

Authentication Against a Myopic Adversary

Allison Beemer, Eric Graves, Jörg Kliewer, Oliver Kosut, and Paul Yu

Abstract

We consider keyless authentication for point-to-point communication in the presence of a myopic adversary. In particular, the adversary has access to a non-causal noisy version of the transmission and uses this knowledge to choose the state of an arbitrarily-varying channel between legitimate users. The receiver succeeds by either decoding accurately or correctly detecting adversarial interference. We show that a channel condition called U -overwritability, which allows the adversary to make its false message appear legitimate and untampered with, is a sufficient condition for zero authentication capacity. We present a useful way to compare adversarial channels, and show that once an AVC becomes U -overwritable, it remains U -overwritable for all “less myopic” adversaries. Finally, we show that stochastic encoders are necessary for positive authentication capacity in some cases, and examine in detail a binary adversarial channel that illustrates this necessity. Namely, for this channel, we show that when the adversarial channel is degraded with respect to the main channel, the no-adversary capacity of the underlying channel is achievable with a deterministic encoder. Otherwise, provided the channel to the adversary is not perfect, a stochastic encoder is necessary for positive authentication capacity; if such an encoder is allowed, the no-adversary capacity is again achievable.

Index Terms

Authentication, keyless authentication, arbitrarily-varying channel, myopic adversary, channel capacity

I. INTRODUCTION

When communicating over unsecured channels, verifying the trustworthiness of a received signal is critical. Thus, it may be useful for a receiver to declare adversarial tampering, even if it cannot decode the message perfectly. This allows for messages to be rejected unless they are confirmed to be trustworthy. This is known as *authentication*. We consider authentication over an arbitrarily-varying channel (AVC) with an adversary who has some noisy version of the transmitted sequence. An AVC is a channel that takes as inputs both the legitimate and an adversarial transmission [2], where the adversarial transmission (called a *state*) may be maliciously chosen with the goal of causing a decoding error at the receiver. A plethora of variations on the AVC appear in the literature, in which the adversary has varying degrees of power and knowledge of the legitimate transmission, and the transmitter and receiver may or may not have access to shared secret information. In the case of authentication, we say that the decoder is successful if one of the following scenarios occurs: (1) the adversary is not active, and the decoder recovers the correct message, or (2) the adversary is active and the decoder either recovers the correct message or declares adversarial interference. This is a relaxation of the classical AVC, where the decoder is successful if and only if it correctly recovers the transmitted message. Indeed, we find that authentication is “easier” in the sense that the AVC capacity can be zero, even though the authentication capacity for the same channel is positive (or even unchanged from the no-adversary setting).

The current work lies at the intersection of two broader areas of previous study: authentication over AVCs (with an adversary oblivious to the transmitted sequence), and traditional error correction over AVCs with myopic adversaries. Lying squarely in the former category are [3], [4]. In [3], a channel condition called *overwritability* is introduced, and it is shown that authentication can be achieved with high probability for non-overwritable AVCs

A. Beemer and J. Kliewer are with the Department of Electrical and Computer Engineering, New Jersey Institute of Technology, Newark, NJ, 07103; O. Kosut is with the School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, AZ 85287; and E. Graves and P. Yu are with the Computer and Information Sciences Division, U.S. Army Research Laboratory, Adelphi, MD 20783.

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at positive (but asymptotically vanishing) rates. The results are extended in [4], where overwritability is shown to completely classify authentication capacity over an (oblivious adversary) AVC, with all non-overwritable channels having capacity equal to the non-adversarial capacity of the AVC. Meanwhile, more knowledgeable adversaries over the AVC were studied in [5], [6], [7], [8]. The capacity of an AVC with a myopic adversary who does not have knowledge of the codebook realization (i.e., the legitimate users share common randomness) is characterized in [5]. The authors of [6] give the capacity of the AVC in the case where the legitimate users have a sufficiently large shared key, and the adversary can see the transmission sequence (i.e. the channel input) perfectly. In [7], the authors give bounds on the capacity of an AVC in the presence of a power-constrained myopic adversary, showing that once the channel to the adversary is bad enough, it is essentially oblivious. In the context of Gaussian channels, [8] examines the case where both the transmitter and myopic adversary are quadratically power-constrained.

In this paper, we focus on keyless authentication when the adversary has full knowledge of the encoding procedure, and some limited knowledge of the transmitted sequence. More specifically, the adversary chooses a state based on a non-causal noisy version of the legitimate transmission, as well as full knowledge of the codebook being used, but no direct knowledge of the message being transmitted. The adversary is not power-constrained, a freedom that is reasonable given we are considering authentication and not pure error-correction, and the possible attacks by the adversary are determined by the AVC between legitimate users.

Prior work on authentication with a variety of more capable adversaries includes [9], [10], [11], [12], [13], [14]. Simmons [9] considered authentication in the case where the adversary knows the encoding procedure and may intercept the transmission perfectly, but the legitimate users are allowed some shared key with which they randomize their choice of codebook. In [10], Maurer considers authentication for secret key agreement in the setting where each of the legitimate users and the adversary have access to a random variable, the three of which are correlated in some way. In that model, the adversary may substitute its transmission for that of the actual transmitter (over an otherwise noiseless channel), with the goal of hoodwinking the legitimate receiver; the (authentication) capacity is characterized by whether the joint distribution is *simulatable* by the adversary. An inner bound on keyless authentication capacity is given by Graves, Yu, and Spasojevic in [12] for the case in which the adversary knows both the encoding procedure and the message, but does not have any additional knowledge of the transmitted sequence, which is allowed to be the output of a stochastic encoder. Lai, El Gamal, and Poor [11] consider authentication in the presence of a myopic adversary who carries out *impersonation* or *substitution* attacks; the transmitter and receiver are allowed a shared secret key. The attacks considered in the current work reduce to substitution attacks when the underlying AVC allows for direct modification of a transmission. This model is extended in [13], where the authors additionally provide inner bounds for the keyed and keyless authentication capacities with impersonation/substitution attacks, and extend the simulatability condition of [10] to characterize nonzero capacity for this set of attacks. Finally, [14] improves on the inner bounds of [11], [13] for keyed authentication.

A myopic adversary model bridges the gap between oblivious and omniscient adversaries. As mentioned above, authentication capacity over the AVC in the presence of an oblivious adversary is characterized in [4] by the channel condition of overwritability. In this work, we show that an analogous condition for a myopic adversary called *U-overwritability*, first introduced in [1], is a sufficient condition for zero authentication capacity for both deterministic and stochastic encoders. When we are restricted to deterministic encoders, a sufficiently capable adversary becomes essentially omniscient, giving more cases of zero authentication capacity; interestingly, when we allow for stochastic encoders, the authentication capacity has the potential to increase to the non-adversarial capacity of the underlying channel, which is illustrated by example. These results naturally lead to the questions about the relationships between the possible channels to the adversary: namely, how capable can an adversary be before the channel becomes *U-overwritable*? We examine this question by endowing the set of channels to the adversary with a partially ordered set structure, showing that the up-set of any *U-overwritable* channel gives a set of channels which also lead to *U-overwritability*. This relationship is perhaps most useful when we know or want to say something about *U-overwritability* for omniscient or oblivious adversaries.

The paper is organized as follows. In Section II we introduce the model and necessary notation. We give several cases for which the authentication capacity is zero in Section III, and demonstrate that there are cases where the deterministic authentication capacity is zero but a stochastic encoder allows for a positive authentication capacity. Section IV gives results derived by comparing different channels to the adversary. In Section V, we examine a specific binary model, proving that when the channel is not *U-overwritable*, the authentication capacity is equal to the underlying non-adversarial capacity, and that stochastic encoders must be used to achieve this in a portion of

cases. Section VI concludes the paper.

II. PRELIMINARIES

Notation: Throughout the paper, an (M, n) code is a code with M codewords and block length n . The notation $H(\cdot)$ will indicate the binary entropy function, $\|\cdot\|$ will denote the Hamming weight of a vector, \mathbb{E} will denote expected value, \oplus will be used to indicate (coordinate-wise) modulo 2 addition, and “ \rightarrow ” will indicate a Markov chain. A random variable will be denoted using a capital letter, with the corresponding alphabet and alphabet elements written with script and lowercase. For example, X is a random variable with alphabet \mathcal{X} such that for all $x \in \mathcal{X}$, $P_X(x)$ (or $P(x)$ when the random variable is understood) is the probability that $X = x$. If $P_X(x)$ is an integer multiple of $1/n$ for all $x \in \mathcal{X}$, we write τ_X to indicate the type class corresponding to P_X (i.e. the set of vectors of length n whose empirical distributions match the distribution P_X). Finally, we define the typical set $\mathcal{T}_\epsilon^{(n)}(X)$ to be the set of sequences $\mathbf{x} \in \mathcal{X}^n$ such that the empirical probability of each $x \in \mathcal{X}$ in \mathbf{x} is within $\epsilon \cdot P_X(x)$ of $P_X(x)$. We may write $\mathcal{T}_\epsilon^{(n)}$ when the random variable(s) are clear from context.

We consider authentication when there is a legitimate transmitter and receiver as well as an active adversary who induces some channel state at each transmission. We assume that the adversary has full knowledge of the codebook of the legitimate parties, though not necessarily of the specific transmission being sent. More formally, let $W_{Y|X,S}$ be a discrete memoryless adversarial channel with the sets \mathcal{X}, \mathcal{S} , and \mathcal{Y} as the input, state, and output alphabets. In the event of multiple channel uses, we write $W(\mathbf{y} | \mathbf{x}, \mathbf{s})$ for the product channel, where the sequence lengths are understood. Similarly, there is a discrete memoryless channel between the legitimate transmitter and the adversary given by $U_{Z|X}$, where \mathcal{Z} is the output alphabet at the adversary, so that the adversary has a noisy version \mathbf{z} of any transmitted sequence \mathbf{x} with probability $U(\mathbf{z} | \mathbf{x})$. See Figure 1 for a depiction of this setup.

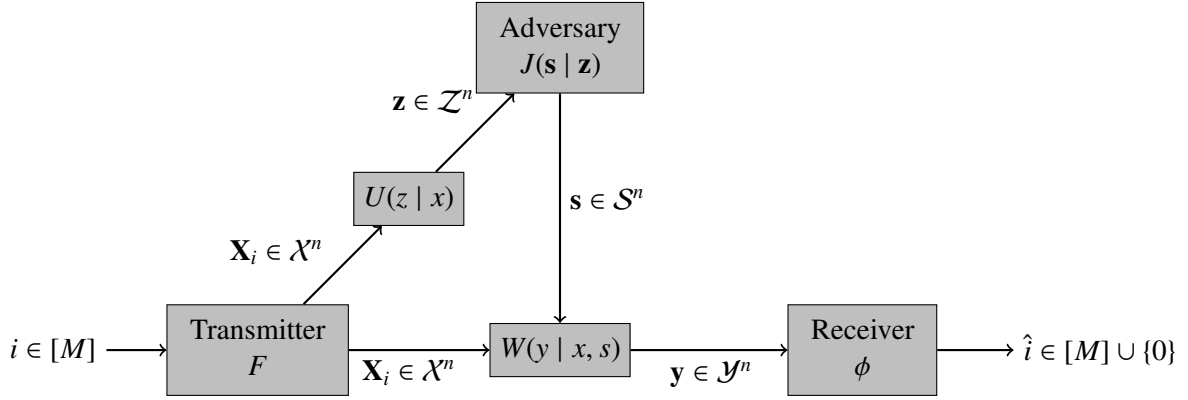


Fig. 1. The transmitter sends a length- n sequence $\mathbf{X}_i := F(i)$, which is the (possibly stochastic) encoding of a message $i \in [M]$. The adversary views a noisy version \mathbf{z} of this transmission and sends state \mathbf{s} to the channel between transmitter and receiver. The receiver then receives \mathbf{y} , whose probability is conditioned on both the legitimate transmission and the adversarial state, and decodes to $\phi(\mathbf{y}) \in [M] \cup \{0\}$, where “0” is a declaration of adversarial interference.

A *myopic adversary* will be formally defined by a distribution $J_{S^n|\mathcal{Z}^n}$, where the adversary’s choice of J may depend on U and on the codebook used by the legitimate users. Notice that the adversary has a noncausal noisy view of the transmission, so that they may view the entire block length before choosing a state. The case in which the adversary’s choice is deterministic, as in [5], is included in the above definition.

Let $s_0 \in \mathcal{S}$ and the corresponding constant state sequence $\mathbf{s}_0 \in \mathcal{S}^n$ represent the state in which there is no adversary: i.e. $W_{Y|X,S=\mathbf{s}_0}$ is a non-adversarial channel. An (M, n) authentication code for W is an encoder/decoder pair, where the encoder is possibly *stochastic*:

$$F : \{1, 2, \dots, M\} \rightarrow \mathcal{X}^n$$

$$\phi : \mathcal{Y}^n \rightarrow \{0, 1, 2, \dots, M\},$$

where an output of “0” under ϕ indicates a declaration of adversarial interference. The decoder ϕ is successful if (1) the output is equal to the input message, or (2) $\mathbf{s} \neq \mathbf{s}_0$ and the output is equal to 0. In other words, the decoder either successfully detects adversarial interference, or decodes correctly.

