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Efficient graph planarization in sensor networks and local routing algorithm

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***Abstract*—In this paper, we propose an efficient planarization algorithm and a routing algorithm dedicated to Unit Disk Graphs whose nodes are localized using the Virtual Raw Anchor Coordinate system (VRAC). Our first algorithm computes a planar 2-spanner under light constraints on the edge lengths and induces a total exchange of at most $6n$ node identifiers. Its total computational complexity is $O(n\Delta)$, with Δ the maximum degree of the communication graph. The second algorithm that we present is a simple and efficient algorithm to route messages in this planar graph that requires routing tables with only three entries. We support these theoretical results by simulations showing the robustness of our algorithms when the coordinates are inaccurate.**

I. Introduction

In many problems on networks, among which the problem of message routing [BMSU01], [KK00], [KWZ08], it is useful to know a planar subgraph of the communication graph. Subgraphs in which the length of a path between two nodes is not much longer than in the original graph are especially interesting. This is captured by the following notion: given two nodes x and y, the ratio between the length of a shortest xy-path (i.e., a path from x to y) in the subgraph and the length of a shortest xy-path in the original graph is called *stretch factor*. A subgraph whose maximum stretch factor is upper-bounded by k is called a k-spanner, and if the original graph is a complete graph embedded on the plane with edges length the Euclidean distance, we speak of geometric k-spanner. In this paper, we first propose a *distributed and simple way to compute such a planar geometric 2-spanner of a unit disk graph* when the nodes of the communication graph are localized using the *Virtual Raw Anchors Coordinate system* (VRAC [HJLR10], [HJLR11b]), instead of the stronger hypothesis of having the nodes localized in a classical 2D coordinate system and that we have specific conditions.

Then, we propose a *simple, efficient and light routing algorithm* dedicated to the constructed subgraph.

A. Related work

1) VRAC: VRAC coordinate system was motivated by the fact that most of the techniques to localized nodes start to measure distances to some anchors and use these measures to compute the 2D-coordinates. One aim of VRAC is to reduce the impact of measurement errors on the performances of routing algorithms. We refer to [HJLR10], [HJLR11b] for further details on the motivation and for numerical experiments. In this paper, we assume that the nodes are localized using this coordinate system. The definition is recalled for the reader convenience (Definition 1), and a simple variant is presented (see Section IV and Figure 5(a)). Localization in these coordinate systems, is a strictly weaker hypothesis than assuming that the nodes are localized in a traditional 2D coordinate system. Indeed, if the nodes are localized in a traditional 2D coordinate system, it is easy to compute their coordinates in the virtual raw anchors coordinate systems, whereas the converse is difficult. Furthermore, this coordinate system is expected to be much easier to implement in practice.

2) Planar graph and poset dimension: in [Sch89], it is proved that a graph $G = (V, E)$ is planar if and only if it has order-dimension at most three, where a graph has order-dimension d if and only if there exists a sequence $<_1, \dots, <_d$ of total orders on V that satisfies

- the intersection of the orders is empty[1]
- for each edge $(xy) \in E$ and for each $z \in V \setminus \{x, y\}$, there is at least one order $<_j$ in the sequence such that $x <_j z$ and $y <_j z$.

Hence, any set of three total orders whose intersection is empty induces a planar graph: the maximal subgraph of the complete graph obtained by keeping only the edges that satisfy the second condition. We refer to this graph as the Schnyder graph of the three total orders $<_1, <_2, <_3$, and we note it $G^{Schnyder}_{<_1,<_2,<_3}$ or G^S for short.

3) Planar spanner: one of the first planarization technique that preserves the connectivity is the Gabriel graph, but an edge may be replaced by a path of unbounded length [BDE07] and the stretch factor of the resulting graph may be large.

Subsequently, k-*spanner* were introduced and it is known that the Delaunay triangulation of a set of

[1] An order relation $>$ is seen as a subset $S_> \subset X \times V$. The intersection of two order relations $>$ and $\tilde{>}$ corresponds to the intersection of the two subsets $S_> \cup S_{\tilde{>}}$.

vertices V is a planar geometric spanner. Its stretch factor is upper bounded by 1.998 [Xia11], and strictly lower bounded by $\pi/2$ [BDL$^+$11], the exact stretch being unknown. In [LCW02], the authors proposed an efficient construction of a planar 2.5-spanner of an UDG that is the intersection of the UDG and the Delaunay triangulation of the set of nodes. The complexity of this construction is improved in [BCSX10], in which an algorithm needing 5 broadcasts per node is proposed. An other construction of a spanner of a Unit Disk Graph is proposed in [BCC$^+$07] whose stretch factor is greater than two. Other spanners exist, in particular a way to construct a 2-spanner from a complete graph is proposed in [Che89]. Interestingly, it is shown in [BGHI10] that three different constructions lead to the same planar geometric 2-spanner. These three constructions are the half-θ_6-graphs, the triangular-distance Delaunay triangulation (TD Delaunay graphs) and the geodesic embedding. In [BGHP10], the authors further proposed a planar spanner with bounded degree. We refer the interested reader to the recent survey of Bose and Smid [BS09].

As mentioned in Section I-A2, a graph of order-dimension three is planar [Sch89]. Hence, to planarize any graph, it is sufficient to select edges that correspond to three total orders. This technique has been used, for instance, to construct the half-θ_6 graph mentioned previously. However, this technique may require large computations: the three orders being total, the computations may not be feasible locally. Another problem is that the computed planar subgraph is a subgraph of the complete graph, and may not be a subgraph of the communication graph. In this paper, we address these issues when the communication graph is a Unit Disk Graph. For this, we propose three total orders based on the VRAC coordinates and construct a *planar subgraph* of the communication graph under simple additional assumptions. Our algorithm uses only comparison (no other operation of any type), it outputs a 2-spanner, and it requires the broadcast of at most $6n$ node identifiers.

In particular, our result improves the result of [LCW02] by constructing a spanner whose stretch factor is 2 versus 2.5. If in this work we use the planarity criterion of Schnyder [Sch89], we stress out that our construction does not necessarily lead to the planar graph induced by the three total orders when we apply Schnyder's theory. This is due to the bound on the length of the edges of the UDG communication model. In the graph obtained following Schnyder's theory, the edges may have unbounded lengths.

4) Greedy embedding: when one consider building a spanner, one usually does not focus on preserving easy routing properties such as the success of the greedy routing algorithm. It means that even if the greedy routing algorithm would deliver a message to the destination in the original communication graph, it may not succeed in the built spanner. Preserving a greedy routing success property is of great interest.

This problem is related to the following conjecture [PR04]: given a 3-connected graph, does an embedding exist such that the greedy routing algorithm is always successful ?

