An index for the "local" influence in social networks

Wilbert Samuel Rossi[1] and Paolo Frasca[1]

Abstract— **In this paper we define a novel index of node centrality in social networks that extends the recently proposed Harmonic Influence Centrality (HIC) and that we call *Local-Harmonic Influence centrality* (L-HIC). Indeed, when compared with the HIC, our index shows a local nature that rules out one pathological behavior of the HIC. Similarly to the HIC, the L-HIC can be approximated by a distributed *message passing algorithm* that is inspired by an analogy between electrical and social networks on tree graphs. We prove a result that guarantees convergence on graphs containing at most one cycle.**

I. Introduction

In the study of networks and dynamical processes therein, one important issue is the identification of the most influential nodes, i.e. those with the higher ability to drive the others towards a desired state. Different notions of centrality have been proposed, depending on the process and the control objective, see [1], [2], [3], [4], [5].

Inspired by works like [6], [7], [8], [9], we describe an opinion dynamics model in a network containing stubborn agents. Regular agents are influenced uniformly by their neighbors and update their opinions following a local consensus rule that includes a bias.

Given a group of stubborn agents, we wish to identify among the remaining regular agents the one that, once turned into a stubborn, would "influence" the network the most. We formulate an *Optimal Stubborn Placement Problem* assuming that, without loss of generality, the given stubborns and the bias have null opinion, while the "test" stubborn has opinion one. The solution of this optimization problem can be used in a greedy sub-optimal routine to place further stubborns in the network thanks to the submodularity of the harmonic influence [8].

In order to measure the influence of the test stubborn, we generalize the *Harmonic Influence Centrality* index, introduced in [1], [10] to account for the bias. We call the new measure *Local-HIC*: indeed, the opinion dynamic model has a useful electrical interpretation that justifies the use of the adjectives *local* and *harmonic* in the index name. In fact, the asymptotic opinion of the regular agents can be computed as the harmonic electrical potential when the social graph is interpreted as an electrical network. In this interpretation, the bias opinion plays the role of reference potential, localizing the decay of the stubborns influence. The electrical interpretation also suggests how to design a *message passing algorithm* (MPA) to approximate the L-HIC. Such an MPA is exact and convergent on trees and converges on regular graphs, by a direct extension of the results in [1]. By using our formulation, in this paper we prove that the MPA converges on any graph containing at most one cycle. Actually, we believe that our approach can be extended to general topologies.

Paper Structure: Section II describes the opinion dynamic model with the external bias, introduces the Local-HIC index, and formulates the optimal stubborn placement problem. Section III discusses the electrical interpretation of the model, gives the expression of the L-HIC, and studies its properties. Section IV presents the MPA to compute the L-HIC, its proof of convergence on unicyclic graphs, and some simulation results. Section V concludes the paper.

Notation: The set of real and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Vectors are denoted with boldface letters and matrices with capital letters. The all-zero and all-one vectors are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. The symbol $\mathbb{I}$ denotes any identity matrix with appropriate dimension. The symbol $\preccurlyeq$ denotes entry-wise $\leq$ for vectors and matrices. The symbol $\prec$ is used if the entry-wise inequality is strict for at least one entry. Given a matrix Q, $Q^\top$ denotes its transpose and Q^{-1} its inverse. Given a vector $\mathbf{v}$, $\mathrm{Diag}(\mathbf{v})$ is the square diagonal matrix with the entries of $\mathbf{v}$ on the main diagonal. The cardinality of the set S is denoted by $|S|$. Given the matrix $Q \in \mathbb{R}^{S\times S}$ and the subsets $T, T' \subseteq S$, $Q_{T,T'}$ denotes the sub-matrix of Q that selects the rows and columns corresponding to T and T', respectively. A non-negative matrix $Q \in \mathbb{R}_+^{S\times S}$ is said to be stochastic, sub-stochastic and strictly sub-stochastic if $Q\mathbf{1} = \mathbf{1}$, $Q\mathbf{1} \preccurlyeq \mathbf{1}$ and $Q\mathbf{1} \prec \mathbf{1}$, respectively. A matrix Q is called Schur stable if all its eigenvalues are strictly inside the unit circle.

II. Opinion Dynamics and Optimal Stubborn Agent Placement

Consider a simple, undirected and connected graph $\mathcal{G} = (I, E)$ with node set I of cardinality N and edge set E. The set $N_i = \{j \in I : \{i,j\} \in E\}$ contains the neighbors of i in $\mathcal{G}$; the degree of i is $d_i = |N_i|$. Each node represents an agent, endowed with a scalar opinion $x_i[t]$ of initial value $x_i[0] \in \mathbb{R}$, updated at discrete time steps $t \in \mathbb{N}$. All opinions are stacked in the vector $\mathbf{x}[t] \in \mathbb{R}^I$.

The agents are partitioned in two subsets: the set R of *regular* agents and the set S of *stubborn* agents. Stubborn agents never change their opinions, thus

$$\mathbf{x}_S[t] = \mathbf{x}_S[0] \quad \forall t \in \mathbb{N}.$$

Instead, each regular agent updates its opinion to a convex combination of its own opinion, the opinions of its neighbors,

[1]Wilbert Samuel Rossi and Paolo Frasca are with the Department of Applied Mathematics, University of Twente, 7500 AE Enschede, The Netherlands w.s.rossi@utwente.nl; p.frasca@utwente.nl

and the constant external bias $x_{\mathfrak{b}} \in \mathbb{R}$. Consider a sub-stochastic matrix $Q \in \mathbb{R}_+^{I\times I}$ and a non-negative vector $\mathbf{q}\in\mathbb{R}_+^I$ such that

$$\begin{cases} Q_{ij}=0 \Leftrightarrow \{i,j\}\notin E \\ \sum_j Q_{ij}+q_i=1 \end{cases} \qquad \forall i\in R\,. \tag{1}$$

The weights Q_{ij} and q_i represent how much agent i trusts agent j and the bias, respectively. The update rule of the regular agents takes the compact form

$$\mathbf{x}_R[t+1]=Q_{R,R}\mathbf{x}_R[t]+Q_{R,S}\mathbf{x}_S[0]+\mathbf{q}_R x_{\mathfrak{b}} \quad \forall t\geq 0,$$

with initial condition $\mathbf{x}_R[0]$ and with the following assumption.

