An Optimal Control Problem of Unmanned Aerial Vehicle

Kahina Louadj, Fethi Demim, Abdelkrim Nemra, Philippe Marthon

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Kahina LOUADJ [1,3], Fethi DEMIM [2], Abdelkrim NEMRA [2] and Philippe MARTHON [3]

***Abstract*— An unmanned aerial vehicle (UAV) has grown rapidly the last years. They are being engaged in many types of missions, ranging from military to agriculture passing from entertainment and rescue or even delivery. This work consist to ensure the convergence of positions and yaw angle, to their desired trajectories while maintaining stability of roll and pitch angle. For this, formulate this problem to an optimal control problem which minimizes the distance between state and desired state in free final time. Our contribution is to use the Bocop software to solve this problem. And to implement Shooting method with Matlab software.**

I. Introduction

Unmanned Aerial Vehicles (UAV) commonly called "Drones" [1], [2], [3]; since the earliest days of aviation, their use has increased dramatically in the first decade of the 21^{st} century, as they have many advantages compared to air vehicles conventional fixed wings. They can take off and land vertically in limited spaces and easily hover above of the target, which allows their use in any which ground, to the opposition of fixed-wing aircraft, which require prepared tracks for them to taking off and to land. Drones can be used for everything from law enforcement, to search and rescue after natural disasters, weather research, pest control, and even for taking aerial shots for real estate developers.

There are several types of drones, in this paper, we are interested in the quad-rotor kind, the choice of this type of drone is motivated by the simplicity of mechanics which facilitates its maintenance, its small size which let fly near obstacles and its dynamic that represents a very attractive control problem [4], [6]. Indeed, these drones are unstable in open loop, hence to drive them, it is necessary to stabilize them using a control algorithm in closed loop.

In our case, We choose quad-rotors drone which is handy, allow vertical takeoff and landing, as well as flying in hard to- Reach areas. The disadvantages are its mass and the consumption of energy caused by motors. The drone can perform three flight modes; hover, vertical flight and translation flight. In our work, we are interested in a translation flight that corresponds to the navigation on a horizontal plane, it is ensured by basing itself on pitch and roll tilting movements [25].

In the present study, we consider an optimal control problem of quad-rotor. In order to illustrate this study, we consider distance minimization problem between state and desired state in free final time. To solve this problem we will use the principle of Pontryaguin [18], [19], [24], [26], and determine the optimality equations resulting from this principle; i.e.; a differential-algebraic system as the state equation is provided and the adjoint equation. Thus, in order to determine the initial condition of the adjoint state; we use Bocop software [28] to ensure the convergence of this method. The work presented in this paper is organized as follows : In section I, a dynamic model of the quad-rotor is considered. In section II, implementation of the Shooting method with Matlab software. Finally, discussion, simulation results and conclusion are provided in Section III.

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II. Quad-rotor flight model

Let

$$X = [\phi, \dot{\phi}, \theta; \dot{\theta}, \psi, \dot{\psi}, x, \dot{x}, y, \dot{y}, z, \dot{z}]^T \tag{1}$$

The model dynamic of the quad-rotor is given under the state space as $\dot{x} = f(x) + g(x,u)$ considering $X = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}]^T$, and $u = [u_1, u_2, u_3, u_4]^T$ be the state, the control vector of the system respectively is given as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = a_1 x_4 x_6 + a_2 x_4 \Omega + b_1 u_1, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = a_3 x_2 x_6 + a_4 x_2 \Omega + b_2 u_2, \\ \dot{x}_5 = x_6, \\ \dot{x}_6 = a_5 x_2 x_4 + b_3 u_3, \\ \dot{x}_7 = x_8, \\ \dot{x}_8 = \frac{u_4}{m}(cos x_1 sin x_3 cos x_5 + sin x_1 sin x_5), \\ \dot{x}_9 = x_{10}, \\ \dot{x}_{10} = \frac{u_4}{m}(cos x_1 sin x_3 sin x_5 - sin x_1 cos x_5), \\ \dot{x}_{11} = x_{12}, \\ \dot{x}_{12} = \frac{cos x_1 cos x_3}{m} u_4 - g, \\ -\pi/2 \leq x_1 \leq \pi/2, -\pi/2 \leq x_3 \leq \pi/2, \\ -\pi \leq x_5 \leq \pi, x_i(0) = 0, i = \overline{1,12}, \\ -20 \leq u_j \leq 20, \ j = \overline{1,4}, \ 0 \leq u_4 \leq 20. \end{cases} \tag{2}$$

