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***Abstract*—This paper considers a Cognitive Radio (CR) channel composed of a secondary user (SU) and a primary user (PU). An analysis of the power minimization over several orthogonal frequency bands at the SU level under the following constraints is provided: a minimum Quality of Service (QoS) constraint, maximum peak and average interference to the PU constraints. The general solution, when it exists, is a water-filling type of solution which can be computed via iterative algorithms. It turns out that, in the case of two orthogonal bands a closed-form analytical solution can be found and a complete analysis of the feasibility of these opposing constraints is presented in details. Several numerical results that sustain and give inside into the analysis are also discussed.**

***Index Terms*—Cognitive Radio channels, power-efficient spectrum allocation problems, green communications**

I. Introduction

The concept of Cognitive Radio [1] has recently emerged as a promising paradigm for a more efficient use of the available spectrum by allowing the coexistence of licensed (primary) and unlicensed (secondary) users in the same bandwidth. Another major concern that is rapidly gaining momentum is the energy consumption for green communications. Recently, there has been an important switch of focus from a data rate maximization vision to an application-based utility optimization one, that takes into account the cost of power consumption to achieve the target data rates [2], [3].

In this paper, we consider a home-automation scenario where several technologies (i.e., WiFi, PLC, Femto, etc.) are able to operate simultaneously and where different appliances may have different hierarchical priorities. The EconHome-box is meant to allocate frequency bands and powers in a green telecommunications sprint. Notice that the analysis we provide is not restricted to this scenario alone and can be applied to any CR system with several orthogonal sub-channels available for communication: OFDM systems, interoperability of several coexisting transmission systems.

To model this home-automation scenario we consider a CR channel composed of a primary user (PU) and a secondary user (SU) that is opportunistically accessing the channel and interfering with the PU. The question under investigation is to know how the SU will optimally allocate the available power over the available bands to minimize its total power consumption under several constraints: target QoS constraint, peak and average maximal interference levels that the SU can create to the PU.

The problem of resource allocation has mainly been studied from a data rate maximization point of view. For exemple, in [4], [5], [6] the authors considered the maximization of the Shannon achievable rate over several sub-channels, subject to transmit powers and mask constraints on each sub-channel. In [4], a centralized approach based on convex optimization (see e.g., [7], [8]) was proposed to compute, under some technical conditions, the achievable rate region of the system.

To the best of the authors' knowledge, the closest works to ours are [9] and [10]. In [10], the authors consider the distributed minimization of transmit powers subject to rate target constraints for the multi-user parallel interference channels. The difference to our work consists in the absence of PU and thus, of the interference constraints that limit the SU transmissions'. In [9], the authors study the multi-users multiple-input multiple-output (MIMO) Cognitive Radio channels. In order to design cognitive MIMO SU transceivers, the authors of [9] model the distributed maximization of the Shannon achievable rates under interference constraints imposed by the presence of the PU. In order to approach this complex problem, the authors study several particular cases by systematically eliminating one of the interference constraints and solving a simpler problem. In this paper, we consider a simpler channel model (the single-SU case and the multiple

This work was sponsored in part by the FUI SYSTEM@TIC Project EconHome.

orthogonal sub-channels). However, the novelty and interest of our work consists in the fact that we study the dual problem of [9] in its generality with respect to the feasibility constraints.

Our contributions can be summarized as follows: i) in the general case of arbitrary number of orthogonal bands, we give the analytical water-filling solution to the power minimization problem under minimum rate and maximum peak and average interference constraints created to the PU, when it exists; ii) in the case of two orthogonal frequency bands, we give an analytical closed-form solution and a complete study of the feasibility of all the constraints is provided; iii) numerical simulations that sustain the analysis are provided for different channel parameters.

The remainder of this paper is organized as follows. In section II, we describe the system model. We provide the water-filling type of solution of the general optimization problem section III. A closed-form solution in the case of two orthogonal bands is given in section IV. In section V, we present some interesting simulation results. We conclude this paper in sectionVI.

II. System Model

The CR channel under study system is depicted in Fig.1. The primary/secondary user consists of a Primary/Secondary Transmitter (PT/ST) and a Primary/Secondary Receiver (PR/ST) respectively. Each transmitter or receiver is equipped with only one antenna. The transmission is done over $N \geq 2$ orthogonal frequency bands. The transmit power of ST in the frequency band $k \in \{1, ...N\}$ is denoted by p_k. We denote the overall power allocation profile by $\underline{p} = (p_1, p_2, ..., p_N)$, $\underline{p} \in \mathbb{R}_+^N$.

The received signal at SR in band k can be written as:

$$y_k = \sqrt{p_k} h_k s_k + i_{k,PU} + n_k, \tag{1}$$

where h_k represents the channel gain, s_k the transmit signal of the ST is given by unit-power variable over each orthogonal frequency band. The instantaneous power gain of the ST-SR direct link, the interfering ST-PR link is denoted by are denoted by $|h_k|^2$, g_k respectively. All the channels are assumed to be stationary, ergodic and independent from the noises. The noise $n_k \backsim \mathcal{N}(0, \sigma_k^2)$ is a zero-mean circularly symmetric complex Gaussian noise vector and the interfering signal from the PU $i_{k,PU} \backsim \mathcal{N}(0, \sigma_{k,PU}^2)$.

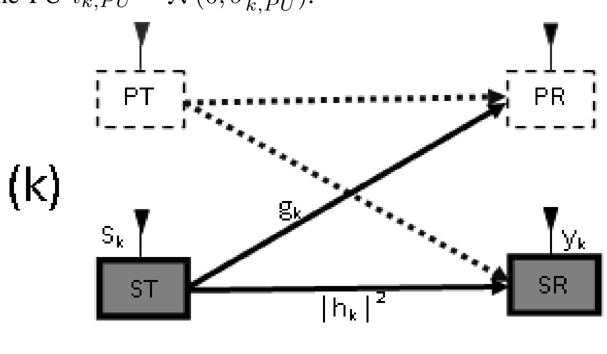


Fig. 1. Channel model for the communication in band k.

In this scenario, the Shannon achievable rate of the SU is given by

$$R(\underline{p}) = \sum_{k=1}^{N} \log_2(1 + c_k p_k), \tag{2}$$

where c_k is related to the interfering ST-PR link by

$$c_k = \frac{|h_k|^2}{\sigma_k^2 + \sigma_{k,PU}^2} \tag{3}$$

where $\sigma_{k,PU}^2$ is the interfering variance from the PU and σ_k^2 is the variance of the thermal noise.

III. General optimization problem

We consider the constrained power minimization problem at the SU level. Three constraints are considered in our optimization problem: a) a QoS constraint (4); b) peak (6) and average (5) interference constraints to protect the PU and limit the interference caused by the SU.

1— *Target rate constraint:*

$$R(\underline{p}) = \sum_{k=1}^{N} \log_2(1 + c_k p_k) \geq R_{min}, \tag{4}$$

which is imposed to achieve a minimum QoS of the SU transmission.

2— *Average interference power shaping constraint:*

$$\sum_{k=1}^{N} g_k p_k \leq \overline{P}, \tag{5}$$

where $\overline{P}$ is the maximum average interference level that can be received at PR. This constraint is an imposed interference limitation at the PR.

3— *Peak interference power shaping constraints:*

$$0 \leq g_k p_k \leq P_k^{peak}, \quad \forall\ k = 1, \dots N, \tag{6}$$

where P_k^{peak} is the maximum peak interference level that can be received at PR in a given band.