Assumption 0: $Q_{R,R}$ is a strictly sub-stochastic matrix. •

Thanks to the connectivity of $\mathcal{G}$, this assumption implies that $Q_{R,R}$ is a Schur stable matrix [11, Lemma 5]. Thus, given $\mathbf{x}_S[0]$ and $x_{\mathfrak{b}}$, the opinions of the regular agents tend to the limit $\mathbf{x}_R[\infty]$, which is the unique solution of

$$\mathbf{x}_R[\infty]=Q_{R,R}\mathbf{x}_R[\infty]+Q_{R,S}\mathbf{x}_S[0]+\mathbf{q}_R x_{\mathfrak{b}}\,. \tag{2}$$

The regular agents' asymptotic opinions $\mathbf{x}_R[\infty]$ are convex combinations of the opinions $\mathbf{x}_S[0]$ and the bias $x_{\mathfrak{b}}$.

In our model we assume that each regular agent weights uniformly the influence of his neighbors in $\mathcal{G}$.

Assumption 1: Row by row, the non-zero entries of the matrix $Q_{R,I}$ have the same value. •

Assumption 1 and (1) together are equivalent to the existence of a non-negative vector $\boldsymbol{\gamma}\in\mathbb{R}_+^I$ such that for every $i\in R$

$$Q_{ij}=\begin{cases} (d_i+\gamma_i)^{-1} & \text{if } j\in N_i \\ 0 & \text{otherwise}\end{cases}$$

and $q_i=1-\sum_j Q_{ij}=\gamma_i(d_i+\gamma_i)^{-1}$. For $i\notin R$, the values Q_{ij} and q_i have no role and will be chosen at convenience.

To formulate the *Optimal Stubborn Agent Placement* (OSAP) problem, given the graph $\mathcal{G}=(I,E)$ we fix a subset $Z\subset I$ and the vector $\boldsymbol{\gamma}\in\mathbb{R}_+^I$. The decision variable is $\ell\in I\setminus Z$: the set of stubborn agents is the disjoint union $S^\ell=Z\cup\{\ell\}$. Without loss of generality we assume the following initial conditions.

Assumption 2: The opinion of the stubborn agents are $\mathbf{x}_Z[0]=\mathbf{0}$ and $x_\ell[0]=1$, while the bias is $x_{\mathfrak{b}}=0$. •

The regular agents form the subset $R^\ell=I\setminus S^\ell$; their asymptotic opinions are collected in the vector $\mathbf{x}_{R^\ell}[\infty]$. Following [1], we define the influence of ℓ on the network as the sum of the asymptotic opinions of all the agents. Thus, the Local Harmonic Influence Centrality (L-HIC) of ℓ is

$$H(\ell):=1+\mathbf{1}^\top\mathbf{x}_{R^\ell}[\infty].$$

The OSAP problem amounts at finding the agent in $I\setminus Z$ with highest L-HIC:

$$\ell^*=\arg\max_{\ell\in I\setminus Z} H(\ell).$$

The agent ℓ^* is the best suited to introduce a new, different opinion in the social network (with respect to the null bias and the null opinion of the stubborn agents in Z).

We stress that our formulation of the opinion dynamic model distinguishes the role of stubborn agents and bias, with the freedom to give a different weight to the bias; this choice will be useful in the proofs of Section IV. Note that the bias can be regarded as an additional stubborn agent, to which every node may be connected. For $\boldsymbol{\gamma}=\mathbf{0}$, we recover the formulation of [1].

III. The Electrical Interpretation

In the light of Assumptions 0 and 1, the opinion dynamic model of Section II is intimately related to the linear circuit theory and the asymptotic opinions $\mathbf{x}_{R^\ell}[\infty]$ can be interpreted as electrical potentials.

In general an *electrical network* is a pair $(\mathcal{N},C)$, where $\mathcal{N}=(J,F)$ is a simple connected graph and $C\in\mathbb{R}_+^{J\times J}$ is a non-negative, symmetric matrix "adapted" to it (i.e. $C_{ij}=C_{ji}=0\Leftrightarrow\{i,j\}\notin F$). To each edge $\{i,j\}\in F$ we assign the electrical conductance C_{ij} and hence we call C *conductance matrix*. In the following we describe the electrical network $(\mathcal{N},C)$ corresponding to the our opinion dynamic model.

Consider the graph $\mathcal{G}=(I,E)$ of the opinion dynamic model and the vector $\boldsymbol{\gamma}$. Given Assumptions 0 and 1, a new vector $\tilde{\boldsymbol{\gamma}}\in\mathbb{R}_+^I$ with $\tilde{\boldsymbol{\gamma}}_R=\boldsymbol{\gamma}_R$ and $\tilde{\boldsymbol{\gamma}}_S=\mathbf{1}$ is always $\tilde{\boldsymbol{\gamma}}\succ\mathbf{0}$. Consider the extended node set $I_{\mathfrak{b}}:=I\cup\{\mathfrak{b}\}$ and the new graph $\mathcal{N}=(I_{\mathfrak{b}},E\cup E_{\mathfrak{b}})$, obtained adding to $\mathcal{G}$ the *reference* node $\mathfrak{b}$ and the edges in $E_{\mathfrak{b}}=\{\{\mathfrak{b},i\}:i\in I,\tilde{\gamma}_i>0\}$. We remark that the graph $\mathcal{N}$ is connected by construction. Corresponding to $\mathcal{G}$ we introduce the *adjacency matrix* $A\in\{0,1\}^{I\times I}$ with $A_{ij}=A_{ji}=1\Leftrightarrow\{i,j\}\in E$ and the *Laplacian matrix* $L=\mathrm{Diag}(A\mathbf{1})-A$. Then, the conductance matrix $C\in\mathbb{R}_+^{I_{\mathfrak{b}}\times I_{\mathfrak{b}}}$ has entries

$$C_{I,I}=A,\ C_{I,\{\mathfrak{b}\}}=\tilde{\boldsymbol{\gamma}},\ C_{\{\mathfrak{b}\},I}=\tilde{\boldsymbol{\gamma}}^\top,\ C_{\{\mathfrak{b}\},\{\mathfrak{b}\}}=0 \tag{3}$$

The pair $(\mathcal{N},C)$ is the electrical network corresponding to our opinion dynamic model: edges between nodes in I have unit conductance while edges between regular nodes in R and node $\mathfrak{b}$ have conductance γ_i. The conductance of edges between S and node $\mathfrak{b}$ will not affect the following analysis.

