2015 European Control Conference (ECC)
July 15-17, 2015. Linz, Austria

Continuous-time consensus dynamics with quantized all-to-all communication

Francesca Ceragioli Paolo Frasca

Abstract— **This paper deals with a quantized version of a consensus dynamics in continuous-time, which is motivated by opinion dynamics applications. Under the assumption of all-to-all communication, we show existence and completeness of solutions, we characterize the equilibria, and we prove asymptotical convergence to a state of quantized consensus. For almost all initial conditions, the consensus value differs from the initial average by at most the quantizer precision. Furthermore, we discuss the implications of more general assumptions on the communication graph.**

I. INTRODUCTION

Consensus dynamics have the goal to steer a group of agents to a common value while only allowing each agent to use information from its immediate neighbors. In the last few years, the effects of quantization in consensus dynamics have attracted the interest of a large number of researchers. Indeed, quantization can be due to coarse sensing capabilities, to digital communication over band-limited communication channels, or to limited precision in computation. As a consequence of this variety of applications, a wide range of quantized consensus systems has been studied: without attempting to be exhaustive, we only mention here some of the early works [1], [2], [3], [4], [5] and some of the most recent developments [6], [7], [8], [9], [10], [11], [12], [13], [14]. Due to the discontinuous nature of the resulting dynamics, convergence results have only been obtained in few cases, sometimes under restrictive assumptions about the topology of the network that describes the communication among the agents. Instead, the most common approach is to compare the quantized dynamics with a corresponding non-quantized dynamics. Quantization induces a deviation between the two: this deviation can be estimated, as a function of the quantizer and of the topology, and can often be made small by suitable design choices.

In this work we consider one specific continuous-time dynamics involving quantization, where the agents can only access a quantized version of the other agents' states, but can access their own state with unlimited precision. We restrict our attention to the case of all-to-all communication, which is amenable to a complete analysis. We characterize the behavior of Carathéodory solutions: we prove their existence and completeness, we describe their equilibria (which coincide with quantized consensus states), and we prove that they converge to an equilibrium that is close to the initial average of the states. Additionally, we provide some preliminary results about the more general case when communication is described by a weighted graph. In general, the limit behavior substantially differs from a consensus: a precise characterization will be the topic of future work.

A quantized consensus dynamics with perfect knowledge of the internal states has been briefly considered, in a discrete-time setting, in one of the earliest works on quantized consensus [15]. In that paper, the authors are seeking an algorithm achieving consensus at the average of the initial values of the agents, in spite of the constraint of using quantized communication. In order to reach that goal, they recommend to use the quantized internal state in the feedback loop, in place of the non-quantized one, because this choice guarantees the preservation of the average state during the dynamics. Instead, using infinite precision for the internal state prevents this preservation. Because of this drawback, subsequent works aiming at average consensus have used quantized internal states [16], [17]. On the contrary, the dynamics with perfect knowledge of the internal states has not been studied much and its qualitative analysis remains open. Indeed, this dynamics is hardly appealing from an engineering perspective. Instead, the motivation for its study comes from social science applications of consensus-seeking systems, which are receiving increasing interest in our community [18], [19], [20], [21]. In these applications, the agents are interpreted as individuals and the states as opinions on some topic: the consensus system describes the effect of interactions between individuals that communicate among them and are influenced by others' opinions. In such context, quantization can arise as a mismatch between perfect self-awareness and imperfect communication, due to limited verbalization capabilities. A similar tension arises between personal attitudes and displayed actions, which typically feature a limited number of options. These phenomena have been observed by many researchers and incorporated in a few models of opinion dynamics: relevant examples range from sociology [22] and behavioral sciences [23] to physics [24] and auction theory [25]. In these contexts, it is natural to assume that the opinions of the individuals do not depend on their own displayed (thus, quantized) opinions, but instead on the real ones.

Paper structure: The dynamics with all-to-all communication is defined in Section II and studied in Section III. Next, in Section IV we elaborate on the case when communication is restricted to certain neighbors. Final remarks are given in Section V.

Francesca Ceragioli is with Dipartimento di Scienze Matematiche, Politecnico di Torino, c.so Duca degli Abruzzi 24, 10129, Torino, Italy. E-mail: francesca.ceragioli@polito.it

Paolo Frasca is with Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. E-mail: p.frasca@utwente.nl

II. Quantized all-to-all communication

We consider a set of agents, indexed in $\mathcal{I} = \{1, \dots, N\}$: each of them has a time-dependent real-valued state $x_i(t)$ that obeys the following dynamics

$$\dot{x}_i(t) = \sum_{j \neq i} [q(x_j(t)) - x_i(t)], \qquad i \in \mathcal{I}, \tag{1}$$

where $q : \mathbb{R} \to \mathbb{Z}$ is defined as $q(t) = \lfloor t + \frac{1}{2} \rfloor$ and is represented in Figure 1. Note that in the above model agent j influences agent i through $q(x_j(t))$, i.e., a quantization of its state.

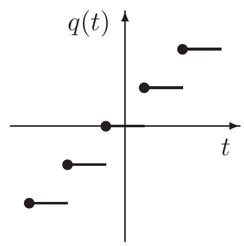

Fig. 1. The uniform quantizer q.

