Laboratoire des Signaux & Systèmes

Robustness margins and robustification of nominal explicit MPC

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29 June 2016

European Control Conference

Outline

- Problem statement
- Robustness margins
  - Polytopic uncertainty
  - Gain margin
  - Neglected dynamics
- Robustification of explicit solutions

Problem statement

- **A nominal discrete time linear systems**

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases}$$

- **A function** $u_{pwa} : X \to \mathbb{R}^p$ **defined over a polyhedral partition** $X = \bigcup_{i \in I_N} X_i \subset \mathbb{R}^n$

$$u_{pwa}(x) = G_i x + g_i \quad for \;\; x \in X_i$$

$$G_i \in \mathbb{R}^{p \times n} \quad g_i \in \mathbb{R}^{p \times 1}$$

Is a piecewise affine control law over the partition X

Problem statement

➢ Polyhedral state partition

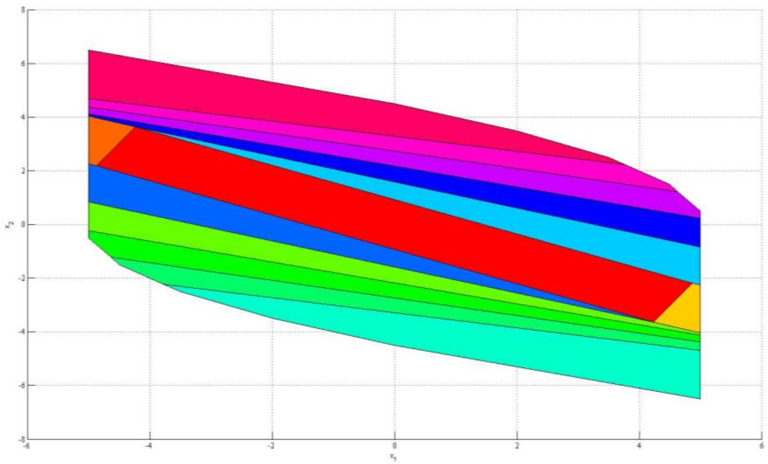
A polyhedral partition of a compact set $X \subset \mathbb{R}^n$ is defined as follows:

1) $X = \bigcup_{i \in I_N} X_i, \ N \in N_+$

2) X_i is polyhedral $\forall i \in I_N$

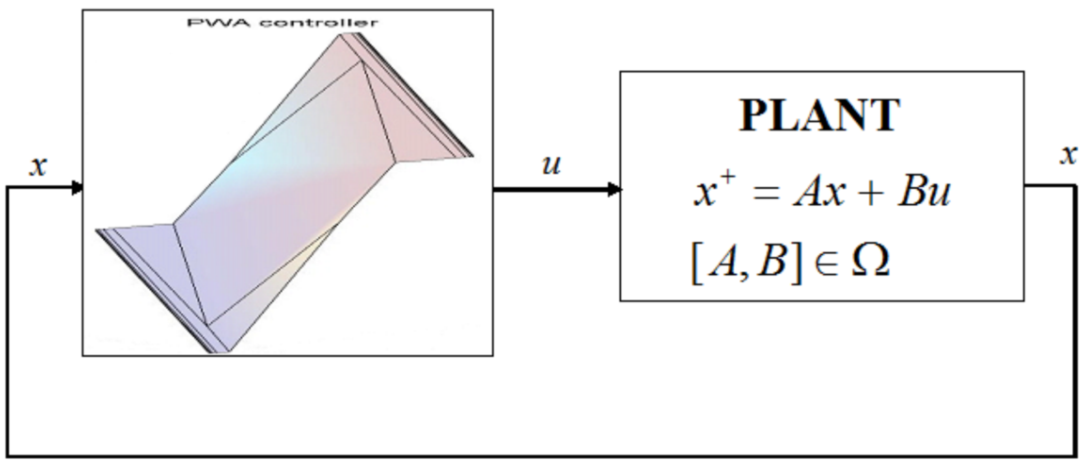
3) $\text{int}(X_i) \bigcap \text{int}(X_j) = \varnothing$ with $i \neq j, \ (i, j) \in I_N^2$

4) (X_i, X_j) are neighbours if $(i, j) \in I_N^2$,
$i \neq j$ and $\dim(X_i \bigcap X_j) = n - 1$



Problem statement

- Several popular techniques lead to such a problem formulation, among which we can mention for example the Model Predictive Control in its explicit form (Bemporad et al, Seron et al, Tondel et al., Olaru and Dumur).



Problem statement

➢ **Goals:**

- To analyse the inherent robustness of PWA control towards model uncertainties → robustness margin problems.
- To further tune the original controller to improve robustness margin.
- Synthesize explicit solution with good robustness margin.

Outline

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Robustness margins

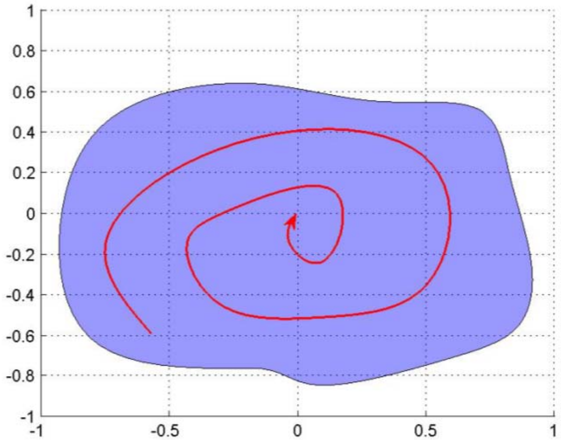
- **Nominal closed loop**

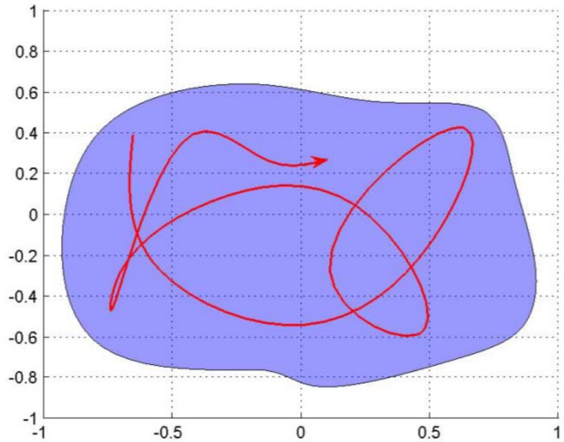
  The nominal closed loop dynamics represent a PWA system:

$$x(k+1) = f_{pwa}\left(x(k)\right) = \left(A_0 + B_0 G_i\right) x(k) + B_0 g_i \quad \text{for } x(k) \in X_i$$