The asymptotic opinions of the regular agents, solution of (2), coincide with the electrical potentials that solve the circuit equations of the network.

Lemma 1: Consider the opinion dynamic model on the graph $\mathcal{G}$ with stubborn agents' set S, initial opinions $\mathbf{x}[0]$ and bias $x_{\mathfrak{b}}$, and let Assumptions 0 and 1 hold. Consider the electrical network $(\mathcal{N},C)$ as defined above and the vector with the nodes' electrical potentials $\mathbf{y}\in\mathbb{R}^{I_{\mathfrak{b}}}$. If the potentials of the node $\mathfrak{b}$ and the nodes in S are held at $y_{\mathfrak{b}}=x_{\mathfrak{b}}$ and $\mathbf{y}_S=\mathbf{x}_S[0]$ respectively, then $\mathbf{y}_R=\mathbf{x}_R[\infty]$.

Proof: Given the Assumptions 0 and 1, the electrical network $(\mathcal{N},C)$ described above is connected, has a symmetric conductance matrix C and contains at least one node with known and fixed potential. The potentials of the nodes in $R\subset I\subset I_{\mathfrak{b}}$ are uniquely determined using the Kirchhoff's current law and the Ohm's law. The system of

$|R|$ independent *node* equations is

$$\forall i \in R \quad \sum_{j \in I_{\mathfrak{b}}} C_{ij}(y_i - y_j) = 0.$$

Substituting the entries of C (3) and using $\tilde{\boldsymbol{\gamma}}_R = \boldsymbol{\gamma}_R$ we get

$$\forall i \in R \quad \sum_{j \in N_i} (y_i - y_j) + \gamma_i (y_i - y_{\mathfrak{b}}) = 0. \tag{4}$$

Dividing each equation by $d_i + \gamma_i > 0$ we recognize the elements Q_{ij} and q_i

$$\forall i \in R \quad \sum_{j \in I} Q_{ij}(y_i - y_j) + q_i (y_i - y_{\mathfrak{b}}) = 0.$$

Since $q_i + \sum_{j \in I} Q_{ij} = 1$ for every $i \in R$, we get

$$\forall i \in R \quad y_i = \sum_{j \in R} Q_{ij} y_j + \sum_{j \in S} Q_{ij} y_j + q_i y_{\mathfrak{b}}$$

that in compact form is $\mathbf{y}_R = Q_{R,R}\mathbf{y}_R + Q_{R,S}\mathbf{y}_S + \mathbf{q}_R y_{\mathfrak{b}}$. If $\mathbf{y}_S = \mathbf{x}_S[0]$ and $y_{\mathfrak{b}} = x_{\mathfrak{b}}$, the vector $\mathbf{y}_R$ coincides with $\mathbf{x}_R[\infty]$ because it satisfies the set of equations (2). ■

Assumption 2 sets the values of bias and stubborn agents' opinion, thus fixing the potentials of the node $\mathfrak{b}$ and the nodes in S^ℓ. The potential of the nodes in R^ℓ can be computed with the sub-matrix L_{R^ℓ,R^ℓ} of the Laplacian of $\mathcal{G}$.

Lemma 2: Consider the above electrical network $(\mathcal{N}, C)$, the subset $Z \subset I$ and the node $\ell \in I \setminus Z$. Let Assumptions 0, 1 and 2 hold. The potentials of the nodes in R^ℓ are

$$\mathbf{y}_{R^\ell} = (L_{R^\ell,R^\ell} + \mathrm{Diag}(\boldsymbol{\gamma}_{R^\ell}))^{-1} A_{R^\ell,\{\ell\}} \, . \tag{5}$$

Proof: We rewrite the system (4) as

$$\forall i \in R \quad (d_i + \gamma_i) y_i - \sum_{j \in N_i \cap R^\ell} y_j = \sum_{j \in N_i \cap S^\ell} y_j + \gamma_i y_{\mathfrak{b}}.$$

Using $y_{\mathfrak{b}} = 0$, $y_\ell = 1$ and $y_i = 0$ if $i \in Z \subset S^\ell$, we get

$$\forall i \in R \quad (d_i + \gamma_i) y_i - \sum_{j \in N_i \cap R^\ell} y_j = A_{i\ell}$$

since $A_{i\ell} = 1 \Leftrightarrow \ell \in N_i$. Using the sub-matrix L_{R^ℓ,R^ℓ} of the Laplacian of $\mathcal{G}$, we recognize that the system above is

$$(L_{R^\ell,R^\ell} + \mathrm{Diag}(\boldsymbol{\gamma}_{R^\ell}))\mathbf{y}_{R^\ell} = A_{R^\ell,\{\ell\}}$$

The result follows because the matrix $L_{R^\ell,R^\ell} + \mathrm{Diag}(\boldsymbol{\gamma}_{R^\ell})$ is invertible: L_{R^ℓ,R^ℓ} is positive definite [12] (the graph $\mathcal{G}$ is connected and $\emptyset \subset R^\ell \subset I$) and $\boldsymbol{\gamma}_{R^\ell}$ is non-negative. ■

Using the electrical interpretation – Lemmas 1 and 2 – we can compute the L-HIC as

$$\begin{aligned} H(\ell) &= 1 + \mathbf{1}^\top \mathbf{y}_{R^\ell} \\ &= 1 + \mathbf{1}^\top \left(L_{R^\ell,R^\ell} + \mathrm{Diag}(\boldsymbol{\gamma}_{R^\ell})\right)^{-1} A_{R^\ell,\{\ell\}}. \end{aligned}$$

The Assumptions 0, 1 and 2 will always hold in the remaining part of the paper.

