Modulo-Limiter Modulation of ARMA Processes

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September 10, 2021

Abstract

In this paper, we study the quantization errors of modulo sigma-delta modulated finite, asymptotically-infinite, infinite causal stable ARMA processes. We prove that the normalized quantization error can be taken as a uniformly distributed white noise for all the cases. Moreover, we find that this nice property is guaranteed by two different mechanisms: the high-enough quantization resolution [2]-[6] and the asymptotic convergence of quantization errors for some quasi-stationary processes [7]-[9], for different cases. But the assumption of the smooth density of the sampled random processes is needed in all the cases.

1 Introduction

The use of high resolution theory for quantization error analysis dates back to late 1940s [1]-[2]. In [2], Bennett demonstrated that under the assumption of high resolution and smooth density of the sampled stochastic process, the quantization error can be treated as an additive white noise. Since then, researcher had further proven the following conclusion: "under most circumstances, the noise is additively white and uncorrelated with the signal being quantized; and it is uniformly distributed between minus half a quanta to plus half a quanta, with a zero mean and a mean square as $\frac{1}{12}$ of the square of a quanta" [6]. There are already some nice surveys in this field, e.g. [3]-[6].

In [7]-[8], Chou and Gray studied the quantization error that is derived for a modulo sigma-delta modulator driven by a quasi-stationary stochastic process. They proved that the quantization errors for a causal stable MA process behaves just as an additive white noise, if the sum of the regression coefficients for the MA processes does not converges to zero.

In a recent report [9], we proved that the conclusion also holds for a fGn process with the Hurst exponent $H \in (0, \frac{1}{2})$. Such a fGn process can also be

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viewed as a special causal stable MA process; while the sum of its regression coefficients converges to 0.

Inspired by this new founding, we will further prove in this short paper that the conclusion holds for any causal stable MA processes. In the rest of this paper, we will sequentially discuss the finite, asymptotically infinite and infinite MA processes, and finally derive the conclusions for the general causal stable ARMA processes.

Suppose the original signal x(n) is bounded within [-b, b] in a finite time horizon [0, t]. An *M*-level uniform quantizer in [-b, b] is applied and the sample rate and the resolution of the quantizer are high enough.

As shown in [8], by defining $\Delta = \frac{2b}{M-1}$, the normalized quantization noise e(n) of x(n) through the modulo-limiter modulator can be written as

$$e(n) \triangleq \frac{1}{2} - \left\langle \sum_{i=0}^{n-1} \left(\frac{x(i)}{\Delta} + \frac{1}{2} \right) \right\rangle \tag{1}$$

where $\langle x \rangle = x \mod 1$ is the fractional part of x.

2 The Results for Finite MA Processes

For the uniform quantizer, we have the following useful lemma.

Lemma 1 [2]-[6] Suppose x(n) is a special MA process

$$x(n) = z(n) \tag{2}$$

where z(n) is a sequence of 1D random variables having an identical distribution with a smooth probability density function (actually z(n) does not need to be independent). The distribution of the normalized quantization error e(n) under modulo-limiter modulation converges to the uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ under the assumption of high resolution. Moreover, the quantization error is additively white and uncorrelated with the signal being quantized.

The term "under the assumption of high resolution" is frequently used in quantization error analysis [6], [8]. Indeed, it indicates the existence of a sufficient condition "under high enough resolution", under which the conclusion is true according to the criteria of uniformity of distribution and whiteness for quantization errors. Normally, we will further determine which resolution level is acceptable by numerical testing, with respect to the practical requirements.

The modulo sigma-delta modulation driven by a input as Eq.(2) can also be viewed as dithering. As pointed out in [7], the limit distribution of the normalized quantization error e(n) after dithering is the uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ regardless of the distribution of the input signal. $Lemma\ 1$ immediately leads to the following result for more general finite MA Processes.

Theorem 1 Define a causal stable MA process x(n)

$$x(n) = \psi(L)z(n) = \sum_{i=0}^{k} \psi_i z(n-i)$$
(3)

where z(n) is 1D i.i.d. stochastic process with a smooth probability density function. k is a constant, $k \in \mathbb{N}$. If ψ_i does not always equal to 0, for i = 1, ..., k, the conclusion in Lemma 1 also holds if the other conditions are the same.