In [Dha08], it is proved that the conjecture is true if the graph is a plane triangulation by using Schnyder's characterization of planar graphs [Sch89]. Later, the conjecture was proved for every 3-connected graphs in [LM10], [GS09]. In this paper, we present a greedy like algorithm dedicated to the planar graph that we build. We prove that it guarantees message delivery under similar connectivity conditions.

B. geographic routing

Among the numerous geographic routing algorithms we mention GPSR and GFG [KK00] that use a planar subgraph to guarantee message delivery. The planarization technique that we present in this paper for an UDG is a first step towards the adaptation of these routing algorithms to the VRAC coordinate system.

We also mention [FGG06] in which the authors propose a routing protocol which guarantees delivery but do not require the planarization of the graph. The algorithm uses the *empty-circle* property of the Delaunay triangulation to characterize the nodes where greedy routing fails in order to locate the border of the holes. The results that we present in this paper show that there is a similar *empty-circle* property with a metric defined by a triangle. Hence, we expect that an approach similar to [FGG06] can be developed in the VRAC coordinate system.

C. Summary of results

1) Planar subgraph and spanner: In Section III, Lemma 4, we distributively construct a subgraph $\widetilde{G}$ of a Unit Disk Graph G. We prove in Lemma 3 that if the length of the edges in $\tilde{G}$ are bounded by $3r/4$, then $\widetilde{G}$ is planar. Thus, we can enforce the planarity of $\tilde{G}$ by restricting its edge set $\tilde{E}$ to the edges of length bounded by $3r/4$. However, this may improperly disconnect the obtained graph.

This leads us to Section IV in which we use a slightly modified version of the VRAC coordinates system. In that section, we introduce virtual edges (edges that are not edges of the connexion graph) and we define a new graph $\tilde{G\prime}$ (see Definition 5). We then prove that $\widetilde{G\prime}$ is a subgraph of $G^{\mathcal{S}}$ together with some properties on the stretch factor:

- the length of a shortest path in $\widetilde{G\prime}$ is at most twice the length of a shortest path in G.
- a virtual edge corresponds to a path of two edges in G.

This implies that a path in G is replaced by a path at most four times longer when the routing is computed in $\widetilde{G\prime}$. In the case where there are no virtual edges[2], $\widetilde{G\prime}$ is a planar 2-spanner of G.

[2]If there are virtual edges, $\tilde{G\prime}$ is not a subgraph of G and this is why we can't say that $\tilde{G\prime}$ is a 4-spanner.

Using the VRAC coordinates, our algorithm induces the broadcast of at most $6n$ node identifiers (excluding the one needed for neighborhood discovery), and has computational complexity $O(n\Delta)$, where Δ is the maximum degree of G. Using traditional Euclidean coordinates, it induces the broadcast of as few as $3n$ node identifiers that can be done in a single communication round; furthermore, when the density is high enough, no broadcast at all is required and the constructed graph is a planar 2-spanner, which *answers open problem number 22 in [BS09].*

2) Routing: given an embedding of the graph in the plane, we look at the problem of designing a routing algorithm which guarantees delivery and which is as close as possible to the greedy routing algorithm. In $\widetilde{G\prime}$, each node is associated to (at most) three out-neighbors (definition 4), which we use to design our local routing algorithm. Under an assumption similar to that $\tilde{G\prime}$ is a triangulation, we prove that it guarantees message delivery (Theorem 2). Our local routing algorithm requires routing tables containing as few as three neighbors together with their VRAC coordinates.

3) Simulations: We present experimental results showing that our algorithms are highly resilient to error measurements on the distances to the anchors. Indeed, they illustrate that even with an error of $+-5\%$ on the distance, the planarization algorithm still provides an almost planar graph and that the performances of the routing algorithm are not significantly degraded.

II. The model

A. Communication model

We consider a wireless network in which two nodes can communicate if they are at distance at most r, the communication radius. We can normalize the distances so that $r = 1$, in which case we have Unit Disk Graphs (UDG). However, we will keep mentioning r, as we believe it carries useful information. The use of the UDG model for the communication links is subject to caution from a practical point of view. We quickly mention the recent paper [LL09] that discusses how protocols that are proved valid under the UDG model can be turned into valid protocols in the more realistic SINR model. Another way of extending the results of this paper to more general communication models is to use basic properties of such models like the convexity of the region where the communication can happen [AEK$^+$09]. Indeed, it seems to us that most of the arguments that we use are related to this property.

The communication graph is given by the structure (V, E) where V is the set of nodes and E, the set of edges, i.e., the set of couples of nodes that can communicate together directly. In this paper we will use virtual edges. A virtual edge is an edge between two nodes x and y such that $(xy) \notin E$, but such that there is a path from x to y with edges in E.

Finally, we do not consider the impact of interferences or collisions during wireless communication.

B. Coordinate system

We use the virtual raw anchor coordinate system [HJLR10], [HJLR11b] with three anchors A_1, A_2, A_3. It means that each node knows its distances to the three anchors, distances which form the node coordinates. I.e., the coordinates of node x is the vector $(d(x, A_1), d(x, A_2), d(x, A_3))$.

Definition 1. *The coordinates of a node x is a vector*

$$(x_1, x_2, x_3) = (d(x, A_1), d(x, A_2), d(x, A_3))$$

Throughout the paper, we suppose that all nodes lay inside the triangle defined by the three anchors on a 2D-plane, this area is denoted $\mathcal{A}$. We use two different distances to define the coordinate system.

In Section III, we use the Euclidean distance for the distance function d. Given two points x and y, we note $|xy|$ the Euclidean distance from x to y, throughout the paper.

In Section IV, we extend the results using for the distance $d(x, A_1), d(x, A_2), d(x, A_3),$ the heights of the triangles $\widehat{A_2xA_3}$, $\widehat{A_1xA_3}$ and,$\widehat{A_1xA_2}$ respectively. We will note this distance $d^h(x, A_i)$ or $d^h_{A_i}(x)$ for $1 \leq i \leq 3$. We further suppose that $\widehat{A_1A_2A_3}$ is equilateral and that all nodes know the distances between the anchors: $|A_1A_2|$, $|A_1A_3|$,$|A_2A_3|$.

III. Distributed graph planarization

In this section, given an UDG G, we build a planar subgraph $\widetilde{G}$. We further extend it to $\widetilde{G\prime}$ by changing some of its edges by virtual edges, where a virtual edge represents a path in G.

Recall that in [Sch89] it is proven that a graph $G = (V, E)$ admits a planar embedding if we have three total order relations, $<_1, <_2, <_3$ (according to which the edges are defined) on the set of nodes and the two following conditions:

- **Condition A:** the intersection of the three order relations is empty,
- **Condition B:** for each edge $(x, y) \in E$ and for each vertex $z \notin \{x, y\}$ there is at least one order $<_i$ such that $x <_i z$ and $y <_i z$.