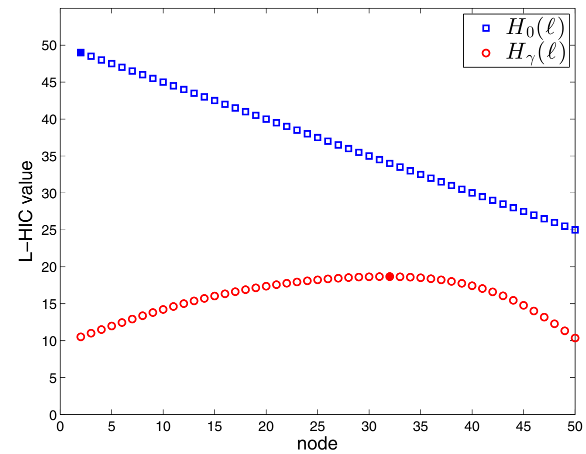

Fig. 1. The L-HIC of the nodes $i \geq 2$ in the line graph with 50 nodes, $Z = \{1\}$ and bias $\gamma\mathbf{1}$. Circles correspond to $\gamma = 0.010$; squares to $\gamma = 0$.

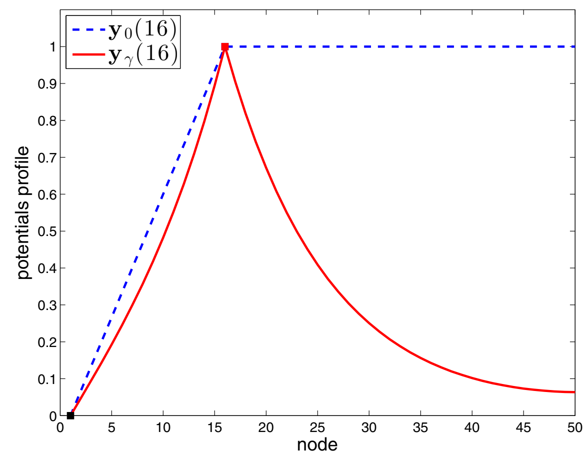

Fig. 2. The node's potential in the line graph with 50 nodes, $Z = \{1\}$, bias $\gamma\mathbf{1}$ and $\ell = 16$. The solid line corresponds to $\gamma = 0.010$; the dashed line to $\gamma = 0$.

A. *The Local Harmonic Influence Centrality*

The potentials $\mathbf{y}_{R^\ell}$ are said to be the *harmonic* extension of the fixed potentials of bias and stubborn nodes to the regular nodes. For this reason, [1] and [10] call the index $H(\ell)$ the *Harmonic Influence Centrality* (HIC) of node ℓ.

We call our index *Local* HIC due to the presence of the bias, as discussed in the following simple example. Let $\mathcal{G} = (I, E)$ be the line graph with $N = 50$ nodes; let the first end node be a zero stubborn, e.g. $Z = \{1\}$. Assume the bias is uniform, i.e. $\boldsymbol{\gamma} = \gamma\mathbf{1}$ with $\gamma \geq 0$. In the electrical network, there is a unitary resistance corresponding to any edge in E while there is a resistance γ^{-1} between the reference node $\mathfrak{b}$ and any node in $I \setminus Z$. The L-HIC of ℓ, for any $\gamma \geq 0$, is

$$H_\gamma(\ell) := 1 + \mathbf{1}^\top (L_{R^\ell,R^\ell} + \gamma\mathbb{I})^{-1} A_{R^\ell,\{\ell\}} \, , \tag{6}$$

the node achieving the highest L-HIC is

$$\ell^*_\gamma = \arg\max_{\ell \in I \setminus Z} H_\gamma(\ell)$$

We stress that $H_0(\ell)$ is the index of [1], [10].

In the Figure 1 we compare $H_\gamma(\ell)$ with $\gamma = 0.010$ (red circles) with $H_0(\ell)$ (blue squares), for $\ell \in I \setminus Z$. The full markers represent the nodes ℓ^*_γ and ℓ^*_0. In Figure 2 we set $\ell = 16$ and draw the harmonic potential profiles $\mathbf{y}_\gamma(16)$ (solid red line) and $\mathbf{y}_0(16)$ (blue dashed line).

If $\gamma = 0$ the bias does not come into play and we recover the result of [1]. The potential of every node i with $i > \ell$ is one, regardless of the distance $|i-\ell|$, because these nodes are only connected to the stubborn ℓ (see Figure 2 for $i > 16$). Using electrical computations it is easy to see that $H_0(\ell) = N - \ell/2$ for $\ell \geq 2$ (as confirmed by Figure 1) and hence $\ell^*_\gamma = 2$. However, fixed ℓ if $N \to \infty$, $H_0(\ell)$ diverges.

Even if $\gamma > 0$ is small, the profile of $H_\gamma(\ell)$ is quite different from that of $H_0(\ell)$, see Figure 1. The nodes in $\{20, \ldots, 40\}$ have a similar value of $H_\gamma(\ell)$ and $\ell^*_\gamma \neq 2$. In Figure 2, the potentials of the nodes with $i > 16$ (red line) decay with the distance $|i - \ell|$. In a very long line graph (N going to infinity), it is not difficult to show that the decay would be exponential in $|i - \ell|$ and hence $H_\gamma(\ell)$ cannot grow unboundedly. This is caused by the uniform bias, which makes every node connected to the reference potential. For this reason we use the adjective *Local*.

Even if the above example shows that HIC and L-HIC can be arbitrarily far apart for large graphs, the following result confirms the intuition that HIC and L-HIC can be arbitrarily close for small $\boldsymbol{\gamma}$.