The optimization problem under study can be written as:

$$\begin{cases} \min\limits_{\{p_k\}_{k=1}^N} \quad \sum\limits_{k=1}^{N} p_k \\ \text{s.t.} \quad R(\underline{p}) = \sum\limits_{k=1}^{N} \log_2(1 + c_k p_k) \geq R_{min} \\ \qquad \sum\limits_{k=1}^{N} g_k p_k \leq \overline{P} \\ \qquad 0 \leq g_k p_k \leq P_k^{peak} \quad \forall\ k \in \{1, ..N\} \end{cases} \tag{7}$$

Notice that without loss of generality, we can consider only the case where $R_{min} > 0$. In the case where $R_{min} = 0$ the solution is trivial and the transmission powers are equal to zero, $p_k^* = 0\ \forall k$, i.e.,no transmission is performed at the SU.

In the case of applying a flat power allocation policy (i.e., $\forall k \in \{1,..N\}$ $p_k = p$), all the constraints become more strict and the feasible set is reduced. The constraints in this case will be given by $p \leq \min\left\{\frac{P_k^{peak}}{g_k}, \frac{\overline{P}}{\sum_k g_k}\right\}$ and $\sum_k \log(1+c_k p) \geq R_{min}$. Therefore, the benefit of applying our optimal allocation policy comparing to this flat power allocation policy, is to have much more flexibility in the feasibility of our constraints and a significant gain in terms of power consumption.

Theorem 1: If the feasible set of the optimization problem (7) is not void, the optimal power allocation policy is given by the following water-filling type solution [1]:

$$p_n^* = \left[\xi_n - \frac{1}{c_n}\right]_0^{\frac{P_n^{peak}}{g_n}}, \ \forall n \in \{1, \dots N\} \tag{8}$$

where $\xi_n = \frac{\lambda}{\ln(2)(1+\beta g_n)}$, *$\lambda$ and β verify the following conditions:*

$$\left\{\lambda > 0 \text{ and} \sum_{k=1}^{N} \log_2(1+c_k p_k^*) = R_{min}\right\}$$

and

$$\left\{\beta > 0 \text{ and } \sum_{k=1}^{N} g_k p_k^* = \overline{P}\right\} \text{ or } \left\{\beta = 0 \text{ and } \sum_{k=1}^{N} g_k p_k^* < \overline{P}\right\}.$$

If the feasible set is void then the problem (7) has no solution.

Proof: We denote the feasible set by $\mathcal{S}_N$ such that:

$$\mathcal{S}_N = \Big\{\underline{p} \in \mathbb{R}_+^N \mid 0 \leq g_k p_k \leq P_k^{peak}, \sum_{k=1}^{N} \log_2(1+c_k p_k) \geq R_{min}, \sum_{k=1}^{N} g_k p_k \leq \overline{P}\Big\}. \tag{9}$$

Two different scenarios are possible:

- $\mathcal{S}_N = \varnothing$: In this case, there is no feasible power allocation policy, i.e., no vector $\underline{p} \in \mathbb{R}_+^N$ that satisfies simultaneously all the constraints in (9).
- $\mathcal{S}_N \neq \varnothing$: Given the concavity of $\log(.)$ function, the linearity of powers constraints and the convexity of the objective function $\sum_{k=1}^{N} p_k$, it is easy to prove that $\mathcal{S}_N$ is convex [8], i.e., $\forall\ \widehat{p}_k, \tilde{p}_k \in \mathcal{S}_N,\ \ \forall\ \theta \in [0,1]$ we have

$$\theta \widehat{p}_k + (1-\theta)\ \tilde{p}_k \in \mathcal{S}_N. \tag{10}$$

Thus, the optimization problem in (7) is a convex problem which has at least an optimal solution. Therefore, we can apply the Lagrangian method to solve (7).
The Lagrangian is given by:

$$\begin{aligned}\mathcal{L}(\underline{p}, \lambda, \beta, \underline{\alpha}, \underline{\mu}) &= \sum_{k=1}^{N} p_k - \lambda\left(\sum_{k=1}^{N} \log_2(1+c_k p_k)\right. \\ &\left.-R_{min}\right) + \beta\left(\sum_{k=1}^{N} g_k p_k - \overline{P}\right) - \sum_{k=1}^{N} \alpha_k g_k p_k \\ &+ \sum_{k=1}^{N} \mu_k \left(g_k p_k - P_k^{peak}\right),\end{aligned} \tag{11}$$

where λ, β, $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_N)$ and $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_N)$ are the non-negative Lagrangian multipliers of the corresponding constraints (4)(5)(6), respectively. We denote $\underline{p}^*$ an optimal solution.
The *Karush-Kuhn-Tucker* (KKT) optimality conditions are:

i) Compute the solution that gives null Lagrangian derivative: $\left.\frac{\partial \mathcal{L}}{\partial p_n}\right|_{p_n = p_n^*} = 0$;
ii) Non-negative Lagrangian multiplier of the rate constraint $\lambda > 0$ and $\sum_{k=1}^{N} \log_2(1+c_k p_k^*) = R_{min}$ or $\lambda = 0$ and $\sum_{k=1}^{N} \log_2(1+c_k p_k^*) > R_{min}$;

The case of null Lagrangian multiplier of the rate constraint cannot occur, because if $\lambda = 0$ then the Lagrangian in (11) becomes:

$$\begin{aligned}\mathcal{L}(\underline{p}, \beta, \underline{\alpha}, \underline{\mu}) &= \sum_{k=1}^{N} p_k + \beta\left(\sum_{k=1}^{N} g_k p_k - \overline{P}\right) \\ &- \sum_{k=1}^{N} \alpha_k g_k p_k + \sum_{k=1}^{N} \mu_k \left(g_k p_k - P_k^{peak}\right),\end{aligned}$$

and the optimization problem in (7) is given by:

$$\left\{\begin{array}{l} \min\limits_{\{p_k\}_{k=1}^N} \ \sum_{k=1}^{N} p_k \\ \text{s.t. } \sum_{k=1}^{N} g_k p_k \leq \overline{P} \\ \quad 0 \leq g_k p_k \leq P_k^{peak} \quad \forall\ k \in \{1,..N\} \end{array}\right.$$

In this power optimization problem the optimal power solution $\underline{p}^*$ is null. Then the SU optimal rate is null also (i.e., $\sum_{k=1}^{N} \log_2(1+c_k p_k^*) = 0$), this is

[1] We denote by $[x]_a^b = \min\{\max\{x,a\}, b\}$ or equivalently

$$[x]_a^b = \begin{cases} x, & \text{if } \ a \leq x \leq b \\ a, & \text{if } \ x < a \\ b, & \text{if } \ x > b. \end{cases}$$

contradictory to the QoS constraint which is strictly positive ($R_{min} > 0$). Thus, the Lagrangian multiplier of the rate constraint is strictly positive and given by: $\lambda > 0$ and $\sum_{k=1}^{N} \log_2(1 + c_k p_k^*) = R_{min}$

iii) Non-negative Lagrangian multiplier of the average interference power constraint $\beta > 0$ and $\sum_{k=1}^{N} g_k p_k^* = \overline{P}$ or $\beta = 0$ and $\sum_{k=1}^{N} g_k p_k^* < \overline{P}$;

iv) Non-negative Lagrangian multiplier of the peak powers constraints for each band:

$$\alpha_n > 0 \text{ and } g_n p_n^* = 0 \text{ or } \alpha_n = 0 \text{ and } g_n p_n^* > 0$$

$$\mu_n > 0 \text{ and } p_n^* = \frac{P_n^{peak}}{g_n} \text{ or } \mu_n = 0 \text{ and } p_n^* < \frac{P_n^{peak}}{g_n}.$$

As we said before, in order to compute the optimal solution of our problem we derive the Lagrangian in (11),

$$\left.\frac{\partial \mathcal{L}}{\partial p_n}\right|_{p_n = p_n^*} = 0,$$

$$\Leftrightarrow \quad 1 - \frac{\lambda c_n}{\ln(2)(1 + c_n p_n^*)} + (\beta - \alpha_n + \mu_n) g_n = 0,$$

$$\Leftrightarrow \quad \frac{\lambda c_n}{\ln(2)(1 + c_n p_n^*)} = 1 + (\beta - \alpha_n + \mu_n) g_n,$$

$$\Leftrightarrow \quad (1 + c_n p_n^*) = \frac{\lambda c_n}{\ln(2)(1 + (\beta - \alpha_n + \mu_n) g_n)},$$

$$\Leftrightarrow \quad p_n^* = \frac{\lambda}{\ln(2)(1 + (\beta - \alpha_n + \mu_n) g_n)} - \frac{1}{c_n}.$$