There are a variety of ways to formally define a stochastic encoder (also called a *randomized encoder*), all of which allow the transmitter to make use of some local randomness. We use the following (see, e.g. [15], pg. 550).

Definition II.1. Given a message set $[2^{nR}]$, a stochastic encoder generates a codeword in \mathcal{X}^n for $m \in [2^{nR}]$ according to some conditional distribution $P_{\mathbf{X}|M}(\mathbf{x} | m)$.

Let $\phi^{-1}(A) \subseteq \mathcal{Y}^n$ represent the set of channel outputs which decode to some $i \in A$ under ϕ , and let $\phi^{-1}(A)^c$ be the complement in \mathcal{Y}^n of this set. Let $\mathbf{X}_i := F(i)$ denote the length- n encoding of message i ; notice that if the encoder is stochastic, $F(i)$ can take on one of several values, and so \mathbf{X}_i is a random variable. Given transmitted message i and choice of myopic adversary J , we define the probability of error for the authentication code (F, ϕ) as:

$$e(i, J) = \mathbb{E}_{\mathbf{X}|i} \left[\sum_{\mathbf{z}} U(\mathbf{z} | \mathbf{X}) J(\mathbf{s}_0 | \mathbf{z}) W(\phi^{-1}(i)^c | \mathbf{X}, \mathbf{s}_0) + \sum_{\mathbf{s} \neq \mathbf{s}_0} \sum_{\mathbf{z}} U(\mathbf{z} | \mathbf{X}) J(\mathbf{s} | \mathbf{z}) W(\phi^{-1}(\{i, 0\})^c | \mathbf{X}, \mathbf{s}) \right] \quad (1)$$

where the expectation is over the realizations of the stochastic encoder conditioned on message i being sent. If the adversary is absent, then $J(\mathbf{s}_0 | \mathbf{z}) = 1$ for all $\mathbf{z} \in \mathcal{Z}^n$. We denote this distribution by $J_{\mathbf{s}_0}$, and observe that with this choice of adversarial action, (1) reduces to the standard error probability given that message i is transmitted.

$$e(i, J_{\mathbf{s}_0}) = \mathbb{E}_{\mathbf{X}|i} [W(\phi^{-1}(i)^c | \mathbf{X}, \mathbf{s}_0)].$$

Similarly, if the adversary decides that $J(\mathbf{s} | \mathbf{z}) = 1$ for all $\mathbf{z} \in \mathcal{Z}^n$ and some particular $\mathbf{s} \neq \mathbf{s}_0$, denoted $J_{\mathbf{s}}$,

$$e(i, J_{\mathbf{s}}) = \mathbb{E}_{\mathbf{X}|i} [W(\phi^{-1}(\{i, 0\})^c | \mathbf{X}, \mathbf{s})].$$

The above two cases show the reduction to the so-called *oblivious* case, in which the adversary has no knowledge about the transmission before choosing a state vector \mathbf{s} , and so should simply choose \mathbf{s} to optimize its chances of causing decoding failure.

We assume that each message in $[M] := \{1, 2, \dots, M\}$ is transmitted with equal probability. Then the average probability of error over all possible messages for a given adversarial choice of J is given by

$$e(J) = \frac{1}{M} \sum_{i=1}^M e(i, J). \quad (2)$$

The maximal probability of error is given by

$$e_{\max}(J) = \max_i e(i, J). \quad (3)$$

Remark II.2. For fixed i and any choice of J , $e(i, J) \leq \max_{\mathbf{s}} e(i, J_{\mathbf{s}})$, implying that $\sup_J e(i, J) = \max_{\mathbf{s}} e(i, J_{\mathbf{s}})$. However, since the maximizing choice of \mathbf{s} may differ by message i , and a myopic adversary has access to some information about i , this does not imply that $e(J) \leq \max_{\mathbf{s}} e(J_{\mathbf{s}})$ for all J . This marks a fundamental difference from the oblivious adversary model: in that case, an adversarial strategy is defined by a specific choice of \mathbf{s} that is made independent of the transmitted message, and the probability of error for that choice is given by $e(\mathbf{s}) = \frac{1}{M} \sum_i e(i, \mathbf{s})$. Here, it is clearly true that $e(\mathbf{s}) \leq \max_{\mathbf{s}} e(\mathbf{s})$ for every choice of adversarial strategy \mathbf{s} .

We say that a rate R is *achievable* if there exists a sequence of $(2^{nR}, n)$ authentication codes such that $\sup_J e(J) \rightarrow 0$ as $n \rightarrow \infty$, or $\sup_J e_{\max}(J) \rightarrow 0$ as $n \rightarrow \infty$. We will primarily consider the average probability of error in this paper, and will explicitly state when we are claiming a stronger achievability result (i.e. with reference to the maximal probability of error). Notice that $\sup_J e(J)$ is the highest error probability the adversary can hope for, achieved by designing J optimally. The *authentication capacity* C_{auth} is the supremum of all achievable rates. Let C denote the capacity in the no-adversary setting (i.e., $J = J_{\mathbf{s}_0}$).

The work in [4] shows that a channel property called *overwritability* exactly determines when the authentication capacity is nonzero for oblivious adversaries.

Definition II.3. An adversarial channel $W_{Y|X,S}$ with no-adversary state s_0 is *overwritable* if there exists a distribution $P_{S|X'}$ such that

$$\sum_{\mathbf{s}} P_{S|X'}(\mathbf{s} | x') W(y | x, \mathbf{s}) = W(y | x', s_0) \text{ for all } x, x', y. \quad (4)$$

Less formally, over an overwritable channel, an adversary can seamlessly make their own false message appear legitimate to the receiver without being detected.

Theorem II.4. [4] *If a channel with an oblivious adversary is not overwritable, then $C_{\text{auth}} = C$; if it is overwritable, $C_{\text{auth}} = 0$.*

Overwritability should be compared with symmetrizability for the standard AVC problem [16]: $W_{Y|X,S}$ is *symmetrizable* if there exists $P_{S|X'}$ such that $\sum_s P_{S|X'}(s | x')W(y | x, s) = \sum_s P_{S|X'}(s | x)W(y | x', s)$ for all x, x', y . In [4] it is shown that overwritability implies symmetrizability, but that the converse does not hold. Since a classical AVC has zero capacity if and only if it is symmetrizable [16], this implies that the authentication capacity can be positive even though the AVC capacity is zero. There is an equivalence between overwritability and *simulatability*, as introduced in [10], when the adversary is oblivious to the transmission but has complete control over the communication channel (i.e. can substitute in its transmission for the legitimate transmission at will).

When the adversary is not oblivious, we propose a modification of overwritability, called *U-overwritability*, that will help us to characterize capacity in the myopic case.

Definition II.5. *We say that an adversarial channel $W_{Y|X,S}$ with no-adversary state s_0 is U-overwritable, where $U_{Z|X}$ is a conditional distribution, if there exists a distribution $P_{S|X',Z}$ such that*

$$\sum_{s,z} U(z | x) P_{S|X',Z}(s | x', z) W(y | x, s) = W(y | x', s_0) \quad \forall x, x', y. \quad (5)$$

Let $\mathbb{1}_{Z|X}$ be the deterministic identity distribution: that is, $\mathbb{1}(z | x) = 1$ if $z = x$, and zero otherwise. This corresponds to the case of a so-called *omniscient* adversary, and induces a special case of *U-overwritability*, as defined below.

Definition II.6. *An adversarial channel $W_{Y|X,S}$ with no-adversary state s_0 is I-overwritable if there exists a distribution $P_{S|X',X}$ such that*

$$\sum_s P_{S|X',X}(s | x', x) W(y | x, s) = W(y | x', s_0) \quad \forall x, x', y. \quad (6)$$

With the introduction of a noisy channel to the adversary, there are several relevant problems in terms of authentication. The first is characterizing the channel capacity given an AVC $W_{Y|X,S}$ and a channel $U_{Z|X}$ to the adversary. Another is determining the robustness of a channel $W_{Y|X,S}$ against myopic adversaries: in other words, what is the “best” channel to the adversary that the legitimate users can withstand (i.e. still transmit reliably with a positive rate). We consider these problems in the remainder of the paper.

III. ZERO AUTHENTICATION CAPACITY

In this section, we characterize the conditions under which the authentication capacity is zero. In Section III-A, we consider the case where we are restricted to deterministic encoders. Section III-B shows that these results do not necessarily hold when the encoder is allowed some local randomness.

First, we give a result that will apply to both deterministic and stochastic encoders.

Theorem III.1. *If the channel to the adversary is given by $U_{Z|X}$ and $W_{Y|X,S}$ is U-overwritable, $C_{\text{auth}} = 0$.*

Proof. Suppose $W_{Y|X,S}$ is U-overwritable, and let $P_{S|X',Z}$ be the distribution guaranteed by Definition II.5. Consider a sequence of $(2^{nR}, n)$ authentication codes with fixed $R > 0$, and let $M := 2^{nR}$. Let $\tilde{J}(\mathbf{s} | \mathbf{z}) = \frac{1}{M} \sum_{j=1}^M \mathbb{E}_{\mathbf{X}'|j} [P_{S|X',Z}(\mathbf{s} | \mathbf{X}', \mathbf{z})]$, where the expectation is over the stochastic encoding of message j . Then,

$$e(\tilde{J}) = \frac{1}{M} \sum_{i=1}^M e(\tilde{J}, i) \quad (7)$$

$$\geq \frac{1}{M^2} \sum_{i,j} \mathbb{E}_{\mathbf{X}|i} \left[\sum_{\mathbf{s}, \mathbf{z}} U(\mathbf{z} | \mathbf{X}) \mathbb{E}_{\mathbf{X}'|j} [P_{S|X',Z}(\mathbf{s} | \mathbf{X}', \mathbf{z})] W(\phi^{-1}(\{i, 0\})^c | \mathbf{X}, \mathbf{s}) \right] \quad (8)$$

$$= \frac{1}{M^2} \sum_{i,j} \mathbb{E}_{\mathbf{X}|i} \left[\mathbb{E}_{\mathbf{X}'|j} [W(\phi^{-1}(\{i, 0\})^c | \mathbf{X}', \mathbf{s}_0)] \right] \quad (9)$$

$$\geq \frac{1}{M^2} \sum_j \sum_{i \neq j} \mathbb{E}_{\mathbf{X}'|j} [W(\phi^{-1}(j) | \mathbf{X}', \mathbf{s}_0)] \quad (10)$$

$$\geq \frac{1}{M^2} \sum_j \sum_{i \neq j} (1 - e(j, J_{\mathbf{s}_0})) \quad (11)$$

$$\geq \frac{M-1}{M} \left(\frac{1}{M} \sum_j (1 - e(j, J_{\mathbf{s}_0})) \right) \quad (12)$$

$$\geq \frac{M-1}{M} (1 - e(J_{\mathbf{s}_0})), \quad (13)$$

where (9) follows from U -overwritability and (10) follows because $j \in \{i, 0\}^c$ as long as $i \neq j$. The deterministic distribution $J_{\mathbf{s}_0}$ is as defined in Section II. Since $\sup_J e(J) \geq e(\tilde{J})$ and $\sup_J e(J) \geq e(J_{\mathbf{s}_0})$,

$$\sup_J e(J) \geq \frac{M-1}{M} (1 - e(J_{\mathbf{s}_0})) \geq \frac{M-1}{M} \left(1 - \sup_J e(J) \right).$$

We conclude that $\sup_J e(J) \geq \frac{M-1}{M} (1 - \sup_J e(J))$. This reduces to $\sup_J e(J) \geq \frac{M-1}{2M-1}$, which approaches $1/2$ as $n \rightarrow \infty$. Thus, $C_{\text{auth}} = 0$. \square

A. Deterministic Encoders

The encoder f of a $(2^{Rn}, n)$ authentication code is deterministic if each message $i \in [2^{Rn}]$ has a single associated codeword $f(i) := \mathbf{x}_i \in \mathcal{X}^n$ that will be transmitted across the channel $W_{Y|X,S}$. That is, if f is a function from $[2^{Rn}]$ to \mathcal{X}^n . For deterministic encoders, knowing the message is synonymous with knowing the transmitted sequence. Thus, in this scenario, if the adversary can reliably decode the message, we can consider them as an essentially omniscient adversary. We show for the set of deterministic encoders that if the no-adversary channel between legitimate users is *stochastically degraded* with respect to the channel to the adversary, then I -overwritability guarantees $C_{\text{auth}} = 0$.

Definition III.2. We say that a channel $P_{Y|X}$ is stochastically degraded with respect to $P_{Z|X}$ if there exists $P_{Y|Z}$ such that for all $x \in \mathcal{X}, y \in \mathcal{Y}$:

$$P_{Y|X}(y|x) = \sum_{z \in \mathcal{Z}} P_{Y|Z}(y|z) P_{Z|X}(z|x).$$

Theorem III.3. Suppose that $W_{Y|X,S=s_0}$ is stochastically degraded with respect to $U_{Z|X}$. Then, if $W_{Y|X,S}$ is I -overwritable, and we are restricted to a deterministic encoder, $C_{\text{auth}} = 0$.