Proof 1 Define a process y(n) as

$$y(n) = \sum_{i=0}^{k} \psi_i z(n-i) \tag{4}$$

Consider the weighted sum of independent random variables [10]-[12], if ψ_i does not always equal to 0, y(n) is therefore a sequence of random variables having a certain identical distribution with a smooth density. Following Lemma 1, we can reach the statement naturally.

It should be pointed out that we can allow $\sum_{i=0}^{k} \psi_i = 0$ in *Theorem 1*.

3 The Results for Asymptotically Infinite MA Processes

On the other side, we have the following lemma for the asymptotically infinite MA processes.

Lemma 2 [7]-[8] Define a causal stable MA process x(n)

$$x(n) = \psi(L)z(n) = \sum_{i=0}^{n} \psi_i z(n-i)$$
(5)

where z(n) is an 1D i.i.d process having a certain distribution with a smooth density. if the regression coefficients ψ_i of this MA process satisfy $\sum_{i=0}^{\infty} \psi_i \neq 0$, then the distribution of the normalized quantization error e(n) under modulo sigma-delta modulation converges to the uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ under the assumption of high resolution, when $n \to \infty$. Moreover, the quantization error is additively white and uncorrelated with the signal being quantized. Based on Lemma 2, we will first study a simple cases, where z(n) has a symmetric stable distribution to illustrate the outline of our main proof. Then, the results will be extended to the general MA processes.

Theorem 2 Define an causal stable MA process x(n) satisfying Eq.(5), where ψ_i does not always equal to 0. If z(n) is an i.i.d stochastic process having a symmetric stable distribution with the stability index (characteristic exponent) $\alpha \in [1, 2)$, the normalized quantization error converges to the uniform distribution in $[-\frac{1}{2}, \frac{1}{2}]$ under the assumption of high resolution, when $n \to \infty$.

Proof 2 According to [13], if z(n) is is an i.i.d stochastic process having a symmetric stable distribution (for simplicity, we assume it is symmetric about 0), the characteristic function of z(n) is written as

$$\varphi_z(t) = E\left(e^{itz}\right) = \exp\left\{-\sigma^\alpha \left|t\right|^\alpha\right\} \tag{6}$$

where t is the variable of the characteristic function. $\sigma > 0$ is the scale parameter. When $\alpha \in [1, 2)$, the distribution function is smooth.

We will discuss three cases in the follows, respectively.

i) If $\sum_{i=0}^{\infty} \psi_i \neq 0$, according to Lemma 2, the conclusion is true.

ii) If $\sum_{i=0}^{\infty} \psi_i = 0$, but ψ_i does not always equal to 0 and $\sum_{i=0}^{\infty} |\psi_i|^{\alpha}$ converges, we will show that x(n) will converge to a sequence of random variables having an identical certain identical distribution with a smooth density, when $n \to \infty$.

According to the definition (5), we have the limit characteristic function of x(n) as

$$\varphi_x(t) = E\left(e^{itx}\right) = \lim_{n \to \infty} \prod_{i=0}^n \varphi_z(\psi_i t) \tag{7}$$

Let $\sum_{i=0}^{\infty} |\psi_i|^{\alpha} = S > 0$, we have

$$\varphi_x(t) = \exp\left\{-\sigma^{\alpha}\left(\sum_{i=0}^{\infty} |\psi_i|^{\alpha}\right) |t|^{\alpha}\right\} = \exp\left\{-\sigma^{\alpha} S |t|^{\alpha}\right\}$$
(8)

which indicates that x(n) converges to an identical symmetric stable distribution [12]-[14] (Actually, this is Theorem 9.8.4 shown on page 328 of [14] and a corollary of Lemma 3 below). Thus, following Lemma 1, z(n) will converge to the uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, when $n \to \infty$.

iii) then, let's consider the situation $\sum_{i=0}^{\infty} \psi_i = 0$, but ψ_i does not always equal to 0 and $\sum_{i=0}^{\infty} |\psi_i|^{\alpha}$ does not converge.

We will first prove that $\sum_{i=0}^{\infty} \left| \sum_{j=0}^{i} \psi_j \right|^{\alpha}$ cannot converge in such a situation.

