A Reduced Order Direct Coupling Coherent Quantum Observer for a Complex Quantum Plant

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Abstract— This paper extends previous results on constructing a direct coupling quantum observer for a quantum harmonic oscillator system. In this case, we consider a complex linear quantum system plant consisting of a network of quantum harmonic oscillators. Conditions are given for which there exists a direct coupling observer which estimates a collection of variables in the quantum plant. It is shown that the order of the observer can be the same as the number of variables to be estimated when this number is even and thus this is a reduced order observer.

I. INTRODUCTION

A number of papers have recently considered the problem of constructing a coherent quantum observer for a quantum system; e.g., see [1]–[4]. In the coherent quantum observer problem, a quantum plant is coupled to a quantum observer which is also a quantum system. The quantum observer is constructed to be a physically realizable quantum system so that the system variables of the quantum observer converge in some suitable sense to the system variables of the quantum plant. The papers [4]-[7] considered the problem of constructing a direct coupling quantum observer for a given closed quantum system. In [4], the proposed observer is shown to be able to estimate some but not all of the plant variables in a time averaged sense. Also, the paper [8] shows that a possible experimental implementation of the augmented quantum plant and quantum observer system considered in [4] may be constructed using a non-degenerate parametric amplifier (NDPA) which is coupled to a beamsplitter by suitable choice of the NDPA and beamsplitter parameters.

In the paper [4], the quantum plant consisted of a number of quantum harmonic oscillators where the number of variables to be estimated was allowed to be at most half of the total number of variables describing the quantum plant. However, the quantum plant was assumed to have very simple dynamics corresponding to a zero Hamiltonian. Then a quantum observer was constructed whose number of variables was equal to twice the number of variables to be estimated. In this paper we extend the results of [4] by first allowing for more general linear quantum plants with non-zero Hamiltonians. Conditions are given on whether a given set of variables of interest can be estimated via a direct coupling quantum observer. Then a direct coupling quantum observer is constructed whose order is the same as the number of variables to be estimated when this number is even. In the case that the number of variables to be estimated is odd, the order of the observer is one more than the number of variables to be estimated. Compared to the result in [4], this is a reduced order observer. As in [4], the convergence of the observer outputs to the plant outputs is a time averaged convergence since the overall plant-observer system is a closed quantum linear system.

II. QUANTUM SYSTEMS

In the quantum observer problem under consideration, both the quantum plant and the quantum observer are linear quantum systems; see also [9]–[11]. We will restrict attention to closed linear quantum systems which do not interact with an external environment. The quantum mechanical behavior of a linear quantum system is described in terms of the system *observables* which are self-adjoint operators on an underlying infinite dimensional complex Hilbert space \mathfrak{H} . The commutator of two operators x and y on \mathfrak{H} is defined as [x, y] = xy - yx. Also, for a vector of operators x on \mathfrak{H} , the commutator of x and a scalar operator y on \mathfrak{H} is the vector of operators [x, y] = xy - yx, and the commutator of x and its adjoint x^{\dagger} is the matrix of operators

$$[x, x^{\dagger}] \triangleq xx^{\dagger} - (x^{\#}x^T)^T$$

where $x^{\#} \triangleq (x_1^* x_2^* \cdots x_n^*)^T$ and * denotes the operator adjoint.

The dynamics of the closed linear quantum systems under consideration are described by non-commutative differential equations of the form

$$\dot{x}(t) = Ax(t); \quad x(0) = x_0$$
 (1)

where A is a real matrix in $\mathbb{R}^{n \times n}$, and $x(t) = \begin{bmatrix} x_1(t) & \dots & x_n(t) \end{bmatrix}^T$ is a vector of system observables; e.g., see [9]. Here n is assumed to be an even number and $\frac{n}{2}$ is the number of modes in the quantum system.

The initial system variables $x(0) = x_0$ are assumed to satisfy the *commutation relations*

$$[x_j(0), x_k(0)] = 2i\Theta_{jk}, \quad j, k = 1, \dots, n,$$
(2)

where Θ is a real skew-symmetric matrix with components Θ_{jk} . In the case of a single quantum harmonic oscillator, we can choose $x = (x_1, x_2)^T$ where $x_1 = q$ is the position operator, and $x_2 = p$ is the momentum operator. The

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commutation relations are [q, p] = 2i. In general, the matrix Θ is assumed to be of the form

$$\Theta = \operatorname{diag}(J, J, \dots, J) \tag{3}$$

where J denotes the real skew-symmetric 2×2 matrix

$$J = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]$$

The system dynamics (1) are determined by the system Hamiltonian which is a self-adjoint operator on the underlying Hilbert space \mathfrak{H} . For the linear quantum systems under consideration, the system Hamiltonian will be a quadratic form $\mathcal{H} = \frac{1}{2}x^T R x$, where R is a real symmetric matrix. Then, the corresponding matrix A in (1) is given by

$$A = 2\Theta R. \tag{4}$$

where Θ is defined as in (3). e.g., see [9]. In this case, the system variables x(t) will satisfy the *commutation relations* at all times:

$$[x(t), x(t)^T] = 2\mathbf{i}\Theta \text{ for all } t \ge 0.$$
(5)

That is, the system will be *physically realizable*; e.g., see [9].