Proof. Suppose that $W_{Y|X,S}$ is I -overwritable, and let $P_{S|X',X}$ be the distribution guaranteed by I -overwritability. Because $W_{Y|X,S=s_0}$ is stochastically degraded with respect to $U_{Z|X}$, there exists $P_{Y|Z}$ such that for all x, y , we have $\sum_z U(z|x) P_{Y|Z}(y|z) = W(y|x, s_0)$. Consider a sequence of $(2^{nR}, n)$ authentication codes with $R > 0$, and let $M := 2^{nR}$. With an abuse of notation, we will let $\phi(\mathbf{y})$ denote either a message or a codeword; they are in one-to-one correspondence due to the assumption of a deterministic decoder. Let

$$\tilde{J}(\mathbf{s} | \mathbf{z}) = \frac{1}{M} \sum_{j=1}^M \sum_{\mathbf{y}} P_{S|X',X}(\mathbf{s} | \mathbf{x}_j, \phi(\mathbf{y})) P_{Y|Z}(\mathbf{y} | \mathbf{z}),$$

where ϕ is the non-adversarial decoding rule of the legitimate receiver. Then,

$$\begin{aligned} e(\tilde{J}) &\geq \frac{1}{M^2} \sum_{i,j,s,\mathbf{z},\mathbf{y}} U(\mathbf{z} | \mathbf{x}_i) P_{S|X',X}(\mathbf{s} | \mathbf{x}_j, \phi(\mathbf{y})) \\ &\quad \cdot P_{Y|Z}(\mathbf{y} | \mathbf{z}) W(\phi^{-1}(\{i, 0\}^c | \mathbf{x}_i, \mathbf{s})) \end{aligned} \quad (14)$$

$$= \frac{1}{M^2} \sum_{i,j,s,\mathbf{y}} W(\mathbf{y} | \mathbf{x}_i, \mathbf{s}_0) P_{S|X',X}(\mathbf{s} | \mathbf{x}_j, \phi(\mathbf{y})) W(\phi^{-1}(\{i, 0\}^c | \mathbf{x}_i, \mathbf{s})) \quad (15)$$

$$\geq \frac{1}{M^2} \sum_{\substack{i,j,s, \\ \mathbf{y} \text{ s.t. } \phi(\mathbf{y})=i}} W(\mathbf{y} | \mathbf{x}_i, \mathbf{s}_0) P_{S|X',X}(\mathbf{s} | \mathbf{x}_j, \phi(\mathbf{y})) W(\phi^{-1}(\{i, 0\}^c | \mathbf{x}_i, \mathbf{s})) \quad (16)$$

$$= \frac{1}{M^2} \sum_{i,j,s} [1 - e(i, J_{s_0})] P_{S|X',X}(s | \mathbf{x}_j, \mathbf{x}_i) W(\phi^{-1}(\{i, 0\})^c | \mathbf{x}_i, s) \quad (17)$$

$$= \frac{1}{M^2} \sum_{i,j} [1 - e(i, J_{s_0})] W(\phi^{-1}(\{i, 0\})^c | \mathbf{x}_j, s_0) \quad (18)$$

$$\geq \frac{1}{M} \sum_i [1 - e(i, J_{s_0})] \left(\frac{1}{M} \sum_{j \neq i} W(\phi^{-1}(j) | \mathbf{x}_j, s_0) \right) \quad (19)$$

$$\geq \frac{1}{M} \sum_i [1 - e(i, J_{s_0})] \left(\frac{1}{M} \sum_{j \neq i} [1 - e(j, J_{s_0})] \right) \quad (20)$$

$$= \left(\frac{1}{M} \sum_i [1 - e(i, J_{s_0})] \right)^2 - \frac{1}{M^2} \sum_i [1 - e(i, J_{s_0})]^2 \quad (21)$$

$$= [1 - e(J_{s_0})]^2 - \frac{1}{M^2} \sum_i [1 - e(i, J_{s_0})]^2 \quad (22)$$

$$\geq [1 - e(J_{s_0})]^2 - \frac{1}{M} \quad (23)$$

where (15) follows from stochastic degradation, and (18) follows from I -overwritability. Thus, $\sup_J e(J) > [1 - e(J_{s_0})]^2 - \frac{1}{M} \geq [1 - \sup_J e(J)]^2 - \frac{1}{M}$, and

$$\sup_J e(J) \geq \frac{3 - \sqrt{5 + \frac{4}{M}}}{2}$$

We conclude that $\sup_J e(J)$ is bounded away from zero as $n \rightarrow \infty$, and thus that $C_{\text{auth}} = 0$. \square

B. Stochastic Encoders

When a stochastic encoder is used, the adversary has knowledge of the distribution $P_{\mathbf{X}|M}$, though not necessarily the specific transmitted message or sequence. In this case, the ability of a myopic adversary to determine the intended message does not imply that it has perfect knowledge of the transmission itself. As the latter may be necessary for successful malicious interference, this constitutes the distinction from the work in Section III-A.

For the model in which the adversary has perfect knowledge of the message and encoding procedure, but is completely oblivious the transmission sequence itself, [12] gives an example of a channel for which stochastic encoding will increase the channel capacity. This example is expanded below to the model in which the adversary has a noisy version of the transmission (from which it may or may not deduce the message). Via this example, we demonstrate the necessity of the deterministic encoder condition in Theorem III.3. In the proof of Theorem III.3, the deterministic encoder makes an appearance in Equation (17), where knowledge of $\phi(\mathbf{y}) = i$ is treated as equivalent to knowledge of \mathbf{x}_i .

Recall that *binary symmetric channel* (BSC) with crossover probability p , denoted $\text{BSC}(p)$, is a binary-input, binary-output channel such that the probability the output differs from the input is equal to p . A *binary erasure channel* (BEC) with erasure probability p outputs an erasure symbol with probability p , and otherwise transmits binary inputs reliably.

Example III.4. In this example, we exhibit a channel with the property that unless the adversary has perfect knowledge of the transmitted codeword, the receiver will be able to authenticate with high probability. Over this channel, a deterministic encoder results in $C_{\text{auth}} = 0$, but we will show that a stochastic encoder will allow for a positive authentication capacity.

Consider a binary-input channel with output alphabet $\{0, 1, \varepsilon\}$, where ε denotes an erasure symbol. The adversary either chooses not to act (denoted s_0), or chooses a state equal to 0 or 1. If the adversary does not act, the channel operates as a $\text{BSC}(p)$. If the adversary inputs a 0 or 1, there are two possibilities: if the chosen state symbol matches the transmitted symbol, the channel operates as a $\text{BSC}(1-p)$. Otherwise, the transmitted symbol is erased. An example is shown in Figure 2. Notice that the adversary causes authentication failure if and only if it is able

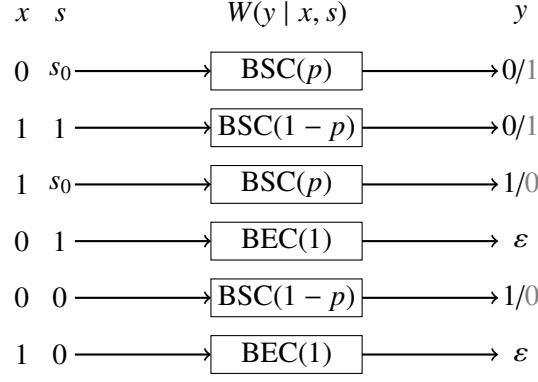


Fig. 2. The word $\mathbf{x}=011001$ is sent. The adversary chooses state sequence $\mathbf{s} = s_0 1 s_0 1 0 0$. For the positions in which the adversary has chosen not to act, the channel is a BSC(p), where we assume without loss of generality that $p < 0.5$. When the adversary's state matches the transmitted symbol, the channel statistics flip. The fourth and the sixth symbols in the output sequence \mathbf{y} will be erasure symbols, since the transmitted symbol and state symbol do not match. Thus, in this example, the receiver can declare with absolute confidence that there has been adversarial interference.

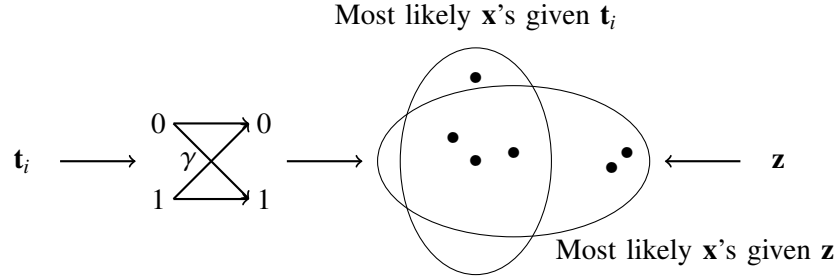


Fig. 3. The message i is encoded as \mathbf{t}_i , and is passed through a BSC(γ), whose output is \mathbf{x} . The adversary observes \mathbf{z} after \mathbf{x} is passed through a BSC(q). Given \mathbf{z} , the adversary can decode reliably to \mathbf{t}_i , but there remains ambiguity about the realization of the sequence \mathbf{x} .

to simultaneously flip enough bits to convince the receiver that a different message was sent, and not trigger any erasures whatsoever.

It is straightforward to see that the channel is I -overwritable. Suppose, then, that the channel to the adversary is a BSC(q), where $q \leq p$. By Theorem III.3, $C_{\text{auth}} = 0$ if the encoder is deterministic. However, we will show that a stochastic encoder will allow for positive authentication capacity. To do so, we design a stochastic encoder as follows. Let $\gamma > 0$, and encode the message i using a deterministic binary code designed for the BSC concatenation of a BSC(γ) and a BSC(p); call this codeword \mathbf{t}_i . Next, simulate an artificial BSC(γ) and send the output of this simulated channel, denoted \mathbf{x} , through the AVC. The non-adversarial channel to the receiver is now the concatenated channel for which the code was designed, and so messages are transmitted reliably in the absence of an adversary. Meanwhile, the channel to the adversary is less noisy than the main channel, and so we assume that given its observation \mathbf{z} , the adversary can determine the codeword \mathbf{t}_i . However, it cannot exactly determine the actual transmitted sequence \mathbf{x} . For an illustration of this, see Figure 3.

In order to avoid being detected, the adversary can either send s_0 or $s = x$, and in order to cause a decoding error, it must manipulate enough symbols that the receiver decodes to an incorrect codeword $\mathbf{t}_{j \neq i}$. Because the adversary cannot be certain of the exact realization of the transmitted sequence, we conclude that if it acts, it is detected with high probability. In other words, this channel has positive capacity. By making γ sufficiently small, the capacity of the concatenated channel approaches the capacity of the underlying BSC(p), allowing our coding scheme to have rates approaching the latter.

Example III.4 is high-stakes: if the adversary makes any mistake whatsoever, it is discovered. Section V examines a more forgiving scenario, where the adversary may flip the BSC channel statistics in whichever time slots it chooses, without needing to know the transmitted sequence. We show that a stochastic encoder also allows for positive (and, in fact, no-adversary) authentication capacity in this case.

IV. RELATIONSHIPS BETWEEN MYOPIC ADVERSARIES

A myopic adversary spans the gap between oblivious adversaries (as in [4]) and omniscient adversaries, who have perfect access to the transmitted sequence. Thus, when the legitimate transmitter and receiver have use of a particular communication channel $W_{Y|X,S}$, there may be some point at which the adversary becomes capable enough that the channel is U -overwritable, bringing the authentication capacity to zero. Since not every pair of potential channels to the adversary are directly comparable, we address this transition with the use of a partial ordering on channels $U_{Z|X}$ given by stochastic degradation: formally, $U_{Z|X}^2 \leq U_{Z|X}^1$ if and only if $U_{Z|X}^2$ is stochastically degraded with respect to $U_{Z|X}^1$. Examples IV.1 and IV.2 illustrate some features of this partial order.

Example IV.1. The channel to the adversary in the oblivious case (i.e. the case in which Z is independent of X) is stochastically degraded with respect to any other channel $U_{Z|X}$. On the other hand, every channel $U_{Z|X}$ is stochastically degraded with respect to the omniscient channel ($U_{Z|X} = \mathbb{1}_{Z|X}$). Thus, these extremes give the unique maximum (omniscient adversary) and minimum (oblivious adversary) elements of our partial order.

Example IV.2. As an example of a totally ordered chain within the partial order, consider the set of binary symmetric channels. The $\text{BSC}(p)$ is stochastically degraded with respect to the $\text{BSC}(q)$ if and only if $0.5 \geq p \geq q \geq 0$. The maximum element of the chain is the $\text{BSC}(0)$ (omniscient adversary), and the minimum is the $\text{BSC}(0.5)$ (oblivious adversary). See Figure 4.

Using the partial order of stochastic degradation, we may compare channels and then use these comparisons to draw conclusions about U -overwritability.