For $\alpha \in [1,2)$, based on the global convexity of $f(x) = |x|^{\alpha}$ for $x \in \mathbb{R}$, $\alpha \in [1,2)$ and Jensen's inequality, we have

$$2\sum_{i=0}^{\infty} \left| \sum_{j=0}^{i} \psi_{j} \right|^{\alpha}$$

$$= |\psi_{0}|^{\alpha} + \sum_{i=1}^{\infty} \left(\left| \sum_{j=0}^{i} \psi_{j} \right|^{\alpha} + \left| -\sum_{j=0}^{i-1} \psi_{j} \right|^{\alpha} \right)$$

$$\geq |\psi_{0}|^{\alpha} + 2\sum_{i=1}^{\infty} \left| \frac{1}{2} \psi_{i} \right|^{\alpha}$$

$$= \frac{1}{2^{\alpha-1}} \sum_{i=0}^{\infty} |\psi_{i}|^{\alpha} + \left[1 - \frac{1}{2^{\alpha-1}} \right] |\psi_{0}|^{\alpha}$$
(9)

which indicates that $\sum_{i=0}^{\infty} \left| \sum_{j=0}^{i} \psi_{j} \right|^{\alpha}$ also diverges in such situations. The limit distribution of the quantization noise can be derived through the

limit of the corresponding characteristic functions. As shown in [8], we can rewrite Eq.(1) as

$$e(n) = 1 - \frac{1}{2} \langle \delta(n) \rangle \tag{10}$$

where $\delta(n) \triangleq \sum_{i=0}^{n-1} \left(\frac{x(i)}{\Delta} + \frac{1}{2}\right)$. Notice that $\sum_{i=0}^{\infty} |\psi_i|^{\alpha}$ does not converge, we can reach the conclusion, if we then prove that [8]

$$\lim_{n \to \infty} \varphi_{\langle \delta(n) \rangle}(t) = \begin{cases} 1 & , t = 0 \\ 0 & , t \neq 0 \end{cases}$$
(11)

The limit characteristic function of $\langle \delta(n+1) \rangle$ can be written as

$$\lim_{n \to \infty} \varphi_{\langle \delta(n+1) \rangle}(2\pi t) \\
= \lim_{n \to \infty} \left| E \left\{ \exp \left[2i\pi t \sum_{i=0}^{n} \left(\frac{x(i)}{\Delta} + \frac{1}{2} \right) \right] \right\} \right| \\
= \lim_{n \to \infty} \left| E \left\{ \exp \left[2i\pi \frac{t}{\Delta} \sum_{i=0}^{n} \sum_{j=0}^{i} \psi_j z(i-j) \right] \right\} \right|$$
(12)

The innermost sum in Eq.(12) can be grouped as

$$\sum_{i=0}^{n} \sum_{j=0}^{i} \psi_j z(i-j) = \psi_0 z(n) + \dots + \left(\sum_{j=0}^{n} \psi_j\right) z(0)$$
(13)

Therefore

$$\lim_{n \to \infty} \varphi_{\langle \delta(n+1) \rangle}(2\pi t) = \lim_{n \to \infty} \prod_{i=0}^{n} \varphi_z \left(2\pi \frac{t}{\Delta} \sum_{j=0}^{i} \psi_j \right)$$
(14)

where by definition of symmetric stable process, we have

$$\prod_{i=0}^{n} \varphi_{z} \left(2\pi \frac{t}{\Delta} \sum_{j=0}^{i} \psi_{j} \right)$$

$$= \prod_{i=0}^{n} \exp \left[-\sigma^{\alpha} \left| 2\pi \frac{t}{\Delta} \sum_{j=0}^{i} \psi_{j} \right|^{\alpha} \right]$$

$$= \prod_{i=0}^{n} \exp \left[-\frac{\sigma^{\alpha} |2\pi t|^{\alpha}}{\Delta^{\alpha}} \left| \sum_{j=0}^{i} \psi_{j} \right|^{\alpha} \right]$$
(15)

Noticing that $\sum_{i=0}^{\infty} \left| \sum_{j=0}^{i} \psi_{j} \right|^{\alpha}$ does not converge, given any a small positive number $\epsilon \in (0,1)$ and a certain $t \in \mathbb{R} - \{0\}$, we can we can always find a large enough integer n^{*} such that

$$\sum_{i=0}^{n} \left| \sum_{j=0}^{i} \psi_{j} \right|^{\alpha} \ge -\ln(\epsilon) \frac{\Delta^{\alpha}}{\sigma^{\alpha} \left| 2\pi t \right|^{\alpha}}$$
(16)

for $n > n^*$, as n goes to infinity. Thus, for $t \neq 0$, we have

$$\prod_{i=0}^{n} \varphi_z \left(2\pi \frac{t}{\Delta} \sum_{j=0}^{i} \psi_j \right) \le \exp\left(\ln \epsilon\right) = \epsilon \tag{17}$$

for $n > n^*$, which clearly indicates $\lim_{n\to\infty} \varphi_{\langle \delta(n) \rangle}(t) = 0$ for $t \neq 0$, if we consider (14).