Due to quantization, the right-hand side of (1) is discontinuous. Classical solutions of (1) may not exist, thus we consider Carathéodory solutions, i.e. solutions to the integral equation

$$x_i(t) = x_{0_i} + \int_0^t \sum_{j \neq i} [q(x_j(s)) - x_i(s)] ds.$$

Other types of generalized solutions may be chosen in order to deal with discontinuities in the system [26]. For example, in [27] a quantized consensus problem was studied using Krasovskii solutions. Often, the choice of generalized solutions as Krasovskii solutions and Filippov solutions is forced by the non-existence of more classical solutions. As we show in the next section, here Carathéodory solutions have good properties and thus can be successfully used. From now on we simply write solutions to mean Carathéodory solutions.

III. Analysis of the all-to-all dynamics

A. Basic properties

System (1) can be equivalently described by the equation

$$\dot{x} = f(x) \tag{2}$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is a discontinuous vector field given by $f_i(x) = \sum_{j \neq i} [q(x_j) - x_i]$. If we define

$$\begin{aligned} \mathbf{q}(x) &= (\textstyle\sum_{j \neq 1} q(x_j), \dots, \sum_{j \neq N} q(x_j))^T = \\ &= (\textstyle\sum_{j=1}^N q(x_j) - q(x_1), \dots, \sum_{j=1}^N q(x_j) - q(x_N))^T, \end{aligned}$$

and we observe that for every $\mathbf{k} \in \mathbb{Z}^N$ the function $\mathbf{q}$ is constant on each set

$$S_{\mathbf{k}} = \{x \in \mathbb{R}^N : k_i - \frac{1}{2} \leq x_i < k_i + \frac{1}{2}, i = 1, \dots, N\}.$$

Then, the discontinuous vector field can be written as

$$f(x) = \mathbf{q}(x) - (N-1)x. \tag{3}$$

This fact makes evident that in each set $S_{\mathbf{k}}$ trajectories are lines segments. Moreover, the formula (3) is useful in the proof of the following basic properties of the solutions.

Proposition 1 (Properties of solutions):

(i) (Existence) For any initial condition there exists a solution of (1).

(ii) (Invariant manifolds) Let $s_{ij}(x) = x_i - x_j$ and $S_{ij} = \{x \in \mathbb{R}^N : s_{ij}(x) = 0\}$. If x is a solution of (1) such that $x(t_0) \in S_{ij}$, then $x(t) \in S_{ij}$ for all $t \geq t_0$.

(iii) (Order preservation) If $x_i(t_0) \leq x_j(t_0)$ for some $t_0 \in \mathbb{R}$ then $x_i(t) \leq x_j(t)$ for all $t \geq t_0$.

(iv) (Boundedness) For any solution x of (1), there exist $m, M \in \mathbb{R}$ such that $m \leq x(t) \leq M$ for all $t \geq t_0$.

(v) (Completeness) Any solution starting at $t_0 \in \mathbb{R}$ is defined on the set $[t_0, +\infty)$.

Proof: (i) First of all, we remark that the righthand side of (1) is continuous at any point in the interior of $S_{\mathbf{k}}$ for any $\mathbf{k} \in \mathbb{Z}^N$, then local solutions with initial conditions in $\mathbb{R}^N \setminus \cup_{\mathbf{k} \in \mathbb{Z}^N} \partial S_{\mathbf{k}}$ do exist, where $\partial S_{\mathbf{k}}$ denotes the border of $S_{\mathbf{k}}$. Then, we consider initial conditions on $\cup_{\mathbf{k} \in \mathbb{Z}^N} \partial S_{\mathbf{k}}$. For any $x_0 \in \mathbb{R}^N$ we denote by $I(x_0)$ the subset of $\{1, \dots, N\}$ of the indices i such that $x_{0i} = k_i + \frac{1}{2}$ for some $k_i \in \mathbb{Z}$ and by M be the cardinality of $I(x_0)$.

We first consider initial conditions x_0 such that $I(x_0) = \{i\}$, i.e. $x_{0i} = k_i + \frac{1}{2}$ for some $k_i \in \mathbb{Z}$ and $x_{0j} \neq h + \frac{1}{2}$ for any $j \neq i$ and any $h \in \mathbb{Z}$. Let us denote

$$s_i(x) = x_i - k_i - \frac{1}{2},$$

$$S_i^+ = \{x \in \mathbb{R}^N : x_i - k_i - \frac{1}{2} \geq 0\},$$

$$S_i^- = \{x \in \mathbb{R}^N : x_i - k_i - \frac{1}{2} < 0\},$$

$$f|_{S_i^-}(x_0) = \lim_{x \in S_i^-, x \to x_0} f(x).$$

We have that

$$\begin{aligned} a(x_0) &= \nabla s_i(x_0) \cdot f|_{S_i^+}(x_0) = \nabla s_i(x_0) \cdot f|_{S_i^-}(x_0) \\ &= \textstyle\sum_{j \neq i} (q(x_{0j}) - k_i - \frac{1}{2}). \end{aligned}$$

If $a(x_0) < 0$ there is a solution starting at x_0 which satisfies the equations $\dot{x} = f|_{S_i^-}(x)$ and stays in S_i^- in an interval of the form $(t_0, t_0 + \epsilon)$ for some $\epsilon > 0$. If $a(x_0) \geq 0$ there is a solution starting at x_0 which satisfies the equation $\dot{x} = f|_{S_i^+}(x)$ and stays in S_i^+ in an interval of the form $(t_0, t_0 + \epsilon)$ for some $\epsilon > 0$. Note in particular that if $a(x_0) = 0$ the vector field $f|_{S_i^+}$ is tangent to S_i^+ in a neighborhood of x_0.