Considering the other KKT conditions, the solution in function of the optimal (α_n,μ_n) is as follows:

- ⋆ $\alpha_n = 0$ and $\mu_n = 0$ (the optimal power solution is strictly [2] between 0 and P_n^{peak}/g_n)

$$p_n^* = \left(\frac{\lambda}{\ln(2)(1 + \beta g_n)} - \frac{1}{c_n} \right)_0^{\frac{P_n^{peak}}{g_n}}, \tag{12}$$

- ⋆ $\alpha_n = 0$ and $\mu_n > 0$,

$$p_n^* = \frac{P_n^{peak}}{g_n}, \tag{13}$$

- ⋆ $\alpha_n > 0$ and $\mu_n = 0$,

$$p_n^* = 0. \tag{14}$$

Finally, the equations (12), (13) and (14) can be re-written in a compact form as follows:

$$p_n^* = \left[\frac{\lambda}{\ln(2)(1 + \beta g_n)} - \frac{1}{c_n} \right]_0^{\frac{P_n^{peak}}{g_n}}, \tag{15}$$

where λ and β verify the following conditions:

$$\left\{ \lambda > 0 \text{ and } \sum_{k=1}^{N} \log_2(1 + c_k p_k^*) = R_{min} \right\}$$

$$\left\{ \beta > 0 \text{ and } \sum_{k=1}^{N} g_k p_k^* = \overline{P} \right\}$$

$$\text{or } \left\{ \beta = 0 \text{ and } \sum_{k=1}^{N} g_k p_k^* < \overline{P} \right\}.$$

Thus, by making a change of variable $\xi_n = \frac{\lambda}{\ln(2)(1 + \beta g_n)}$, we obtain the analytical solution of the problem (7), which is expressed by the well known water-filling:

$$p_n^* = \left[\xi_n - \frac{1}{c_n} \right]_0^{\frac{P_n^{peak}}{g_n}}. \tag{16}$$

When the feasible set is void, one or more constraints need to be relaxed if the system owner wishes to allow a non-trivial secondary transmission. ■

IV. The case of two Orthogonal frequency bands

In this section, we focus on the specific case where only two orthogonal frequency bands are available, i.e., $N = 2$. This is motivated by the fact that an analytical study of the feasible set in the general scenario (9) is very difficult. As we will show, this difficulty is overcome in this case. We provide a complete and closed-form analysis of the feasibility of the opposing constraints: the QoS constraint on one hand, and the power constraints on the other. Moreover, this two bands case models the systems where two different technologies can co-exist, i.e., (WiFi+PLC), or (WiFi+Femto), which is pertinent in current home-automation systems. Because these technologies are inherently different or heterogeneous, it is realistic to assume the orthogonality property of the sub-channels.

In what follows, we focus on the simpler case of $N = 2$ for the following reasons: a) it can be realistic in several scenarios among which we have mentioned here above two of them; b) it can be solved in an analytical closed-form way; c) it gives insight and intuition on what happens in a general system with $N > 2$ sub-channels which can be solved only numerically.

[2] The notation $(x)_a^b$ is defined by:

$$(x)_a^b = \begin{cases} x, & \text{if } a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the optimization problem in (7) can be reformulated as:

$$\begin{aligned} &\min\{p_1 + p_2\} \\ &\text{s.t } \underline{p} \in \mathcal{S}_2, \end{aligned} \tag{17}$$

where, the feasible set $\mathcal{S}_2$ is given by

$$\mathcal{S}_2 = \Big\{ \underline{p} \in \mathbb{R}_+^2 \mid 0 \le g_1 p_1 \le P_1^{peak}, 0 \le g_2 p_2 \le P_2^{peak}, \\ \sum_{k=1}^{2} \log_2(1 + c_k p_k) \ge R_{min}, \ \sum_{k=1}^{2} g_k p_k \le \overline{P} \Big\}.$$

Theorem 2: In the case of two orthogonal frequency bands, when a solution to (17) exists, it is unique and its closed-form solution is given here under.

If $\overline{P} < -\dfrac{g_1}{c_1} - \dfrac{g_2}{c_2} + 2\sqrt{(\dfrac{g_1}{c_1}\dfrac{g_2}{c_2}) \exp(R_{min})}$ *or otherwise if* $\max\{\widehat{p_1}, \widetilde{p_1}\} < 0$ *then the constraints (4) and (5) can not be simultaneously satisfied. Also if at least one of the following conditions is met:* $\dfrac{P_1^{peak}}{g_1} < \widehat{p_1}$ *or* $\dfrac{P_2^{peak}}{g_1} < \min\{\widehat{p_2}, \widetilde{p_2}\}$ *or* $\left\{ f_1(\dfrac{P_1^{peak}}{g_1}) > P_2^{peak} \text{ and } f_1^{-1}(\dfrac{P_2^{peak}}{g_2}) > \dfrac{P_1^{peak}}{g_1} \right\}$ *then constraint (6) can not be satisfied. In all the aforementioned cases, a feasible solution does not exist. In all other possible scenarios an optimal solution exists and is given by*

$$\begin{cases} p_1^* &= \left[\dfrac{-1}{c_1} + \sqrt{\dfrac{\exp(R_{min})}{c_1 c_2}} \right]_x^y, \\ p_2^* &= \left[\dfrac{-1}{c_2} + \sqrt{\dfrac{\exp(R_{min})}{c_1 c_2}} \right]_w^z, \end{cases}$$

where x, y, w, z *are given by*

$$\begin{cases} x &= \max\left\{ \max\{\widehat{p_1}, 0\}, f_1^{-1}\left(\dfrac{P_2^{peak}}{g_2}\right) \right\}, \\ y &= \min\left\{ \min\{\widetilde{p_1}, f_1^{-1}(0)\}, \dfrac{P_1^{peak}}{g_1} \right\}, \\ w &= \max\left\{ \max\{\widetilde{p_2}, 0\}, f_1\left(\dfrac{P_1^{peak}}{g_1}\right) \right\}, \\ z &= \min\left\{ \min\{\widehat{p_2}, f_1(0)\}, \dfrac{P_2^{peak}}{g_2} \right\}. \end{cases} \tag{18}$$

The parameters $\widehat{\underline{p}}$, $\widetilde{\underline{p}}$ *are the intersection points of the functions* $f_1(.)$ *and* $f_2(.)$ *defined as:* $f_1 : \mathbb{R} \to \mathbb{R}$ *(derived from the rate constraint(4)) such that:*

$$f_1(p_1) = \frac{1}{c_2}\left(\frac{\exp(R_{min})}{1 + c_1 p_1} - 1 \right) \tag{19}$$

and $f_2 : \mathbb{R} \to \mathbb{R}$ *(derived from the average interference power constraint(5)) such that:*

$$f_2(p_1) = \frac{\overline{P} - p_1 g_1}{g_2}. \tag{20}$$

Proof:

In order to solve the optimization problem (17), we rewrite the feasible set using the two functions defined in (19) and (20) in the following manner:

$$\begin{aligned} \mathcal{S}_2 &= \mathcal{A} \bigcap \mathcal{B} \\ \mathcal{A} &= \{(p_1, p_2) \in \mathbb{R}^2 \mid f_1(p_1) \le p_2, \\ &\quad\ f_2(p_1) \ge p_2\} \\ \mathcal{B} &= \left[0, \frac{P_1^{peak}}{g_1}\right] \times \left[0, \frac{P_2^{peak}}{g_2}\right]. \end{aligned} \tag{21}$$

The set $\mathcal{B}$ represents simply the peak interference constraints in each band which restricts the feasible set $\mathcal{S}_2$. Notice that the set $\mathcal{B}$ is not void, however the set $\mathcal{A}$ can be void.