In this paper, we adapt this result to UDGs. In particular, we assume that the nodes are positioned i.e., the embedding is fixed, and we show how a reasoning similar to that in [Sch89] can be applied to construct a planar subgraph of an UDG. This leads to a simple and *localized distributed* algorithm to planarize a communication graph of a wireless network and to a simple description of the communication graph that accepts an efficient routing algorithm.

Given the VRAC coordinates of the nodes, we define three total order relations, $<_1, <_2, <_3$ on the set of nodes V in the following way:

Definition 2. *For $k \in \{1, 2, 3\}$, nodes x and y with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) satisfy the relation $x \ <_k \ y$, if and only if $x_k < y_k$.*

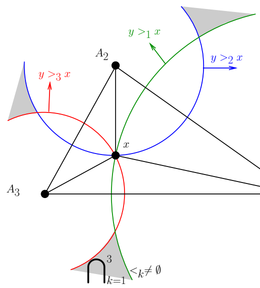


Fig. 1. If we do not restrict ourselves to the region $\mathcal{A}$, $\bigcap_{k=1}^{3} <_k \neq \emptyset$. $\bigcap_{k=1}^{3}$ is represented in gray

Lemma 1. *Given three non-collinear anchors, we consider the set of nodes that are inside $\mathcal{A}$, see Figure 1. If we consider the restriction of the order relations $<_k$ on $\mathcal{A} \times \mathcal{A}$ denoted $<_k |_{\mathcal{A}}$ then their intersection is empty.*

$$\bigcap_{k=1}^{3} <_k |_{\mathcal{A}} = \emptyset. \qquad (1)$$

Proof: To prove that the intersection is empty is equivalent to prove that given any point x that belongs to the convex hull of the three anchors the triangular area $\mathcal{A}$ is covered by the three circles centered on the anchors and passing through x. Indeed, if the intersection is not empty there is a point $y \in \mathcal{A}$ that belongs outside of the three circles (and reciprocally), i.e., $x \ <_k \ y$, for $k \in \{1, 2, 3\}$.

Because the area $\mathcal{A}$ is the union of the three triangles $\widehat{A_1xA_3}$, $\widehat{A_1xA_2}$ and $\widehat{A_2xA_3}$, see Figure 2(b), it is sufficient to show that the three triangles are covered by the circles. We consider $\widehat{A_1xA_3}$ particularly and the proof extends to the others triangles. We decompose the triangle $\widehat{A_1xA_3}$ into two sub-triangles $\widehat{A_1xx\prime}$ and $\widehat{A_3xx\prime}$, where $x\prime$ is such that the line $xx\prime$ crosses the line A_1A_3 perpendicularly. Because the length of the segment A_1x is larger than the length of the segment $A_1x\prime$ the sub-triangle $\widehat{A_1xx\prime}$ is covered by the circle centered in A_1 and passing through x. The same argument applies to the sub-triangle $\widehat{A_3xx\prime}$ and this concludes the proof. ∎

Remark 1. *Notice that if we do not assume that the nodes belong to the area $\mathcal{A}$ then, the intersection (1) may not be empty. Indeed there are point y whose distances to the three anchors are larger than the distances of x to the three anchors, i.e., $x <_k y$, $\forall k = 1, 2, 3$, see Figure 1.*

Definition 3. *We define the three binary relations $\tilde{<}_1, \tilde{<}_2, \tilde{<}_3$ by $\forall x, y \in V$, $k = 1, 2, 3$, $x \tilde{<}_k y \iff x <_k y$ and $y <_j x$ for $j \neq k$.*

Moreover, if the original order relations $<_1, <_2, <_3$ have an empty intersection, for each edge $(x, y) \in E$ there exists a unique $k \in \{1, 2, 3\}$ such that $x \tilde{<}_k y$ or $y \tilde{<}_k x$. This induces an orientation on the edges.

Definition 4. *We assume given a graph $G = (V, E)$ and three total order relations on V with empty intersection. An edge $(x, y) \in E$ is* **outgoing** *at x if $\exists k \in \{1, 2, 3\}$ such that $x \tilde{<}_k y$ in which case y is an out-neighbor of x. The edge is either outgoing at x or at y. We say that an edge is* **incoming** *at x if it is not outgoing in which case y is an in-neighbor of x.*

There are at most three outgoing edges at a node.

Lemma 2. [3] *Provided that* **Condition A** *is satisfied,* **Condition B** *is equivalent to the requirement that for each edge (x, y), if $x \tilde{<}_k y$ then $y = min_k\{z \mid x \tilde{<}_k z\}$*[4] *and if $y \tilde{<}_k x$ then $x = min_k\{z \mid y \tilde{<}_k z\}$,*

Proof: We consider an edge (x, y) such that $x \tilde{<}_k y$ and $y = min_k\{z \mid x \tilde{<}_k z\}$, the others cases are similar. This amounts to assume $x <_k y$, $x >_i y$, $x >_j y$ for $i, j, k \in \{1, 2, 3\}$ and i, j, k all different. Let $z \in V \setminus \{x, y\}$. If $z >_i x$ or $z >_j x$ we obtain that $z >_i y$ or $z >_j y$ and the result is valid. If $z <_i x$ then Condition A implies that $z >_k x$ or $z >_j x$. We only need to consider the first case and then z satisfies $x <_k z$, $x >_i z$ or $x >_j z$ (the same inequality as y). By assumption we have $y = min_k\{z \mid x \tilde{<}_k z\}$, and then $x <_k z$ and $y <_k z$. This proves the implication.

On the other side, let us assume that Condition B is satisfied and (x, y) is an edge of the graph. Condition A implies that there exist $i, j, k \in \{1, 2, 3\}$ all different such that $x <_k y$, $x >_i y$ $x >_j y$ (or $y <_k x$, $y >_i x$ $y >_j x$, and the proof is similar in both cases). Let us assume that there exists $z \in V$ such that $x <_k z <_k y$ then z is larger than x in the three orders contradicting Condition A. ∎

The graph $G^{Schnyder}_{\tilde{<}_1, \tilde{<}_2, \tilde{<}_3}$ induced by these three total orders is planar. However, as we mentioned in the introduction, there are some major issues: 1) $G^{Schnyder}_{\tilde{<}_1, \tilde{<}_2, \tilde{<}_3}$ may not be a subgraph of an UDG, and 2) $G^{Schnyder}_{\tilde{<}_1, \tilde{<}_2, \tilde{<}_3}$ can not be computed locally because the length of the edges is unbounded.