Proposition 3: Let the graph $\mathcal{G} = (I, E)$ and the subset $Z \subset I$ be given and assume $\boldsymbol{\gamma} = \gamma\mathbf{1}$. For all $\ell \in I \setminus Z$, the L-HIC $H_\gamma(\ell)$ is a Lipschitz continuous function of γ, with Lipschitz constant independent of ℓ.

Proof: The matrix L_{R^ℓ,R^ℓ} is symmetric and positive definite. We denote by λ^ℓ_i and $\mathbf{v}^{\ell,i}$, with $i \in \{1, \ldots, |R^\ell|\}$, its i^{th} eigenvalue and orthonormal eigenvector, respectively. The matrix $(L_{R^\ell,R^\ell} + \gamma\mathbb{I})^{-1}$ has spectral decomposition

$$(L_{R^\ell,R^\ell} + \gamma\mathbb{I})^{-1} = \textstyle\sum_i (\lambda^\ell_i + \gamma)^{-1} \mathbf{v}^{\ell,i} \mathbf{v}^{\ell,i\top}$$

that, plugged into the L-HIC $H_\gamma(\ell)$ of ℓ (6), gives

$$\begin{aligned} H_\gamma(\ell) &= 1 + \textstyle\sum_i (\lambda^\ell_i + \gamma)^{-1} \mathbf{1}^\top \mathbf{v}^{\ell,i} \mathbf{v}^{\ell,i\top} A_{R^\ell,\{\ell\}} \\ &= 1 + \textstyle\sum_i c^\ell_i (\lambda^\ell_i + \gamma)^{-1} \end{aligned}$$

where the coefficients c^ℓ_i depend on ℓ. Since $\lambda^\ell_i > 0$ for every i and ℓ, the L-HIC is a continuous, differentiable function of $\gamma \geq 0$. Moreover, given a pair $\gamma, \gamma' \geq 0$ we have

$$\begin{aligned} \left|H_\gamma(\ell) - H_{\gamma'}(\ell)\right| &= \left| \sum_i c^\ell_i \left[\frac{1}{\lambda^\ell_i + \gamma} - \frac{1}{\lambda^\ell_i + \gamma'} \right] \right| \\ &\leq \sum_i \left|c^\ell_i\right| \frac{|\gamma - \gamma'|}{(\lambda^\ell_i + \gamma)(\lambda^\ell_i + \gamma')} \\ &\leq |\gamma - \gamma'| \sum_i \left|c^\ell_i\right| \left(\lambda^\ell_i\right)^{-2}. \end{aligned}$$

Therefore, for any $\ell \in I \setminus Z$

$$|H_\gamma(\ell) - H_{\gamma'}(\ell)| \leq \alpha\, |\gamma - \gamma'|$$

with $\alpha := \max_\ell \left(\sum_i \left|c^\ell_i\right| \left(\lambda^\ell_i\right)^{-2} \right)$ and the thesis follows. ∎

IV. Distributed computation of the L-HIC

This section is devoted to the distributed computation of the L-HIC index.

The computation of the L-HIC index $H(\ell)$ by its definition (6) requires the inversion of the matrix $(L_{R^\ell,R^\ell} + \mathrm{Diag}(\boldsymbol{\gamma}_{R^\ell}))$: this procedure has to be repeated for every node $\ell \in I \setminus Z$ to solve the Optimal Stubborn Agent Placement problem. Thus, this direct method requires global knowledge of the graph and becomes computationally demanding for large networks. Hence, it is interesting to develop a distributed approach, to allow every node in $I \setminus Z$ to estimate its own L-HIC index.

We shall make the following assumption.

Assumption 3: Either $Z \neq \emptyset$ or $\boldsymbol{\gamma} \succ \mathbf{0}$ or both. Moreover, the degree of any node $z \in Z$ is $d_z = 1$. •

Indeed, the case with $Z = \emptyset$ and $\boldsymbol{\gamma} = \mathbf{0}$ is trivial: regardless of the choice of ℓ, the opinion of every agent in $I \setminus \{\ell\}$ tends asymptotically to one and hence $H(\ell) = N$. Moreover, nodes in Z can be assumed to have degree one without loss of generality, because otherwise one can simply replace the stubborn node with degree k with k "copies" of the same node, each only connected to one of the original neighbors.

We briefly recall some terminology. A *tree* is a connected graph $\mathcal{T} = (I, E)$ with $|I| = |E| - 1$. A connected graph $\mathcal{G} = (I, E)$ is *unicyclic* if $|I| = |E|$; it is a *cycle* if moreover $d_i = 2$, $\forall i \in I$.

Inspired by the electrical analogy, the authors of [1], [10] proposed a *Message Passing Algorithm* (MPA) for the distributed estimation of the HIC index, i.e. the L-HIC with $\boldsymbol{\gamma} = \mathbf{0}$ and $Z \neq \emptyset$. If the graph $\mathcal{G}$ is a tree, the MPA converges in a finite number of iterations to the exact HIC values. Moreover, convergence was proved on regular graphs and conjectured on general graphs.

Using the electrical analogy, we extend the MPA to the presence of a bias $\boldsymbol{\gamma} \succcurlyeq \mathbf{0}$. Consider a connected graph $\mathcal{G} = (I, E)$, the subset $Z \subset I$ and the bias vector $\boldsymbol{\gamma} \in \mathbb{R}^I_+$. Let $\{i, j\}$ be an edge in E and let $t \in \{0, 1, \ldots\}$ be the iteration counter. At each step, node i sends to node j two messages: $W^{i\to j}[t] \in [0,1]$ and $H^{i\to j}[t] \in \mathbb{R}_+$. These two messages can be interpreted as follows [1].