In order to find the conditions under which the set $\mathcal{S}_2$ is void or not, i.e., a solution exists or not, we have first to discuss the set $\mathcal{A}$. Thus, we analyse the feasible set $\mathcal{A}$ in the plane $p_1 \mathcal{O} p_2$, where $\mathcal{O}$ is the origin point of coordinates (0,0) and p_1 and p_2 are the axes of our plane. In order to do that, we investigate the intersection points $(p_1, p_2) \in \mathcal{A}$, of the functions $f_1(.)$ and $f_2(.)$ by solving the system of equations:

$$\begin{cases} p_2 &= \dfrac{1}{c_2}\left(\dfrac{\exp(R_{min})}{1 + c_1 p_1} - 1 \right), \\ p_2 &= \dfrac{\overline{P} - g_1 p_1}{g_2}, \\ p_1 &\in \left[0, \dfrac{P_1^{peak}}{g_1}\right], \end{cases} \tag{22}$$

We further have:

$$\begin{cases} \dfrac{\overline{P} - g_1 p_1}{g_2} = \left(\dfrac{1}{c_2}\right)\left(\dfrac{\exp(R_{min})}{1 + c_1 p_1} - 1 \right), \\ p_1 \in \left[0, \dfrac{P_1^{peak}}{g_1}\right], \end{cases} \tag{23}$$

$$\Leftrightarrow \left(\frac{\overline{P}}{g_2} - \frac{g_1}{g_2} p_1 \right) = \frac{1}{c_2}\frac{\exp(R_{min})}{(1 + c_1 p_1)} - \frac{1}{c_2},$$

$$\Leftrightarrow \left(\overline{P} c_2 - g_1 c_2 p_1 + g_2 \right)(1 + c_1 p_1) = g_2 \exp(R_{min}).$$

We obtain a quadratic equation w.r.t p_1

$$\begin{aligned} -g_1 c_2 c_1 p_1^2 + \left(\overline{P} c_1 c_2 - g_1 c_2 + g_2 c_1 \right) p_1 \\ + \overline{P} c_2 + g_2(1 - \exp(R_{min})) = 0. \end{aligned} \tag{24}$$

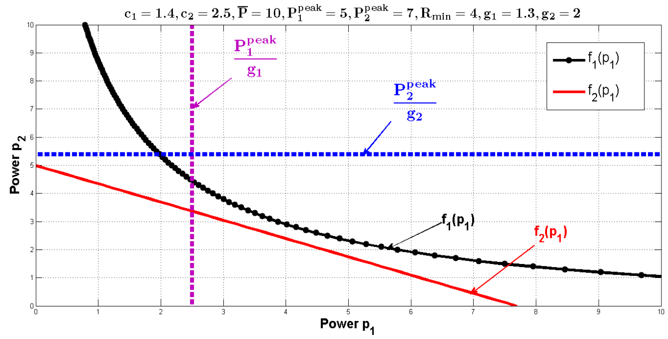
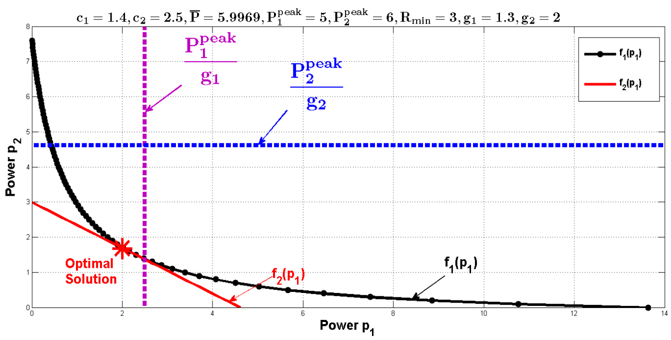


Fig. 2. No Solution because the set $\mathcal{A} = \varnothing \Rightarrow \mathcal{S}_2 = \varnothing$

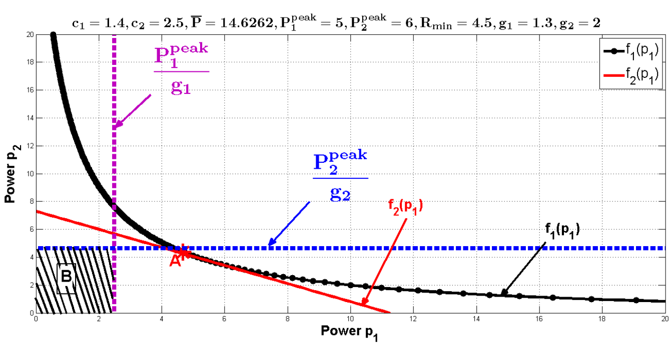
Thus, we obtain in (24) a quadratic equation where its discriminant is given by

$$\Delta_{p_1} = \left(\overline{P}c_1c_2 - g_1c_2 + g_2c_1\right)^2$$
$$+4g_1c_2c_1(\overline{P}c_2 + g_2 - g_2\exp(R_{min})). \qquad (25)$$

(a) If $\Delta_{p_1} < 0$ we have that $f_1(p_1) > f_2(p_1)$ $\forall p_1 \in \mathbb{R}$, this means that the functions $f_1(.)$ and $f_2(.)$ do not intersect i.e the two functions in this case are disjoint (Fig.2) and the set $\mathcal{A} = \varnothing$. Thus, $\mathcal{S}_2 = \varnothing$ and we have no solution.



(a) One Solution: $\mathcal{S}_2 = \{(p_1^*, p_2^*)\}$.



(b) No Solution because $\mathcal{A} = \{(\widehat{p_1}, \widehat{p_2})\} \cap \mathcal{B} = \varnothing$

Fig. 3. $\Delta_{p_1} = 0$ and the function $f_2(.)$ is tangent to the function $f_1(.)$ where (a) : One Solution, (b): No Solution

(b) If $\Delta_{p_1} = 0$, then another equation in the second degree is required. To find $\overline{P}$ in this case we calculate the discriminant $\Delta_{\overline{P}}$, which is given by

$$\Delta_{\overline{P}} \;=\; 16c_1^3c_2^3g_1g_2\exp(R_{min}) > 0. \qquad (26)$$

Thus, we obtain two solutions:

$$\overline{P}_{1,2} = -\frac{g_1}{c_1} - \frac{g_2}{c_2} \pm 2\sqrt{\frac{g_1g_2}{c_1c_2}\exp(R_{min})}. \qquad (27)$$

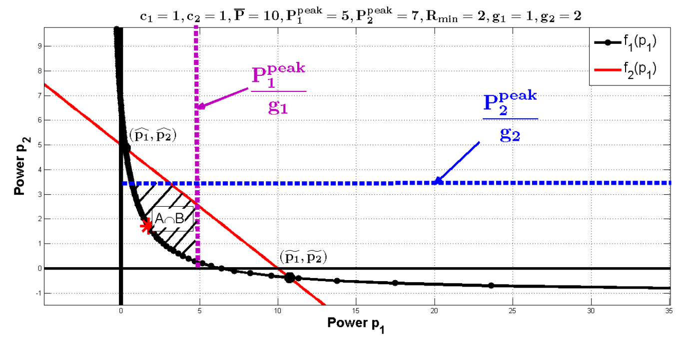
Note that one of the two solutions is negative, so it is rejected. Thus, the constraint of average interference powers must satisfy this condition:

$$\overline{P} = -\frac{g_1}{c_1} - \frac{g_2}{c_2} + 2\sqrt{\frac{g_1g_2}{c_1c_2}\exp(R_{min})}. \qquad (28)$$