Lemma 3. *Given a graph $G = (V, E)$ where the nodes are localized in the VRAC coordinate system, if $\forall (x, y) \in E$ and $\forall z \in V \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ such that $x <_k z$ and $y <_k z$ then the graph is planar.*[5]

Proof: We consider two edges (xy) and (uv). By assumption, there exists $k_1, k_2, k_3, k_4 \in \{1, 2, 3\}$ such that:

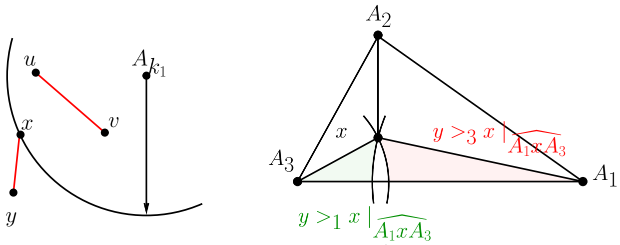
$$u, v \ <_{k_1} \ x, \ u, v \ <_{k_2} \ y, \ x, y \ <_{k_3} \ u, \ x, y \ <_{k_4} \ v.$$

It is clear that $k_1 \neq k_3, k_4$ and $k_2 \neq k_3, k_4$ and wlog, we can assume that $k_1 = k_2$ and then $u, v \ <_{k_1} \ \min(x, y)$. Indeed, if $k_3 \neq k_4$ we have $k_1 = k_2$ because $k_i \in \{1, 2, 3\}$. If $k_3 = k_4$ we apply the same argument to u, v instead of x, y.

[3]This result was presented in [Sch89] without proof. We include it for the reader convenience.

[4]We denote min_k the minimal z with respect to the order relation $<_k$.

[5]In [Sch89], the author proves that given three total order conditions, there exists a planar embedding of the graph. Our result is different as we select edges to obtain a planar subgraph using the VRAC coordinate system, and this for the given embedding.



(a) $u, v <_{k_1} \min(x, y)$ implies that (uv) and (xy) do not intersect.

(b) $\bigcap_{k=1}^{3} <_k |_{\mathcal{A}} = \emptyset$.

Fig. 2.

We conclude that (uv) does not cross (xy) as each point of (uv) are $<_{k_1}$ smaller than x and y, see Figure 2(a). ■

Lemma 4 gives a *local* condition to ensure planarity.

Lemma 4. *(Local planarization.) Given an UDG $G = (V, E)$ and three anchors A_1, A_2, A_3. We denote $\mathcal{N}_x^2$ the two hops neighborhood of $x \in V$ i.e., the set of nodes y such that $(x, y) \in E$ or such that there exists $z \in V$ such that $(x, z) \in E$ and $(z, y) \in E$. We define the subgraph $\widetilde{G} = \left(V, \widetilde{E}\right)$ of G by $\forall x, y \in V, (xy) \in \widetilde{E} \iff (xy) \in E$ and $\exists k \in \{1, 2, 3\}$ such that $y = min_k\{z \in \mathcal{N}_x^2 \mid x \lesssim_k z\}$ or $x = min_k\{z \in \mathcal{N}_y^2 \mid y \lesssim_k z\}$. Provided that the length of the edges in $\widetilde{E}$ is bounded by $3/4r$, where r is the radius of the UDG, the resulting graph $\widetilde{G} = \left(\widetilde{V}, \widetilde{E}\right)$ is planar.*

Proof: Let us assume that there are two edges (x, y) and (u, v) in $\tilde{E}$ that intersect. Without loss of generality we can assume $y = min_k\{z \in \mathcal{N}_x^2 \mid x \lesssim_k z\}$ and $v = min_k\{z \in \mathcal{N}_u^2 \mid u \lesssim_k z\}$. By Lemma 3 and the equivalence between Conditions A and B (Lemma 2), this can happen only if x does not know about u and v and u does not know about x and y (otherwise, by Lemma 3 the intersection would be detected). But if the length of the edges is bounded by $3/4r$ then it must be that all three nodes belong to the second neighborhood of the fourth.[6] ■

This lemma is illustrated by Figure 3 which is itself sub-divided into three figures. They all represent subgraphs of the same randomly generated UDG. In the top representation, all the edges of the UDG are represented in the background in light grey. The edges satisfying the order relations without any consideration on their lengths are in color and among them, we observe two crossing edges. In the representation below, the edges longer than $3r/4$ are removed and no more crossing are observed. However, a crossing may still appear when for a node x, the node y' which satisfies $y' = min_k\{z \mid x \tilde{<}_k z\}$ is out of range, i.e., the edge (x, y') does not belong to E. This is illustrated in Figure 4 where we represent a node x in the middle

[6]This is classical and due to the fact that the perimeter of a (convex) quadrilateral is bounded by 4 times the sum of the diagonal. In our setting, this means that the perimeter is bounded by 3 r.

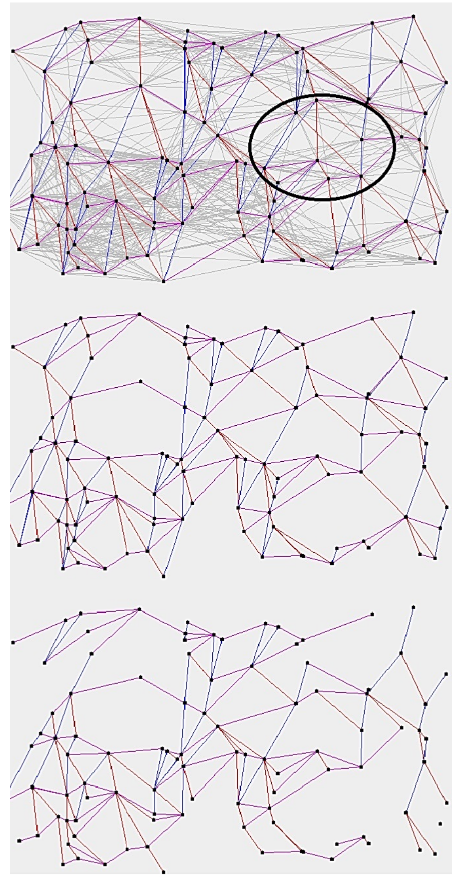


Fig. 3. Illustration of Lemma 4. The top the graph is obtained by considering only the edges $(xy) \in E$ such that $x \tilde{<}_k y$ or $y \tilde{<}_k x$. We draw an oval around crossing edges. In the middle, the length of the edges is bounded by $3/4r$. The bottom graph results from the application of Lemma 4.

of a circle representing its communication range. x is connected to a node y, and the black region is the region that is out of range of x and such that if a node y' belongs to this area, x should be connected to y' instead of y to apply Schnyder's criterion. Such regions are small and can be bounded if we make assumptions on the anchors positions. This is what we do in Section IV. In particular, we introduce *virtual edges* in order to cover these harmful regions. In the figure at the bottom of Figure 3, we exclusively kept the edges satisfying the hypothesis of Lemma 4. There we proved that no crossing can appear.

