the continuous differentiability of $V(x)$, allowing for Lipschitz continuous candidate Lyapunov functions, which are necessary in several applications (e.g. in sliding mode control [6]).

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