- **Positive invariance**

  A set $X \in \mathbb{R}^n$ is posive invariant with respect to the system $x(k+1) = f_{pwa}\left(x(k), w\right)$ if for any $x(k_0) \in X$, $x(k) \in X \;\; \forall k > \mathbb{N}$





Robustness margins

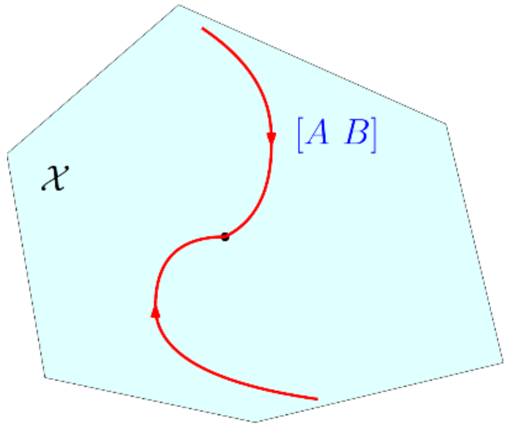
- **Assumptions**
  - The set X is a polytope
  - The set X is positive invariant with respect to the nominal model
  - The control $u_{pwa} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous
  - The origin is the only fixed point of the nominal dynamics
  - The origin is asymptotically stable with X as basin of attraction

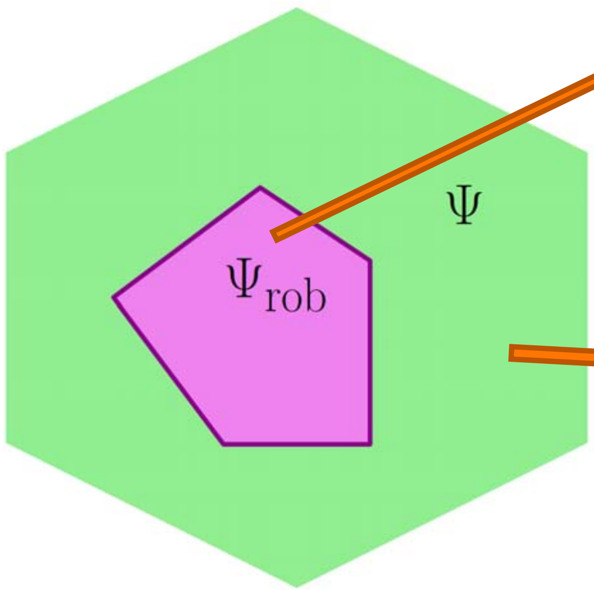
Robustness margins

- **Robust invariance for polytopic uncertainty**

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = Cx(k) \end{cases}$$

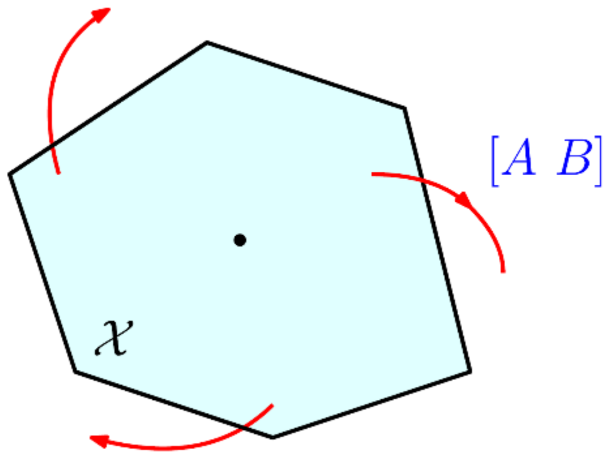
$$[A(k)\,B(k)] \in \Psi = conv([\mathrm{A}_1, \mathrm{B}_1], \ldots [\mathrm{A}_L, \mathrm{B}_L])$$





$$[A(k)\,B(k)] \in \Psi_{rob}$$

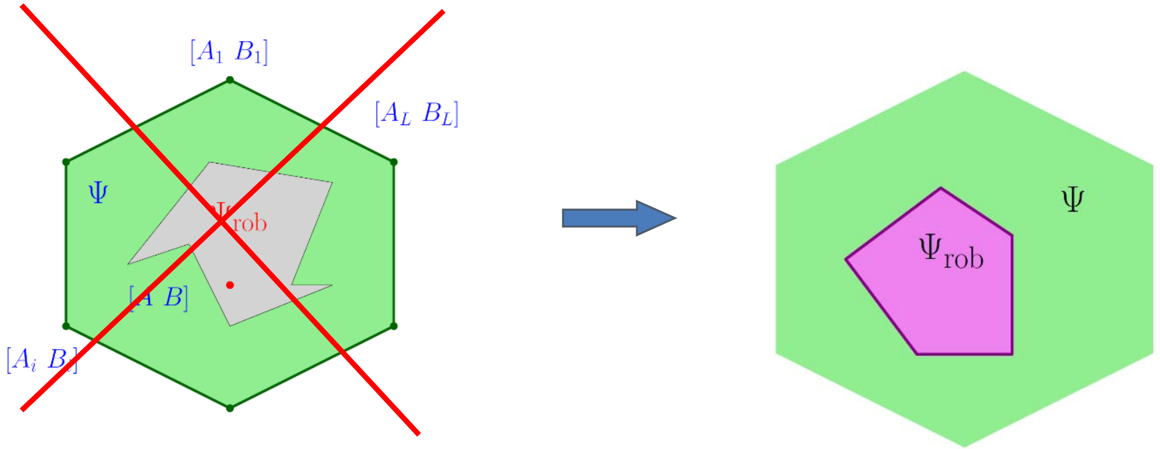
$$[A(k)\,B(k)] \notin \Psi_{rob}$$



Robustness margins

- **Robust invariance for polytopic uncertainty**

➡ Ψ_{rob} is polytopic



$$\Psi^{\alpha}_{\text{rob}} = \left\{ \alpha \in \mathbb{R}^L_+ \,\middle|\, \sum_{j=1}^{L} \alpha_j \left(A_j + B_j G_i\right) X_i \oplus \alpha_j B_j g_i \subset X,\ \forall_i \in I_N, \mathbf{1}^T \alpha = 1 \right\}$$