We now consider initial conditions x_0 such that $1 < M \leq N$. The vector field f has 2^M limit values at x_0 corresponding to the 2^M sectors defined by the inqualities $x_i - k_i - \frac{1}{2} \geq 0$ and $x_i - k_i - \frac{1}{2} < 0$. We describe these sectors by means of vectors $Q \in \{0,1\}^N \subset \mathbb{R}^N$. Let $H(t) = 1$ if $t \geq 0$ and $H(t) = 0$ if $t < 0$. We define $Q_i = H(x_i - k_i - \frac{1}{2})$ if $i \in I(x_0)$ and $Q_i = 0$ if $i \notin I(x_0)$.

Let $f_Q = \lim_{x\to x_0, x\in Q} f(x)$. We want to prove that there exists Q such that $H((f_Q)_i) = Q_i$ for all $i \in I(x_0)$, that is, that at x_0 the vector field f_Q points inside the sector Q.

Preliminarily, note that the ith component of f_Q can be written as

$$\begin{aligned}(f_Q)_i &= \sum_{j\neq i} q(x_j) - (N-1)x_i \\ &= \sum_{j\neq i}(k_j + Q_j) - (N-1)(k_i + \tfrac{1}{2}) \\ &= \sum_j k_j - Nk_i - \tfrac{N-1}{2} + \sum_{j\neq i} Q_j.\end{aligned}$$

Now, we start by considering the sector Q^1 such that $(Q^1)_i = 0$ for all $i \in I(x_0)$. If $H((f_{Q^1})_i) = 0$ for all $i \in I(x_0)$, we have finished. Otherwise, there exists $i \in I(x_0)$ such that $H((f_{Q^1})_i) = 1$. Assume without loss of generality that such i is 1, i.e. $H((f_{Q^1})_1) = 1$. Since $(f_{Q^1})_1 = \sum_j k_j - Nk_i - \frac{N-1}{2} > 0$, then also for all $Q \neq Q^1$ we have $(f_Q)_1 = \sum_j k_j - Nk_i - \frac{N-1}{2} + \sum_{j\neq i} Q_j > 0$ and thus $H((f_Q)_1) = 1$. We then examine only those Q such that $H(Q_1) = 1$. In particular the next Q we consider, that we call Q^2, is such that $H((Q^2)_1) = 1$ and $H((Q^2)_i) = 0$ for all other $i \in I(x_0)$. If $H((f_{Q^2})_i) = 0$ for all $i \in I(x_0)$, then we have finished. Otherwise, there exists $i \in I(x_0) \setminus \{1\}$ such that $H((f_{Q^1})_i) = 1$. Assume that such i is 2, i.e. $H((f_{Q^2})_2) = 1$. Then for all $Q \neq Q^1, Q^2$ we have $H((f_Q)_2) = 1$. We can then restrict our attention to those Q such that $H(Q_1) = H(Q_2) = 1$, and so forth. By proceeding in this way, in M step at most we find the sector Q with the desired property.

As already mentioned, the meaning of the condition $H((f_Q)_i) = Q_i$ for all i is that the vector field $f_Q(x_0)$ is directed inside Q. Then, there exist a solution φ of (1), and $t_0 \in \mathbb{R}, \epsilon > 0$ such that $\varphi(t_0) = x_0$ and $\dot\varphi(t) = f_Q(\varphi(t))$ for almost every $t \in (t_0, t_0+\epsilon)$. Let us denote by $\tilde I(x_0)$ the subset of $I(x_0)$ such that $(f_Q(x_0))_i = 0$ if $i \in \tilde I(x_0)$. There exists a neighborhood of x_0 –denoted by $N(x_0)$– such that the vector field f_Q is tangent to $[\cap_{i\in\tilde I(x_0)}\{x_i = k_i + \frac{1}{2}\}] \cap N(x_0) \cap Q$.

(ii) Assume that the claim is not true, i.e., that for some pair (i,j) and for some solution x of (1) such that $x(t_0) \in S_{ij}$, one has $s_{ij}(x(t^*)) > 0$ for some $t^* > t_0$. This implies –by the absolute continuity of Carathéodory solutions– that there exists some positive measure set $I \subset (t_0, t^*)$ such that $s_{ij}(x(t)) > 0$ and $\frac{d}{dt}s_{ij}(x(t)) > 0$ for all $t \in I$. On the other hand for almost all $t \in I$,

$$\begin{aligned}&\tfrac{d}{dt}s_{ij}(x(t)) = \\ &= \nabla s_{ij}(x(t))\cdot \dot x(t) \\ &= \nabla s_{ij}(x(t))\cdot f(x(t)) = e_i\cdot f - e_j \cdot f \\ &= \sum_{h\neq i}[q(x_h(t)) - x_i(t)] - \sum_{h\neq j}[q(x_h(t)) - x_j(t)] \\ &= -[q(x_i(t)) - q(x_j(t))] - (N-1)[x_i(t) - x_j(t)] < 0,\end{aligned}$$

which gives a contradiction.