$\widetilde{G}$ may not be a subgraph of $G^{Schnyder}_{\lesssim_1, \lesssim_2, \lesssim_3}$, however, Lemma 4 provides a sufficient condition ensuring that $\widetilde{G}$ is planar. We point out that a node x uses only its second neighborhood. Using it, each node decides independently to keep an edge (xy) given its length (that must be estimated) and the criterion enunciated in Lemma 4. We also mention that once this algorithm has been executed, a node x needs only to memorize 3 nodes: its out-neighbors.

To obtain stronger results, in the next section, we consider a simple variation of the VRAC coordinate system and we assume that the three anchors form an equilateral triangle.

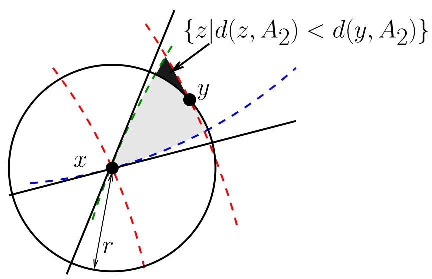
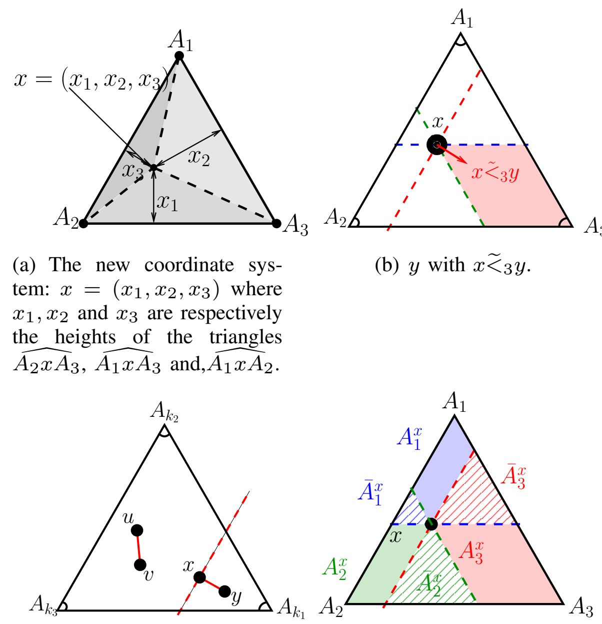


Fig. 4. Illustration of the regions where the existence of a node may lead to crossing edges.



(a) The new coordinate system: $x = (x_1, x_2, x_3)$ where x_1, x_2 and x_3 are respectively the heights of the triangles $\widehat{A_2xA_3}$, $\widehat{A_1xA_3}$ and,$\widehat{A_1xA_2}$.

(b) y with $x\tilde{<}_3y$.

(c) New proof of Lemma 3

(d) Definition of the greedy regions

Fig. 5. Adapting proofs to the new settings.

IV. Properties of the Planar Embedding

In this section we discuss a simple modification of the VRAC coordinate system that makes computations easier and we make the following hypothesis:

- There are three anchors A_1, A_2, A_3, the nodes belong to the convex hull $\mathcal{A}$ of the anchors and they know their distances to all three anchors.
- $\widehat{A_1A_2A_3}$ is equilateral.
- the nodes know the distances between the anchors ($|A_1A_2| = |A_1A_3| = |A_2A_3|$).

With respect to the first part of the paper, the two last hypothesis are new. By using the distances between the anchors, each node x can compute the heights of the triangles $\widehat{A_2xA_3}$, $\widehat{A_1xA_3}$ and, $\widehat{A_1xA_2}$. We denote these values (x_1, x_2, x_3), see Figure 5(a).

A. Adapting results of Section III and further

1) Results of Section III: Using the coordinates defined above, we define the order relations $<_1, <_2, <_3$ and $\tilde{<}_1, \tilde{<}_2, \tilde{<}_3$ the same way we did in Section III. In Section III, given a node x, the nodes satisfying $y\tilde{>}_kx$ were outside the circle centered at A_k of radius $|xA_k|$. With the new definition of the distance function d (c.f. Section II-B), the nodes y satisfying $y\tilde{>}_kx$ are contained in the half plane containing A_k defined by the line parallel to $(A_{(k+1 \mod 3)+1}, A_{(k \mod 3)+1})$ going through x, as illustrated in Figure 5(b).

Using this observation, it is easy to see that the intersection of the three order relations is empty, so Lemma 1 is still valid in this new coordinate system. Lemma 2 is independent of the metric and thus still valid. Similarly, Lemma 3 remains true, the argument is the same as in Figure 2(a) except that the circle is replaced by a straight line. The proof of Lemma 4 is similar in this new coordinate system since it mainly depends on the previous lemmas that are still valid.

In summary, **all the results that we have proved previously are valid with the new coordinate system**.

2) Definition of the new graph $\tilde{G'}$: The main difficulty in handling the Conditions A and B of Section III to planarize the UDG is the fact that for each node x the set $\{z \mid x \tilde{<} z\}$ may not be included in its set of neighbors and some extra communication is required if x desires to know this set. An advantage of our assumption on the position of the anchors is that this set can be easily computed. Indeed, it is possible to locally compute a subgraph of $G^{Schnyder}_{\tilde{<}_1,\tilde{<}_2,\tilde{<}_3}$. For this we introduce **virtual links**. If we are in the situation where $x \tilde{<}_k y$ and $x \tilde{<}_k y'$, $y' <_k y$ but y' is out of range of communication of x, from Schnyder's theory, x should be connected to y' rather than to y. However, due to the limited communication range of x, it does not occur and, the edge (xy) can potentially cross an edge incident to y'. We prove that such a problem can only occur with a node y' in the second neighborhood of x. So we proceed as follow: the node y that knows its neighborhood informs x of the existence of such a y'. In the case there is such a y' a virtual edge between x and y' replaces the edge (xy). (xy') is said to be a virtual edge, it corresponds to the path x, y, y' in G. In the following, we prove that the resulting graph is planar and that it has a good stretch factor.

The algorithm below (cf. also Algorithm 1) computes a new graph $\tilde{G'}$ from G. The computations are all local and distributed.