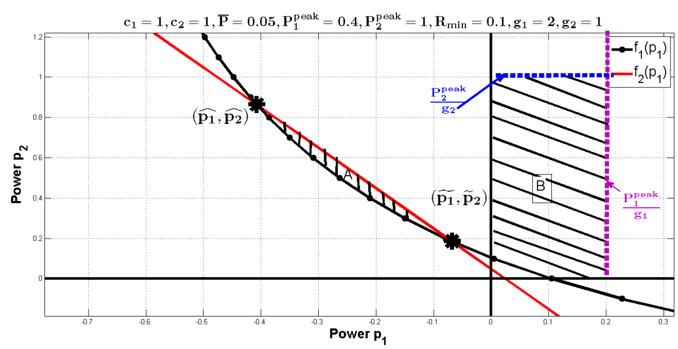
Thus the intersection point in this case ($\Delta_{p_1} = 0$) where $f_1(.)$ is tangent to $f_2(.)$, is unique $\underline{\widehat{p}} = \underline{\widetilde{p}}$ with

$$\begin{cases} \widehat{p_1} & = & \dfrac{\overline{P}}{2g_1} \\ \widehat{p_2} & = & \dfrac{\overline{P}}{2g_2} \end{cases} \qquad (29)$$

Note that in this case $\mathcal{A} = \{(\widehat{p_1}, \widehat{p_2})\}$ and if this singleton is in $\mathcal{B}$ (Fig.3. (a)), it is the solution of (17) i.e., $\mathcal{S}_2 = \{(\widehat{p_1}, \widehat{p_2})\}$ and $\underline{p}^* = \underline{\widehat{p}}$. However, if $\{(\widehat{p_1}, \widehat{p_2})\} \notin \mathcal{B}$ see Fig.3.(b), then, we have no solution and $\mathcal{S}_2 = \varnothing$.



(a) If $\widetilde{p_1} < 0$ and $\widehat{p_1} > 0$



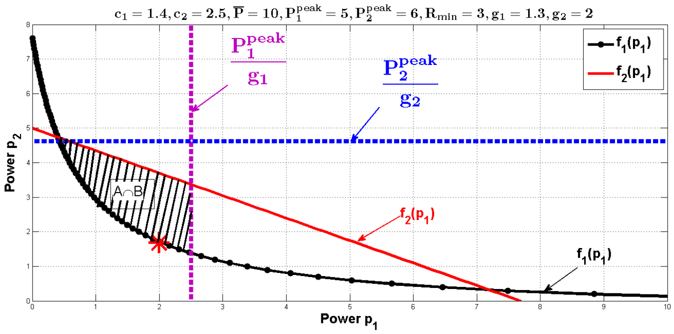
(b) No solution: if $max\{\widehat{p_1}, \widetilde{p_1}\} < 0$ because $\mathcal{A} \cap \mathcal{B} = \varnothing$

Fig. 4. Particular cases: either (a): One solution or (b):No solution

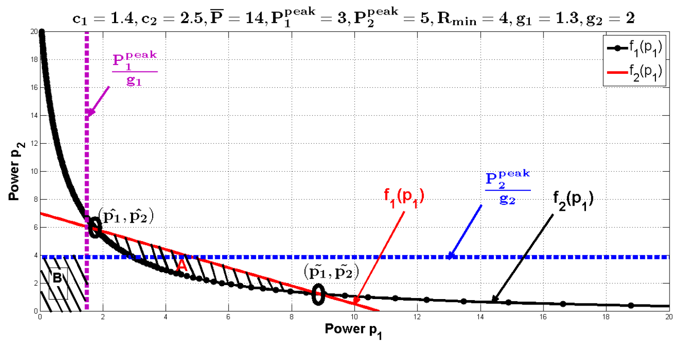
(c) If $\Delta_{p_1} > 0$ which is equivalent to $\overline{P} > -\frac{g_1}{c_1} - \frac{g_2}{c_2} + 2\sqrt{\frac{g_1g_2}{c_1c_2}\exp(R_{min})}$, then we have two distinct intersection points (see Fig.4 and Fig.5) $(\widehat{p_1}, \widehat{p_2}), (\tilde{p_1}, \tilde{p_2}) \in \mathcal{A}$, where

$$\widehat{p_1} = \frac{(\overline{P}c_1c_2 - g_1c_2 + g_2c_1) - \sqrt{\Delta_{p_1}}}{2g_1c_1c_2},$$
$$\widetilde{p_1} = \frac{(\overline{P}c_1c_2 - g_1c_2 + g_2c_1) + \sqrt{\Delta_{p_1}}}{2g_1c_1c_2}, \tag{30}$$

where Δ_{p_1} is expressed in (25), $\widehat{p_1} < \widetilde{p_1}$, $\widehat{p_2} = f_1(\widehat{p_1})$ and $\widetilde{p_2} = f_1(\widetilde{p_1})$.



(a) One Solution: if $\mathcal{A} \bigcap \mathcal{B} \neq \varnothing$



(b) No solution because $\mathcal{A} \cap \mathcal{B} = \varnothing$

Fig. 5. $\Delta_{p_1} > 0$, $\mathcal{A} \neq \varnothing$ either (a):One Solution, or (b):No Solution

The discussion in this case is done as a function of the sign of the intersection points $(\widehat{p_1}, \widehat{p_2})$ and $(\widetilde{p_1}, \widetilde{p_2})$.

- If $\max\{\widehat{p_1}, \widetilde{p_1}\} < 0$ (see Fig.4(b)), then we are back to the case $f_1(p_1) > f_2(p_1)$, $\forall\ p_1 \in \mathbb{R}$. Thus, $\mathcal{S}_2 = \varnothing$ and we have no solution.
- If $\widehat{p_1}$ or $\widetilde{p_1}$ is negative, (see Fig.4 (a) when $\widetilde{p_1} < 0$), then we have an unique solution if $\mathcal{A} \cap \mathcal{B} \neq \varnothing$. In this case, see Fig.4(a), the optimal solution (p_1^*, p_2^*) of the problem (17) is not between the intersection points $(\widehat{p_1}, \widehat{p_2})$and $(\widetilde{p_1}, \widetilde{p_2})$, but it is computed between $(\widehat{p_1}, \widehat{p_2})$ and $(0, f_1(0))$. These two last cases (if $max\{\widehat{p_1}, \widetilde{p_1}\} < 0$ and if $\widehat{p_1}$ or $\widetilde{p_1}$ is negative) are met, only for small values of R_{min}.
- If both of the two abscissas of the intersection points $\widehat{p_1}$ and $\widetilde{p_1}$ are positive, (see Fig.5), then we have an unique solution if $\mathcal{A} \cap \mathcal{B} \neq \varnothing$ (see Fig.5(a)) and in this case, the optimal solution (p_1^*, p_2^*) exists. However, if $\mathcal{A} \cap \mathcal{B} = \varnothing$, see Fig.5(b), because of the peak interference constraint P_k^{peak}, then we have no solution.

∎

We give a short sketch of this analysis in Summary 1.