➡ Ψ_{rob} can be computed using vertex or half-space representation of

$X = \bigcup_{i \in I_N} X_i,\ N \in N_+$

Robustness margins

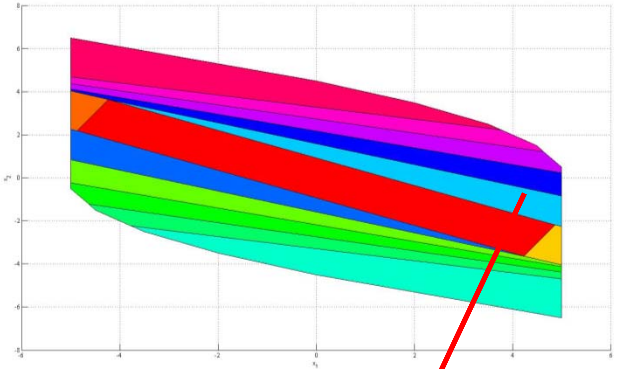
- **For half-space representation:**

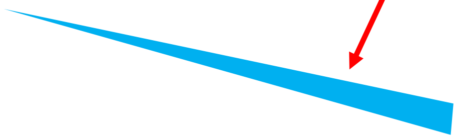
$$X = \left\{ x \in \mathbb{R}^n \,\middle|\, Fx \le h \right\} \qquad X_i = \left\{ x \in \mathbb{R}^n \,\middle|\, F_i x \le h_i \right\}$$

$$\Psi_{\mathrm{rob}}^{\alpha} = \mathrm{Proj}_{\mathbb{R}^L} P$$

$$P = \left\{ \left(\alpha, M_1, \ldots M_N\right) \middle| \begin{array}{l} \sum_{j=1}^{L} \alpha_j F\left(A_j + B_j G_i\right) = M_i F_i, \\ M_i h_i \le h - F \sum_{j=1}^{L} \alpha_j B_j g_i, \end{array} \quad \forall i \in I_N \right\}$$

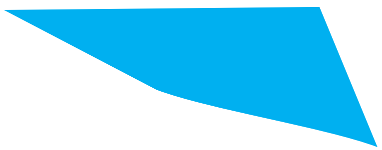
Robustness margins





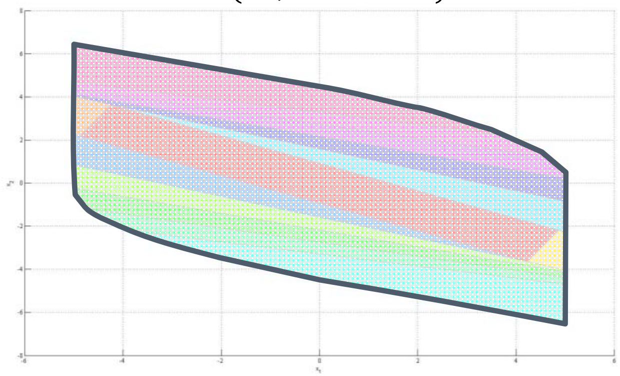
$$F_i x \le h_i$$

$$P_1 = \left\{ x \middle| F_i \left(A(k) + B(k)(G_i x + g_i) \right) \le h_i \right\}$$



Positive Invariance if : $P_1 \subseteq P_2$

$$P_2 = \left\{ x \middle| Fx \le h \right\}$$



$$P_1 \subseteq P_2 \Leftrightarrow \exists M \ s.t.$$

$$\begin{cases} vec(M) \ge 0, \\ MF_1 = F_2 \\ Mh_1 \le h_2 \end{cases}$$

Robustness margins

- **Numerical example**

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0.9 & 0.5 \end{bmatrix} A_2 = \begin{bmatrix} 2 & 1 \\ 1.5 & 1.5 \end{bmatrix} A_3 = \begin{bmatrix} 1.5 & 1 \\ 3.8 & 1 \end{bmatrix}$$

$$B_1 = B_2 = B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In presence of constraints on the control variable and the output variable