Where $g(m/s^2)$:gravity acceleration;$I_x, I_y, I_z(kg/m^2)$:roll, pitch and yaw inertia moments respectively,$J_r(kg/m^2)$: the rotor inertia; $m(kg)$: mass; $x, y, z(m)$: longitudinal, lateral and vertical motions respectively; $\phi, \theta, \psi(rad)$: roll, pitch and yaw angles, respectively; $w_k(rad/s)$: rotor angular velocity, where, k equal to 1, 2, 3 and 4; $d(m)$: the distance between the quadrotor center of mass and the propeller rotation axis; $u_1, u_2, u_3(N.m)$: aerodynamical roll, pitch and yaw moments respectively; $u_4(N)$: lift force.

With $\Omega = w_1 - w_2 + w_3 - w_4$, $a_1 = \frac{I_y - I_z}{I_x}$, $a_2 = \frac{-J_r}{I_x}$,$a_3 = \frac{I_z - I_x}{I_y}$,$a_4 = \frac{J_r}{I_y}$,$a_5 = \frac{I_x - I_y}{I_z}$,$b_1 = \frac{d}{I_x}$,$b_2 = \frac{d}{I_y}$, $b_3 = \frac{d}{I_z}$.

The one of the main objectives of this paper is to find control which ensure the convergence of positions $\{x, y, z\}$ and yaw angle ψ, to their desired trajectories respectively $\{x_d, y_d, z_d\}$, and yaw angle ψ_d while maintaining stability of roll and pitch angle $\{\phi, \theta\}$ in free final time.

Then, the criterion is formulate as follows:

$$J=\int_0^{t_f}(\frac{1}{\rho}(x_5-\psi_d)^2+(x_7-x_d)^2+(x_9-y_d)^2+(x_{11}-z_d)^2dt \tag{3}$$

where $t_f(second)$: free final time ; $\{x_d, y_d, z_d\}$: desired trajectory of $\{x, y, z\}$; ψ_d : desired trajectory of ψ.

III. Shooting Indirect method

The shooting indirect method is used to obtain the value of $p(0)$ necessary to the solution of the problem characterized by the Pontryaguin principle. If it is possible, from the condition of minimization of the Hamiltonian to express the control extremal function of $(x(t), p(t))$ then the extremal system is a differential system of the form $\dot{\upsilon}(t) = G(t, \upsilon(t))$ where $\upsilon(t) = (x(t), p(t))$.
The Hamiltonian of the system 2 is given by:
$H = p_1x_2 + p_2(a_1x_4x_6 + a_2x_4\Omega + b_1u_1) + p_3(x_4) + p_4(a_3x_2x_6 + a_4x_2\Omega + b_2u_2) + p_5(x_6) + p_6(a_5x_2x_4 + b_3u_3) + p_7(x_8) + p_8(\frac{u_4}{m}(cosx_1sinx_3cosx_5 + sinx_1sinx_5)) + p_9(x_{10}) + p_{10}(\frac{u_4}{m}(cosx_1sinx_3sinx_5 - sinx_1cosx_5)) + p_{11}(x_{12}) + p_{12}(\frac{cosx_1cosx_3}{m}u_4 - g) + \frac{1}{\rho}[(x_5 - \psi_d)^2 + (x_7 - x_d)^2 + (x_9 - y_d)^2 + (x_{11} - z_d)^2]$.
The Euler-Lagrange equations leads to:

$$\begin{cases}
\dot{x}_1 = \frac{\partial H}{\partial p_1} = x_2,\\
\dot{x}_2 = \frac{\partial H}{\partial p_2} = a_1x_4x_6 + a_2x_4\Omega + b_1u_1,\\
\dot{x}_3 = \frac{\partial H}{\partial p_3} = x_4,\\
\dot{x}_4 = \frac{\partial H}{\partial p_4} = a_3x_2x_6 + a_4x_2\Omega + b_2u_2,\\
\dot{x}_5 = \frac{\partial H}{\partial p_5} = x_6,\\
\dot{x}_6 = \frac{\partial H}{\partial p_6} = a_5x_2x_4 + b_3u_3,\\
\dot{x}_7 = \frac{\partial H}{\partial p_7} = x_8,\\
\dot{x}_8 = \frac{\partial H}{\partial p_8} = \frac{u_4}{m}(cosx_1sinx_3cosx_5 + sinx_1sinx_5),\\
\dot{x}_9 = \frac{\partial H}{\partial p_9} = x_{10},\\
\dot{x}_{10} = \frac{\partial H}{\partial p_{10}} = \frac{u_4}{m}(cosx_1sinx_3sinx_5 - sinx_1cosx_5),\\
\dot{x}_{11} = \frac{\partial H}{\partial p_{11}} = x_{12},\\
\dot{x}_{12} = \frac{\partial H}{\partial p_{12}} = \frac{cosx_1cosx_3}{m}u_4 - g,\\
\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -p_8\frac{u_4}{m}(-sinx_1sinx_3cosx_5 + cosx_1sinx_5)\\
-p_{10}\frac{u_4}{m}(-sinx_1sinx_3sinx_5 - cosx_1cosx_5)\\
-p_{12}(\frac{-sinx_1cosx_3}{m}u_4),\\
\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1 - p_4(a_3x_6 + a_4\Omega) - p_6(a_5x_4),\\
\dot{p}_3 = -\frac{\partial H}{\partial x_3} = -p_8\frac{u_4}{m}(cosx_1cosx_3cosx_5)\\
-p_{10}(\frac{u_4}{m}(cosx_1cosx_3sinx_5) - p_{12}(\frac{-cosx_1sinx_3}{m}u_4),\\
\dot{p}_4 = -\frac{\partial H}{\partial x_4} = -p_2(a_1x_6 + a_2\Omega) - p_3 - p_6(a_5x_2)\\
\dot{p}_5 = -\frac{\partial H}{\partial x_5} = -p_8\frac{u_4}{m}(-cosx_1sinx_3sinx_5 + sinx_1sinx_5)\\
-p_{10}\frac{u_4}{m}(cosx_1sinx_3sinx_5 + sinx_1cosx_5) - \frac{2}{\rho}(x_5 - \psi_d),\\
\dot{p}_6 = -\frac{\partial H}{\partial x_6} = -p_2(a_1x_4) - p_4(a_3x_2) - p_5,\\
\dot{p}_7 = -\frac{\partial H}{\partial x_7} = -\frac{2}{\rho}(x_7 - x_d),\\
\dot{p}_8 = -\frac{\partial H}{\partial x_8} = -p_7,\\
\dot{p}_9 = -\frac{\partial H}{\partial x_9} = -\frac{2}{\rho}(x_9 - y_d),\\
\dot{p}_{10} = -\frac{\partial H}{\partial x_{10}} = -p_9,\\
\dot{p}_{11} = -\frac{\partial H}{\partial x_{11}} = -\frac{2}{\rho}(x_{11} - z_d),\\
\dot{p}_{12} = -\frac{\partial H}{\partial x_{12}} = -p_{11},
\end{cases} \tag{4}$$

Let us defined a control law:

$$\begin{cases}
\frac{\partial H}{\partial u_1} = p_2b_1 + \frac{1}{2\rho}u_1,\\
\frac{\partial H}{\partial u_2} = p_4b_2 + \frac{1}{2\rho}u_2,\\
\frac{\partial H}{\partial u_3} = p_6b_3 + \frac{1}{2\rho}u_3,\\
\frac{\partial H}{\partial u_4} = \frac{p_8}{m}(cosx_1sinx_3cosx_5 + sinx_1sinx_5)\\
+\frac{p_{10}}{m}(cosx_1sinx_3sinx_5 - sinx_1cosx_5) + p_{12}(\frac{cosx_1cosx_3}{m})\\
+\frac{1}{2\rho}u_4,
\end{cases} \tag{5}$$