- *If $\mathcal{G}$ is a tree and $j \equiv \ell$, then $W^{i\to j}[t]$ is the time-t estimate of the potential of i.*
- *If $\mathcal{G}$ is a tree, remove the edge $\{i, j\}$ and consider the connected subgraph containing i. Then, $H^{i\to j}[t]$ is the time-t estimate of the L-HIC index of i in its subgraph.*

At $t = 0$ the MPA starts with the following messages

$$\begin{aligned} &\text{if } i \notin Z: \quad W^{i\to j}[0] = 1, \quad H^{i\to j}[0] = 1; \\ &\text{if } i \in Z: \quad W^{i\to j}[0] = 0, \quad H^{i\to j}[0] = 0. \end{aligned}$$

At every iteration, if $i \notin Z$, the messages update with

$$W^{i\to j}[t+1] = \left(1 + \gamma_i + \textstyle\sum_{k\in N_i\setminus\{j\}} \left(1 - W^{k\to i}[t]\right)\right)^{-1} \quad (7)$$

$$H^{i\to j}[t+1] = 1 + \textstyle\sum_{k\in N_i\setminus\{j\}} W^{k\to i}[t]\, H^{k\to i}[t], \quad (8)$$

whereas there is no update if $i \in Z$: $W^{i\to j}[t+1] = 0$ and $H^{i\to j}[t+1] = 0$. The estimate of the L-HIC index $H(\ell)$ is

$$H^{\ell}[t] = 1 + \sum_{i \in N_\ell} W^{i\to\ell}[t]\, H^{i\to\ell}[t].$$

Next, we show the convergence of the MPA on trees and prove a theorem that implies the convergence of the MPA on cycles. Combining the two results, we will finally show the convergence of the MPA on unicyclic graphs.

We need the following notation. Given the graph $\mathcal{G} = (I, E)$, we denote its size by $|\mathcal{G}| = |I|$. We call *leaf* any node of degree one. We call *simple path* any ordered sequence of distinct nodes $(k_1, k_2, \dots, k_q)$ (excluding k_1 and k_q at most), such that $\{k_p, k_{p+1}\} \in E$ for all $p \in \{1, \dots, q-1\}$. Let $\mathcal{G}$ be a tree and for $\{i, j\} \in E$ consider all the simple paths having form $(j, i, \dots)$. We let $\mathcal{T}^{ij<}$ denote the smallest subgraph of $\mathcal{G}$ containing all such paths, while $\mathcal{T}^{<ij}$ is the subgraph of $\mathcal{T}^{ij<}$ without the edge $\{i, j\}$ and the node j.

Lemma 4: Consider a tree $\mathcal{T} = (I, E)$ and let j be a leaf, with $\{i, j\} \in E$. The messages $W^{i\to j}[t]$ and $H^{i\to j}[t]$ converge after a finite number of steps. Moreover, $W^{i\to j}[t]$ is non-increasing; $\lim_t W^{i\to j}[t] = 1$ if and only if $Z = \emptyset$ and $\boldsymbol{\gamma}_{I\setminus\{j\}} = \mathbf{0}$; and $\lim_t H^{i\to j}[t] \leq |\mathcal{T}| - 1$.

Proof: Consider any node $b \neq j$ and let $a \in N_b$ be the unique neighbor of b such that there exists a simple path from b to j containing a. If $N_b \setminus \{a\} = \emptyset$ then b is a leaf and the messages $W^{b\to a}[t]$ and $H^{b\to a}[t]$ converge by time $t = 1$. Note that in any case $W^{b\to a}[1] \leq W^{b\to a}[0]$ while $W^{b\to a}[1] = 1$ if and only if $b \notin Z$ and $\gamma_b = 0$. Moreover $H^{b\to a}[0] \leq 1 = |\mathcal{T}^{<ba}|$.

If $N_b \setminus \{a\}$ is non-empty, assume for the induction hypothesis that $\forall c \in N_b \setminus \{a\}$ the messages $W^{c\to b}[t]$ and $H^{c\to b}[t]$ converge by time s_c, $W^{c\to b}[t] \leq W^{c\to b}[t-1]$, $H^{c\to b}[s_c] \leq |\mathcal{T}^{<cb}|$ and finally $W^{c\to b}[t] = 1$ if and only if for every node k in $\mathcal{T}^{<cb}$, $\gamma_k = 0$ and $k \notin Z$. Then, the messages $W^{b\to a}[t]$ and $H^{b\to a}[t]$ converge at time $s_b = 1 + \max s_c$, and $W^{b\to a}[t+1] \leq W^{b\to a}[t]$ because the update function is monotone increasing in the messages $W^{c\to b}[t]$. Moreover $W^{b\to a}[s_b] = 1$ if and only if for every node k in $\mathcal{T}^{<ba}$, $\gamma_k = 0$ and $k \notin Z$, whereas $H^{b\to a}[t] \leq |\mathcal{T}^{<ba}|$ since $H^{c\to b}[s_c] \leq |\mathcal{T}^{<cb}|$ and $|\mathcal{T}^{<ba}| = 1 + \sum_c |\mathcal{T}^{<cb}|$. The thesis then follows by induction. ∎

Lemma 4 guarantees that the L-HIC estimate $H^j[t]$ of any node j in any tree converges after a finite number of steps. In fact it is sufficient to apply the lemma to any sub-tree $\mathcal{T}^{<ij}$ with $i \in N_j$.

We prove a theorem that guarantees the convergence of the MPA on any cycle graph $\mathcal{C}$ where $Z = \emptyset$ and $\boldsymbol{\gamma} \succ \mathbf{0}$. For $i \notin Z$, we consider the following generalization of the MPA update laws (7) and (8):

$$W^{i\to j}[t+1] = \Big(1 + \gamma_i[t] + \textstyle\sum_{k\in N_i\setminus\{j\}} \big(1 - W^{k\to i}[t]\big)\Big)^{-1} \quad (9)$$

$$H^{i\to j}[t+1] = 1 + \beta_i[t] + \textstyle\sum_{k\in N_i\setminus\{j\}} W^{k\to i}[t] H^{k\to i}[t] \quad (10)$$

where $\gamma_i[t] \in \mathbb{R}_+$ is monotonically increasing for every i and non identically zero for at least one i, while $\beta_i[t] \in \mathbb{R}_+$ converges for every i.