Theorem IV.3. *If $U_{Z'|X}^2$ is stochastically degraded with respect to $U_{Z'|X}^1$, and $W_{Y|X,S}$ is U^2 -overwritable, then $W_{Y|X,S}$ is also U^1 -overwritable.*

Proof. Let $P_{S|X',Z'}$ be the distribution guaranteed by U^2 -overwritability, and let $P_{S|X',Z}(s|x', z) = \sum_{z'} P_{S|X',Z'}(s|x', z') P_{Z'|Z}(z'|z)$, where $P_{Z'|Z}$ is the distribution guaranteed by stochastic degradation. Then,

$$\begin{aligned} \sum_{s,z} U^1(z|x) P_{S|X',Z}(s|x', z) W(y|x, s) &= \sum_{s,z,z'} P_{Z'|Z}(z'|z) U^1(z|x) P_{S|X',Z'}(s|x', z') W(y|x, s) \\ &= \sum_{s,z'} U^2(z'|x) P_{S|X',Z'}(s|x', z') W(y|x, s) \\ &= W(y|x', s_0). \end{aligned}$$

We conclude that $W_{Y|X,S}$ is U^1 -overwritable. \square

Theorem IV.3 implies that overwritability is a stronger condition than U -overwritability, formalized in the following corollary.

Corollary IV.4. *If a channel $W_{Y|X,S}$ is overwritable, then it is U -overwritable for every channel $U_{Z|X}$.*

Proof. The channel to an oblivious adversary is stochastically degraded with respect to every channel $U_{Z|X}$. The result follows by Theorem IV.3. \square

Clearly, if the adversary can successfully generate a false message given a noisy version of a transmission, it is also successful in the noiseless case. This is formalized in the following corollary.

Corollary IV.5. *If a channel $W_{Y|X,S}$ is U -overwritable for some $U_{Z|X}$, then it is also I -overwritable.*

Proof. Every channel $U_{Z|X}$ is stochastically degraded with respect to $\mathbb{1}_{Z|X}$. The result follows by Theorem IV.3. \square

Example IV.6. Extending Example IV.2, we see that if $W_{Y|X,S}$ is U -overwritable for $U_{Z|X} = \text{BSC}(p)$, then it is U -overwritable for $U_{Z|X} = \text{BSC}(q)$ for all $q \leq p$, and the authentication capacity is equal to zero whenever the channel to the adversary is a $\text{BSC}(q)$ for $q \leq p$. See Figure 4.

Example IV.7. By the contrapositive of Corollary IV.5, if a channel is not I -overwritable, then it is not U -overwritable for any $U_{Z|X}$. As an example, consider the following AVC: $\mathcal{X} = \mathbb{F}_2$, $\mathcal{Y} = \mathbb{F}_2 \cup \{\varepsilon\}$, $\mathcal{S} = \{0, 1, s_0\}$, and $W(y|x, s)$ is a binary erasure channel (BEC) operating on x with erasure probability s if $s = 0, 1$. If $s = s_0$, we define the erasure probability to be equal to p .

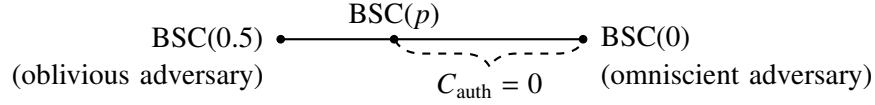


Fig. 4. The binary symmetric channels are totally ordered by stochastic degradation. If $W_{Y|X,S}$ is U -overwritable for $U_{Z|X} = \text{BSC}(p)$, then the authentication capacity C_{auth} is equal to zero for that $U_{Z|X}$ and all binary symmetric channels with smaller crossover probability.

First, we observe that if the channel is I -overwritable, then $p = 1$: indeed, if there exists $P_{S|X',X}$ such that

$$\sum_s P_{S|X',X}(s | x', x) W(y | x, s) = W(y | x', s_0)$$

for all x, x', y , then, letting $x' = y = 0$ and $x = 1$, we have $0 = 1 - p$; we conclude $p = 1$. Thus, for all $p < 1$, the channel is not I -overwritable. It cannot then be U -overwritable for any channel $U_{Z|X}$ to the adversary.

In fact, we can show that the authentication capacity is equal to $C = 1 - p$ for any value of p and any channel to the adversary. The converse follows from the capacity of the underlying $\text{BEC}(p)$. To see achievability, design a code of rate $1 - p - \delta$ for a $\text{BEC}(p)$ with vanishing error probability, and decode as follows:

- (1) If there is a single codeword consistent with the observed sequence, decode to that codeword.
- (2) If there is more than one codeword consistent with the observed sequence, declare adversarial interference.

The only error that may occur is when the adversary is not present, and we declare an error in step (2). However, this would constitute a regular decoding error in our code for the $\text{BEC}(p)$. Thus, as $n \rightarrow \infty$, the probability of this type of error goes to zero. In all, then, we have shown that rates arbitrarily close to $1 - p$ are achievable.

Remark IV.8. In general, it is straightforward to see that the existence of a distribution $P_{S|X',Z}$ (resp. $P_{S|X',X}$) satisfying the equality in Definition II.5 (resp. Definition II.6), and thus the U -overwritability (resp. I -overwritability) of a channel, may be determined via linear programming. Once U -overwritability has been determined for a particular choice of $U_{Z|X}$, Theorem IV.3 allows us to determine U -overwritability for less degraded channels.

V. A MYOPIC BINARY ADVERSARIAL CHANNEL

In this section, we examine in detail a binary model that we believe will provide insight into the more general case. In this model, the adversary views the transmitted codeword through a $\text{BSC}(q)$, and decides on a binary state sequence \mathbf{s} , which is added to the transmission \mathbf{x} . The sequence $\mathbf{x} \oplus \mathbf{s}$ is then transmitted across a $\text{BSC}(p)$. We call this the myopic binary adversarial channel with parameters p and q , and denote it by $\text{MBAC}_{p,q}$.

Remark V.1. Consider an $\text{MBAC}_{p,q}$ such that $0 \leq p, q \leq 0.5$. We first claim that $W_{Y|X,S}$ is not U -overwritable as long as $q > 0$ and $p < 0.5$.

Indeed, suppose $W_{Y|X,S}$ is overwritable. Then there exists $P_{S|X',Z}$ such that for any choice of x, x' , and y ,

$$\sum_{s,z} U(z | x) P_{S|X',Z}(s | x', z) W(y | x, s) = W(y | x', s_0).$$

If $x = x' = y = 0$, then the above reduces to:

$$\sum_{s,z} U(z | 0) P_{S|X',Z}(s | 0, z) W(0 | 0, s) = W(0 | 0, 0).$$

We then have

$$\begin{aligned} 1 - p &= \sum_{s,z} U(z | 0) P_{S|X',Z}(s | 0, z) W(0 | 0, s) \\ &= (1 - 2p)(1 - q)P_{S|X',Z}(0 | 0, 0) + q(1 - 2p)P_{S|X',Z}(0 | 0, 1) + p. \end{aligned}$$

So,

$$1 = (1 - q)P_{S|X',Z}(0 | 0, 0) + qP_{S|X',Z}(0 | 0, 1).$$

If $q > 0$, this can only occur if $P_{S|X',Z}(0 | 0, 0) = P_{S|X',Z}(0 | 0, 1) = 1$, and thus, $P_{S|X',Z}(1 | 0, 0) = P_{S|X',Z}(1 | 0, 1) = 0$. With this in mind, let $x = 1$ and $x' = y = 0$. In this case, we may show that

$$1 - p = p = 0.5.$$

Thus, if $q > 0$ and $p < 0.5$, the channel is not U -overwritable, where $U_{Z|X}$ is a $BSC(q)$.

Next, consider the boundary cases: if $q > 0$ and $p = 0.5$, the channel has non-adversarial capacity $1 - H(0.5) = 0$. The final case is that in which $q = 0$. In this case, the channel is U -overwritable, or equivalently, I -overwritable: indeed, choose $P_{S|X',Z=X}$ to be deterministic such that $s = x' + x$.

A. When the adversary is more myopic than the receiver

We begin with the case in which the channel to the adversary is stochastically degraded with respect to the non-adversarial channel $W_{Y|X,S=s_0}$ between legitimate users. That is, $0 \leq p < q \leq 1/2$. By our arguments in Remark V.1, the channel is never $BSC(q)$ -overwritable in this case, and so we cannot make use of Theorem III.1. In fact, we show that the authentication capacity is not only nonzero when $q > p$, but that we can achieve the non-adversarial capacity with a deterministic encoder.

Theorem V.2. *If $0 \leq p < q \leq 1/2$, the authentication capacity C_{auth} is equal to the non-adversarial capacity $C = 1 - H(p)$. Moreover, this rate can be achieved with a deterministic encoder.*

Theorem V.2 converse proof. Since any authentication code must also be an error-correcting code for the underlying non-adversarial channel, we have $C_{\text{auth}} \leq C_{BSC(p)} = 1 - H(p)$. \square

Per the $MBAC_{p,q}$ model, the adversary can see a noisy version of the transmitted codeword. In our proof of achievability, we will strengthen the adversary in order to simplify some arguments. Proving achievability for a stronger adversary simultaneously proves the result for any weaker adversary. Specifically, we introduce an oracle who will reveal to the adversary the exact distance d of the transmitted codeword, \mathbf{x}_i , from the received word \mathbf{z} . Generally speaking, even given this information, there remain enough potentially transmitted words to make the adversary's task difficult. We will also allow the adversary to be aware of the exact error pattern, \mathbf{e} , of the $BSC(p)$ between the transmitter and receiver, so that it can design the state knowing the exact difference between the transmission and what the receiver will see. That is, the adversary can add the state $\mathbf{s}' = \mathbf{s} \oplus \mathbf{e}$, so that the receiver will see $\mathbf{x}_i \oplus \mathbf{s}' \oplus \mathbf{e} = \mathbf{x}_i \oplus \mathbf{s}$. To simplify our analysis, we will simply assume the adversary adds \mathbf{s} and there is no additional channel noise from the $BSC(p)$. However, the decoder will still assume there has been some channel noise and decode appropriately (i.e. the receiver does not have any increased knowledge).

We first present two lemmas that will allow us to choose a good codebook. The first is a variation of Lemma 3 from [16]. In each, let $M := 2^{nR}$.

Lemma V.3 ([16]). *Let $\epsilon_1 > 0$ and let $\mathbf{x}_1, \dots, \mathbf{x}_M$ be drawn uniformly at random from the type class of type P_X . With high probability, this codebook satisfies the following. For any type class $\tau_{XX'S}$ and any sequence \mathbf{s} ,*

$$|\{i : \exists j \neq i \text{ s.t. } (\mathbf{x}_i, \mathbf{x}_j, \mathbf{s}) \in \tau_{XX'S}\}| \leq 2^{n[R - I(X;X'S) + |R - I(X';S)|^+ + n\epsilon_1]}. \quad (24)$$

The proof of the following lemma appears in Appendix A.

Lemma V.4. *Let codewords $\mathbf{x}_1, \dots, \mathbf{x}_M$ be drawn independently and uniformly at random from some type class τ_X . For each type class τ_Z , and each $\mathbf{z} \in \tau_Z$, the number of messages i such that $\|\mathbf{x}_i \oplus \mathbf{z}\| = d$ is, with high probability in n , bounded below by*

$$\left\lfloor \frac{1}{(n+1)^8} 2^{n(R - I(X;Z))} \right\rfloor,$$

where we let $p_{X|Z}$ be the conditional distribution of pairs of words in their respective type classes that are distance d apart, as follows:

$$p_{X|Z}(1|0) = \frac{d}{2np_Z(0)} + \frac{1}{2} - \frac{p_X(0)}{2p_Z(0)}$$

$$p_{X|Z}(0|1) = \frac{d}{2np_Z(1)} + \frac{1}{2} - \frac{p_X(1)}{2p_Z(1)}.$$

Combining Lemma V.4 with the fact that $|H(t) - H(t \pm \epsilon)| \leq -\epsilon \log_2(\epsilon/2)$ for $\epsilon < 1/4$ yields the following corollary.

Corollary V.5. *Given $\epsilon < 1/4$ and $X, Z \sim \text{Bernoulli}(1/2)$, if $\mathbf{z} \in \mathcal{T}_\epsilon^{(n)}(Z)$, $|d - nq| \leq n\epsilon$, and $R = 1 - H(p) + 5\epsilon \log_2 \epsilon/2$, then the number of codewords messages distance d from \mathbf{z} is, with high probability, bounded below by*

$$\left\lfloor \frac{1}{(n+1)^8} 2^{n(R-1+H(d/n)+\epsilon \log_2 \epsilon/2)} \right\rfloor \geq \left\lfloor \frac{1}{(n+1)^8} 2^{n(H(q)-H(p)+7\epsilon \log_2 \epsilon/2)} \right\rfloor.$$

We now prove achievability with a deterministic encoder.

Theorem V.2 achievability proof. Let $\tilde{\delta}_n$ be an arbitrary sequence such that $n\tilde{\delta}_n \geq \sqrt{n} \log_2 n$ and $\lim_{n \rightarrow \infty} \tilde{\delta}_n = 0$. For convenience, set $\delta_n := -\tilde{\delta}_n \log_2 \tilde{\delta}_n/2$.