Then, the control is:

$$\begin{cases}
u_1 = \frac{-1}{2\rho}p_1b_1,\\
u_2 = \frac{-1}{2\rho}p_4b_2,\\
u_3 = \frac{-1}{2\rho}p_6b_3,\\
u_4 = \frac{1}{2\rho}(\frac{p_8}{m}(cosx_1sinx_3cosx_5 + sinx_1sinx_5)\\
+\frac{p_{10}}{m}(cosx_1sinx_3sinx_5 - sinx_1cosx_5)\\
+p_{12}(\frac{cosx_1cosx_3}{m})),
\end{cases} \tag{6}$$

The transversality condition of the Hamiltonian is defined as follows:

$$H(t_f, x_i, p_i, u_j, i = \overline{1,12}, j = \overline{1,4}) = 0 \tag{7}$$

To implement Shooting method, consider $|u_j| \leq 1, \quad j = \overline{1,4}$. Then, we have to solve the following system :

$$\begin{cases}
\dot{\upsilon}_1 = \upsilon_2,\\
\dot{\upsilon}_2 = a_1\upsilon_4\upsilon_6 + a_2\upsilon_4\Omega + b_1u_1,\\
\dot{\upsilon}_3 = \upsilon_4,\\
\dot{\upsilon}_4 = a_3\upsilon_2\upsilon_6 + a_4\upsilon_2\Omega + b_2u_2,\\
\dot{\upsilon}_5 = \upsilon_6,\\
\dot{\upsilon}_6 = a_5\upsilon_2\upsilon_4 + b_3u_3,\\
\dot{\upsilon}_7 = \upsilon_8,\\
\dot{\upsilon}_8 = \frac{u_4}{m}(cos\upsilon_1sin\upsilon_3cos\upsilon_5 + sin\upsilon_1sin\upsilon_5),\\
\dot{\upsilon}_9 = x_{10},\\
\dot{\upsilon}_{10} = \frac{u_4}{m}(cos\upsilon_1sin\upsilon_3sin\upsilon_5 - sin\upsilon_1cos\upsilon_5),\\
\dot{\upsilon}_{11} = \upsilon_{12},\\
\dot{\upsilon}_{12} = \frac{cos\upsilon_1cos\upsilon_3}{m}u_4 - g,\\
\dot{\upsilon}_{13} = -\upsilon_{20}\frac{u_4}{m}(-sin\upsilon_1sin\upsilon_3cos\upsilon_5 + cos\upsilon_1sin\upsilon_5)\\
-\upsilon_{22}\frac{u_4}{m}(-sin\upsilon_1sin\upsilon_3sin\upsilon_5 - cos\upsilon_1cos\upsilon_5)\\
-\upsilon_{24}(\frac{-sin\upsilon_1cos\upsilon_3}{m}u_4),\\
\dot{\upsilon}_{14} = -\upsilon_{13} - \upsilon_{16}(a_3\upsilon_6 + a_4\Omega) - \upsilon_{18}(a_5\upsilon_4),\\
\dot{\upsilon}_{15} = -\upsilon_{20}\frac{u_4}{m}(cos\upsilon_1cos\upsilon_3cos\upsilon_5)\\
-\upsilon_{22}(\frac{u_4}{m}(cos\upsilon_1cos\upsilon_3sin\upsilon_5) - \upsilon_{24}(\frac{-cos\upsilon_1sin\upsilon_3}{m}u_4),\\
\dot{\upsilon}_{16} = -\upsilon_{14}(a_1\upsilon_6 + a_2\Omega) - \upsilon_{18} - \upsilon_{18}(a_5\upsilon_2)\\
\dot{\upsilon}_{17} = -\upsilon_{20}\frac{u_4}{m}(-cos\upsilon_1sin\upsilon_3sin\upsilon_5 + sin\upsilon_1sin\upsilon_5)\\
-\upsilon_{22}\frac{u_4}{m}(cos\upsilon_1sin\upsilon_3sin\upsilon_5 + sin\upsilon_1cos\upsilon_5) - \frac{2}{\rho}(\upsilon_5 - \psi_d),\\
\dot{\upsilon}_{18} = -\upsilon_{14}(a_1\upsilon_4) - \upsilon_{16}(a_3\upsilon_2) - \upsilon_{17},\\
\dot{\upsilon}_{19} = -\frac{2}{\rho}(\upsilon_7 - x_d),\\
\dot{\upsilon}_{20} = -\upsilon_{19},\\
\dot{\upsilon}_{21} = -\frac{2}{\rho}(\upsilon_9 - y_d),\\
\dot{\upsilon}_{22} = -\upsilon_{21},\\
\dot{\upsilon}_{23} = -\frac{2}{\rho}(\upsilon_{11} - z_d),\\
\dot{\upsilon}_{24} = -\upsilon_{23},\\
\upsilon_i(0) \in \mathbf{R}, \quad i = \overline{1,24}
\end{cases} \tag{8}$$