Theorem 5: Consider the cycle $\mathcal{C} = (J, F)$ with $J = \{0, \dots, N-1\}$ and let $Z = \emptyset$. Let $\boldsymbol{\gamma}[t] \in \mathbb{R}_+^J$ be a sequence of vectors with $\boldsymbol{\gamma}[t+1] \succcurlyeq \boldsymbol{\gamma}[t]$ for every t, such that $\gamma_i[t]$ is not identically zero at least for one i. Let $\boldsymbol{\beta}[t] \in \mathbb{R}_+^J$ be a converging sequence of vectors. The modified MPA with update rules (9) and (10) converges on $\mathcal{C}$.

Proof: The messages flowing in the edges of $\mathcal{C}$ can be grouped in two independent families: those flowing "clockwise" and those "counterclockwise". It is sufficient to prove that every clockwise message converges to prove that the modified dynamic (9) and (10) converges. Consider the clockwise messages and stack them in the two vectors $\mathbf{w}[t] \in [0, 1]^J$ and $\mathbf{h}[t] \in \mathbb{R}_+^J$ with

$$w_i[t] = W^{i\to i+1}[t], \quad h_i[t] = H^{i\to i+1}[t]$$

with index addition "modulo N" (also in the following).

First, we show that $\lim_{t\to\infty} \mathbf{w}[t] \prec \mathbf{1}$. Given the assumption on $\boldsymbol{\gamma}[t]$, it exists $s \geq 0$ such that for $t < s$, $\boldsymbol{\gamma}[t] = \mathbf{0}$ while $\boldsymbol{\gamma}[s] \succ \mathbf{0}$. Hence, $\mathbf{w}[t] = \mathbf{1}$ for $t \leq s$, while $\mathbf{w}[s+1] \prec \mathbf{1}$. Indeed, it exists j such that $\gamma_j[s] > 0$, which implies

$$w_j[s+1] = \frac{1}{1 + \gamma_j[s] + (1 - w_j[s])} = \frac{1}{1 + \gamma_j[s]} < 1\,.$$

Now, let $t \geq s+1$ be such that $\mathbf{w}[t] \prec \mathbf{w}[t-1]$ and let i such that $w_i[t] < w_i[t-1]$. Given the monotonicity of $\gamma_{i+1}[t]$, the strict inequality on node i implies a strict inequality on node $i+1$ one step later, because

$$\begin{aligned} w_{i+1}[t+1] &= \frac{1}{2 + \gamma_{i+1}[t] - w_i[t]} \\ &< \frac{1}{2 + \gamma_{i+1}[t-1] - w_i[t-1]} = w_{i+1}[t]\,. \end{aligned}$$

Hence $\mathbf{w}[t+1] \prec \mathbf{w}[t]$, and by induction this inequality holds for every $t \geq s+1$. The vector $\mathbf{w}[t] \in [0, 1]^J$ thus converges with $w_i[\infty] < 1$ for every i.

Now, consider the matrix $M[t] \in \mathbb{R}_+^{J\times J}$ such that

$$\forall i \in J \quad M_{i,i-1}[t] = w_{i-1}[t]$$

while all the other entries are zero. Each matrix $M[t]$ is non-negative, irreducible and for $t \geq s+1$, $M[t]\mathbf{1} \prec \mathbf{1}$, hence $\lim_t M[t]$ is a Schur stable matrix. The modified update rule (10) for the vector $\mathbf{h}[t]$ has compact form

$$\mathbf{h}[t+1] = \mathbf{1} + \boldsymbol{\beta}[t] + M[t]\mathbf{h}[t]$$

with $\mathbf{h}[0] = \mathbf{1}$. Using the convergence of $\boldsymbol{\beta}[t]$ and of $M(t)$, the convergence of $\mathbf{h}(t)$ follows from straightforward calculus considerations. ∎

Observe that with $\boldsymbol{\gamma}[t] = \boldsymbol{\gamma} \succ \mathbf{0}$ and $\boldsymbol{\beta}[t] = \mathbf{0}$ in the statement of the Theorem 5 we obtain the MPA with update rules (7) and (8). Hence, the theorem proves the convergence of the MPA on cycles.

We are now ready to prove our result on the convergence of the MPA on unicyclic graphs. Any unicyclic graph $\mathcal{G}$ can be decomposed in a "core" cycle to which several appendages are connected. Each of these appendages is a tree and has just one node in common with the cycle. The key idea is the following. All the messages coming from the appendages toward the cycle converge in finite time. Assume j is a node in the cycle and $i \in N_j$ is a node in one appendage: the message $W^{i \to j}[t]$ can be seen as an additional contribution to the bias of j, while the message $H^{i \to j}[t]$ just add up in the cycle computation. Finally, after the messages on the cycle have converged, also the messages flowing back in to the appendages will converge.

Corollary 6: Consider a connected unicyclic graph $\mathcal{G} = (I, E)$, the subset $Z \subset I$ and the bias vector $\boldsymbol{\gamma} \in \mathbb{R}_+^I$. The MPA with update rules (7) and (8) converges on $\mathcal{G}$.