Suppose $0 \leq p < q \leq 1/2$. For n sufficiently large, and without loss of generality, let $R = 1 - H(p) - 5\delta_n > 1 - H(q) + \frac{8}{n} \log_2(n+1) + 2\delta_n$. We construct a $(M := 2^{nR}, n)$ code family with vanishing probability of error.

Encoding: By Lemmas V.3 and V.4, for n sufficiently large, there exist codewords $\mathbf{x}_1, \dots, \mathbf{x}_M$ from the type class τ_X , where $X \sim \text{Bernoulli}(1/2)$, such that: (1) for every $\mathbf{z} \in \mathcal{T}_{\tilde{\delta}_n}^{(n)}(Z)$ where $Z \sim \text{Bernoulli}(1/2)$, the number of codewords that are distance d from \mathbf{z} , where $|d - nq| \leq n\tilde{\delta}_n$, is bounded below by $2^{n(H(q)-H(p)-7\delta_n-\frac{8}{n} \log_2(n+1))}$, and (2) for any type class $\tau_{X|X^S}$ and any sequence \mathbf{s} , (24) holds. Given a message $i \in [M]$, transmit \mathbf{x}_i .

Decoding: Let $\epsilon > 0$ be sufficiently small. Given an output $\mathbf{y} \in \{0, 1\}^n$, decode to message $i \in [M]$ if i is unique with the property that $\|\mathbf{x}_i \oplus \mathbf{y}\| < n(p + \tilde{\delta}_n)$. Otherwise, declare adversarial interference by outputting “0”.

Probability of error analysis: Define

$$S(\mathbf{z}, d) := \{i \in [M] : \|\mathbf{z} \oplus \mathbf{x}_i\| = d\},$$

$$E(\mathbf{s}) := \{i \in [M] : \exists j \neq i \text{ s.t. } \|\mathbf{x}_i \oplus \mathbf{s} \oplus \mathbf{x}_j\| < n(p + \tilde{\delta}_n)\}.$$

That is, $S(\mathbf{z}, d)$ is the set of messages in $[M]$ whose corresponding codewords are distance d from $\mathbf{z} \in \{0, 1\}^n$, and $E(\mathbf{s})$ is the set of messages i in $[M]$ such that adding \mathbf{s} to \mathbf{x}_i results in the decoder potentially confusing the intended message with a false message j . Let $J_{\mathbf{s}|\mathbf{z}, d}$ be adversary's choice of distribution given knowledge of the distance to the transmitted codeword, which is given by the random variable D . For fixed J , $e(i, J)$ is the probability of decoding error given that message i was sent. Then,

$$e(J) = \frac{1}{M} \sum_{i=1}^M P(\text{error} | i) \tag{25}$$

$$= \sum_{i, \mathbf{s}, \mathbf{z}, d} P(\text{error} | i, \mathbf{s}, \mathbf{z}, d) P(i | \mathbf{s}, \mathbf{z}, d) J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d) \tag{26}$$

$$= \sum_{i, \mathbf{s}, \mathbf{z}, d} P(\text{error} | i, \mathbf{s}, \mathbf{z}, d) P(i | \mathbf{z}, d) J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d) \tag{27}$$

Since every message whose codeword is distance d from \mathbf{z} is equiprobable

$$P(i | \mathbf{z}, d) = \begin{cases} \frac{1}{|S(\mathbf{z}, d)|} & \text{if } i \in S(\mathbf{z}, d), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $P(\text{error} | i, \mathbf{s}, \mathbf{z}, d) = 0$ if for every $j \neq i$, $\|\mathbf{x}_i \oplus \mathbf{s} \oplus \mathbf{x}_j\| \geq n(p + \tilde{\delta}_n)$. In other words, $P(\text{error} | i, \mathbf{z}, \mathbf{s}, d) = 0$ if $i \notin E(\mathbf{s})$. Then, (27) is bounded as follows:

$$\sum_{\mathbf{s}, \mathbf{z}, d} J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d) \sum_{i=1}^M P(\text{error} | i, \mathbf{s}, \mathbf{z}, d) P(i | \mathbf{z}, d) \leq \sum_{\mathbf{s}, \mathbf{z}, d} J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d) \cdot \frac{|S(\mathbf{z}, d) \cap E(\mathbf{s})|}{|S(\mathbf{z}, d)|} \tag{28}$$

$$\leq \sum_{\substack{\mathbf{s}, \\ d: |d-nq| \leq n\tilde{\delta}_n \\ \mathbf{z} \in \mathcal{T}_{\tilde{\delta}_n}^{(n)}}} J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d) \cdot \frac{|S(\mathbf{z}, d) \cap E(\mathbf{s})|}{|S(\mathbf{z}, d)|} \tag{29}$$

$$\begin{aligned}
& + \sum_{\substack{\mathbf{s}, d \\ \mathbf{z} \notin \mathcal{T}_{\tilde{\delta}_n}^{(n)}}} J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d) \\
& + \sum_{\substack{\mathbf{s}, \mathbf{z} \\ d: |d-nq| > n\tilde{\delta}_n}} J(\mathbf{s} | \mathbf{z}, d) P(\mathbf{z}, d),
\end{aligned} \tag{30}$$

$$\tag{31}$$

since $\frac{|S(\mathbf{z}, d) \cap E(\mathbf{s})|}{|S(\mathbf{z}, d)|} \leq 1$.

Now, observe that we can choose $\tilde{\delta}_n$ such that (30) and (31) are typicality conditions, and more specifically

$$\lim_{n \rightarrow \infty} ((30) + (31)) \leq \lim_{n \rightarrow \infty} \left(\Pr(\mathbf{Z} \notin \mathcal{T}_{\tilde{\delta}_n}^{(n)}) + \Pr(|D - nq| > n\tilde{\delta}_n) \right) = 0.$$

On the other hand, to show that (29) (and thus (27)) also approaches zero as n increases, we make use of the following lemma, whose proof appears in Appendix B.

Lemma V.6. *Let the codewords $\mathbf{x}_1, \dots, \mathbf{x}_M$ be constructed as above. Then, for any $\mathbf{z} \in \mathcal{T}_{\tilde{\delta}_n}^{(n)}$ and $\mathbf{s} \in \mathbb{F}_2^n$, n sufficiently large, and d such that $|d - nq| \leq \tilde{\delta}_n n$,*

$$|S(\mathbf{z}, d) \cap E(\mathbf{s})| \leq 2^{n(H(q) - H(p) - 7.5\tilde{\delta}_n)}. \tag{32}$$

Combining Corollary V.5 and Lemma V.6, we see that

$$\frac{|S(\mathbf{z}, d) \cap E(\mathbf{s})|}{|S(\mathbf{z}, d)|} \leq (n+1)^8 2^{-0.5n\tilde{\delta}_n} \tag{33}$$

which converges to 0 since $0.5n\tilde{\delta}_n = -0.5n\tilde{\delta}_n \log_2 \tilde{\delta}_n/2 \geq 0.5\sqrt{n} \log_2 n$. Since (33) also serves as an upper bound for (29), we are done. \square

Remark V.7. *The authors of [7] examine capacity under the following model: the adversary views the transmitted codeword through a BSC(q), and decides on a state sequence \mathbf{s} such that $\|\mathbf{s}\| \leq tn$ for fixed parameter t (to avoid confusion, we use “ t ” rather than “ p ” as used in [7]). This state sequence is added to \mathbf{x} and sent noiselessly to the receiver. This differs from our model in two significant ways: (1) the adversary is power constrained and the no-adversary case is noiseless, and (2) error correction rather than authentication is considered.*

Consider this model in the authentication setting. Because the channel is not overwritable for any power constraint t , the oblivious case of $q = 1/2$ yields an authentication capacity of $C_{\text{auth}} = 1$, the non-adversarial capacity, by Theorem II.4. In fact, Theorem V.2 shows that even if the adversary can flip any number of bits (i.e. $t = 1$), as long as $q > 0$, the authentication capacity is equal to 1. Interestingly, the power constraint t , which is instrumental in the general AVC case, is immaterial in the authentication case.

B. When the adversary is less myopic than the receiver

Now, suppose that the non-adversarial channel between legitimate users is the worse channel. In this case, the authentication capacity is dependent on whether the encoder must be deterministic, or allowed to be stochastic.

Theorem V.8. *If $0 \leq q \leq p \leq 1/2$, and the encoder is deterministic, $C_{\text{auth}} = 0$.*

Proof. It is straightforward to show that if $0 \leq q \leq p \leq 1/2$, $U_{Z|X}$ is stochastically degraded with respect to $W_{Y|X, S=s_0}$ and the channel is I -overwritable: to see I -overwritability, choose $P_{S|X', X}$ to be deterministic such that $s = x' \oplus x$. Then, by Theorem III.3, $C_{\text{auth}} = 0$. \square

Remark V.9. *The authors of [7] show that for their similar model (which was detailed above in Remark V.7), if $q < p$, then the deterministic coding capacity is equal to the capacity of the channel with an omniscient adversary. This is also what we have shown for authentication (and our modified model) in Theorem V.8, since the authentication capacity of the MBAC $_{p,q=0}$ is equal to zero.*

By the comments at the end of Remark V.1 and Theorem III.1, if $q = 0$ or $p = 1/2$, the authentication capacity of the MBAC $_{p,q}$ is zero, regardless of the type of encoder. However, allowing a stochastic encoder, we can achieve positive capacity as long as $q > 0$ and $p < 1/2$. Similarly to the strategy in Example III.4, we will send our

initial codeword, \mathbf{t} , through an artificial channel, before sending that through the channels $U_{Z|X}$ and $W_{Y|X,S}$. Here, however, there must be some asymmetry to our artificial channel; otherwise the error pattern of the artificial channel is independent of \mathbf{t}_i , and the adversary may choose $\mathbf{s} = \mathbf{t}_i \oplus \mathbf{t}_j$ to reliably deceive the decoder.

Theorem V.10. *If $0 < q \leq p < 1/2$, and the encoder is allowed to be stochastic, $C_{\text{auth}} = 1 - H(p)$, where this authentication capacity holds with error probability measured either as average or maximum over all messages.*

Notice that we state the capacity holds even if we consider maximum error probability over all messages: that is, considering error to be $\sup_j e_{\max}(J)$. Since the maximum error approaching zero with increasing block length implies average error doing the same, this is a stronger statement. We are able to make such a statement here because we may assume the adversary knows exactly which message is being sent, and claim that we can authenticate even in this case.

Theorem V.10 converse proof. As before, since any authentication code must also be an error-correcting code for the underlying non-adversarial channel, we have $C_{\text{auth}} \leq C_{\text{BSC}(p)} = 1 - H(p)$. \square

For this case, we will again strengthen the adversary slightly in order to simplify some arguments. Specifically, we assume that the adversary can determine the message i with perfect accuracy, and we allow the adversary's choice of distribution, $J_{S^n|Z^n}$, to be a function of the exact joint type of \mathbf{t}_i , the transmitted word \mathbf{x} , and the observed sequence \mathbf{z} .

We first present a result that will allow us to choose a good codebook; the proof of the following lemma may be found in Appendix C.

Lemma V.11. *Let $P_{Y|T}$ be a discrete memoryless channel with capacity $C > 0$ and capacity-achieving input distribution P_T . Let $R > 0$ such that $R < C$, let $\epsilon > 0$ be sufficiently small, let n be sufficiently large, and let $M := 2^{nR}$. Then there exist codewords $\mathbf{t}_1, \dots, \mathbf{t}_M \in \tau_T$ such that*

(a) *for any type class $\tau_{T'T'S}$, $i \in [M]$, and any sequence \mathbf{s} ,*

$$|\{j : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{s}) \in \tau_{T'T'S}\}| \leq 2^{n[R - I(T';TS)]^+ + n\epsilon}, \quad (34)$$

(b) *$H(T_i | T_j) \geq \epsilon$ for all $i \neq j$, where T_i and T_j are artificial random variables with joint distribution given by the empirical distribution of \mathbf{t}_i and \mathbf{t}_j , and*

(c) *with a typical set decoder, the maximum error probability for transmission over $P_{Y|T}$ is bounded above by ϵ .*

We now prove achievability.

Theorem V.10 achievability proof. Let $0 < q \leq p < 1/2$. Let $\gamma, \delta > 0$ be sufficiently small, and let $R := C' - \delta$ where C' is the capacity of a binary asymmetric channel, $\text{BAC}(p, \gamma + p - 2\gamma p)$. The $\text{BAC}(p, \gamma + p - 2\gamma p)$ is equivalent to a Z-channel with crossover probability γ followed by a $\text{BSC}(p)$. We construct a $(M := 2^{nR}, n)$ code family with vanishing probability of error using Lemma V.11.

Encoding: Let $P_T \sim \text{Bernoulli}(\alpha)$, with α chosen according to the capacity-achieving distribution of the $\text{BAC}(p, \gamma + p - 2\gamma p)$, and choose a codebook $\mathbf{t}_1, \dots, \mathbf{t}_M$ as in Lemma V.11. In order to send message $i \in [M]$, we pass \mathbf{t}_i through $V_{X|T}$, a Z-channel with $V(0 | 1) = \gamma$, and $V(1 | 0) = 0$.