Let $\upsilon(t, x_i, p_i, i = \overline{1,12})$ be the solution of the previous system at time t with the initial condition $(\upsilon_i(0), i = \overline{1,24})$. We construct a shooting function given by:

(a) (b)

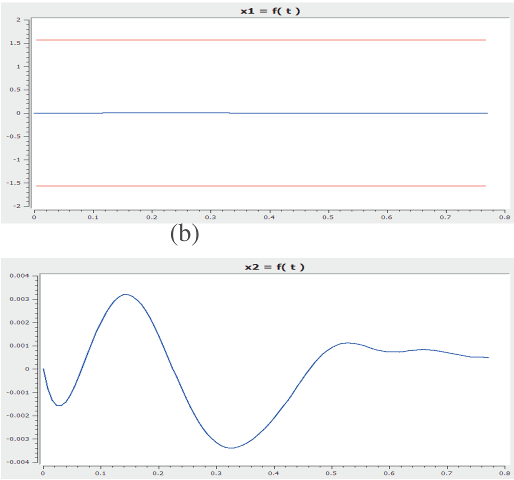


Fig. 1. Trajectory state of x_1 and x_2 respectively .

(a) (b)

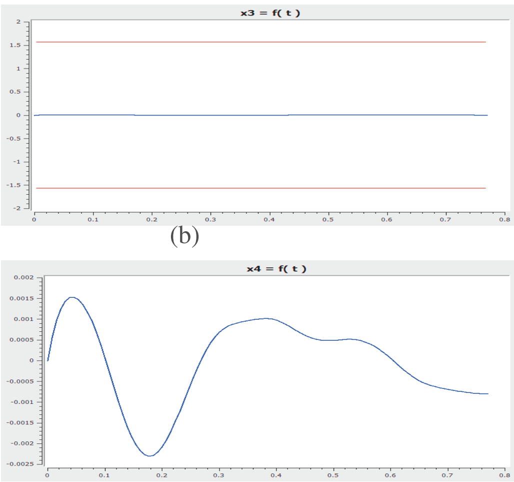


Fig. 2. Trajectory state of x_3 and x_4 respectively .

$$\varphi(p) = \begin{pmatrix} \upsilon_5(t_f, x_i, p_i, i = \overline{1,12}) \\ \upsilon_7(t_f, x_i, p_i, i = \overline{1,12}) \\ \upsilon_9(t_f, x_i, p_i, i = \overline{1,12}) \\ \upsilon_{11}(t_f, x_i, p_i, i = \overline{1,12}) - 2 \end{pmatrix}$$

In the solution, we used the Newton's method.The solution of $\varphi(p(0)) = 0$ is to find $p(0)$ such that $\varphi(p(0))$ gives the desired value of $x(t_f) = (\psi_d, x_d, y_d, z_d)$, which in our case uses the method quasi- newton (implemented in 'fsolve' of Matlab).