(iii) This is an immediate consequence of the previous statement. Assume first that $x_i(t_0) < x_j(t_0)$ and $x_i(t^*) > x_j(t^*)$ for some $t^* > t_0$. Then there exist $\bar t \in (t_0, t^*)$ such that $x_i(\bar t) = x_j(\bar t)$ and $t^* > t_0$ such that $x_i(t^*) > x_j(t^*)$, which contradicts (ii). In the case $x_i(t_0) = x_j(t_0)$ and $x_i(t^*) > x_j(t^*)$ for some $t^* > t_0$ the contradiction with (ii) is immediate.

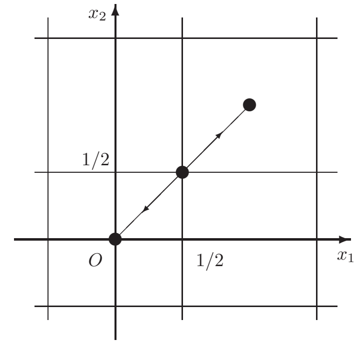

Fig. 2. Non-uniqueness of the solution through the point $(1/2, 1/2)$.

(iv) Let $x(t)$ be any solution of (1). Assume that $x_1(t_0) \le x_2(t_0) \le .... \le x_N(t_0)$. By (iii) $x_1(t) = \min\{x_1(t), ..., x_N(t)\}$ and $x_N(t) = \max\{x_1(t), ..., x_N(t)\}$ for all $t \ge t_0$. Let us point the attention on $x_N(t)$. Note that at any time t such that $q(x_N(t)) \le x_N(t) \le q(x_N(t)) + \frac{1}{2}$, one has $\dot x_N(t) = \sum_{j\neq N} q(x_j(t)) - x_N(t) \le 0$, then, if $q(x_N(t_0)) \le x_N(t_0) \le q(x_N(t_0)) + \frac{1}{2}$, one gets that $x_N(t) \le x_N(t_0)$ for all $t \ge t_0$.

On the other hand if $q(x_N(t)) - \frac{1}{2} \le x_N(t) < q(x_N(t))$ for some time t, then $\dot x_N(t)$ can be positive, depending on $x_1(t), ..., x_{N-1}(t)$. Nethertheless, if $x_N(t)$ increases, it can not overcome $q(x_N(t))$ because if at some time $\bar t$, one has $x_N(\bar t) > q(x_N(\bar t))$ then $\dot x_N(\bar t) \le 0$. Finally in this case $x_N(t) \le q(x_N(t_0))$ for all $t \ge t_0$. Since $x_1(t)$ can be analogously treated we get that the solution $x(t)$ is bounded.

(v) Local existence and boundedness of solutions imply that all solutions with initial condition at $t_0 \in \mathbb{R}$ are defined on $[t_0, +\infty)$. ■

In general uniqueness of solutions is not guaranteed, as shown in the following example illustrated in Figure 2.

Example 1 (Multiple solutions): Consider system (1) with $N = 2$ and initial condition $(x_1(0), x_2(0)) = (1/2, 1/2)$. There are two solutions issuing from this point, whose trajectories are the line segments joining the initial condition with the equilibria $(0,0)$ and $(1,1)$. ■

B. Equilibria

We now describe the equilibria of the vector field $f(x)$, which are states of quantized consensus.

Proposition 2 (Equilibria): The set of equilibria of (1) is

$$E = \{x \in \mathbb{Z}^N : \ \exists h \in \mathbb{Z} \text{ such that } \ x_i = h \ \forall i = 1, ..., N\}.$$

Proof: Clearly, any point of E is an equilibrium of (1). Let now x^* be an equilibrium of (1). This means that

$$x^* = \frac{1}{N-1}\mathbf{q}(x^*).$$

Let $\mathbf{k} \in \mathbb{Z}^N$ be such that x^* belongs to $S_{\mathbf{k}}$ and let $Q = \sum_{j=1}^N k_j$. One has

$$k_i - \frac{1}{2} \le x_i^* = \frac{Q - k_i}{N-1} < k_i + \frac{1}{2}$$

and by multiplying by $N-1$, for all $i = 1, \ldots, N$, we get

$$(N-1)\left(k_i - \frac{1}{2}\right) \le Q - k_i < (N-1)\left(k_i + \frac{1}{2}\right).$$

which implies

$$\frac{Q}{N} - \frac{N-1}{2N} < k_i \le \frac{Q}{N} + \frac{N-1}{2N}.$$

Since $k_i \in \mathbb{Z}$ and there is a unique integer number in the interval $\left(\frac{Q}{N} - \frac{N-1}{2N}, \frac{Q}{N} + \frac{N-1}{2N}\right]$, this implies that $k_1 = \ldots = k_n$. This means that if x^* is an equilibrium then $x^* \in S_{(h,\ldots,h)}$ for some $h \in \mathbb{Z}$. In this case $f(x^*)_i = (N-1)h - (N-1)x_i^*$ for all $i = 1, \ldots, N$, and finally $x_i^* = h$ for all $i = 1, \ldots, N$. ■

Remark 1 (Local stability): Note that equilibria are Lyapunov stable and locally asymptotically stable. In fact for any equilibrium x^* there exists a neighborhood where the vector field is given by $f(x) = -(N-1)(x - x^*)$.