Decoding: Let $\epsilon > 0$ be sufficiently small. Upon receiving vector $\mathbf{y} \in \mathbb{F}_2^n$, decode to message i if and only if i is unique such that $(\mathbf{t}_i, \mathbf{y}) \in \mathcal{T}_\epsilon^{(n)}(T, Y)$, where $(T, Y) \sim P_T \times P_{Y|T}$, where $P_T \sim \text{Bernoulli}(\alpha)$ and $P_{Y|T}$ is given by $W_{Y|X,S=s_0} \times V_{X|T}$.

Probability of error analysis: Let $e_1(\mathbf{t}_i, \mathbf{x}, \mathbf{s})$ be the probability i does not satisfy the decoding requirement, assuming message i is chosen, \mathbf{x} is sent, and \mathbf{s} is transmitted by the adversary, and let $e_2(\mathbf{t}_i, \mathbf{x}, \mathbf{s})$ be the probability that some $j \in [M]$, $j \neq i$ satisfies the decoding requirement, assuming message i is chosen, \mathbf{x} is sent, and \mathbf{s} is transmitted by the adversary. In other words,

$$e_1(\mathbf{t}_i, \mathbf{x}, \mathbf{s}) = W(\phi^{-1}(i)^c | \mathbf{x}, \mathbf{s}) \text{ and } e_2(\mathbf{t}_i, \mathbf{x}, \mathbf{s}) = W(\phi^{-1}(0, i)^c | \mathbf{x}, \mathbf{s}).$$

Allowing the adversary to know \mathbf{t}_i, \mathbf{z} and $P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}$, we have

$$e(i, J) = \mathbb{E}_{\mathbf{X}|\mathbf{t}_i} \left[\sum_{\mathbf{z}} U(\mathbf{z} | \mathbf{X}) J(\mathbf{s}_0 | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_1(\mathbf{t}_i, \mathbf{X}, \mathbf{s}_0) + \sum_{\mathbf{s} \neq \mathbf{s}_0} \sum_{\mathbf{z}} U(\mathbf{z} | \mathbf{X}) J(\mathbf{s} | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_2(\mathbf{t}_i, \mathbf{X}, \mathbf{s}) \right] \quad (35)$$

$$= \sum_{\mathbf{x}, \mathbf{z}} P(\mathbf{x} | \mathbf{t}_i) U(\mathbf{z} | \mathbf{x}) \left(J(\mathbf{s}_0 | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_1(\mathbf{t}_i, \mathbf{x}, \mathbf{s}_0) + \sum_{\mathbf{s} \neq \mathbf{s}_0} J(\mathbf{s} | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_2(\mathbf{t}_i, \mathbf{x}, \mathbf{s}) \right) \quad (36)$$

$$= \sum_{\mathbf{x}, \mathbf{z}} P(\mathbf{x}, \mathbf{z} | \mathbf{t}_i) \left(J(\mathbf{s}_0 | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_1(\mathbf{t}_i, \mathbf{x}, \mathbf{s}_0) + \sum_{\mathbf{s} \neq \mathbf{s}_0} J(\mathbf{s} | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_2(\mathbf{t}_i, \mathbf{x}, \mathbf{s}) \right) \quad (37)$$

$$= \sum_{\substack{\mathbf{x}, \mathbf{z} \\ (\mathbf{t}_i, \mathbf{x}, \mathbf{z}) \in \mathcal{T}_\epsilon^{(n)}(T, X, Z)}} P(\mathbf{x}, \mathbf{z} | \mathbf{t}_i) \left(J(\mathbf{s}_0 | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_1(\mathbf{t}_i, \mathbf{x}, \mathbf{s}_0) + \sum_{\mathbf{s} \neq \mathbf{s}_0} J(\mathbf{s} | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_2(\mathbf{t}_i, \mathbf{x}, \mathbf{s}) \right) \quad (38)$$

$$+ \sum_{\substack{\mathbf{x}, \mathbf{z} \\ (\mathbf{t}_i, \mathbf{x}, \mathbf{z}) \notin \mathcal{T}_\epsilon^{(n)}(T, X, Z)}} P(\mathbf{x}, \mathbf{z} | \mathbf{t}_i) \left(J(\mathbf{s}_0 | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_1(\mathbf{t}_i, \mathbf{x}, \mathbf{s}_0) + \sum_{\mathbf{s} \neq \mathbf{s}_0} J(\mathbf{s} | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) e_2(\mathbf{t}_i, \mathbf{x}, \mathbf{s}) \right), \quad (39)$$

where (37) follows because $T \rightleftharpoons X \rightleftharpoons Z$ is a Markov chain. Consider (39). With high probability in n , $(\mathbf{T}_i, \mathbf{X}, \mathbf{Z})$ lies in the typical set. Thus, since

$$(39) \leq \sum_{\substack{\mathbf{x}, \mathbf{z} \\ (\mathbf{t}_i, \mathbf{x}, \mathbf{z}) \notin \mathcal{T}_\epsilon^{(n)}(T, X, Z)}} P(\mathbf{x}, \mathbf{z} | \mathbf{t}_i) = P\left\{(\mathbf{T}_i, \mathbf{X}, \mathbf{Z}) \notin \mathcal{T}_\epsilon^{(n)}(T, X, Z) \mid \mathbf{T}_i = \mathbf{t}_i\right\},$$

we have that (39) approaches 0 as $n \rightarrow \infty$. Notice that

$$(38) \leq \sum_{\substack{\mathbf{s}, \mathbf{x}, \mathbf{z} \\ (\mathbf{t}_i, \mathbf{x}, \mathbf{z}) \in \mathcal{T}_\epsilon^{(n)}(T, X, Z)}} P(\mathbf{z} | \mathbf{t}_i) J(\mathbf{s} | \mathbf{z}, \mathbf{t}_i, P_{\mathbf{t}_i, \mathbf{X}, \mathbf{Z}}) P(\mathbf{x} | \mathbf{t}_i, \mathbf{z}) W(\phi^{-1}(i)^c | \mathbf{x}, \mathbf{s}) \quad (40)$$

If for every fixed choice of \mathbf{s} and \mathbf{z} we have $\sum_{\mathbf{x}} P(\mathbf{x} | \mathbf{t}_i, \mathbf{z}) W(\phi^{-1}(i)^c | \mathbf{x}, \mathbf{s}) \rightarrow 0$ as $n \rightarrow \infty$, where the sum is over \mathbf{x} 's such that $(\mathbf{t}_i, \mathbf{x}, \mathbf{z}) \in \mathcal{T}_\epsilon^{(n)}(T, X, Z)$, then (38) approaches zero asymptotically. Thus, it is sufficient to prove the following lemma, which states that any choice of \mathbf{s} given knowledge of \mathbf{t}_i and \mathbf{z} causes decoding failure for a vanishing fraction of the transmissions \mathbf{x} that are consistent with \mathbf{t}_i and \mathbf{z} . A proof of the lemma appears in Appendix D.

Lemma V.12. *Let i, \mathbf{z} , and \mathbf{s} be fixed such that $(\mathbf{t}_i, \mathbf{z}) \in \mathcal{T}_\epsilon^{(n)}$, and suppose \mathbf{X} and \mathbf{Y} are drawn from the distribution $P(\mathbf{x} | \mathbf{t}_i, \mathbf{z}) W(\mathbf{y} | \mathbf{x}, \mathbf{s})$, where $P(\mathbf{x} | \mathbf{t}_i, \mathbf{z})$ is the conditional distribution for \mathbf{X} from the Markov chain $T \rightleftharpoons X \rightleftharpoons Z$. Then,*

$$P\left\{\exists j \neq i : (\mathbf{t}_j, \mathbf{Y}) \in \mathcal{T}_\epsilon^{(n)}\right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

With Lemma V.12 in hand, letting δ and γ approach zero gives us a code with rate arbitrarily close to $\lim_{\delta, \gamma \rightarrow 0} C' - \delta = 1 - H(p)$ and vanishing error probability, proving achievability. \square

VI. CONCLUSIONS

In this paper, we considered keyless authentication over an AVC where the adversary sees the transmitted sequence through a noisy channel $U_{Z|X}$. We introduced the channel condition U -overwritability as a generalization of the oblivious-adversary condition overwritability, and showed that U -overwritability is a sufficient condition for zero authentication capacity. We also showed that if users are restricted to deterministic encoders, there are additional cases in which the authentication capacity is zero: namely, when the adversary is able to reliably decode, and the AVC is vulnerable to an omniscient adversary (I -overwritable). However, allowing for stochastic encoders can allow for positive authentication capacity in these cases.

Next, we compared adversaries to one another, and showed that once an adversary has a channel $U_{Z|X}$ such that the AVC is U -overwritable, the AVC is also U -overwritable for any less degraded channel to the adversary. As

a consequence, if an AVC is overwritable, it is also U -overwritable for every $U_{Z|X}$. This can allow us to quickly determine U -overwritability for a large group of channels to the adversary.

Finally, we examined a myopic binary adversarial channel in detail. Interestingly, for this case the authentication capacity is always equal to the non-adversarial capacity of the underlying channel as long as the channel to the adversary is not perfect and we allow stochastic encoders. Furthermore, in this case the maximum error authentication capacity is equal to the average error authentication capacity. If we restrict to deterministic encoders, the authentication capacity drops to zero for the cases in which the non-adversarial channel between users is stochastically degraded with respect to the channel to the adversary, as the adversary is essentially omniscient in this scenario. An open question is whether U -overwritability is more generally a necessary condition for zero authentication capacity when stochastic encoders are allowed, and whether the authentication capacity is always equal to the non-adversarial capacity when it is positive.

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APPENDIX A

LEMMA V.4

First we derive two lemmas which will help considerably in establishing Lemma V.4 by simplifying the analysis that results from using Bernstein's trick when bounding sums of random variables.

Lemma A.1. *Let $0 < t, p < 1$, let $s \in \{-1, 1\}$, and let B_1, \dots, B_n be independent Bernoulli(p) random variables.*

$$\min_{h>0} \frac{\mathbb{E} \left[e^{h \sum_{i=1}^n s B_i} \right]}{e^{hsnt}} \leq e^{-nD(t||p)},$$

where $D(t||p) := t \ln \left(\frac{t}{p} \right) + (1-t) \ln \left(\frac{1-t}{1-p} \right)$.

Proof. First, note that since the B_i 's are independent, for fixed $h > 0$ we have

$$\mathbb{E} \left[e^{h \sum_{i=1}^n s B_i} \right] = \prod_{i=1}^n \mathbb{E} \left[e^{hs B_i} \right] = (1 - p + p e^{hs})^n.$$

Thus,

$$\min_{h>0} \frac{\mathbb{E} \left[e^{h \sum_{i=1}^n s B_i} \right]}{e^{hsnt}} = \min_{h>0} \left((1-p + pe^{hs})^n e^{-hsnt} \right). \quad (41)$$

Solving, we find that as long as $t < 1$,

$$\min_{h>0} \frac{\mathbb{E} \left[e^{h \sum_{i=1}^n s B_i} \right]}{e^{hsnt}} \leq e^{-nD(t\|p)}. \quad (42)$$

since the only critical point of (41) occurs at $h = s \ln \frac{t(1-p)}{p(1-t)}$ and the second derivative is always positive. \square

In order to apply Lemma A.1 to obtain Lemma V.4 it will be helpful to lower bound $MD(t\|p)$.