Summary 1 Closed-form analytical solution for the $N = 2$ case

If $\overline{P} < -\frac{g_1}{c_1} - \frac{g_2}{c_2} + 2\sqrt{\frac{g_1}{c_1}\frac{g_2}{c_2}\exp(R_{min})}$, then $\mathcal{A} = \varnothing$:

No Solution: Average interference power $\overline{P}$ and QoS constraints are never simultaneously satisfied

Else $\mathcal{A} \neq \varnothing$,

If $\max\{\widehat{p_1}, \widetilde{p_1}\} < 0 \Leftrightarrow \mathcal{A} \subset (-\infty, 0] \times [0, +\infty)$, then $\mathcal{A} \cap \mathcal{B} = \varnothing$:

No Solution: Average interference power $\overline{P}$ and QoS constraints are not simultaneously satisfied in $\mathcal{B}$

Else $\max\{\widehat{p_1}, \widetilde{p_1}\} > 0 \Leftrightarrow \mathcal{A} \not\subset (-\infty, 0] \times [0, +\infty)$,

If $\frac{P_1^{peak}}{g_1} < \widehat{p_1}$ or $\frac{P_2^{peak}}{g_2} < \min\{\widehat{p_2}, \widetilde{p_2}\}$ or $\left\{f_1\left(\frac{P_1^{peak}}{g_1}\right) > \frac{P_2^{peak}}{g_2}\right.$ and $\left.f_1^{-1}\left(\frac{P_2^{peak}}{g_2}\right) > \frac{P_1^{peak}}{g_1}\right\}$, then $\mathcal{A} \cap \mathcal{B} = \varnothing$:

No Solution: P_k^{peak} constraints are not satisfied

Else $\mathcal{A} \cap \mathcal{B} \neq \varnothing$:

Exists a solution given by:

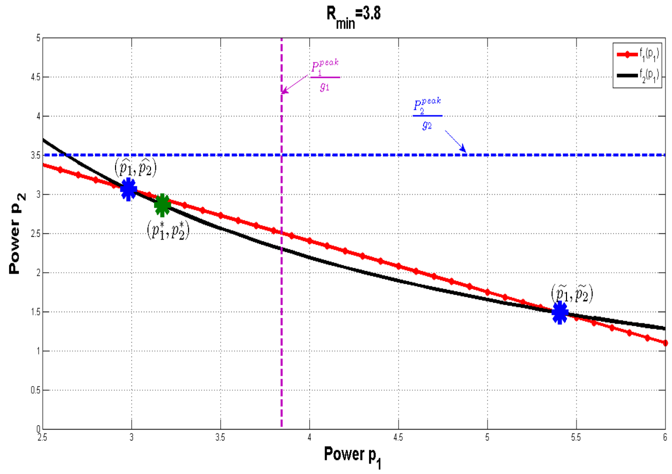
$$\left\{\begin{aligned} p_1^* &= \left[-\frac{1}{c_1} + \sqrt{\frac{\exp(R_{min})}{c_1c_2}}\right]_x^y, \\ p_2^* &= \left[-\frac{1}{c_2} + \sqrt{\frac{\exp(R_{min})}{c_1c_2}}\right]_w^z. \end{aligned}\right.$$

EndIf

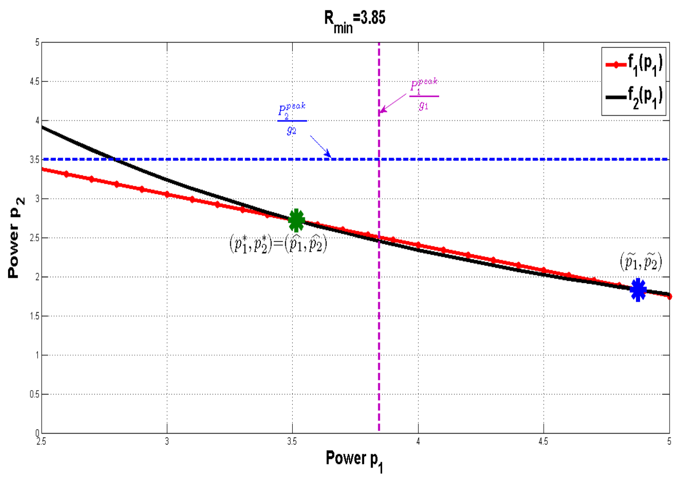
EndIf

EndIf

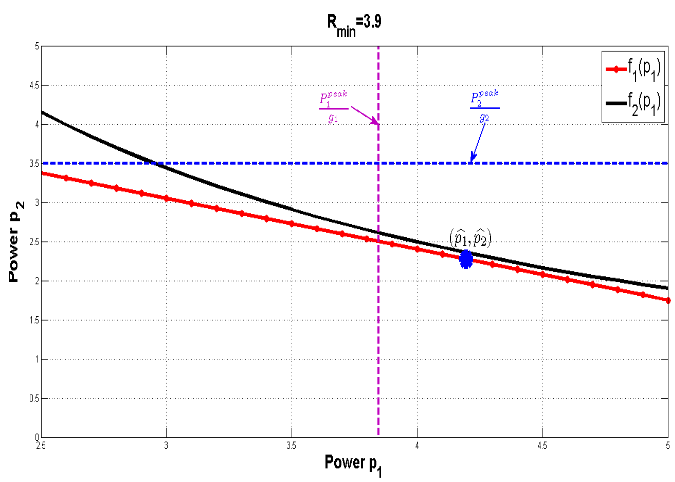
where x, y, w, z are given in Theorem 2 in equation (18), and $(\widehat{p_1}, \widehat{p_2})$, $(\widetilde{p_1}, \widetilde{p_2})$ are given in (30) and Δ_{p_1} is given in (25).



(a) One solution (p_1^*, p_2^*) gets closer to $(\widehat{p_1}, \widehat{p_2})$



(b) One solution $(p_1^*, p_2^*) = (\widehat{p_1}, \widehat{p_2})$



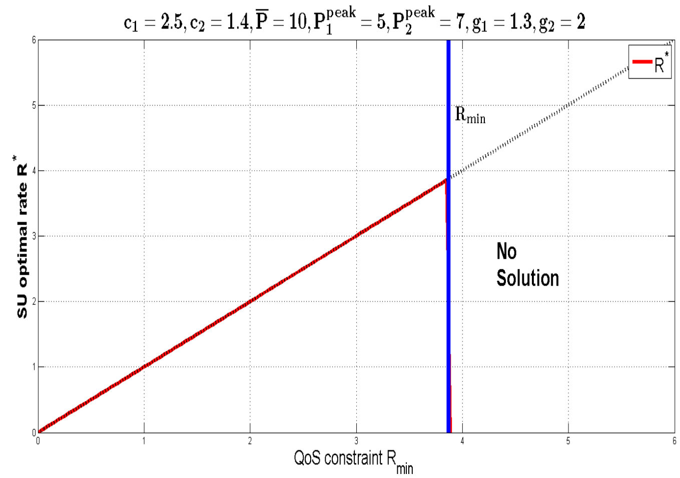
(c) No solution (p_1^*, p_2^*) gets out of the set $\mathcal{B}$

Fig. 6. The average interference constraint is satisfied with a strict inequality in ?? and (a), or with equality in (b). However in (c) the feasible set does not exist

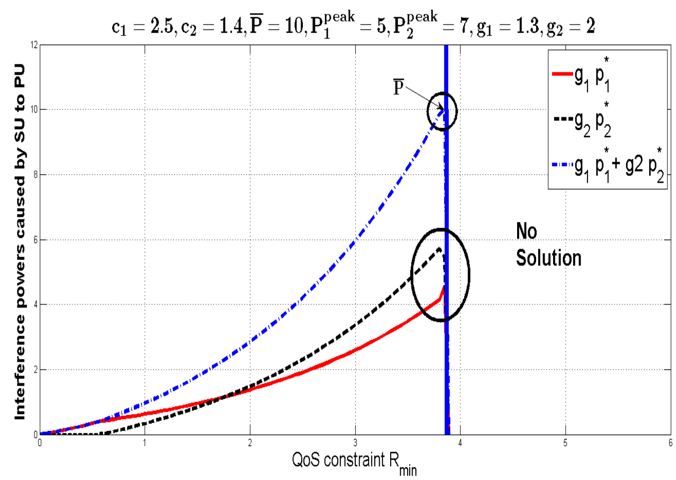
V. NUMERICAL SIMULATIONS

Our simulation parameters are generic and, thus, all the observations we make here under remain valid for any values of the parameters and any practical system.