On the other hand, we can easily have $\lim_{n\to\infty} \varphi_{\langle \delta(n) \rangle}(0) = 1$ by definition. Therefore, the limit distribution of e(n) is the uniform distribution among

 $\left[-\frac{1}{2},\frac{1}{2}\right]$ due to the definition Eq.(10).

The proof for the additively whiteness and non-correlated property of the quantization error is similar to what had been given in [2]-[9] and is thus omitted here. So are the rest.

The proof for *Theorem 2* can be extended to the general cases by using the following useful lemma.

Lemma 3 (Lévy Continuity Theorem) [14] If P_n are probability laws on \mathbb{R}^k whose characteristic functions $f_n(t)$ converge for all t to some f(t), where f is continuous at 0 along each coordinate axis, then $P_n \xrightarrow{} P$ for a probability law P with characteristic function f.

Based on *Lemma 3*, we have the following general conjecture.

Theorem 3 Define a causal stable MA process x(n) satisfying Eq.(5), where ψ_i does not always equal to 0. If z(n) is an *i.i.d* stochastic process having a certain smooth density function, the normalized quantization error converges to the uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ under the assumption of high resolution. Moreover, the quantization error is additively white and uncorrelated with the signal being quantized.

Proof 3 If $\sum_{i=0}^{\infty} \psi_i \neq 0$, according to Lemma 2, the conclusion is true. In the follows, we will focus on the cases with $\sum_{i=0}^{\infty} \psi_i = 0$. Noticing that φ_z is bounded as $|\varphi_z| \leq 1$ and $\prod_{i=0}^{n} \varphi_z(\psi_i t)$ is a monotonic

series in terms of n for any given $t \in \mathbb{R}$, we can see that $\prod_{i=0}^{\infty} \varphi_z(\psi_i t)$ must converge pointwise as

$$\varphi_x(t) = \lim_{n \to \infty} \varphi_{x_n}(t) = \prod_{i=0}^{\infty} \varphi_z(\psi_i t) = \hat{\varphi}(t)$$
(18)

Notice that function φ_z is sufficiently smooth, for any $t \neq 0$, we can have the Taylor's expansion of $\varphi_z(\psi_i t)$ around 0 in Lagrange form as

$$\varphi_{z}(\psi_{i}t) = \varphi_{z}(0) + \varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2}$$

$$= 1 + \varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2}$$
(19)

where $\xi_i \in [0, \psi_i t]$ if $\psi_i t \ge 0$, or $\xi_i \in [\psi_i t, 0]$ if $\psi_i t < 0$. Since $\varphi'_z(0) \sum_{i=0}^{\infty} (\psi_i t) =$

0, we have $\psi_n \to 0$ as $n \to \infty$, and thus $\xi_n \to 0$. Clearly, we have $-1 \leq \varphi'_z(0)(\psi_i t) + \varphi''_z(\xi_i)(\psi_i t)^2 \leq 0$ due to $|\varphi_z| \leq 1$. For any $t \neq 0$, we have

$$\ln \prod_{i=0}^{n} \varphi_{z}(\psi_{i}t)$$

$$= \sum_{i=0}^{n} \ln \left[1 + \varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2} \right]$$

$$= \sum_{i=0}^{n} \frac{\varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$
(20)

where we use Taylor's expansion of $f(x) = \ln(1+x)$ in the last row.