Remark 2 (Finite-time exit): From the proof of the previous proposition it follows that in each $S_{\mathbf{k}}$ with $\mathbf{k} \neq (h, \ldots, h)$ for any $h \in \mathbb{Z}$, trajectories are lines segments that are solutions to a vector field whose equilibrium is out of $S_{\mathbf{k}}$. This consideration implies that all solutions to (1) escape from every set $S_{\mathbf{k}}$ with $\mathbf{k} \neq (h, \ldots, h)$ in finite time. On the other hand, once inside some $S_{(h,h,\ldots,h)}$, the equilibrium is reached in "infinite" time.

C. Convergence to consensus

We now prove that each solution converges to an equilibrium, that is, to consensus at a common integer value. The convergence is illustrated in Figure 3.

Proposition 3 (Convergence to consensus): Any solution to (1) converges to a point in E.

Proof: Let x be any solution of (1) and, without loss of generality, assume that $x_1(t_0) \le \cdots \le x_N(t_0)$. By order preservation we have $x_1(t) \le \ldots \le x_N(t)$ for all $t \ge t_0$ and $x_N(t) - x_1(t) \ge 0$ for all $t \ge t_0$. We prove that $x_N(t) - x_1(t) \to 0$ as $t \to +\infty$. In fact

$$\begin{aligned} &\tfrac{d}{dt}(x_N(t) - x_1(t)) = \\ &\quad = \textstyle\sum_{h \neq N}[q(x_h(t)) - x_N(t)] - \sum_{h \neq 1}[q(x_h(t)) - x_1(t)] \\ &\quad = -[q(x_N) - q(x_1)] - (N-1)(x_N - x_1) \end{aligned}$$

and then

$$0 \le x_N(t) - x_1(t) \le (x_N(t_0) - x_1(t_0))e^{-(N-1)(t-t_0)} \to 0$$

as $t \to +\infty$ and $\mathrm{dist}(x(t), S) \to 0$ as $t \to +\infty$, where $S = \{x \in \mathbb{R}^N : x_1 = \ldots = x_N\}$.

We now distinguish two cases. If $x(t)$ is in the interior of $S_{(h,\ldots,h)}$ for some t and some $h \in \mathbb{Z}$, then

$$\dot{x}(t) = f(x(t)) = -(N-1)(x(t) - (h, \ldots, h)^T)$$

and $x(t) \to (h, \ldots, h)^T$ as $t \to +\infty$. Otherwise, $x(t) \to (h+\frac{1}{2}, \ldots, h+\frac{1}{2})^T$ for some $h \in \mathbb{Z}$. In this case, $x(t)$ actually reaches $(h+\frac{1}{2}, \ldots, h+\frac{1}{2})^T$ in finite time, because each $S_{\mathbf{k}}$ is crossed in finite time, as observed in Remark 2; afterwards, $x(t)$ converges either to $(h, \ldots, h)$ or to $(h+1, \ldots, h+1)$. Note that in general convergence is not in finite time. ■

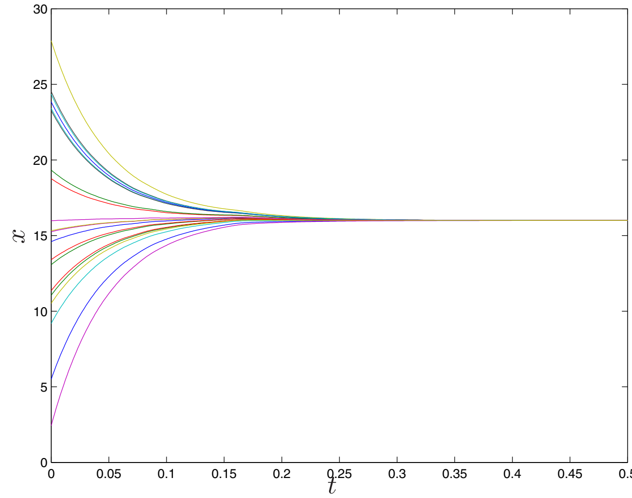

Fig. 3. Simulation of (1) for $N = 20$ from random initial conditions in $[0, 30]$, showing convergence of the states to the consensus value $h = 16$.

D. Average (almost) preservation

In general, solutions to (1) do not preserve the average of the states. However, the following result shows that the deviation from the average is bounded by the precision of the quantizer, with the possible exception of solutions that originate from a negligible set of initial conditions. This result is illustrated by simulations in Figure 4.

Proposition 4 (Consensus value): Let $x_0 \in \mathbb{R}^N$, $K \in \mathbb{Z}$ be such that

$$K < \frac{(x_0)_1 + \ldots + (x_0)_N}{N} < (K+1)$$

and let ϕ_{x_0} be any solution of (1) such that $\phi_{x_0}(t_0) = x_0$. For almost all $x_0 \in \mathbb{R}^N$, $\phi_{x_0}(t)$ tends either to $(K, \ldots, K)$ or to $(K+1, \ldots, K+1)$.