Lemma A.2. *Let $t := \zeta p$ for some positive number $\zeta \in (0, 1)$. Then,*

$$D(t\|p) \geq p(\zeta \ln \zeta - \zeta + 1).$$

Proof. Borrowing from [17, Lemma 17.9], observe that $D(\zeta p\|p)$ is a convex function of p , and thus a linear approximation at $p = 0$ will provide a lower bound. Therefore,

$$D(t\|p) \geq D(0\|0) + \left. \frac{dD(\zeta p\|p)}{dp} \right|_{p=0} p = (\zeta \ln \zeta - \zeta + 1)p. \quad \square$$

Proof of Lemma V.4. Fix $\mathbf{z} \in \tau_Z$ and suppose all codewords \mathbf{X}_i are independently chosen uniformly at random from τ_X . Before continuing further, we will show that there exists a conditional type set $\tau_{X|Z}(\mathbf{z})$ such that the set of all messages Hamming distance d from \mathbf{z} is equal to $\{i : \mathbf{x}_i \in \tau_{X|Z}(\mathbf{z})\}$ since τ_X , τ_Z , and d are fixed. Indeed, define $\tau_{X|Z}(\mathbf{z})$ by

$$p_{X|Z}(1|0) = \frac{d}{2np_Z(0)} + \frac{1}{2} - \frac{p_X(0)}{2p_Z(0)} \quad (43)$$

$$p_{X|Z}(0|1) = \frac{d}{2np_Z(1)} + \frac{1}{2} - \frac{p_X(1)}{2p_Z(1)}. \quad (44)$$

For any $\mathbf{x} \in \tau_X$ that is distance d from \mathbf{z} , observe that d can be written as $d_0 + d_1$, where d_0 is the number on indices where \mathbf{z} is 0 and \mathbf{x} is 1, and d_1 is the be the number of indices where \mathbf{z} is 1 and \mathbf{x} is 0. Furthermore, because d_0 and d_1 must satisfy the linear equations

$$np_Z(0) + d_1 - d_0 = np_X(0) \quad (45)$$

$$d_1 + d_0 = d, \quad (46)$$

we have

$$d_0 = \frac{d + n[p_Z(0) - p_X(0)]}{2} \quad (47)$$

$$d_1 = \frac{d + n[p_Z(1) - p_X(1)]}{2}, \quad (48)$$

and we see that the empirical distributions $p_{X|Z}(1|0)$ and $p_{X|Z}(0|1)$ are as in equations (43) and (44) above. We conclude that $\mathbf{x} \in \tau_{X|Z}(\mathbf{z})$. Because \mathbf{x} was chosen arbitrarily from the type class, we see that any codeword distance d from \mathbf{z} must also be in $\tau_{X|Z}(\mathbf{z})$. Now, let

$$A_i = \begin{cases} 1 & \text{if } \mathbf{X}_i \in \tau_{X|Z}(\mathbf{z}), \\ 0 & \text{else,} \end{cases}$$

so that we may write $|\{i : \|\mathbf{z} \oplus \mathbf{X}_i\| = d\}| = \sum_{i=1}^M A_i$. Recall that $\mathbf{X}_1, \dots, \mathbf{X}_M$ are independent, and note that by [17, Lemma 2.5],

$$2^{n(H(X|Z) - \epsilon)} \leq |\tau_{X|Z}(\mathbf{z})|, \quad \text{and} \quad |\tau_X| \leq 2^{nH(X)},$$

where $\epsilon = 4 \frac{\log_2(n+1)}{n}$. We conclude that the A_i are independent Bernoulli(b) random variables, where

$$b \geq \frac{|\tau_{X|Z}(\mathbf{z})|}{|\tau_X|} \geq 2^{-n(I(X;Z)+\epsilon)}.$$

Hence, the probability that there are fewer than $\lfloor 2^{n(R-I(X;Z)-2\epsilon)} \rfloor$ messages distance d from $\mathbf{z} \in \tau_Z$ is equal to $\Pr\left(\sum_{i=1}^M A_i < \lfloor 2^{n(R-I(X;Z)-2\epsilon)} \rfloor\right)$.

Notice that if $R < I(X;Z) + 2\epsilon$, then $\lfloor 2^{n(R-I(X;Z)-2\epsilon)} \rfloor = 0$. Since the A_i 's are nonnegative, we have

$$\Pr\left(\sum_{i=1}^M A_i < 0\right) = 0. \quad (49)$$

On the other hand, if $R > I(X;Z) + 2\epsilon$ then:

$$\Pr\left(\sum_{i=1}^M A_i < \lfloor 2^{n(R-I(X;Z)-2\epsilon)} \rfloor\right) \leq \Pr\left(\sum_{i=1}^M A_i < 2^{n(R-I(X;Z)-2\epsilon)}\right) \quad (50)$$

$$= \min_{h>0} \Pr\left(e^{-h \sum_{i=1}^M A_i} > e^{-h 2^{n(R-I(X;Z)-2\epsilon)}}\right) \quad (51)$$

$$\leq \min_{h>0} \mathbb{E}\left[e^{-h \sum_{i=1}^M A_i}\right] e^{h 2^{n(R-I(X;Z)-2\epsilon)}} \quad (52)$$

$$\leq e^{-MD(2^{-n(I(X;Z)+2\epsilon)} \| 2^{-n(I(X;Z)+\epsilon)})} \quad (53)$$

$$\leq e^{-2^{n(R-I(X;Z)-\epsilon)} [1-2^{-n\epsilon} (1+n\epsilon \ln 2)]} \quad (54)$$

where (51) is from Bernstein's trick; (52) is from Markov's Inequality; (53) is from Lemma A.1 and the convexity of divergence (since $2^{-n(I(X;Z)+2\epsilon)} < 2^{-n(I(X;Z)+\epsilon)} < b$); and (54) is Lemma A.2. Combining the bounds (49) and (54), we obtain

$$\Pr\left(\left|\{i : \|\mathbf{z} \oplus \mathbf{X}_i\| = d\}\right| < \lfloor 2^{n(R-I(X;Z)-2\epsilon)} \rfloor\right) \leq e^{-2^{n\epsilon} [1-2^{-n\epsilon} (1+n\epsilon \ln 2)]} = e^{-(n+1)^4 + 1 + 4 \ln(n+1)}. \quad (55)$$

From the union bound, then,

$$\Pr\left(\exists \mathbf{z} \text{ such that } \left|\{i : \|\mathbf{z} \oplus \mathbf{X}_i\| = d\}\right| < \lfloor 2^{n(R-I(X;Z)-2\epsilon)} \rfloor\right) \leq e^{-(n+1)^4 + nR + 1 + 4 \ln(n+1)}. \quad (56)$$

Since (56) goes to zero as n goes to infinity, we find that, with high probability, the number of messages that are distance d from any $\mathbf{z} \in \tau_Z$ is at least

$$\left\lfloor 2^{n(R-I(X;Z)-2\epsilon)} \right\rfloor = \left\lfloor \frac{1}{(n+1)^8} 2^{n(R-I(X;Z))} \right\rfloor,$$

where $p_{X,Z}$ is defined by the conditional distributions in (43) and (44). \square

APPENDIX B

LEMMA V.6

Proof of Lemma V.6. We can upper bound the left-hand side of (32) by splitting into different type classes. Namely,

$$|S(\mathbf{z}, d) \cap E(\mathbf{s})| \leq \sum_{X, X', S, Z} |\{i : (\mathbf{x}_i, \mathbf{x}_j, \mathbf{s}, \mathbf{z}) \in \tau_{XX'SZ} \text{ for some } j \neq i\}|,$$

where the binary random variables X, X', S, Z satisfy

$$\mathbb{E}[X] = 1/2, \quad \mathbb{E}[X'] = 1/2, \quad \mathbb{E}[X \oplus Z] = d/n, \quad \mathbb{E}[X \oplus S \oplus X'] \leq p + \tilde{\delta}_n. \quad (57)$$

Let $\epsilon_{1,n} > 0$ be such that $\epsilon_{1,n} < 0.5\delta_n$. Applying Lemma V.3 with (S, Z) in place of S , we have

$$|\{i : (\mathbf{x}_i, \mathbf{x}_j, \mathbf{s}, \mathbf{z}) \in \tau_{XX'SZ} \text{ for some } j \neq i\}| \leq 2^{n|R-I(X;X'SZ)+|R-I(X';SZ)|^+} + n\epsilon_{1,n}. \quad (58)$$

We now want to bound this for random variables satisfying (57). In particular, consider two cases. First, if $R \leq I(X';SZ)$, then we have that $|R-I(X;X'SZ)+|R-I(X';SZ)|^+|^+$ is equal to

$$= |R-I(X;X'SZ)|^+$$

$$\leq |R - I(X; S \oplus X')|^+ \quad (59)$$

$$\leq |R - (1 - H(p + \delta_n))|^+ \quad (60)$$

$$\leq |R - 1 + H(p) + \delta_n|^+ \quad (61)$$

$$= 0, \quad (62)$$

where (59) follows from the data processing inequality, (60) from the last condition of (57), and (62) holds by the definition of R . Second, if $R > I(X'; SZ)$, then we have that $|R - I(X; X'SZ) + |R - I(X'; SZ)|^+|^+$ is equal to

$$\begin{aligned} &= |2R - I(X; X'SZ) - I(X'; SZ)|^+ \\ &= |2R - I(X; SZ) - I(X; X'|SZ) - I(X'; SZ)|^+ \end{aligned} \quad (63)$$

$$= |2R - I(X; SZ) - I(X'; XSZ)|^+ \quad (64)$$

$$\leq |2R - I(X; Z) - I(X'; X \oplus S)|^+ \quad (65)$$

$$\leq |2R - (1 - H(d/n)) - (1 - H(p + \delta_n))|^+ \quad (66)$$

$$\leq |2R + H(d/n) + H(p) - 2 + \delta_n|^+ \quad (67)$$

$$\leq |H(d/n) - H(p) - 9\delta_n|^+ \quad (68)$$

$$\leq H(d/n) - H(p) - 9\delta_n \quad (69)$$

$$\leq H(q) - H(p) - 8\delta_n, \quad (70)$$

where (63) and (64) follow from the chain rule, (65) from the data processing inequality, (66) from the last two conditions of (57), and (69) holds for n sufficiently large. In either case, (58) is upper bounded by $2^{n(H(q) - H(p) - 8\delta_n + \epsilon_{1,n})}$. Since $\epsilon_{1,n} < 0.5\delta_n$ and there are only a polynomial number of types, for sufficiently large n , we have shown (32). \square

APPENDIX C

LEMMA V.11

In order to prove Lemma V.11, we first give an extension of Lemma 3 of [16].

Lemma C.1. *Let $P_{Y|T}$ be a discrete memoryless channel with capacity $C > 0$ and capacity-achieving input distribution P_T . Let $\epsilon'' > 0$, let $R', \epsilon' > 0$ such that $\epsilon' < R' < C$, let n be sufficiently large, and let $M := 2^{nR'}$. Then there exist $\mathbf{t}_1, \dots, \mathbf{t}_M$ of length n from type class τ_T that simultaneously satisfy the following for any type class $\tau_{T'T'S}$:*

(1) *for any $i \in [M]$, and any sequence \mathbf{s} ,*

$$|\{j : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{s}) \in \tau_{T'T'S}\}| \leq 2^{n(R' - I(T'; TS))^+ + n\epsilon'}, \quad (71)$$

(2) *if $I(T; T') - R' > \epsilon'$, then*

$$\frac{1}{M} |\{i : (\mathbf{t}_i, \mathbf{t}_j) \in \tau_{T'T'} \text{ for some } j \neq i\}| \leq 2^{-(\epsilon'/2)n}, \text{ and} \quad (72)$$

(3) *with a typical set decoder, the average error probability for transmission over $P_{Y|T}$ is bounded above by ϵ'' .*

Proof. The existence of a codebook having properties (1) and (2) follows directly from Lemma 3 of [16]. In fact, not only does there exist such a codebook, but a codebook whose codewords are chosen uniformly at random from τ_T has both properties with high probability. Furthermore, choosing codewords uniformly at random from τ_T will result with high probability in a code whose average error probability over $P_{Y|T}$ vanishes as block length goes to infinity. Thus, a randomly chosen codebook simultaneously possesses properties (1)-(3) with high probability. \square

To construct a codebook satisfying (a)-(c) of Lemma V.11, we will take a codebook with properties (1)-(3) of Lemma C.1, and eliminate all codewords \mathbf{t}_i such that the decoding error probability given \mathbf{t}_i was sent is bounded away from zero, and also all \mathbf{t}_i for which there exists $j \neq i$ with the property that $H(T_i | T_j) < \epsilon$, where T_i and T_j are artificial random variables with joint distribution given by the empirical distribution of \mathbf{t}_i and \mathbf{t}_j .

Proof of Lemma V.11. Let $\epsilon, \delta > 0$ be sufficiently small, with $\delta < \epsilon < \delta'$, where $\delta' := \frac{C-R}{2}$ and Y is distributed according to $P_{Y|T}$. Let $\epsilon' > 0$ be such that $\epsilon' < \min\{R + \delta, \epsilon - \delta, \delta' - \epsilon\}$. Letting $R' = R + \delta$, Lemma C.1 states that there exists a codebook of size $2^{n(R+\delta)}$ from the type class τ_T that satisfies properties (1)-(3), letting $\epsilon'' := \epsilon/2$ in property (3).

For each type class $\tau_{TT'}$ such that $H(T|T') < \epsilon$,

$$I(T; T') - (R + \delta) > H(T) - H(T|T') - (R + \delta) \quad (73)$$

$$> H(T) - H(T|T') - (C - \delta') \quad (74)$$

$$= H(T) - H(T|T') - I(T; Y) + \delta' \quad (75)$$

$$= H(T|Y) - H(T|T') + \delta' \quad (76)$$

$$> H(T|Y) + \delta' - \epsilon \quad (77)$$

$$> \epsilon'. \quad (78)$$

Thus, by property (2) of Lemma C.1, for each such $\tau_{TT'}$,

$$\frac{1}{M} |\{i : (\mathbf{t}_i, \mathbf{t}_j) \in \tau_{TT'} \text{ for some } j \neq i\}| \leq 2^{-(\epsilon'/2)n}.$$

Since there are a polynomial number of joint type classes $\tau_{TT'}$ with $H(T|T') < \epsilon$,

$$\frac{1}{M} |\{i : (\mathbf{t}_i, \mathbf{t}_j) \in \tau_{TT'} \text{ for some } T, T' \text{ s.t. } H(T|T') < \epsilon, \text{ and some } j \neq i\}| \leq n^{O(1)} 2^{-(\epsilon'/2)n}. \quad (79)$$

Removing all codewords \mathbf{t}_i such that i falls in the set on the left hand side of (79), we have at least $2^{n(R+\delta)} (1 - n^{O(1)} 2^{-(\epsilon'/2)n})$ codewords remaining. For n sufficiently large,

$$1 - n^{O(1)} 2^{-(\epsilon'/2)n} \geq 2^{-(\delta/2)n}.$$

Thus, the number of remaining codewords is at least $2^{n(R+\delta/2)}$.