Here, η_i is between 0 and $\left(\varphi_z'(0)(\psi_i t) + \varphi_z''(\xi_i)(\psi_i t)^2\right)$. Similarly, we have $\eta_i \to 0, \ \varphi_z''(\xi_i) \to \varphi_z''(0)$ as $n \to \infty$. Given any a small enough non-positive number x, we always have $2x \leq \frac{x}{1+\epsilon x} \leq x$, when $\epsilon \in (0,1)$. Thus, given a $t \neq 0$, there must exit a large enough $N^* \in \mathbb{N}$, constants $C_1, C_2 \in \mathbb{R}, C_3, C_4 \in \mathbb{R}^+$ that

$$\sum_{i=0}^{n} \frac{\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$= \left(\sum_{i=0}^{N^{*}} + \sum_{i=N^{*}+1}^{n}\right) \frac{\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$= C_{1} + \sum_{i=N^{*}+1}^{n} \frac{\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$\leq C_{1} + \sum_{i=N^{*}+1}^{n} \left[\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}\right]$$

$$\leq C_{1} + C_{2} + \sum_{i=N^{*}+1}^{n} \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}$$

$$\leq C_{1} + C_{2} + C_{3} \sum_{i=N^{*}+1}^{n} (\psi_{i}t)^{2}$$
(21)

and

$$\sum_{i=0}^{n} \frac{\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$= C_{1} + \sum_{i=N^{*}+1}^{n} \frac{\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$\geq C_{1} + 2\sum_{i=N^{*}+1}^{n} \left[\varphi'_{z}(0)(\psi_{i}t) + \varphi''_{z}(\xi_{i})(\psi_{i}t)^{2}\right]$$

$$\geq C_{1} + 2C_{2} + C_{4}\sum_{i=N^{*}+1}^{n} (\psi_{i}t)^{2}$$
(22)

Therefore, we have two situations:

i) if $\sum_{i=0}^{\infty} (\psi_i)^2$ converges, $\sum_{i=0}^{\infty} (\psi_i t)^2$ converges for any a given $t \neq 0$. Based on Ineq.(21), we can see that $\lim_{n\to\infty} \ln \prod_{i=0}^n \varphi_z(\psi_i t)$ also converges. Moreover, given a $t \neq 0$, $\frac{\varphi_z'(0)\psi_i}{1+\eta_i}$, $\frac{\varphi_z''(\xi_i)(\psi_i)^2}{1+\eta_i}$ are bounded. From Ineq.(22),

there exist two constants C_5 to $C_8 \in \mathbb{R}$ such that

$$\ln \prod_{i=0}^{\infty} \varphi_{z}(\psi_{i}t)$$

$$= \left(\sum_{i=0}^{N^{*}} + \sum_{i=N^{*}+1}^{\infty}\right) \frac{\varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$\geq C_{5}t + C_{6}t^{2} + \sum_{i=N^{*}+1}^{\infty} \frac{\varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2}}{1 + \eta_{i}}$$

$$\geq C_{5}t + C_{6}t^{2} + 2\sum_{i=N^{*}+1}^{\infty} \left[\varphi_{z}^{'}(0)(\psi_{i}t) + \varphi_{z}^{''}(\xi_{i})(\psi_{i}t)^{2}\right]$$

$$= C_{7}t + C_{8}t^{2}$$
(23)

Similarly, we have $\ln \prod_{i=0}^{\infty} \varphi_z(\psi_i t) \leq C_9 t + C_{10} t^2$ based on Ineq.(21), where $C_9, C_{10} \in \mathbb{R}$.

Thus, $\ln \prod_{i=0}^{n} \varphi_z(\psi_i t) \to 0$ as $t \to 0$. This shows that $\prod_{i=0}^{n} \varphi_z(\psi_i t) \to 1$ and equivalently $\hat{\varphi}(t)$ is continuous around 0.

According to Lemma 3, if $\hat{\varphi}(t)$ is continuous around 0, it is a characteristic function to a certain probability law. Thus, x(n) converges to a sequence of random variables having a certain identical distribution with a smooth density. Following Lemma 1, z(n) will converge to the uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, when $n \to \infty$.

ii) otherwise, $\sum_{i=0}^{\infty} (\psi_i)^2$ diverges, $\sum_{i=0}^{\infty} (\psi_i t)^2$ diverges for any a given $t \neq 0$, which indicates $\hat{\varphi}(t)$ is not continuous around 0. More precisely, based on Ineq.(22), we can easily have

$$\lim_{n \to \infty} \prod_{i=0}^{n} \varphi_z(\psi_i t) = \begin{cases} 1 & , t = 0 \\ 0 & , t \neq 0 \end{cases}$$
(24)

for $t \in \mathbb{R}$.