A. SU Rate and Interference Powers Caused by SU to PU vs. QoS Constraint R_{min}



(a) SU optimal rate vs. rate constraint R_{min}



(b) Interference levels caused by SU to PU vs. rate constraint R_{min}

Fig. 7. SU optimal rate (a) and interference levels caused by SU to PU (b) vs. rate constraint R_{min}

In Fig.7, we plot the SU optimal rate $R^* \triangleq R(\underline{p}^*)$ defined in (2)) and the interference powers levels caused by SU to PU ($g_1 p_1^*$, $g_2 p_2^*$ and $g_1 p_1^* + g_2 p_2^*$) versus the minimum rate constraint R_{min} over two orthogonal frequency channels with the system parameters: $c_1 = 2.5$, $c_2 = 1.4$, $\overline{P} = 10$, $P_1^{peak} = 5$, $P_2^{peak} = 7$, $g_1 = 2$ and $g_2 = 1.3$.

In the two sub-figures, we observe that there is a zone where our problem does not admit a solution where the average interference power and QoS constraints are not simultaneously satisfied (for example the case of Fig.2). However, there is a critical value when $R_{min} \approx 3.866$ below which an optimal solution exists. Therefore, if the QoS constraint is too restrictive, i.e., R_{min} is above this threshold, a feasible solution does not exist.

In Fig.7.(a), we see that the optimal rate is equal to the minimum rate constraint R_{min} which is trivial, because of the

KKT condition of $\lambda > 0$ in (15).

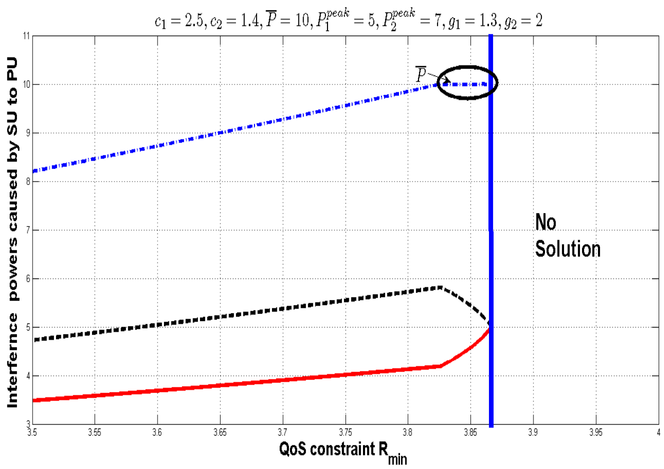


Fig. 8. Zoom of Fig.7 (b) for some values of R_{min}

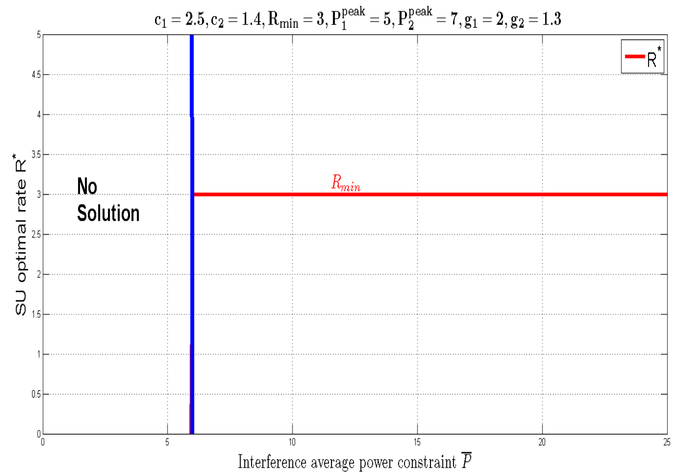
In Fig.7.(b) zoomed in Fig.8, we remark that in the interval where $R_{min} \in [0, 8.26]$ the average interference constraint is satisfied with a strict inequality ($\sum_{k=1}^{N} g_k p_k^* < \overline{P}$) which are illustrated in Fig.6.(a). We observe that the transmit powers required to achieve the QoS constraint R_{min}, increase exponentially. Also the optimal point (p_1^*, p_2^*) is on the curve $f_1(\underline{p})$ in between $\widehat{\underline{p}}$ and $\widetilde{\underline{p}}$. By increasing R_{min} we observe that $\underline{p}^*$ gets closer to $\widehat{\underline{p}}$. It turns out that in this scenario, when $R_{min} > 8.26$ the average interference constraint is satisfied with equality and the optimal solution is equal to the intersection point $\underline{p}^* = \widehat{\underline{p}}$ which is illustrated in Fig.6.(b). Finally when R_{min} exceeds the critical value of 3.866 then a feasible solution does not exist as we see in Fig.6.(c) because $\widehat{\underline{p}}$ gets out of the set $\mathcal{B}$.

B. SU Rate and Interference Powers Caused by SU to PU vs. Interference Power Constraint $\overline{P}$

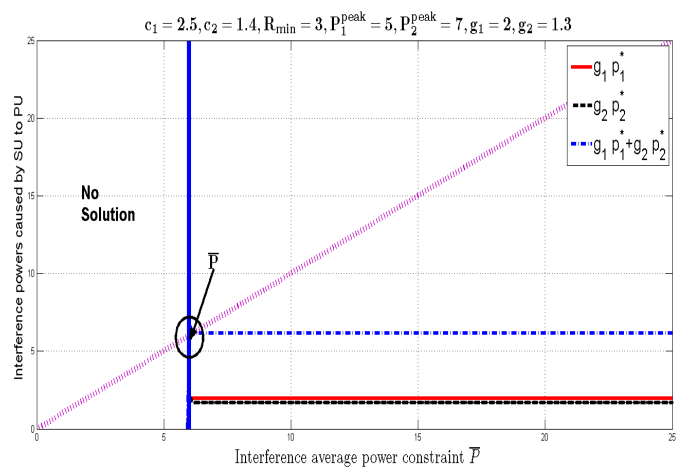
In Fig.9, we plot the SU rate R^* and the interference powers levels caused by SU to PU ($g_1 p_1^*$, $g_2 p_2^*$ and $g_1 p_1^* + g_2 p_2^*$) versus the average interference power $\overline{P}$ over two orthogonal frequency channel with the system parameters: $c_1 = 2.5$, $c_2 = 1.4$, $R_{min} = 3$, $P_1^{peak} = 5$, $P_2^{peak} = 7$, $g_1 = 2$ and $g_2 = 1.3$. Similarly to the previous figure there is a zone where there is no feasible solution.

In Fig.9.(a) the SU rate is equal to the minimum rate constraint R_{min} for $\overline{P} > 6$. In conclusion, if $\overline{P}$ is below a certain threshold then the average interference constraint is too restrictive and a solution does not exist.

In Fig.9.(b) we note that the interference power level caused by SU to PU for each orthogonal frequency band is constant and independent from $\overline{P}$. Therefore, $\overline{P}$ does not influence the expressions of the optimal power $\underline{p}^*$ once it exists. For example, if $\overline{P} \simeq 6$, the average interference constraint is satisfied with equality. However, if $\overline{P} > 6$ we obtain $g_1 p_1^* + g_2 p_2^* < \overline{P}$, then the average interference constraint is satisfied with strict inequality.