Proof: Let us first remark that the sets

$$\Delta_K = \{x \in \mathbb{R}^N : x_1 + \ldots + x_N = KN\}, K \in \mathbb{Z}$$

are weakly invariant. In fact the vector $\mathbf{1} = (1, \ldots, 1)$ is normal to Δ_K and $\mathbf{1} \cdot f|_{S_{\mathbf{k}}} = 0$ if

$$f|_{S_{\mathbf{k}}} = (\sum_{j \neq i} k_j - (N-1)x_i)_i.$$

Since solutions starting in the interior of $S_{\mathbf{k}}$ are locally unique, solutions that do not converge either to $(K, \ldots, K)$ or to $(K+1, \ldots, K+1)$ must cross $\partial S_{\mathbf{k}} \cap (\Delta_K \cup \Delta_{K+1})$ at some time. We prove that the set of initial conditions, whose corresponding solutions reach $\partial S_{\mathbf{k}} \cap (\Delta_K \cup \Delta_{K+1})$ at some time, has zero measure.

Fix any $\mathbf{k} \in \mathbb{Z}^N, K \in \mathbb{Z}$ an consider the set $\partial S_{\mathbf{k}} \cap \Delta_K$. This is the intersection of the boundary of a hypercube with a hyperplane, and has dimension at most $(N-2)$. For any point $\bar{x} \in \partial S_{\mathbf{k}} \cap \Delta_K$, the vector field has a finite number of limit values corresponding to the indices $\mathbf{l} \in N_{\mathbf{k}} \subset \mathbb{Z}^N$ such that $S_{\mathbf{l}} \cap S_{\mathbf{k}} \neq \emptyset$. Then, a finite number of local backward solutions of (1) through $\bar{x}$ can exist. Let $\phi_{\mathbf{l}}$ be such solutions. The set $G_{\mathbf{l}} = \cup_{\bar{x} \in \partial S_{\mathbf{k}} \cap \Delta_K}\{x \in \mathbb{R}^N : x =$

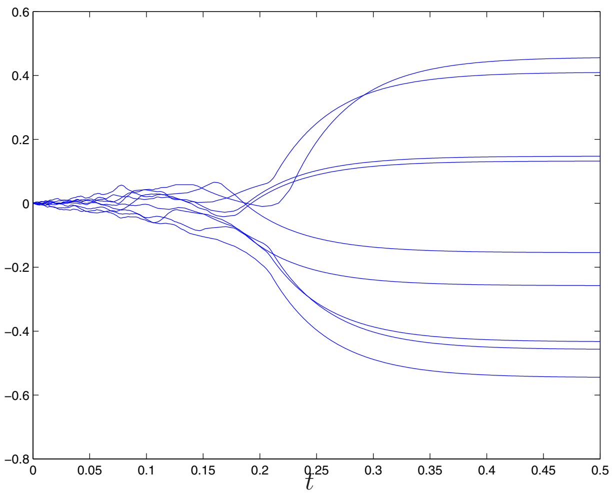

Fig. 4. Sample evolutions of $\frac{1}{N}\sum_i x_i(t) - \frac{1}{N}\sum_i x_i(0)$ for the dynamics (1) with random initial conditions and $N = 20$.

$\phi_{\mathbf{l}}(t) \in S_{\mathbf{l}}, t \in (-\infty, \bar{t}), \phi_{\mathbf{l}}(\bar{t}) = \bar{x}\}$ is then a $(N-1)$-dimensional differential manifold and has null measure. Moreover $\mathrm{cl}(G_{\mathbf{l}}) \cap \left(\cup_{\mathbf{k}\in\mathbb{Z}^N} \partial S_{\mathbf{k}}\right)$ is a $(N-2)$-dimensional manifold. We can then consider solutions in neighboring hypercubes that can reach point in $\mathrm{cl}(G_{\mathbf{l}}) \cap \left(\cup_{\mathbf{k}\in\mathbb{Z}^N} \partial S_{\mathbf{k}}\right)$ and repeat considerations analogous to those already done. In this way it results that the set of points that can reach some point in some $\partial S_{\mathbf{k}} \cap \Delta_K$ is the union of a denumerable union of null-measure sets and then has null measure. ∎

IV. Non-complete graphs

In the previous two sections we have defined and studied a system achieving consensus in case quantized states can be communicated. This system requires all-to-all communication and can thus be seen as a dynamics on a complete graph. However, the engineering and social applications would suggest to consider more general assumptions on the available communication. If the links between the agents are described by a non-complete undirected graph, such an extended model is then

$$\dot{x}_i(t) = \sum_{j\neq i} a_{ij}[q(x_j(t)) - x_i(t)], \qquad i \in \mathcal{I}, \tag{4}$$

where a_{ij}s are the entries of the symmetric adjacency matrix $A \in \{0,1\}^{N\times N}$ encoding the graph topology ($a_{ij} = 0$ if j does not influence i). We will denote to the right-hand side of (4) as $\bar{f}(x)$.

Some of our results about (1) carry on to (4): Carathéodory solutions exist from every initial conditions and they all are bounded and complete. Moreover, simulations suggest that the average of the states is only slightly perturbed during the evolution; some examples are provided in Figure 5. Beyond these basic facts, however, the dynamics are very different from the complete graph. Indeed, they do not preserve the order between solutions and –more important– solutions to (4) can converge to points that are not consensus. This fact can be observed in the simulation of Figure 6, as well as in simple examples.