- (As in Lemma 4) Each node x knows its neighbors and selects the nodes $y_k,\ k = 1, 2, 3$ such that $y_k = \min\{z \mid x \tilde{<}_k z\}$.
- (Virtual edge) Each node x checks with its selected neighbors $y_k,\ k = 1, 2, 3$ that there is no node $y_{k'}$ in its second neighborhood such that $y_{k'} <_k y_k$, $x \tilde{<}_k y'$ ($d(y', y) < r$) and $y_{k'}$ is out of range from x.
  - If no such node exists, the edge (xy) becomes active.
  - Otherwise a virtual edge is created between x and $y_{k'}$ while the original edge (xy_k) is removed.

Definition 5. *Given a graph $G = (V, E)$ we denote $\tilde{G'} = (V, \tilde{E'})$ the graph obtained by the above proce-*

dure, c.f. also Algorithm 1.

3) Connectivity results and stretch:

Definition (Figure 5(d)): Given a node x, we call the **greedy regions** of x the three regions $A_i^x = \{z \mid x \tilde{<}_i z\}$, for $i \in \{1,2,3\}$.

Definition (Figure 5(d)): Given a node x, for $i \in \{1,2,3\}$, we denote the region between the two regions A_i^x and $A_{i \bmod 3+1}^x$ by $\bar{A}_i^x$.

Lemma 5. *Given an edge $(x,y) \in E$ in G, there is a path P from x to y in $\widetilde{G\prime}$. The path is contained in either $\{z \in A_i^x | z \leq_i y\}$ or $\{z \in A_i^y | z \leq_i x\}$ for some $i \in \{1,2,3\}$, and it satisfies: $\sum_{e \in P} |e| \leq 2|xy|$.*

Proof: The proof can be found in [HJLR11a], Lemma 4. ∎

Corollary 1. *If G is connected then $\widetilde{G\prime}$ is connected.*

The proof of Lemma 5 uses the assumption on the position of the anchors and some geometrical properties of the coordinate system.

Lemma 6. *$\widetilde{G\prime}$ is a subgraph of G^S.*

Proof: (sketch) The proof consists in showing that given an edge $(x,y) \in E$ and assuming wlog that $x \tilde{<}_k y$ for a given k, then the node y is a neighbor of all the nodes z (if any) such that $x \tilde{<}_k z$ and $z <_k y$, i.e., the transmission range of y covers the black region in Figure 4. Remember that such a node z is connected to x in the Schnyder graph G^S. Then, with our particular assumptions on the positions of the anchors we deduce that such a node z is connected to x in $\widetilde{G\prime}$ by an edge or a virtual edge (through y) and, by definition, this edge is also an edge of the Schnyder graph G^S. However, it can be that an edge in G^S is not an edge in G[7]. This shows that $\widetilde{G\prime}$ may be a strict subgraph of G^S. ∎

From this lemma, we immediately obtain that $\widetilde{G\prime}$ is planar. When $\widetilde{G\prime} = G^S$, we also deduce that it is a geometric 2-spanner. More generally, a repeated application of Lemma 5 shows that for a path in G between the nodes x and y, there exists a path in $\widetilde{G\prime}$ between x and y whose length in $\widetilde{G\prime}$ is at most twice the length of the path in G.

Theorem 1. *Given a connected graph G, the graph $\widetilde{G\prime}$ is planar, and for any two nodes x and y, if there is a path of length ℓ from x to y in G, there is one of length at most 2ℓ in $\widetilde{G\prime}$.*

Notice that the previous theorem applies to $\widetilde{G\prime}$ which may contains virtual edges that are not edges of G, still, a virtual edge represents a path in G. The next lemma bounds the length of these edges. Notice further that the virtual edges are edges of a Unit Hexagonal Graph (c.f. [BGHI10]). Finally, simulations in Section VI show that the virtual edges are rare and disappear as the node density increases.

Lemma 7. *A virtual link $(xy) \in \widetilde{E\prime}$ represent a path of two edges of G and has length at most $2r/\sqrt{3}$.*

Proof: Without loss of generality, we can suppose that the link is oriented from x to y and with respect to minimizing coordinate y_1. The virtual links represents a path $x, z, \ldots, y$ where z is inside the communication range of x, so $d(x,z) \leq r$. We know by construction that considering the first coordinates, we have $y_1 \leq z_1$. It means that y is in the triangle T_1 delimited by the three lines $\{u|u_1 = z_1\}$, $\{v|v_2 = x_2\}$ and $\{w|w_3 = x_3\}$ as depicted in Figure 6. The furthest points of this triangle are the two summits other than x. This triangle is equilateral and the edges are of length $2r/\sqrt{3}$, so $|xy| \leq 2/\sqrt{3}r$. ∎

From this lemma, we obtain that Algorithm 1 constructs correctly $\tilde{G\prime}$.

> Input: A Unit Disk Graph G.
> Output: $\tilde{G\prime}$.
> **for** all $x \in V$ **do**
>   **for** $k \in 1,2,3$, $y_k = \min_k\{y \in A_k^x, |y_k x| < r\}$ **do**
> x broadcast "activate (xy_k)"
> $y'_k = \min_k\{y \in A_k^x, |y_k y| < r\}$
>   **if** $y_k \neq y'_k$ **then**
> y_k broadcasts "disable (xy_k) and activate (xy'_k)"
>   **end if**
>   **end for**
> **end for**

Algorithm 1: Distributed construction of $\tilde{G\prime}$.

Corollary 2. *Given a connected graph G, if there are no virtual edges in $\widetilde{G\prime}$, then $\widetilde{G\prime}$ is a planar 2-spanner of G.*

Proof: The assumption that there are no virtual edges implies that $\widetilde{G\prime}$ is a subgraph of G, and then the result follows from Theorem 1. ∎

Notice that a common setting is that the nodes of the UDG are randomly distributed in an area. In this case, when the density of the nodes increases the number of virtual edges decreases and the assumption of Corollary 2 is verified.

To summarize, $\widetilde{G\prime}$ is a planar graph such that 1) the length of a shortest path in this graph is at most twice the length of a shortest path in the communication graph, and 2) virtual edges correspond to a path of length two in the communication graph. Furthermore, to construct $\widetilde{G\prime}$ (Algorithm 1), the communication complexity in terms of bits at a node x is as follow: each node broadcasts once its Id and coordinates, then, each node broadcasts the Id of its out-neighbors, and finally, each of this neighbors may send to x an other node Id which is the extremity of a virtual

[7] We ommit the details but the idea is the following: if there is a node z in a greedy region of x and that z is out of range of x in the UDG model but within range in the Unit Hexagonal Graph model, if there are no path from x to z included in the greedy region in which z lays, then the edge xz is in G^S but not in $\widetilde{G\prime}$.

edges starting from x. Hence, each node induces an exchange of at most six nodes' Id (excluding the neighborhood discovery), which gives a total of at most $6n$ nodes' ids that are broadcasted. The computational complexity is $O(n\Delta)$, with Δ the maximum degree of the communication graph: each node x computes which of its neighbors minimizes each order, plus, x requires each of the selected neighbors y_1, y_2, y_3 to verify if there should be a virtual edge, which also consists in computing the minimum of a set of at most Δ elements.