(a) SU optimal rate vs. interference power constraint $\overline{P}$



(b) Interference levels caused by SU to PU vs. interference power constraint $\overline{P}$

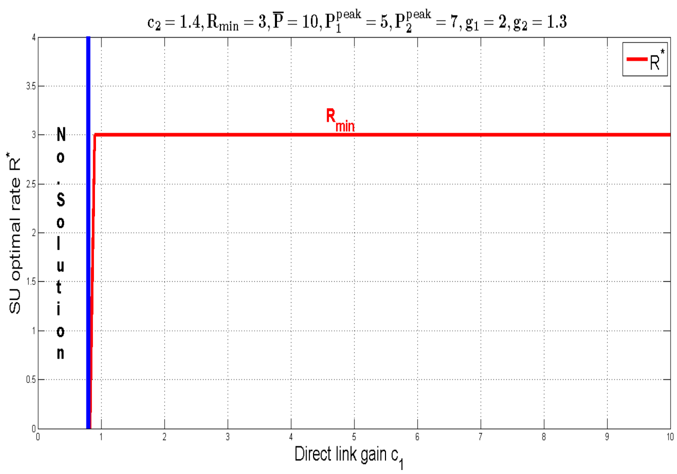
Fig. 9. SU optimal rate (a) and interference levels caused by SU to PU (b) vs. interference power constraint $\overline{P}$

C. SU Rate and Interference Powers Caused by SU to PU vs. the Direct Link Gain c_1

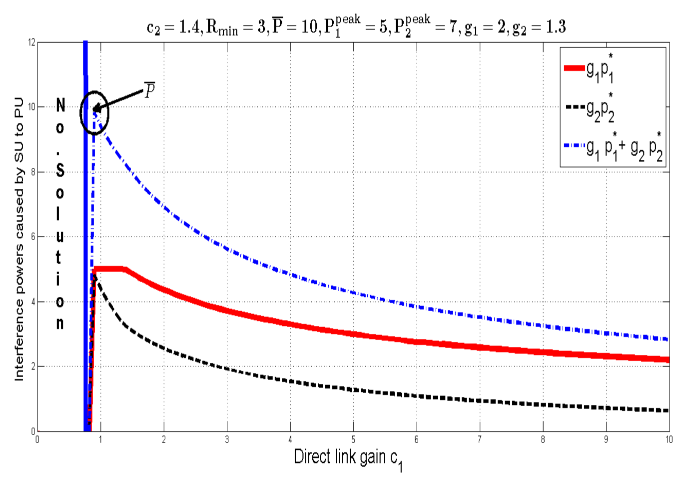
In Fig.10, we plot the SU rate R^* and the interference power level caused by SU to PU ($g_1 p_1^*$, $g_2 p_2^*$ and $g_1 p_1^* + g_2 p_2^*$) versus the direct link gain in the first band c_1 in the scenario: $c_2 = 1.4$, $R_{min} = 3$,$\overline{P} = 10$, $P_1^{peak} = 5$, $P_2^{peak} = 7$, $g_1 = 2$and $g_2 = 1.3$.

In Fig.10.(a), the optimal rate is equal to the minimum rate constraint R_{min} for all c_1 values ($\forall c_1 > 0.9$). We see that if the quality of one of the direct links (i.e., small values of the direct link gain c_1) is very poor, then the QoS constraint cannot be satisfy and thus, a feasible solution does not exist.

In Fig.10.(b), we note that the optimal power for each orthogonal frequency band decreases exponentially with the direct link gain c_1. The higher the quality of the direct links, the lower are the powers required to achieve the QoS constraint. For example, if $c_1 \simeq 0.9$, we see that the average interference constraint is satisfied with equality. However, if $c_1 > 0.9$ the average interference constraint is satisfied with

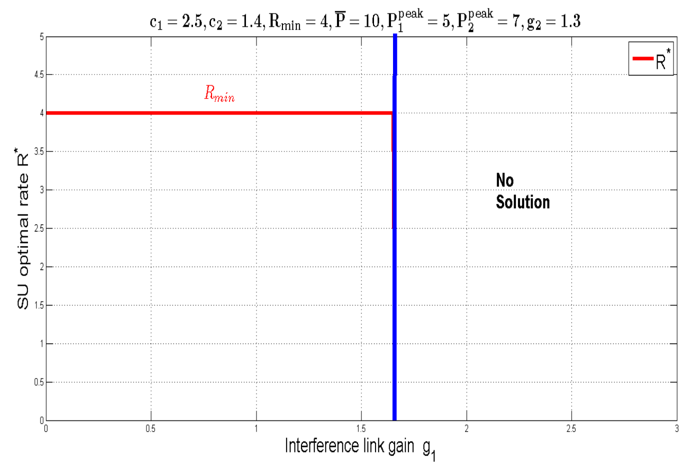


(a) SU optimal rate vs. the direct link gain c_1

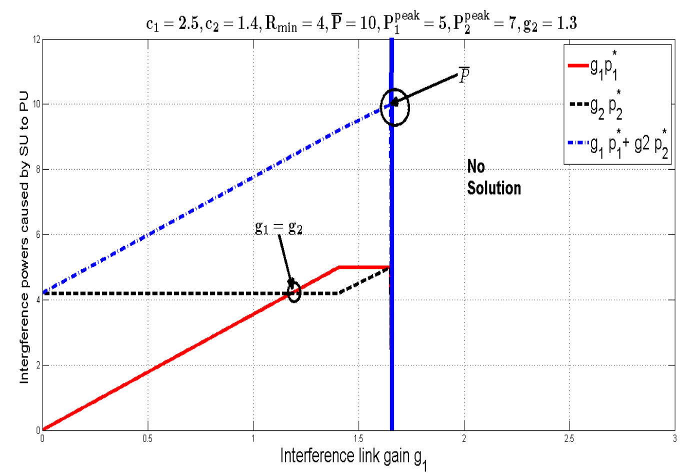


(b) Interference levels caused by SU to PU vs. the direct link gain c_1

Fig. 10. SU optimal rate (a) and interference levels caused by SU to PU (b) vs. the direct link gain c_1



(a) SU optimal rate vs. the interference link gain g_1



(b) Interference levels caused by SU to PU vs. the interference link gain g_1

Fig. 11. SU optimal rate (b) and interference levels caused by SU to PU (b) vs. the interference link gain g_1

strict inequality i.e. $g_1 p_1^* + g_2 p_2^* < \overline{P}$.

D. SU Rate and Interference Powers Caused by SU to PU vs. the interference link gain g_1

In Fig.11, we plot the optimal rate R^* and SU rate and interference powers caused by SU to PU ($g_1 p_1^*$, $g_2 p_2^*$ and $g_1 p_1^* + g_2 p_2^*$) versus the interference link gain g_1 over two orthogonal frequency channels with the system parameters: $c_1 = 2.5$, $c_2 = 1.4$, $R_{min} = 4$, $P_1^{peak} = 5$, $P_2^{peak} = 7$, $\overline{P} = 10$ and $g_2 = 1.3$.

In Fig.11.(a), the optimal rate is equal to the minimum rate constraint R_{min} for values of the interference link gain g_1 below a certain threshold. We see that when the quality of the interfering link is high (i.e. high values of g_1) the maximal interference constraint are not met and a feasible solution does not exist.

In Fig.11.(b), we observe that, except for some borders values (nearly the critical threshold) where $p_1^* = \dfrac{P_1^{peak}}{g_1}$ (see equation (18)) the optimal transmit power value p_1^* does not depend on g_1 . The optimal transmit power p_2^* is independent of the interference link gain g_1 except for the same region where p_2^* is proportional to the interference link gain g_1 (see equation (18)). For example, if $g_1 \simeq 1.7$, we see that the average interference constraint is satisfied with equality. However, if $c_1 < 1.7$ the average interference constraint is satisfied with strict inequality i.e. $g_1 p_1^* + g_2 p_2^* < \overline{P}$.

In the case where $N > 2$, the difficulty lies in the fact that the iterative water-filling algorithm converges to the optimal solution provided that the feasibility set of all the constraints is non void. Finding the conditions on the system parameters that ensure that the feasible set is non void, is a difficult problem since the number of constraints and the degrees of freedom becomes larger.

VI. CONCLUSIONS

In this paper, we have analysed the power allocation problem over orthogonal bands at the opportunistic user's level under a target QoS constraint and maximal interference caused to the primary user of the spectrum constraints.

We have seen that, depending on the system parameters, a feasible solution does not always exists. This is mainly caused by the simultaneous requirements of minimum target achievable rate at the SU and maximum allowed interference level caused to the PU. When a solution exist, it takes the form of water-filling and the closed-form solution is provided for the particular case of two available frequency bands.

Future work include the case of multiple secondary and primary users for which efficient distributed algorithms have to be proposed. Also, we will consider the effect of imperfect parameters estimation or knowledge, and investigate the robustness of our power allocation problem to these imperfections.

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