Example 2 (Non-consensus in line graphs): We consider $N = 4$ agents whose communications are described by a line graph, that is, such as their states evolve according to

$$\begin{aligned} \dot{x}_1 &= q(x_2) - x_1, \\ \dot{x}_i &= q(x_{i-1}) + q(x_{i+1}) - 2x_i \quad i \in \{2, \dots, N-1\}, \\ \dot{x}_N &= q(x_{N-1}) - x_N. \end{aligned} \tag{5}$$

If we take the initial condition $(0, 0.49, 0.51, 1)$, then the corresponding Carathéodory solution tends to the point $x^* = (0, 0.5, 0.5, 1)$, showing no consensus in the limit, but a clusterization of opinions instead. ∎

Interestingly, the limit point x^* in Example 2 is such that $\bar{f}(x^*) \neq 0$ and nevertheless it is attractive for the dynamics: this pathological behavior is allowed by the discountinuity of the vector field $\bar{f}(x)$. At a closer look, however, we can observe the following fact: x^* is not an equilibrium, that is, $\bar{f}(x^*) \neq 0$, but $\bar{f}|_{S_{[0,0,1,1]}}(x^*) = 0$ and x^* belongs to the topological closure of $S_{[0,0,1,1]}$. Hence, solutions that start in a given quantization bin $S_{\mathbf{k}}$ can converge to points that are not equilibria of the piecewise-defined $\bar{f}(x)$, but are both equilibria of the "piece" $\bar{f}|_{S_{\mathbf{k}}}(x)$ and accumulation points of $S_{\mathbf{k}}$. Whether this property is shared by all attractors of (4) is currently an open question.

Furthermore, the following construction –which is a generalization of Example 2– shows that these pathological attractors can be significantly far from consensus.

Example 3 (Linear disagreement): Let ℓ be a positive integer, $N = 2\ell + 2$, and $\varepsilon > 0$ a small number. Assume interactions as in (5). Then, any initial condition in the form

$$[0, \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, \dots, \ell - \frac{1}{2} - \varepsilon, \ell - \frac{1}{2} + \varepsilon, \ell]$$

(modulo translations by an integer) converges to a point $\overline{x}$ such that $\max_i \overline{x}_i - \min_i \overline{x}_i = \frac{N-2}{2}$. ∎

Hence, the distance from consensus can grow linearly in the number of agents.

These preliminary observations lead us to two comments. First, it is plain that the analysis of (4) is not an immediate extension of the analysis for the complete graph, in view of the complexity of its set of attractors. Second, the significant disagreement that characterizes some of such attractors discourages any attempt to simplify the problem by proving that the quantized dynamics is "close" to the corresponding unquantized consensus system.

V. Future work

From the mathematical perspective, the natural prosecution of this work is the thorough analysis of the general model presented in Section IV, including a proof of convergence and the characterization of the limit points. On this matter, we note that the attractors highlighted in Example 2 are equilibria for a suitable convexification of the vector field. It is then a natural question whether solutions according to Krasovskii or Filippov are useful in this context.

From the perspective of the social science applications, we recall a key question in the opinion dynamics literature: identifying the causes of the persistence of disagreement in

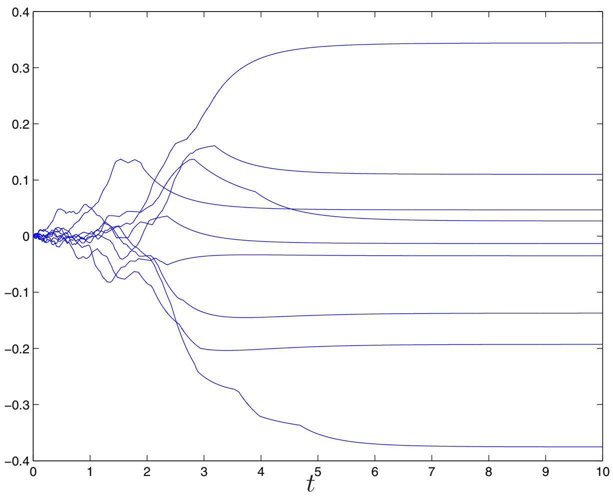

Fig. 5. Sample evolutions of $\frac{1}{N}\sum_i x_i(t) - \frac{1}{N}\sum_i x_i(0)$ for the dynamics (5) with random initial conditions and $N = 20$.

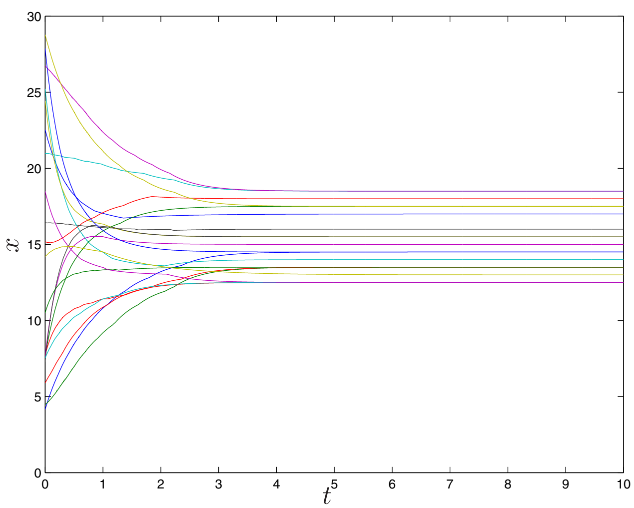

Fig. 6. Simulation of (5) for $N = 20$ from random initial conditions in $[0, 30]$, showing convergence to a non-consensus state.

social networks, in spite of the imitative forces that tend to bring opinions closer to each other. Different answers have already been given, including bounded confidence [28] and the presence of stubborn agents in the network [19]. Our model suggests that limited verbalization can be another cause of persistent disagreement. Consequently, our work motivates further studies on the interplay between discrete and continuous variables in opinion dynamics.