Now, if instead of using the VRAC coordinates, we use the Euclidean coordinates, each node y can computes on its own if it is minimum for one of the three orders for one of its neighbors. It means that we can avoid the statement x *broadcasts "activate* (xy_k)*"* in Algorithm 1. The modified version of Algorithmr̃efalgo needs the broadcasts of at most $3n$ nodes identifiers that can all be performed in a single round of communication. In case there are no virtual edges, the resulting graph is a 2-spanner of the Unit Disk Graph, and in this case no messages are exchanged. It answers the open question 22 of [BS09] under the hypothesis that the density is high enough (so that there are no virtual edges). Recall that $\tilde{G}\prime$ is always a subgraph of a Unit Hexagonal Graph (c.f. [BGHI10]), and hence a planar spanner for such a graph.

V. A LOCAL ROUTING ALGORITHM

We now propose a local routing algorithm. This algorithm has two modes depending on whether the destination is in a greedy region of the sender or not.

Lemma 8. *We assume that the anchors form an equilateral triangle. Let x be a node with a message for y. If y belongs to a greedy region of x, and if x has an out-neighbor in this greedy region, the algorithm proceeds as follow:*

- **(Data delivery)** $|xy| \leq r$*: x sends the message to y.*
- **(Greedy routing)** $|xy| > r$*: x sends the message to its neighbor x' that belongs to the same greedy region as y. We have $|xy| \geq |x'y|$.*

Proof: If x transmits directly to y there is nothing to prove. For the other case, let x' be the node that receives the message. We note the vector $xy = a\mathrm{e}^{i\alpha_1}$ and $xx' = b\mathrm{e}^{i\alpha_2}$ ($|xy| = a$ and $a \geq b$). The vector $x'y = a\mathrm{e}^{i\alpha_1} - b\mathrm{e}^{i\alpha_2}$, $|x'y| = a^2 + b^2 - 2ab\cos(\alpha_1 - \alpha_2)$ and, $0 \leq \alpha_1, \alpha_2 \leq \alpha$ because x' and y belong to the same greedy region. Then, $-\alpha \leq \alpha_1 - \alpha_2 \leq \alpha$ and $\cos(\alpha_1 - \alpha_2) > 1/2$, so $|x'y| \leq |xy| = a$. ■

The next lemma deals with the situation where the destination node is not in a greedy region of x. In this case, x has no out-neighbor in the region where y is located. However, we prove that there is a path that zig-zag around this region and finishes inside. The node at which the path ends is closer to y than x which ensures progress.

Lemma 9. *We assume that the anchors form an equilateral triangle. Let x be a node with a message for y. We assume that y does not belong to a greedy region of x. Without loss of generality, y is in $\bar{A}_2^x$. We further assume that x has three out-neighbors and so do any node in the equilateral triangle T with base the segment parallel to (A_2A_3), centered at x, of length $4/\sqrt{3}r$ and with the other summit in $\bar{A}_2^x$ (c.f. Figure 6). Under those hypothesis, we have two paths (without consideration on the orientation of the edges) $P_1 = x, u_0, P_1^0, u_1, ..., u_{k-1}, P_1^{k-1}, u_k, z$ and $P_2 = x, v_0, P_2^0, v_1, ..., v_{l-1}, P_2^{l-1}, v_l, z$, without loss of generality $u_0 >_1 v_0$ (when $P_2 \neq x, z$, as if $P_2 = x, z$, there are no v_0), and we have:*

- *$k - 1 \leq l \leq k$.*
- *the P_i^j, for $i \in \{1, 2\}$ and $j \leq k$ are monotone paths with respect to $>_1$, potentially of length 0.*
- *$\forall u \in P_1 \setminus \{z\},\ u \in A_2^x$.*
- *$\forall v \in P_2 \setminus \{z\},\ v \in A_3^x$.*
- *$\forall 0 \leq i \leq l$, there is an oriented edge from u_i to v_i and, $\forall u \in \{u_i,\ P_1^i\}$, $u >_1 v_i$.*
- *$\forall 0 \leq i \leq l-1$, there is an oriented edge from v_i to u_{i+1} and, $\forall v \in \{v_i,\ P_2^i\}$, $v >_1 v_{i+1}$.*
- *z is in $\bar{A}_2^x$.*

Given these two paths, either a node from $\{x, u_0, ...u_k, v_0, ...v_l, z\}$ has y within its communication range, or $|zy| < |xy|$.

Proof: The proof can be found in [HJLR11a]. ■

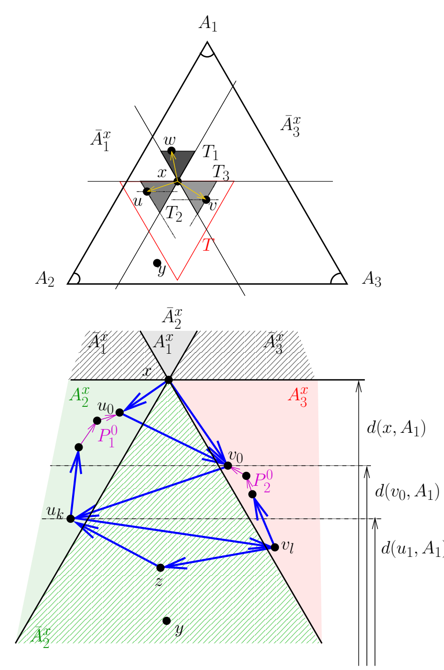


Fig. 6. On the top figure, the three greedy regions associated with x and the edges to the three nodes $u = min_2\{z \mid x \tilde{<}_2 z\}$, $v = min_3\{z \mid x \tilde{<}_3 z\}$ *and,* $w = min_1\{z \mid x \tilde{<}_1 z\}$. On the bottom figure, a path leading to z.