IV. Simulation and Discussion

$w_1 = 150; d = 0.213;\ m = 0.6;\ I_x = I_y = \frac{I_z}{2} = 5.66 \times 10^{-3};\ J_r = 3.4 \times 10^{-5};\ g = 9.8,\ \psi_d = 0,\ x_d = y_d = 0;\ z_d = 2.$

The red line is the delimiter of $x_1,\ x_3,\ x_5,\ u_1,\ u_2,\ u_3,\ u_4$ respectively. And the blue line is the trajectories.

The results given by Bocop software are presented in figures 1-8. This figures shows that the distance between the

(a) (b)

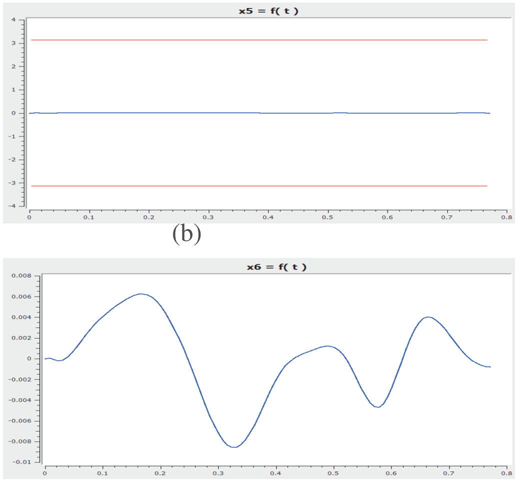


Fig. 3. Trajectory state of x_5 and x_6 respectively .

(a) (b)

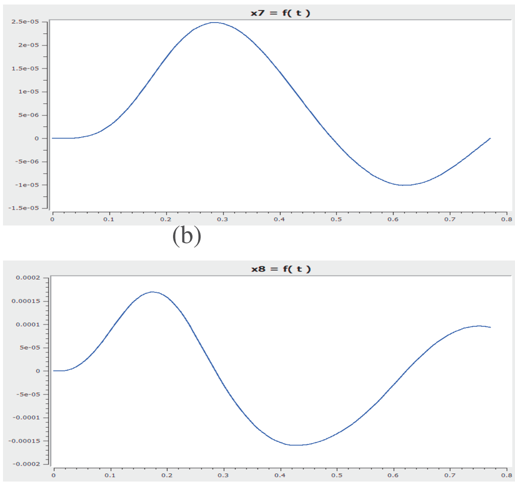


Fig. 4. Trajectory state of x_7 and x_8 respectively .

(a) (b)

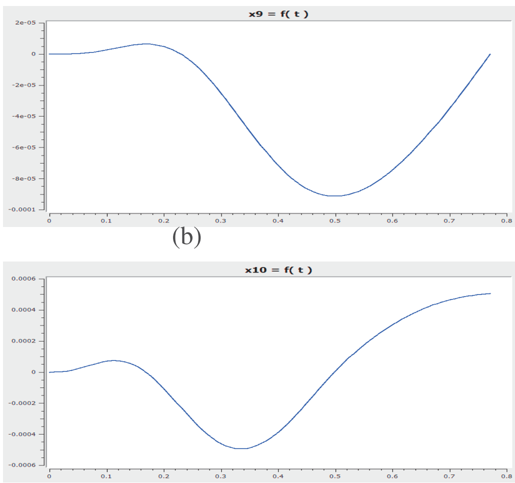
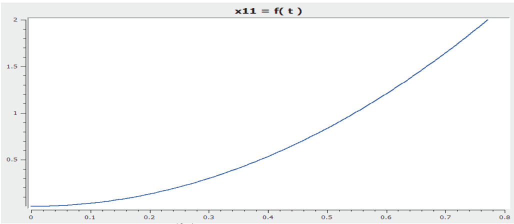


Fig. 5. Trajectory state of x_9 and x_{10} respectively .