According to Eq.(14), we only need to prove that

$$\lim_{n \to \infty} \prod_{i=0}^{n} \varphi_z \left(2\pi \frac{t}{\Delta} \sum_{j=0}^{i} \psi_j \right) = \begin{cases} 1 & , t = 0 \\ 0 & , t \neq 0 \end{cases}$$
(25)

or equivalently

$$\lim_{n \to \infty} \prod_{i=0}^{n} \varphi_z \left(t \sum_{j=0}^{i} \psi_j \right) = \begin{cases} 1 & , t = 0 \\ 0 & , t \neq 0 \end{cases}$$
(26)

to reach the major conclusion.

Based on the global convexity of $f(x) = x^2$ and Jensen's inequality, we know that

$$2\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \psi_{j}\right)^{2}$$

$$= |\psi_{0}|^{2} + \sum_{i=1}^{\infty} \left(\left| \sum_{j=0}^{i} \psi_{j} \right|^{2} + \left| -\sum_{j=0}^{i-1} \psi_{j} \right|^{2} \right)$$

$$\geq |\psi_{0}|^{2} + 2\sum_{i=1}^{\infty} \left| \frac{1}{2} \psi_{i} \right|^{2} = \frac{1}{2} \sum_{i=0}^{\infty} |\psi_{i}|^{2} + \frac{1}{2} |\psi_{0}|^{2}$$
(27)

which implies that $\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \psi_{j}\right)^{2}$ will also diverge if $\sum_{i=0}^{\infty} (\psi_{i})^{2}$ diverges. Therefore, using the similar skills in the proof for Theorem 2, we can draw

Therefore, using the similar skills in the proof for Theorem 2, we can draw the conclusion based on Eq.(14) and Eq.(24).

4 The Results for Infinite MA Processes

It should be pointed out that *Theorem 3* can be extended to infinite causal stable MA processes. Actually, we have

Theorem 4 Define a causal stable MA process x(n)

$$x(n) = \psi(L)z(n) = \sum_{i=0}^{\infty} \psi_i z(n-i)$$
 (28)

The quantization error e(n) also converges to the uniform distribution in $\left[-\frac{1}{2},\frac{1}{2}\right]$ under the assumption of high resolution.

The proof is almost the same to that given for infinite cases except

i) we always have $\sum_{i=0}^{\infty} \psi_i$ converges for infinite cases; otherwise, it is not a well defined MA processes;

ii) Eq.(14) is changed to

$$\lim_{n \to \infty} \varphi_{\langle \delta(n+1) \rangle}(2\pi t) = \lim_{n \to \infty} \prod_{i=0}^{n} \varphi_z \left(2\pi \frac{t}{\Delta} \sum_{j=0}^{i} \psi_j \right) \cdot \prod_{i=1}^{\infty} \varphi_z \left(2\pi \frac{t}{\Delta} \sum_{j=i}^{n+i} \psi_j \right)$$
(29)

Noticing that $|\varphi_z| \leq 1$, we can still apply the above proof, because we can check $\lim_{n\to\infty} \prod_{i=0}^n \varphi_z \left(2\pi \frac{t}{\Delta} \sum_{j=0}^i \psi_j \right)$ instead.

Indeed, *Theorem* 4 is a general case to the conclusion that we had drawn for fGn processes with Hurst exponent $H \in (0, \frac{1}{2})$ in [9].

5 Conclusion

Based on the theory of ARMA processes [15]-[16], we can see that any a causal stable ARMA process can be formulated into a corresponding causal stable MA process. Thus, the above conclusions can be extended to causal stable ARMA processes.

Reviewing the above discussions, we can find that the nice property of quantization error is guaranteed by two different mechanisms: when $\varphi_x(t) = \prod_{i=0}^{\infty} \varphi_z(\psi_i t)$ converges to a continuous characteristic function pointwise, the asymptotically convergence to uniformly distributed white noise can be guaranteed by the high-enough quantization resolution [2], [6]; otherwise, this property is guaranteed by the asymptotic convergence of quantization errors for certain ARMA processes [7]-[9]. But in both cases, the asymption of the smooth density of the sampled random processes is needed.

It should also be pointed out that in many applications, the cases $\sum_{i=0}^{\infty} \psi_i = 0$ are non-trivial. Some interesting yet important examples can be found in [9].

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