Proof: Let $\mathcal{C} = (J, F)$ be the unique cycle contained in $\mathcal{G}$. Take $\{i, j\} \in E$ such that $j \in J$ and $i \notin J$. Observe that the subgraph $\mathcal{T}^{ij<}$ is a tree and apply Lemma 4 to $\mathcal{T}^{ij<}$ using the edge $\{i, j\}$: we obtain that the messages $W^{i \to j}[t]$ and $H^{i \to j}[t]$ converge in finite time. Let the vectors $\boldsymbol{\gamma}'[t], \boldsymbol{\beta}'[t] \in \mathbb{R}_+^J$ be defined by

$$\gamma'_j[t] = \gamma_j + \textstyle\sum_{i \in N_j \setminus J} (1 - W^{i \to j}[t])$$
$$\beta'_j[t] = \textstyle\sum_{i \in N_j \setminus J} H^{i \to j}[t].$$

These vectors satisfy the hypothesis of Theorem 5: indeed, by Assumption 3 there exists in $\mathcal{G}$ a node i such that either $i \in Z$ or $\gamma_i > 0$. We thus obtain that any message in $\mathcal{C}$ converge. It remains to prove that the messages going back in the appendages converge. For every $\{i, j\} \in E$ such that $j \in J$ and $i \notin J$, consider the messages $W^{j \to i}[t]$ and $H^{j \to i}[t]$. Observe that they are continuous functions of convergent messages, therefore they converge. By an inductive argument, all the message in $\mathcal{T}^{ij<}$ converge. ■

Finally, we present some simulation results to compare the L-HIC and its estimate obtained with the distributed MPA. To produce the simulation, we generate an Erdős-Rényi random graph with $N = 50$ nodes and link probability 0.100 and we place one null stubborn: the bias vector is uniform with $\gamma = 0.100$. The MPA converges and returns an approximation that overestimates the L-HIC of all the nodes, but correctly identifies the nodes with highest L-HIC. We illustrate this fact by Figure 3, where every point has coordinates $(H_\gamma(\ell), H_\gamma(\ell)$ MPA). In the plot, the approximate ranking is correct if the points form a strictly increasing function: note that the ranking of the 7 nodes with largest L-HIC is correctly identified by the MPA computation. Furthermore, Spearman's rank-order correlation coefficient [13] between the two series is 0.99549, implying a very high correlation between the two rankings. This is good news because, in view of the Optimal Stubborn Placement Problem, we are more interested in ranking the nodes by their L-HIC than in obtaining a precise approximation.

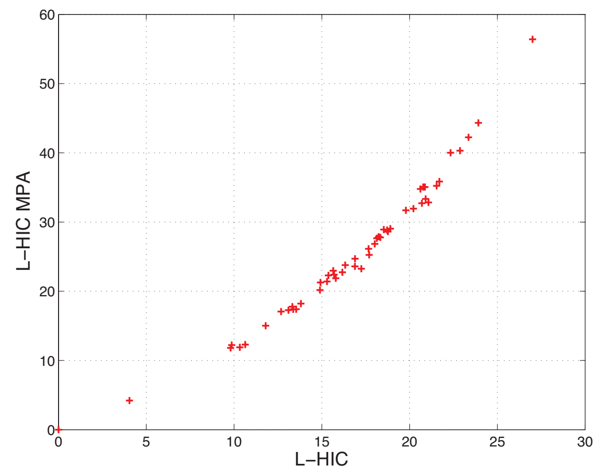

Fig. 3. Ranking correlation of the L-HIC and its estimate with the MPA on a Erdős-Rényi random graph, with $N = 50$ nodes, link probability 0.100, one null stubborn and bias $\boldsymbol{\gamma} = 0.100\,\mathbf{1}$. The coordinates of the crosses are $(H_\gamma(\ell), H_\gamma(\ell)$ MPA).

V. Conclusion

Extending some recent work in distributed influence maximization [1], in this paper we have introduced the *Local-Harmonic Influence Centrality* index and a distributed message passing algorithm to compute it. We proved the convergence of this algorithm on any graph that contains at most one cycle. Our convergence analysis subsumes the results in [1] and, based on preliminary results that could not be included in this paper, we envision the possibility to extend it to general graph topologies.

References

[1] L. Vassio, F. Fagnani, P. Frasca, and A. Ozdaglar, "Message passing optimization of harmonic influence centrality," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 1, pp. 109–120, 2014.

[2] D. Kempe, J. Kleinberg, and É. Tardos, "Maximizing the spread of influence through a social network," in *Proc. of the 9th ACM SIGKDD Int. Conf. on Knowledge Discovery and Data Mining*, 2003, pp. 137–146.

[3] M. Pirani and S. Sundaram, "Spectral properties of the grounded Laplacian matrix with applications to consensus in the presence of stubborn agents," in *American Control Conference (ACC), 2014*, June 2014, pp. 2160–2165.

[4] E. Estrada and N. Hatano, "A vibrational approach to node centrality and vulnerability in complex networks," *Physica A: Statistical Mechanics and its Applications*, vol. 389, no. 17, pp. 3648 – 3660, 2010.

[5] D. Easley and J. Kleinberg, *Networks, crowds, and markets: Reasoning about a highly connected world*. Cambridge University Press, 2010.

[6] N. E. Friedkin and E. C. Johnsen, "Social influence networks and opinion change," in *Advances in Group Processes*, E. J. Lawler and M. W. Macy, Eds. JAI Press, 1999, vol. 16, pp. 1–29.

[7] D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar, "Opinion fluctuations and disagreement in social networks," *Mathematics of Operations Research*, vol. 38, no. 1, pp. 1–27, 2013.

[8] E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione, "Binary opinion dynamics with stubborn agents," *ACM Trans. Econ. Comput.*, vol. 1, no. 4, pp. 19:1–19:30, Dec. 2013.

[9] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, "A new model of opinion dynamics for social actors with multiple interdependent attitudes and prejudices," in *Decision and Control (CDC), 2015 IEEE 54th Conference on*, Dec 2015, pp. 3475–3480.

[10] L. Vassio, F. Fagnani, P. Frasca, and A. Ozdaglar, "A message passing algorithm for the evaluation of social influence," in *Control Conference (ECC), 2014 European*, June 2014, pp. 190–195.

[11] P. Frasca, C. Ravazzi, R. Tempo, and H. Ishii, "Gossips and prejudices: Ergodic randomized dynamics in social networks," in *IFAC Workshop on Estimation and Control of Networked Systems*, Koblenz, Germany, Sept. 2013, pp. 212–219.

[12] P. Barooah and J. Hespanha, "Graph effective resistance and distributed control: Spectral properties and applications," in *Decision and Control, 2006 45th IEEE Conference on*, Dec 2006, pp. 3479–3485.

[13] J. H. Zar, *Spearman Rank Correlation*. Wiley, 2005.