With our initial choice of codebook sequence, the average error probability of the code for transmission over $P_{Y|T}$ with typical set decoding was bounded above by $\epsilon/2$; this remains true for our now-smaller codebook of size $2^{n(R+\delta/2)}$. Denote the average error probability for this new codebook by \overline{P}_e^n . For each codeword \mathbf{t}_i remaining in the codebook, there is an associated decoding error probability $P_e^n(\mathbf{t}_i)$. Remove from the codebook half of the codewords: in particular, those that have the highest error probabilities. We claim that for each remaining codeword, we have $P_e^n(\mathbf{t}_i) \leq 2\overline{P}_e^n \leq \epsilon$. Indeed, were this not the case, then

$$\overline{P}_e^n = \frac{\sum_{i \in S} P_e^n(\mathbf{t}_i) + \sum_{i \in [2^{n(R+\delta/2)}] \setminus S} P_e^n(\mathbf{t}_i)}{2^{n(R+\delta/2)}} > \frac{\epsilon + (2^{n(R+\delta/2)-1} \cdot \epsilon)}{2^{n(R+\delta/2)}} > \epsilon/2,$$

where S is the set of $2^{n(R+\delta/2)-1}$ codewords with smallest error probability. Thus, we conclude that each remaining codeword in our codebook of size $2^{n(R+\delta/2)-1}$ has error probability bounded above by ϵ . Since $2^{n(R+\delta/2)-1} > 2^{nR}$ for n sufficiently large, we may select $M := 2^{nR}$ codewords from those remaining, and these have maximum error probability bounded above by ϵ . Call these $\mathbf{t}_1, \dots, \mathbf{t}_M$.

Finally, we show (a). Let $\tau_{TT'S}$ be some type class. If $R + \delta > I(T'; TS)$, Lemma C.1 gives that for any $i \in [M]$, and any sequence \mathbf{s} ,

$$|\{j : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{s}) \in \tau_{TT'S}\}| \leq 2^{n[(R+\delta)-I(T'; TS)]^+ + n\epsilon'} \quad (80)$$

$$= 2^{n[R-I(T'; TS)] + n(\delta+\epsilon')} \quad (81)$$

$$< 2^{n[R-I(T'; TS)]^+ + n\epsilon}, \quad (82)$$

where (82) follows from the upper bound $\epsilon' < \epsilon - \delta$. On the other hand, if $R + \delta \leq I(T'; TS)$, then $R < I(T'; TS)$, and

$$|\{j : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{s}) \in \tau_{TT'S}\}| \leq 2^{n[(R+\delta)-I(T'; TS)]^+ + n\epsilon'} \quad (83)$$

$$= 2^{n[R-I(T'; TS)]^+ + n\epsilon'} \quad (84)$$

$$< 2^{n[R-I(T'; TS)]^+ + n\epsilon}. \quad (85)$$

□

APPENDIX D
LEMMA V.12

Proof of Lemma V.12. We will use the following notation: if $|A - B| < \epsilon$, we write $A \stackrel{\epsilon}{=} B$. For distributions, “ $\stackrel{\epsilon}{=}$ ” indicates that this holds for each set of realizations of the random variables involved.

Let τ_{TZY} be the joint type class for the sequences \mathbf{t}_i, \mathbf{z} , and \mathbf{s} . Note that here T, Z, S represent artificial random variables. We will use Q to denote the distribution of these artificial random variables (e.g. $Q(t, z, s)$ is the joint type), and P to denote the actual distribution that variables are drawn from. So, for example, we can assume that \mathbf{t}_i, \mathbf{z} are jointly typical with respect to $P(t, z)$, i.e., $Q(t, z) \stackrel{\epsilon}{=} P(t, z)$. Let \mathbf{X}, \mathbf{Y} be random vectors drawn from the distribution

$$P(\mathbf{x}|\mathbf{t}_i, \mathbf{z})W(\mathbf{y}|\mathbf{x}, \mathbf{s}) \quad (86)$$

where $P(\mathbf{x}|\mathbf{t}_i, \mathbf{z})$ is the conditional distribution for \mathbf{X} from the Markov chain $T \text{---} X \text{---} Z$. We wish to upper bound

$$P\{\exists j \neq i : (\mathbf{t}_j, \mathbf{Y}) \in T_\epsilon^{(n)}\}. \quad (87)$$

We can split this probability based on the joint type class of $(\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}, \mathbf{X}, \mathbf{Y})$, restricted to those for which \mathbf{t}_j is jointly typical with \mathbf{Y} :

$$\sum_{\substack{\tau_{TT'ZSY} \\ \text{s.t. } \tau_{T'Y} \subset T_\epsilon^{(n)}}} P\{\exists j \neq i : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}, \mathbf{X}, \mathbf{Y}) \in \tau_{TT'ZSY}\}. \quad (88)$$

Since there are only polynomially many types, we only need to show that each of the probabilities in (88) is exponentially vanishing. Note that, by the law of large numbers, for n sufficiently large, $Q(x, y|t, z, s) \stackrel{\epsilon}{=} P(x|t, z)W(y|x, s)$. Given any joint type, we may write

$$P\{\exists j \neq i : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}, \mathbf{X}, \mathbf{Y}) \in \tau_{TT'ZSY}\} \leq \sum_{\substack{j \neq i \text{ s.t.} \\ (\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}) \in \tau_{TT'ZS}}} P\{(\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}, \mathbf{X}, \mathbf{Y}) \in \tau_{TT'ZSY}\} \\ \leq |\{j \neq i : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}) \in \tau_{TT'ZS}\}| 2^{n(-I(XY; T'|TZS) + \epsilon)} \quad (89)$$

$$\leq 2^{n(|R - I(T'; STZ)|^+ - I(XY; T'|STZ) + 2\epsilon)} \quad (90)$$

where (89) follows from the joint typicality lemma, and (90) follows from our choice of codebook satisfying Lemma V.11.

We now have two cases: the first is that $R \geq I(T'; STZ)$. If this holds, then

$$P\{\exists j \neq i : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}, \mathbf{X}, \mathbf{Y}) \in \tau_{TT'ZSY}\} \leq 2^{n(R - I(T'; STZ) - I(XY; T'|STZ) + 2\epsilon)} \quad (91)$$

$$= 2^{n(R - I(T'; STZXY) + 2\epsilon)} \quad (92)$$

$$\leq 2^{n(R - I(T'; Y) + 2\epsilon)}. \quad (93)$$

Recall that $\tau_{T'Y} \subset T_\epsilon^{(n)}$, and so $I(T'; Y) \geq C' - \epsilon'$ for some $\epsilon' > 0$ such that $\epsilon' \rightarrow 0$ as $\epsilon \rightarrow 0$. Thus, for ϵ sufficiently small, $R < I(T'; Y)$ and $I(T'; Y) - R > 2\epsilon$ simultaneously, so (93) vanishes exponentially, and we are done.

The second case is that $R < I(T'; STZ)$. Here, (90) reduces to

$$P\{\exists j \neq i : (\mathbf{t}_i, \mathbf{t}_j, \mathbf{z}, \mathbf{s}, \mathbf{X}, \mathbf{Y}) \in \tau_{TT'ZSY}\} \leq 2^{n(-I(XY; T'|STZ) + 2\epsilon)}. \quad (94)$$

If $I(XY; T'|TZS) > 2\epsilon$, (94) will be exponentially vanishing, and we are again done. So, now consider just the type classes $\tau_{TT'ZSY}$ such that $I(XY; T'|TZS) \leq 2\epsilon$. That is, the Markov chain $T' \rightarrow STZ \rightarrow XY$ approximately holds for the empirical distribution, so that

$$Q(t, t', z, s, x, y) \stackrel{\epsilon'}{=} Q(t, t', z, s)Q(x, y|t, z, s). \quad (95)$$

Where ϵ' goes to 0 with ϵ . By the assumption that $Q(x, y|t, z, s) \stackrel{\epsilon}{=} P(x|t, z)W(y|x, s)$, we have

$$Q(t, t', z, s, x, y) \stackrel{\epsilon}{=} Q(t, t', z, s)P(x|t, z)W(y|x, s) = Q(t, z)P(x|t, z)Q(t', s|t, z)W(y|x, s) \quad (96)$$

where $\epsilon_2 = \epsilon' + \epsilon$. We can also assume that $Q(t, x, z) \stackrel{\epsilon}{=} P(t, x)U(z|x)$, and $Q(x|t, z) \stackrel{\epsilon}{=} P(x|t, z)$ so we have

$$Q(t, t', z, s, x, y) \stackrel{\epsilon}{=} P(t, x)U(z|x)Q(t', s|t, z)W(y|x, s), \quad (97)$$

where $\epsilon_3 = \epsilon' + 3\epsilon$. In addition, we know that

$$Q(t', y) \stackrel{\epsilon}{=} \sum_x P(t', x) W(y|x, s_0). \quad (98)$$

Therefore,

$$\sum_{t,x,z,s} P(t, x) U(z|x) Q(t', s|t, z) W(y|x, s) \stackrel{\epsilon_4}{=} \sum_x P(t', x) W(y|x, s_0) \quad (99)$$

where $\epsilon_4 = 16(\epsilon' + 3\epsilon) + \epsilon$. However, we will show that as a consequence of our code design, there is in fact no conditional distribution $Q(t', s|t, z)$ that satisfies the above. In particular, we can make ϵ as small as we like, so it is enough to show that if $Q(t', s|t, z)$ satisfies

$$\sum_{t,x,z,s} P(t, x) U(z|x) Q(t', s|t, z) W(y|x, s) = \sum_x P(t', x) W(y|x, s_0) \quad (100)$$

then $Q(t'|t) = \mathbf{1}(t' = t)$, which is a contradiction to our choice of codebook.

Define $\tilde{Y} = X \oplus S$, so the channel $W_{Y|X,S}$ is broken down into a deterministic part from (X, S) to \tilde{Y} , followed by a BSC(p) from \tilde{Y} to Y . We may then rewrite (100) as

$$\sum_{t,x,z,s,\tilde{y}} P(t, x) U(z|x) Q(t', s|t, z) \mathbf{1}(\tilde{y} = x \oplus s) W(y|\tilde{y}) = \sum_x P(t', x) W(y|x, s_0). \quad (101)$$

Consider $t' = 0$. By our encoding procedure, it is then the case that $X = 0$ deterministically, and we can write

$$\sum_{t,x,z,s,\tilde{y}} P(t, x) U(z|x) Q(t' = 0, s|t, z) \mathbf{1}(\tilde{y} = x \oplus s) W(y|\tilde{y}) = \begin{cases} 1-p & y = 0, \\ p & y = 1. \end{cases} \quad (102)$$

Define $a(\tilde{y}) := \sum_{t,x,z,s} P(t, x) U(z|x) Q(t' = 0, s|t, z) \mathbf{1}(\tilde{y} = x \oplus s)$. Since $W(y|\tilde{y})$ is a BSC(p), and $p \neq 0, 1/2$, then the above equation simultaneously holding for $y = 0$ and $y = 1$ implies that $a(0)(1-p) + a(1)p = 1-p$ and $a(0)p + a(1)(1-p) = p$. Solving this system yields $a(1) = 0$, or, expanded:

$$\sum_{t,x,z,s} P(t, x) U(z|x) Q(t' = 0, s|t, z) \mathbf{1}(1 = x \oplus s) = 0. \quad (103)$$

Since each term in this sum is non-negative, they all must equal 0. In particular, suppose there exists a pair (z, s) where $Q(t' = 0, s | t = 1, z) > 0$. Then, noting that we only need to consider $x = s \oplus 1$, so we have

$$P(t = 1, x = s \oplus 1) U(z | x = s \oplus 1) Q(t' = 0, s | t = 1, z) = 0 \quad \text{which implies} \quad P(t = 1, x = s \oplus 1) U(z | x = s \oplus 1) = 0. \quad (104)$$

But this is impossible, since $P(t = 1, x) > 0$ for each x , and $U(z | x) > 0$ for each x (here we use the assumption that $0 < q < 1$). Thus, we must have $Q(t' = 0, s | t = 1, z) = 0$ for all s, z . Therefore, $Q(t' = 0 | t = 1) = 0$, so $Q(t' = 1 | t = 1) = 1$. Since T and T' must have the same marginal distribution of $Q(t) = Q(t') = 0.5$ for all t, t' , we have $Q(t = 1 | t' = 1) = 1$, which in turn implies that $Q(t = 0 | t' = 1) = 0$, and $Q(t' = 1 | t = 0) = 0$. In other words, $Q(t'|t) = \mathbf{1}(t' = t)$. However, this is impossible based on the fact that our codebook satisfies Lemma V.11. In other words, there is no consistent type class $\tau_{TT'ZSXY}$ such that $I(XY; T'|STZ) \leq 2\epsilon$. This completes the proof of the lemma. \square