$$-0.2 \leq u_k \leq 0.2$$

$$-0.5 \leq y_k \leq 0.5$$

With the nominal model chosen to synthesize a PWA controller law

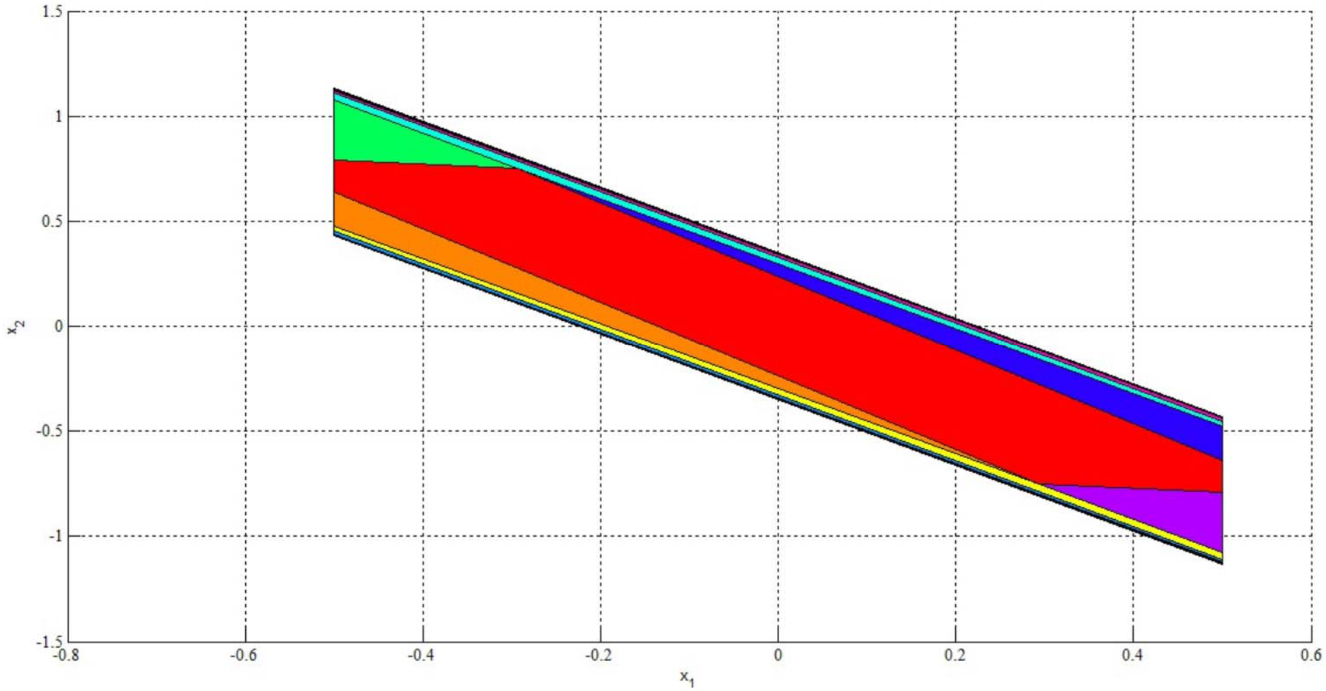
$$A_0 = 0.5A_1 + 0.3A_2 + 0.2A_3$$

$$B_0 = B_1$$

Robustness margins

- **Numerical example**

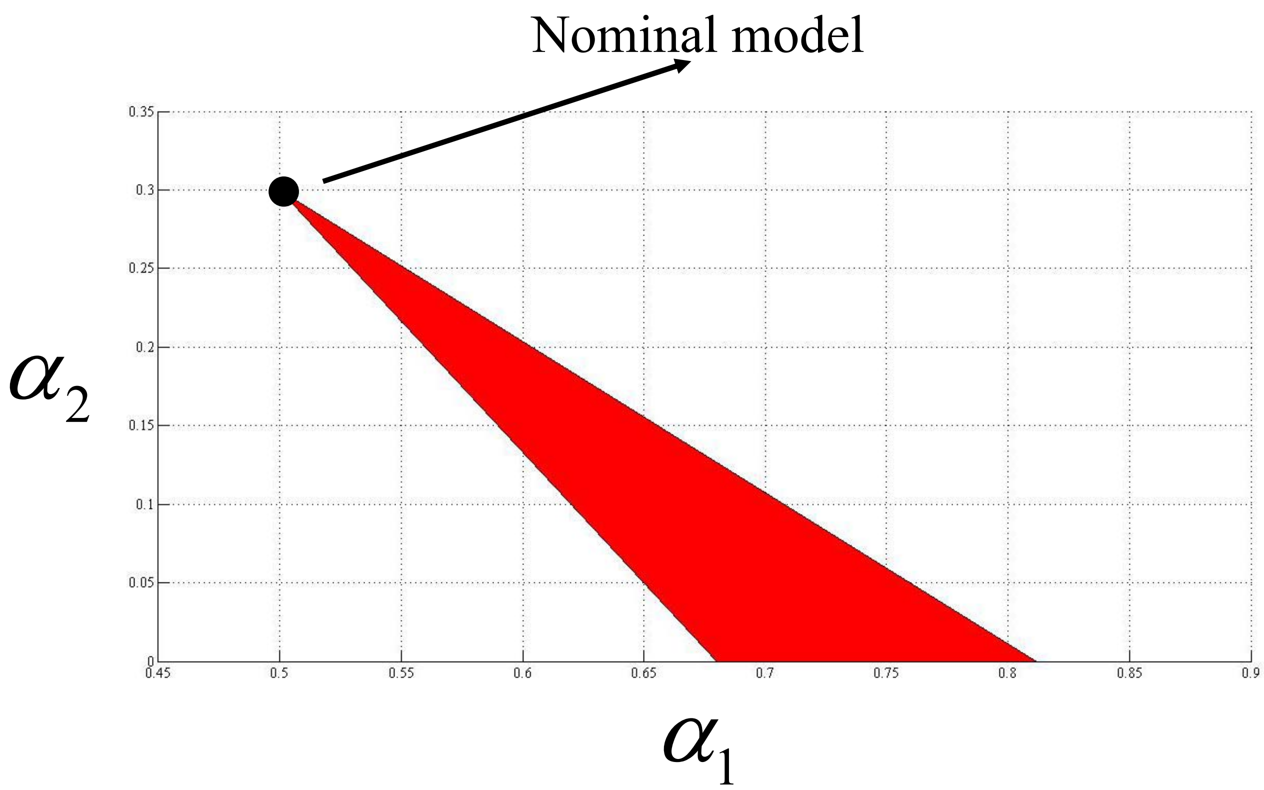
The partition of the obtained PWA control law



Robustness margins

- **Numerical example**

Ψrob projected on the plane $[\alpha_1,\alpha_2]$. Note that $\alpha_3 = 1-\alpha_1-\alpha_2$



Robustness margins

- **Gain margin**

  The gain margin is represented by the set $K \subset \mathbb{R}^m$, such that:

$$x(k+1) = Ax(k) + B\left(I_m + diag(\delta_K)\right)u_{pwa}\left(x(k)\right) \in X, \ \forall x(k) \in X \ \ and \ \ \delta_K \in K$$

$$K = \bigcap_{q=1}^{p} K_q$$

$$K_q = \left\{ z \in \mathbb{R}^m \,\middle|\, \exists u \in \Delta U_q,\ Mz = u \right\}$$

$$\boxed{\begin{array}{l} \Delta U_q = \mathrm{Proj}_U H_q \\ H_q = \left\{ (\delta u, \lambda) \in \mathbb{R}^m \times \mathbb{R}^r,\ and\ [A\ B]\begin{bmatrix} x(k) \\ u_{pwa}\left(x(k)\right) \end{bmatrix} + B\delta u = W\lambda \right\} \end{array}}$$

W stores the vertices of X : $\quad W \in \mathbb{R}^{n \times r}, W = [w_1, \dots w_r]$

Robustness margins

- **Gain margin: numerical example**

Consider a linear discrete system with two inputs and two outputs

$$x(k+1) = \begin{bmatrix} 1.2 & -1.0 & 0 \\ 0 & -1.2 & 0.5 \\ 0.2 & 0.4 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1.0 & 0.2 \\ 0.5 & 0 \\ 0 & 0.7 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k).$$