(a)



(b)

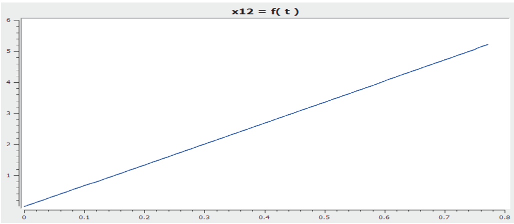
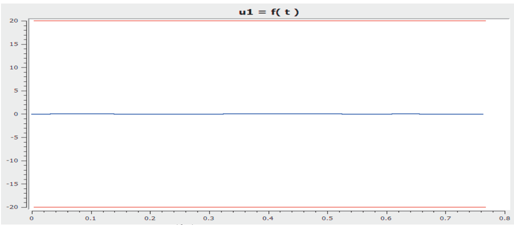


Fig. 6. Trajectory state of x_{11} and x_{12} respectively .

(a)



(b)

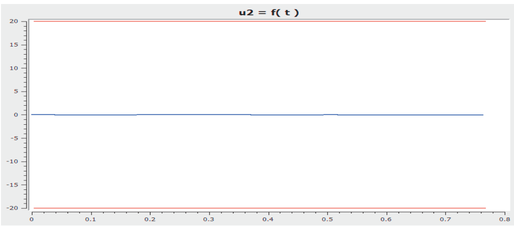
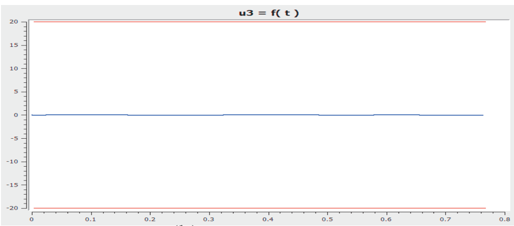


Fig. 7. Control u_1 and u_2 respectively .

(a)



(b)

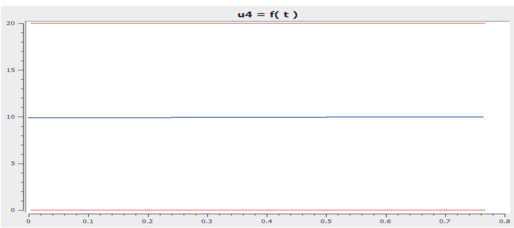
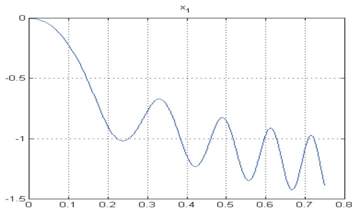


Fig. 8. Control of u_3 and u_4 respectively .

(a)



(b)

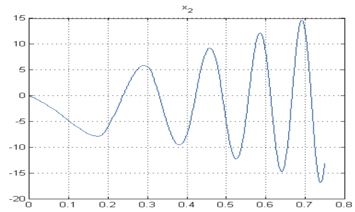
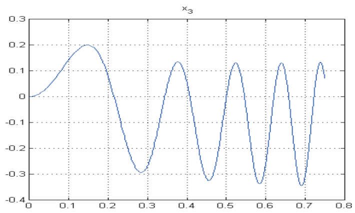


Fig. 9. Trajectory state of x_1 and x_2 respectively .

(a)



(b)

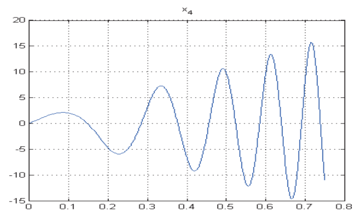
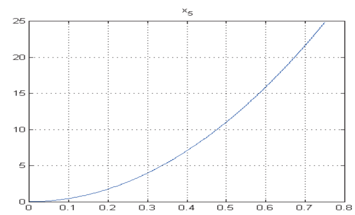


Fig. 10. Trajectory state of x_3 and x_4 respectively .

(a)



(b)

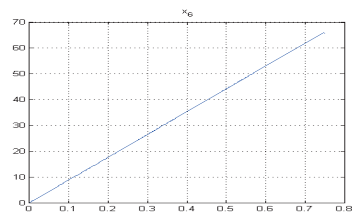
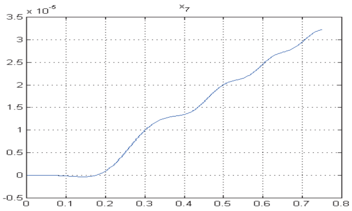


Fig. 11. Trajectory state of x_5 and x_6 respectively .