References

[1] A. Kashyap, T. Başar, and R. Srikant, "Quantized consensus," *Automatica*, vol. 43, no. 7, pp. 1192–1203, 2007.

[2] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, "Communication constraints in the average consensus problem," *Automatica*, vol. 44, no. 3, pp. 671–684, 2008.

[3] T. C. Aysal, M. J. Coates, and M. G. Rabbat, "Distributed average consensus with dithered quantization," *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 4905–4918, 2008.

[4] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri, "Average consensus on networks with quantized communication," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 16, pp. 1787–1816, 2009.

[5] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis, "On distributed averaging algorithms and quantization effects," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2506–2517, 2009.

[6] J. Lavaei and R. M. Murray, "Quantized consensus by means of gossip algorithm," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 19–32, 2012.

[7] C. De Persis and B. Jayawardhana, "Coordination of passive systems under quantized measurements," *SIAM Journal on Control and Optimization*, vol. 50, no. 6, pp. 3155–3177, 2012.

[8] W. Chen, X. Li, and L. C. Jiao, "Quantized consensus of second-order continuous-time multi-agent systems with a directed topology via sampled data," *Automatica*, vol. 49, no. 7, pp. 2236–2242, 2013.

[9] S. Liu, T. Li, L. Xie, M. Fu, and J. F. Zhang, "Continuous-time and sampled-data-based average consensus with logarithmic quantizers," *Automatica*, vol. 49, no. 11, pp. 3329–3336, 2013.

[10] M. Guo and D. V. Dimarogonas, "Consensus with quantized relative state measurements," *Automatica*, vol. 49, no. 8, pp. 2531–2537, 2013.

[11] Q. Zhang and J.-F. Zhang, "Quantized data-based distributed consensus under directed time-varying communication topology," *SIAM Journal on Control and Optimization*, vol. 51, no. 1, pp. 332–352, 2013.

[12] C. De Persis and P. Frasca, "Robust self-triggered coordination with ternary controllers," *IEEE Transactions on Automatic Control*, vol. 58, no. 12, pp. 3024–3038, 2013.

[13] M. El Chamie, J. Liu, and T. Başar, "Design and analysis of distributed averaging with quantized communication," in *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, Dec. 2014, pp. 3860–3865.

[14] M. Jafarian and C. De Persis, "Formation control using binary information," *Automatica*, vol. 53, pp. 125–135, 2015.

[15] R. Carli, F. Fagnani, P. Frasca, T. Taylor, and S. Zampieri, "Average consensus on networks with transmission noise or quantization," in *European Control Conference*, Kos, Greece, Jun. 2007, pp. 1852–1857.

[16] F. Ceragioli, C. De Persis, and P. Frasca, "Discontinuities and hysteresis in quantized average consensus," *Automatica*, vol. 47, no. 9, pp. 1916–1928, 2011.

[17] Y. Wang, Q. Wu, Y. Wang, and D. Yu, "Quantized consensus on first-order integrator networks," *Systems & Control Letters*, vol. 61, no. 12, pp. 1145–1150, 2012.

[18] F. Ceragioli and P. Frasca, "Continuous and discontinuous opinion dynamics with bounded confidence," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 3, pp. 1239–1251, 2012.

[19] D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar, "Opinion fluctuations and disagreement in social networks," *Mathematics of Operations Research*, vol. 38, no. 1, pp. 1–27, 2013.

[20] J. Hendrickx and A. Olshevsky, "Symmetric continuum opinion dynamics: Convergence, but sometimes only in distribution," in *IEEE Conference on Decision and Control*, Florence, Italy, Dec. 2013, pp. 1989–1994.

[21] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2013.

[22] N. E. Friedkin and E. C. Johnsen, *Social Influence Network Theory: A Sociological Examination of Small Group Dynamics*. Cambridge University Press, 2011.

[23] D. Urbig, "Attitude dynamics with limited verbalisation capabilities," *Journal of Artificial Societies and Social Simulation*, vol. 6, no. 1, pp. 1–23, 2003.

[24] A. Martins, "Continuous opinions and discrete actions in opinion dynamics problems," *International Journal of Modern Physics C*, vol. 19, no. 4, pp. 617–624, 2008.

[25] P. Jia and P. E. Caines, "Analysis of a class of decentralized dynamical systems: rapid convergence and efficiency of dynamical quantized auctions," *IMA Journal of Mathematical Control and Information*, pp. 329–372, 2010.

[26] J. Cortés, "Discontinuous dynamical systems – a tutorial on solutions, nonsmooth analysis, and stability," *IEEE Control Systems Magazine*, vol. 28, no. 3, pp. 36–73, 2008.

[27] P. Frasca, "Continuous-time quantized consensus: Convergence of Krasovskii solutions," *Systems & Control Letters*, vol. 61, no. 2, pp. 273–278, 2012.

[28] R. Hegselmann and U. Krause, "Opinion dynamics driven by various ways of averaging," *Computational Economics*, vol. 25, no. 4, pp. 381–405, 2005.