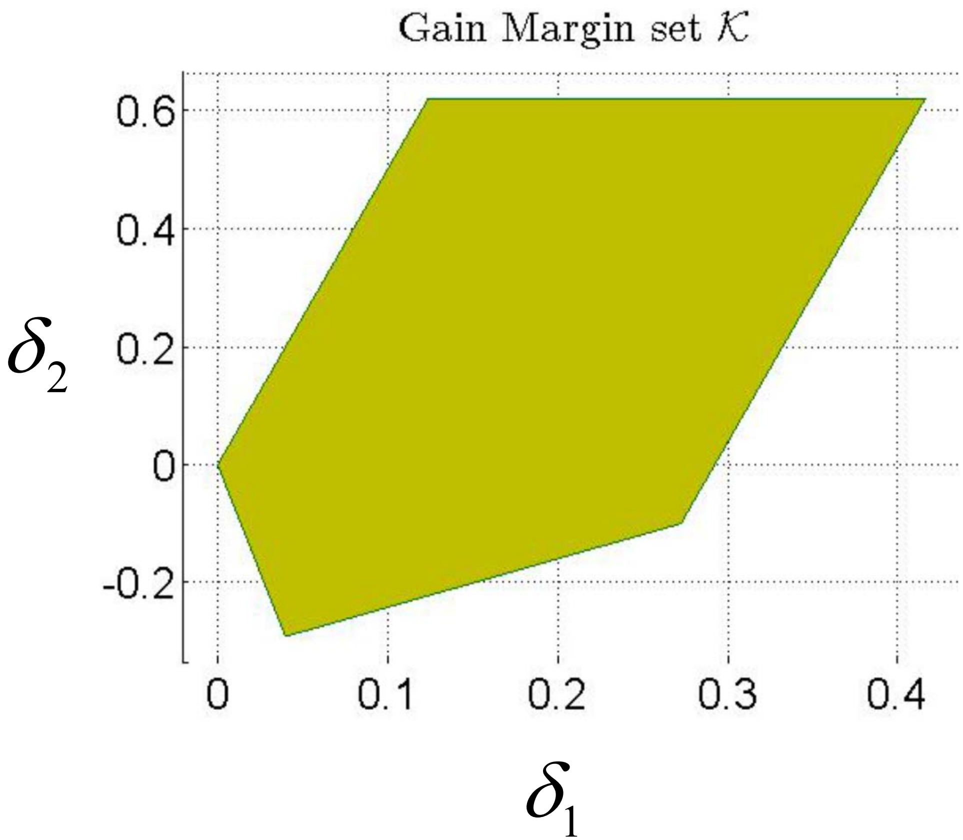
The weight applied on the control inputs and state vectors are

$$Q = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}.$$

The input constraints, $\begin{bmatrix} -2 \\ -2 \end{bmatrix} \leq u(k) \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and the output constraints $\begin{bmatrix} -2 \\ -2 \end{bmatrix} \leq y(k) \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Prediction horizon $N_p = 2$

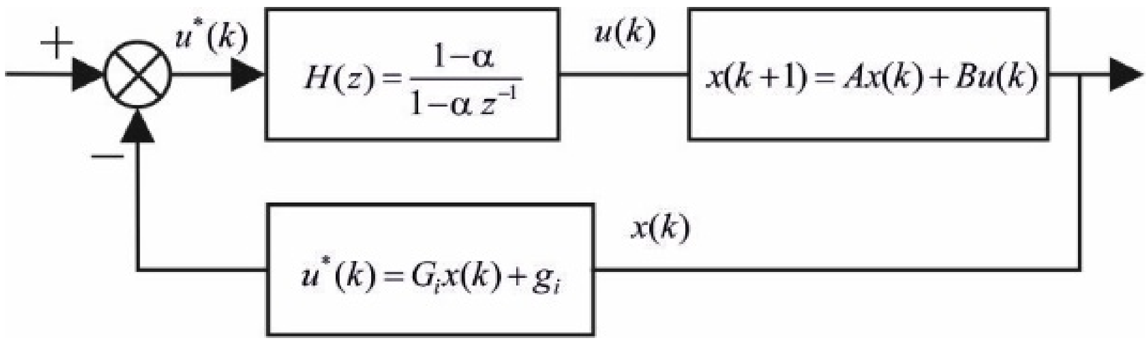
Robustness margins

- **Gain margin: numerical example**



Robustness margins

- **Neglected dynamics**



The augmented model is:

$$x_e(k+1)=\begin{bmatrix} A & \alpha B \\ 0 & \alpha \end{bmatrix} x_e(k)+\begin{bmatrix} B(1-\alpha) \\ \alpha \end{bmatrix} u(k)$$

$$u(k)=\left(x_e(k)\right)=\begin{bmatrix} G_i & 0 \end{bmatrix} x_e(k)+g_i \text{ for } x(k)\in X_i$$

Robustness margins

- **Neglected dynamics**

$$x_e(k+1)=\begin{bmatrix} A & \alpha B \\ 0 & \alpha \end{bmatrix} x_e(k)+\begin{bmatrix} B(1-\alpha) \\ \alpha \end{bmatrix} u(k)$$

Polytopic description of extended model

$$A_1=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} B_1=\begin{bmatrix} B \\ 1 \end{bmatrix} A_2=\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} B_1=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[A_e \ B_e]=(1-\alpha)[A_1 \ B_1]+\alpha[A_2 \ B_2]$$

$$\boxed{\begin{aligned} &\Omega^{\alpha}=\mathrm{Proj}_{\alpha}\ T \\ &T=\left\{(\alpha,\Gamma)\in\mathbb{R}\times\mathbb{R}^{r\times p}\middle|(1-\alpha)\left(A_1V_e+B_1U_e\right)+\alpha\left(A_2V_e+B_2U_e\right)=W\Gamma\right\} \end{aligned}}$$

W stores the vertices of X, V_e stores de vertices of X_i :

Robustness margins

- **Neglected dynamics : numerical example**

Consider the discrete time system given in the previous section. The margin for the first order neglected dynamics is represented by the set $\Omega^{\alpha} = \{0, 0.195\}$.