(a)



(b)

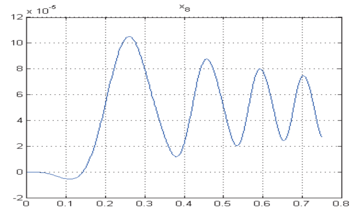
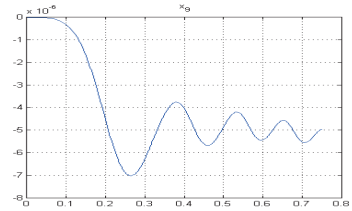


Fig. 12. Trajectory state of x_7 and x_8 respectively .

(a)



(b)

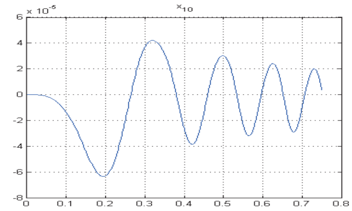
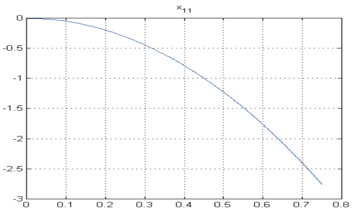


Fig. 13. Trajectory state of x_9 and x_{10} respectively .

(a)



(b)

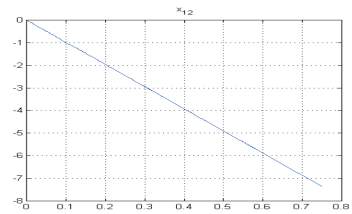
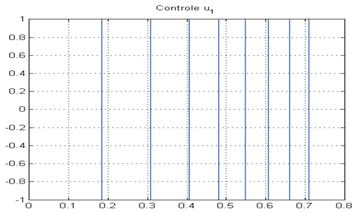


Fig. 14. Trajectory state of x_{11} and x_{12} respectively .

(a)



(b)

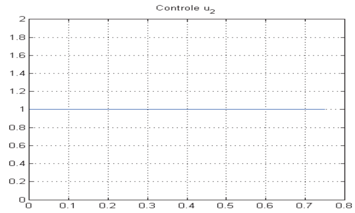
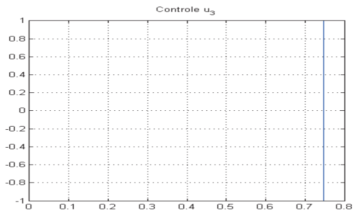


Fig. 15. Control u_1 and u_2 respectively .

(a)



(b)

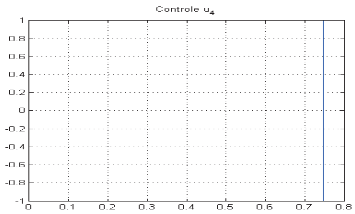


Fig. 16. Control of u_3 and u_4 respectively .

state $\{x, y, , z, \psi\}$ and their desired trajectories $\{0, 0, 0, 2\}$ is ensured in 141 iterations with 9.63 second, and final time $t_f = 0.77036 second$. And the results which are given by Shooting method are presented in figures 9-14. Final time with Shooting method is 0.7889 second, in 2 iterations with 15.869417 seconds. To ensure the convergence of Newton method which is the method's used to founded the zero of the Shooting function, we uses a result of the book of Ortega and Rheinboldt [27]; indeed if the discretization step h_{ij} are small and tend to zero.

V. CONCLUSION

In this work, we have solved an optimal control problem of unmanned aerial vehicle to minimize the distance between the state and the desired state in free final time.

The results are adequate for our purpose in the computational time is 9.63s in 141 iterations with Bocop software. The convergence is fast and the computational time is small. But, with Shooting method, the sensibility of the initial condition of the adjoint state, go slower time with a lot of iterations to find an optimal solution.

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