Theorem 2 (Zig-Zag : an extended greedy routing)**.** *We assume that the anchors form an equilateral triangle. Let x be a node with a message for y. If each node at distance less than $4/\sqrt{3}r$ of x has three outgoing* *neighbors*[8], *then the following strategy delivers the data either to y or to a node z closer to y than x.*

- *If y is in the communication range of x, x sends the message to y.*
- *If y is in a greedy region of x, x sends the message to its out-neighbor which is in the same greedy region.*
- *Otherwise, use the restricted greedy routing process starting at x. Wlog $y \in \bar{A}_2^x$.*
  - *x sends the message to its out-neighbor in $A_2^x \cup A_3^x$ which has the highest first coordinate.*
  - *A node $u \in A_2^x$ sends the message to y if possible or to its out-neighbor v satisfying $u >_1 v$ and $v >_3 x$. If $v >_3 x$ and $v >_2 x$ (i.e., $u \in \bar{A}_2^x$), end the restricted greedy routing process.*
  - *A node $v \in A_3^x$ sends the message to y if possible or to its out-neighbor u satisfying $v >_1 u$ and $u >_2 x$. If $u >_3 x$ and $u >_2 x$ (i.e., $u \in \bar{A}_2^x$), end the restricted greedy routing process.*

Proof: By applying Lemmas 8 and 9 we see that the routing strategy leads to y or to a node that is closer to y than x (z in Lemma 9). Indeed, Lemma 9 ensures that a restricted greedy routing process starting at x follows the path $u_0, v_0, u_1, ..., z$ using the same notation. ■

VI. Simulations

We implemented both the planarization algorithm and Zig-Zag, the routing algorithm. We present below the results of the simulations. We consider a network composed of 300 sensors spread in a square area $[0;1] \times [0;1]$ with three anchors at position $(0.5, 3.5)$, $(-\frac{5}{\sqrt{3}} + 0.5, -1.5)$ and $(\frac{5}{\sqrt{3}} + 0.5, -1.5)$. We consider a communication radius r for the sensors which ranges from $r = 0.137$ to $r = 0.225$. For each value of the communication radius, we performed the average over 10000 networks and successfully routed messages. Notice that values are plotted with respect to the average degree of the nodes in the UDG, and not in the planar graph $\tilde{G}\prime$ where the average degree is upper bounded by six. We compare the efficiency of our algorithms when the VRAC coordinates are exact to when some error is inserted: we either consider the nodes knowing their exact VRAC coordinates, or the nodes knowing their distances to the anchors with a random error of +-1%, +-2.5% and +-5%.

We first plot the number of virtual edges in Figure 7(a). The simulations indicate that the number of virtual edges converges to 0 when the node density increases, and this even when the coordinates are inaccurate. The number of virtual edges remains limited when the error is of less than 5%. When it is of 10%, we have an average of 90 virtual edges out of 1200 edges[9]. When there are no error, we observe that the average number of virtual edges decreases from 1.3 when the network is sparse, to 0.02 (out of ≈ 3500 edges) when it is dense.

In Figure 7(b), we plot the average stretch of the path computed in $\tilde{G}\prime$ by Zig-Zag. We observe that we have a stretch which is between 1.3 and 1.4 which is better than the theoretical stretch factor of 2 when the coordinates are exact. Introducing errors on the coordinates does not degrade this too much.

In Figure 7(c), we plot the average success rate. We observe that it tends to one when the communication radius increases. Indeed in this case, the hypothesis of Theorem 2 are verified.

In Figure 7(d), we plot the number of intersections that occur on the computed graph. As expected, there are no intersection when we use the exact coordinates. When we add 2% of errors, we witness few intersections (less than one every 1000 networks) and this increases as the error introduced increases. However, it still remains low. We measure only an average of 1.8 intersections (this is outside Figure 7(d)) per network with 10% of errors when the networks are sparse (r=0.137, which is the worst case). This show that our technique is highly resilient to measurement errors.

VII. Conclusion

Using the VRAC coordinates system, we propose an efficient planarization algorithm, which from an UDG G, computes a planar graph $\tilde{G}\prime$ which is a 2-spanner of G when the density is high enough or more precisely when there are no virtual edges. We further propose a routing algorithm dedicated to this planar graph and which guarantees delivery when the out-degree of the nodes involved in the routing is three.This is a first contribution towards devising a routing algorithm that guarantees delivery in the spirit of GPSR or GFG. Special attention will be given to the error on the nodes' coordinates.

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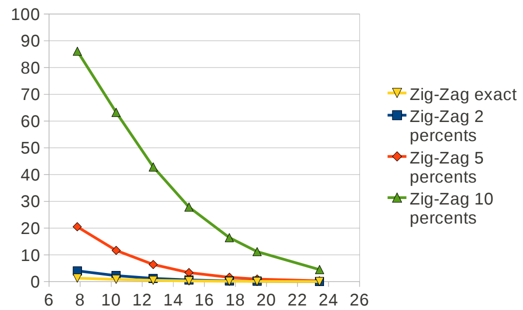
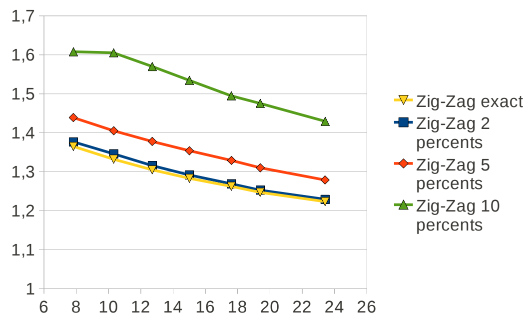
[8]This assumption amounts to require that the routing algorithm is building a path in a region where the graph is triangular. The motivation for this routing algorithm comes from the results in [Dha08] that there exists a greedy embeddings of a triangular graph. We do not assume that **all** the nodes have three out-neighbors because this cannot be true in the border of the equilateral triangle formed by the anchors.

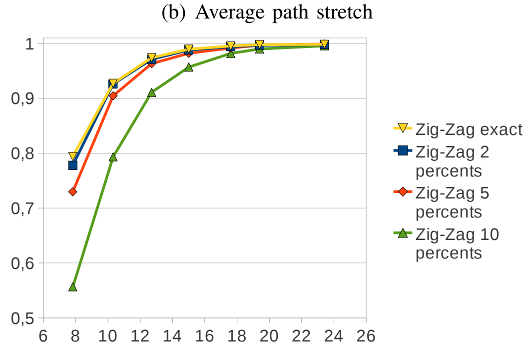
[9]There are 300 nodes, so the number of edges is $300 * degree/2$, which is 1200 for a degree of approximately 8



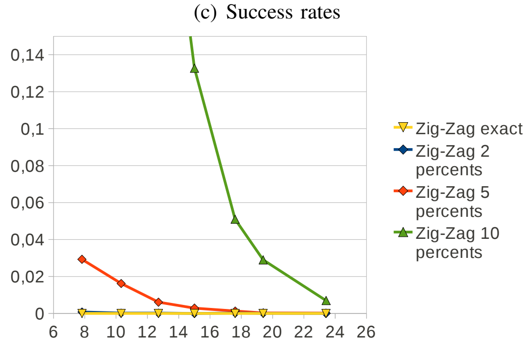
(a) Average number of virtual edges



(b) Average path stretch



(c) Success rates



(d) Number of intersecting edges

Fig. 7. Simulations

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