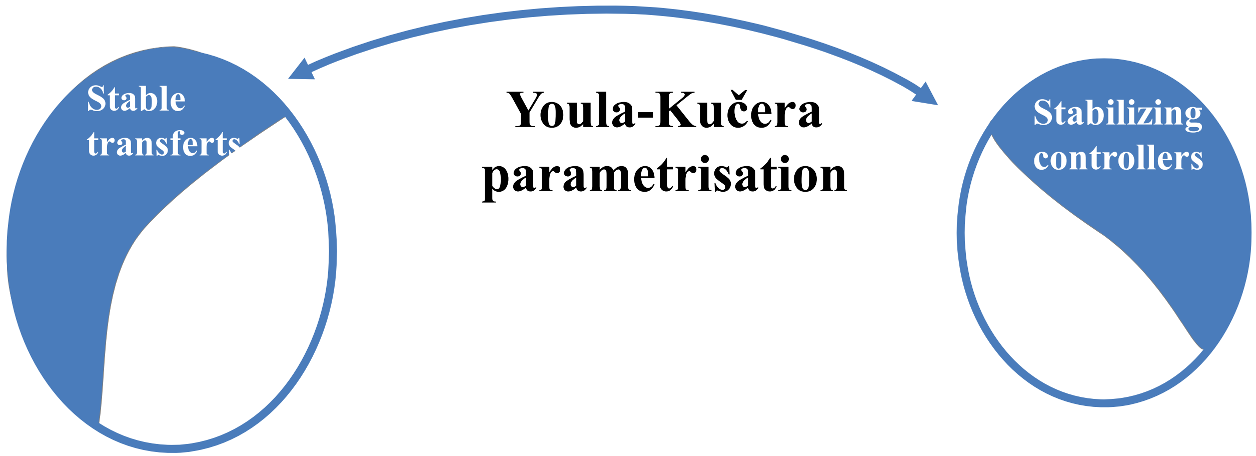
Outline

- Problem statement
- Robustness margins
  - Polytopic uncertainty
  - Gain margin
  - Neglected dynamics
- **Robustification of explicit solutions**

Robustification of explicit solutions

Robustification of linear control law

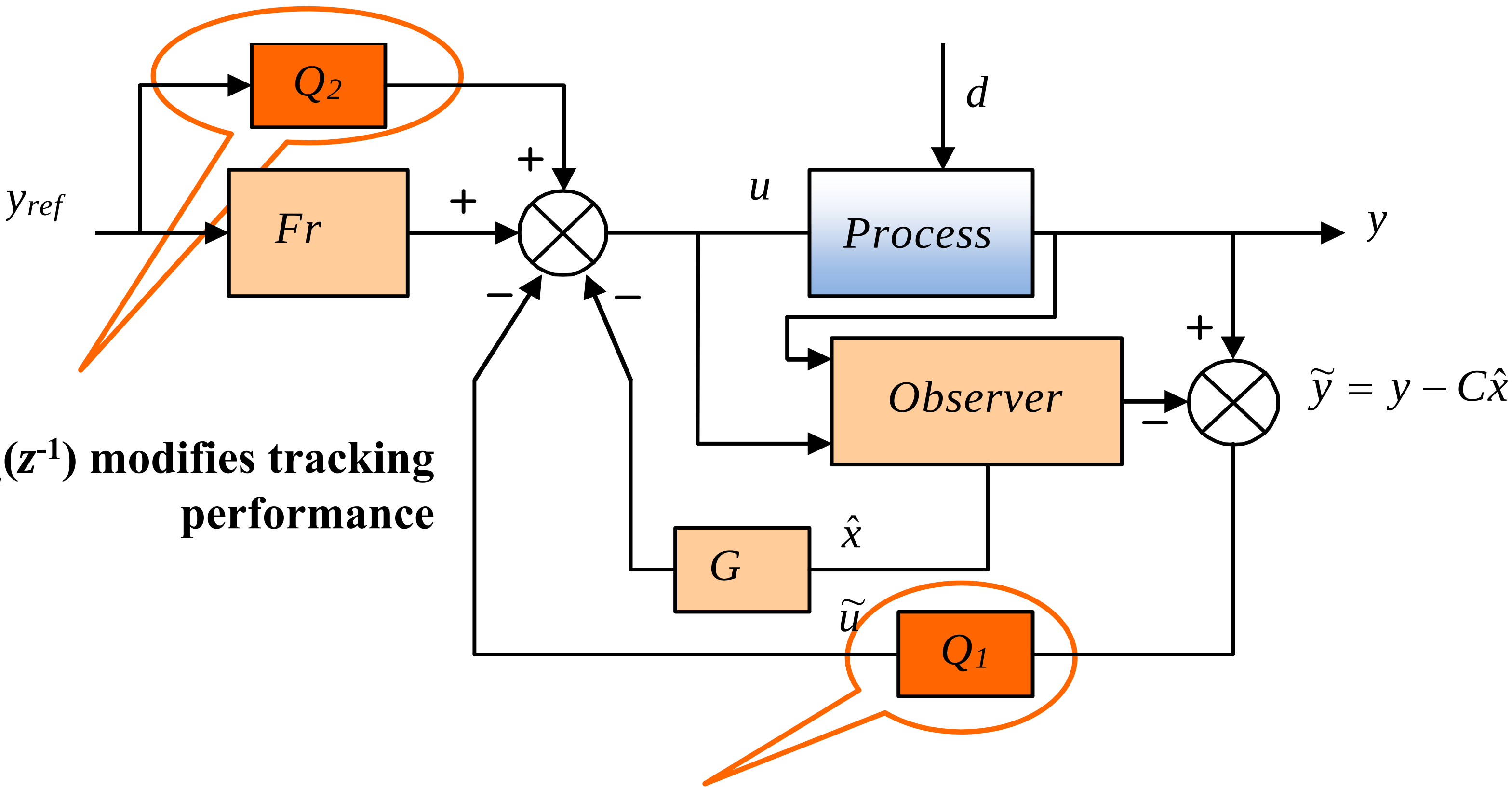
Youla-Kučera parameterisation



Linear time invariant transfers set

Linear time invariant transfers set

Robustification of explicit solutions
Youla-Kučera parameterisation



$Q_2(z^{-1})$ modifies tracking performance

$Q_1(z^{-1})$ modifies regulation dynamics

Robustification of explicit solutions

- Youla-Kučera parametrisation can not be directly used to robustify a PWA controller because the continuity of the control law is lost
- The following approach is follow:

1. Robustification of unconstrained controller
2. Find the perturbance/noise model associated to the obtained controller
3. Use the new model to modify or the initial PWA controller
   - This modification conserves the initial tracking behaviour
   - The number of partitions is kept, as depends on the constraints

Robustification of explicit solutions

Initial model

$$x_{t+1} = Ax_t + Bu_t + Ke_t$$
$$y_t = Cx_t + e_t$$

With e_t white noise

Cost function minimization without constraints

$$\mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} \left\| P\, x_{t+N|t} \right\|_p + \sum_{k=1}^{N-1} \left\{ \left\| Qx_{t+k|t} \right\|_p + \left\| R\, u_{t+k|t} \right\|_p \right\}$$

$$\hat{x}_{t+1} = (A - KC - BL)\hat{x}_t + Ky_t$$
$$u_t = -L\hat{x}_t$$

Initial unconstrained controller

Q Parameter

Robustified controller

$$\left\{ \begin{array}{l} \begin{bmatrix} \hat{x}_{t+1} \\ X_{Q_{t+1}} \end{bmatrix} = \begin{bmatrix} A - KC - BL - BD_Q C & -BC_Q \\ -B_Q C & A_Q \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ X_{Q_t} \end{bmatrix} + \begin{bmatrix} K - BD_Q \\ B_Q \end{bmatrix} y_t \\ u_t = \begin{bmatrix} -L + D_Q C & -C_Q \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ X_{Q_t} \end{bmatrix} - D_Q y_t \end{array} \right.$$

(C1)

Augmented model

$$\underbrace{\begin{bmatrix} x_{t+1} \\ x_{vt+1} \end{bmatrix}}_{x_{et+1}} = \underbrace{\begin{bmatrix} A & KC_v \\ 0 & A_v \end{bmatrix}}_{A_e} \begin{bmatrix} x_t \\ x_{vt+1} \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_e} u_t + \underbrace{\begin{bmatrix} K \\ B_v \end{bmatrix}}_{K_e} e'_t$$

$$y_t = \underbrace{\begin{bmatrix} C & C_v \end{bmatrix}}_{C_e} \begin{bmatrix} x_t \\ x_{vt+1} \end{bmatrix} + e'_t$$

With e'_t white noise

Cost function minimization without constraints

$$\mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} \left\| P\, x_{t+N|t} \right\|_p + \sum_{k=1}^{N-1} \left\{ \left\| Qx_{t+k|t} \right\|_p + \left\| R\, u_{t+k|t} \right\|_p \right\}$$

Unconstrained controller

$$\hat{x}_{et+1} = (A_e - K_e C_e - B_e L_e)\hat{x}_{et} + K_e y_t$$
$$u_t = -L_e \hat{x}_{et}$$

(C2)

Find Av, Bv, Cv to obtain $(C_1) \equiv (C_2)$

Robustification of explicit solutions

Two controllers are equivalents if the following equations are verified :

$$D_Q - L_e K_2 = 0$$
$$A_Q - A_v + B_v C_v = 0$$
$$C_Q - L_v + D_Q C_v = 0$$
$$L_v = H^{-1} F_v(A_v, C_v)$$

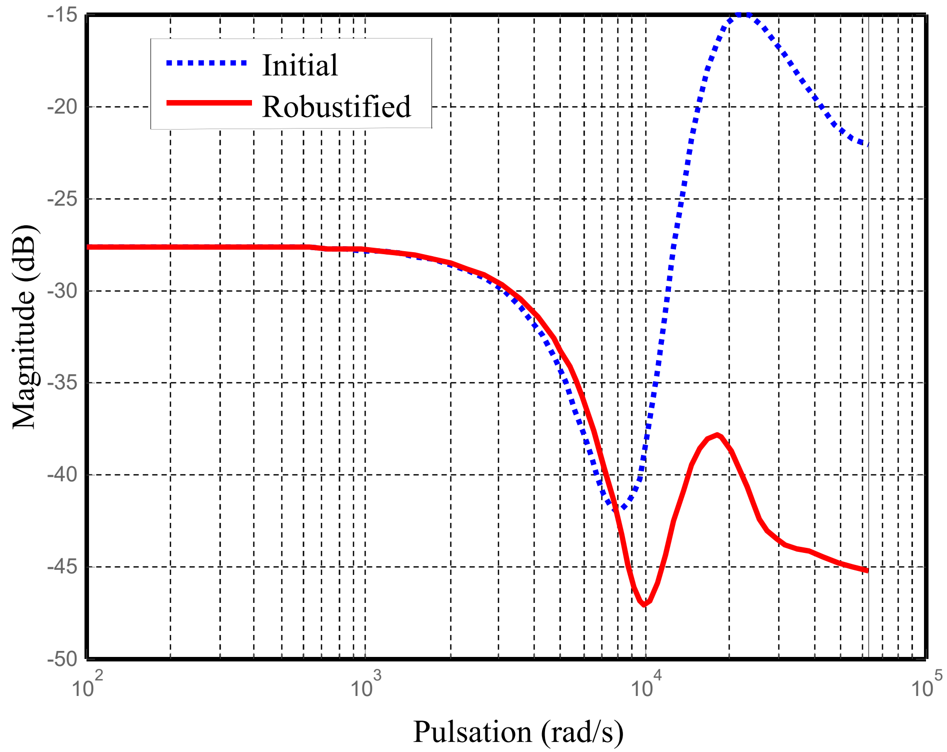
Non-linear equations :

- $F_v(A_v, C_v)$ depends on cost function
- Optimisation techniques can be used to solve the non-linear equation system
- The existence of the solution haven't been proved yet

Robustification of explicit solutions

Buck converter example

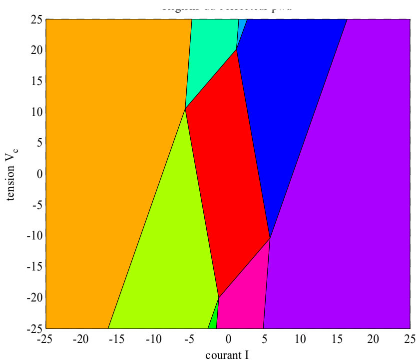
Unconstrained controller robustification towards additive disturbances



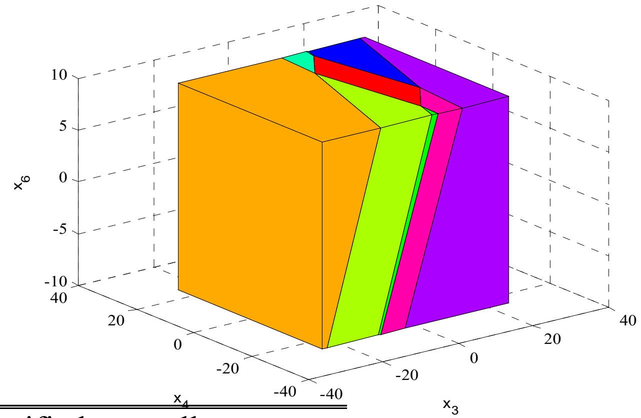
Robustification of explicit solutions

Buck converter example

Initial PWA controller, x1=0, x2=0, x5=0



Robustified PWA controller, x1=0, x2=0, x5=0, x7=0

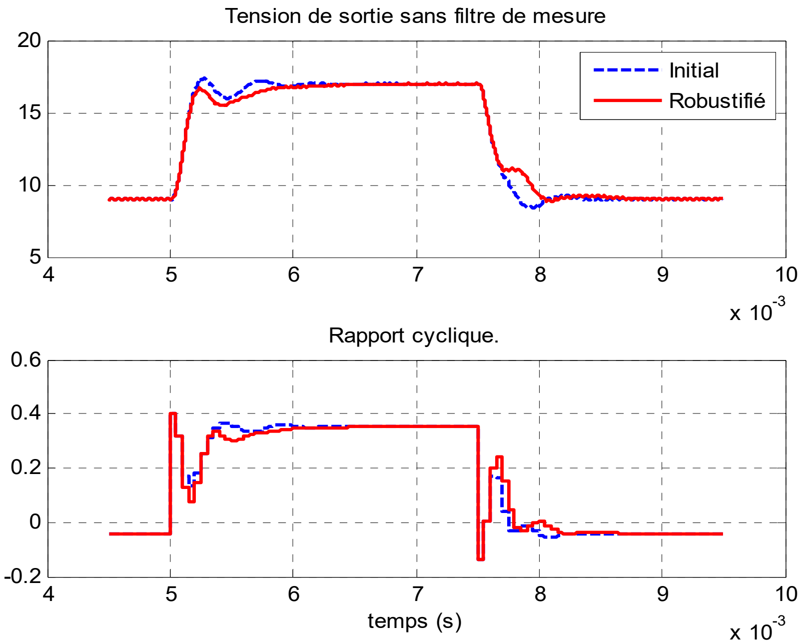


Active region	Initial controller L_i	Initial controller l_i	Robustified controller L_i	Robustified controller l_i
1	$[0 \quad 0{,}061 \quad -0{,}093 \quad -0{,}014 \quad -0{,}76]$	0	$[0 \quad 0{,}061 \quad -0{,}093 \quad -0{,}014 \quad -0{,}76 \quad 0{,}074 \quad -0{,}0139]$	0
2	$[-1 \quad 0 \quad 0 \quad 0 \quad 0]$	0,4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$	0,4
3	$[1{,}005 \quad 0{,}089 \quad -0{,}123 \quad 0{,}007 \quad -1{,}839]$	−0,4	$[1{,}005 \quad 0{,}089 \quad -0{,}123 \quad 0{,}007 \quad -1{,}839 \quad 0{,}093 \quad -0{,}024]$	−0,4
4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0]$	−0,4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$	−0,4
5	$[1{,}005 \quad 0{,}089 \quad -0{,}123 \quad 0{,}007 \quad -1{,}839]$	0,4	$[1{,}005 \quad 0{,}089 \quad -0{,}123 \quad 0{,}007 \quad -1{,}839 \quad 0{,}093 \quad -0{,}024]$	0,4
6	$[-1 \quad 0 \quad 0 \quad 0 \quad 0]$	0,4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$	0,4
7	$[-1 \quad 0 \quad 0 \quad 0 \quad 0]$	0,4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$	0,4
8	$[-1 \quad 0 \quad 0 \quad 0 \quad 0]$	−0,4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$	−0,4
9	$[-1 \quad 0 \quad 0 \quad 0 \quad 0]$	−0,4	$[-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$	−0,4

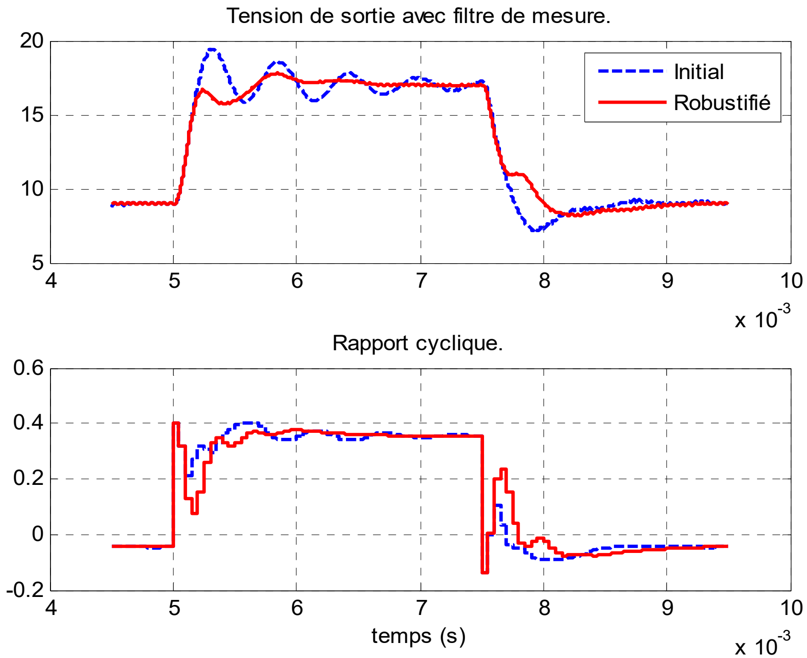
Robustification of explicit solutions
Buck converter example

Output voltage step response

Nominal process



Perturbed process



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P. Rodriguez-Ayerbe and S. Olaru, "On the disturbance model in the robustification of explicit predictive control", International Journal of Systems Science, 2013, Vol. 44(5), pp. 853–864.