III. ANALYSIS OF THE QUANTUM PLANT

In this section we will describe the class of quantum linear systems which will be considered as quantum plants. Also, we will analyse these quantum plants in order to provide conditions under which there exists a direct coupling observer which can estimate the quantum plant outputs.

We consider general *closed linear quantum plants* described by linear quantum system models of the following form:

$$\dot{x}_p(t) = A_p x_p(t); \quad x_p(0) = x_{0p};$$

 $z_p(t) = C_p x_p(t)$ (6)

where z_p denotes the vector of system variables to be estimated by the observer and $A_p \in \mathbb{R}^{n_p \times n_p}$, $C_p \in \mathbb{R}^{m \times n_p}$. It is assumed that this quantum plant is physically realizable and corresponds to a plant Hamiltonian $\mathcal{H}_p = \frac{1}{2}x_p^T R_p x_p$ where R_p is a symmetric matrix and $A_p = 2\Theta_p R_p$. Here Θ_p is of the form (3). Unlike the case in [4], we will not require that R_p is zero. However, we will assume that det $R_p = 0$ so that R_p has a non-trivial null space. In addition, we assume that the matrices R_p and C_p satisfy the following conditions:

$$C_p(sI - \Theta_p)^{-1}R_p \equiv 0; \qquad (7)$$

$$C_p \Theta_p C_p^T = 0; (8)$$

The matrix
$$C_p$$
 is of rank m . (9)

Note that if the matrix C_p is not full rank then some of the components of z_p can be expressed as linear combinations of the other components of C_p . Hence, without loss of generality, we can eliminate these components of z_p to obtain a full rank C_p .

In the sequel, we will show that these conditions imply that there exists a direct coupling quantum observer which can estimate the variables z_p . However, we first analyse the quantum plant satisfying these conditions. Indeed, we first consider the controllability of the pair (Θ_p, R_p) . Since $\Theta_p^2 = -I$, it follows that the corresponding controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} R_p & \Theta_p R_p & \Theta_p^2 R_p & \dots & \Theta_p^{n_p-1} R_p \end{bmatrix}$$

=
$$\begin{bmatrix} R_p & \Theta_p R_p & -R_p & -\Theta_p R_p & R_p & \dots \end{bmatrix};$$

e.g., see [12]. This matrix has the same range space as the matrix

$$\mathcal{C}_r = \left[\begin{array}{cc} R_p & \Theta_p R_p \end{array} \right]. \tag{10}$$

The range space of the matrix C_r will determine which variables of the quantum plant remain constant if the plant is not coupled to the quantum observer. These variables are ones which can be estimated by the quantum observer. We can use the matrix C_r to transform the pair (Θ_p, R_p) into a form corresponding to controllable and uncontrollable subsystems; e.g., see [12]. Indeed, we construct an orthogonal matrix P using the svd of the matrix C_r as $C_r = PSV^T$ where V is also an orthogonal matrix and S is a diagonal matrix. This construction of P yields

$$P^T \Theta_p P = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{22} \end{bmatrix}, \quad P^T R_p = \begin{bmatrix} R_{p1} \\ 0 \end{bmatrix}$$

where the pair (Θ_{11}, R_{p1}) is controllable. Here $\Theta_{11} \in \mathbb{R}^{n_{p1} \times n_{p1}}$ and $\Theta_{22} \in \mathbb{R}^{n_{p2} \times n_{p2}}$ such that $n_{p1} + n_{p2} = n_p$.

We now use the fact that Θ_p is a skew-symmetric matrix and hence $P^T \Theta_p P$ is a skew-symmetric matrix. Therefore, we must have

$$P^T \Theta_p P = \begin{bmatrix} \Theta_{11} & 0\\ 0 & \Theta_{22} \end{bmatrix}$$
(11)

where Θ_{11} is skew-symmetric and Θ_{22} is skew-symmetric. Also, since the matrix Θ_p is non-singular, the matrices Θ_{11} and Θ_{22} must be non-singular.

We also use the fact that R_p is a symmetric matrix. To do this, we first write $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and $P^T R_p = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$.

Hence,

$$P^{T}R_{p}P = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$
$$= \begin{bmatrix} R_{11}P_{11} + R_{12}P_{21} & R_{11}P_{12} + R_{12}P_{22} \\ 0 & 0 \end{bmatrix}.$$

However, R_p is symmetric and hence $P^T R_p P$ is symmetric. Thus, the matrix $P^T R_p P$ must be of the form

$$P^T R_p P = \begin{bmatrix} R_{p11} & 0\\ 0 & 0 \end{bmatrix}$$
(12)

where the matrix R_{p11} is symmetric. Also, since the pair $(\Theta_{11}, \begin{bmatrix} R_{11} & 0 \end{bmatrix})$ is controllable, the condition (7) implies that the matrix $\tilde{C}_p = C_p P$ must be of the form

$$\tilde{C}_p = \begin{bmatrix} 0 & \tilde{C}_{p2} \end{bmatrix}.$$
(13)

where the matrix $\tilde{C}_{p2} \in \mathbb{R}^{m \times n_{p2}}$ is of rank m. From this, it follows that the condition (8) reduces to the condition

$$\tilde{C}_{p2}\Theta_{22}\tilde{C}_{p2}^{T} = 0.$$
 (14)

Now since Θ_{22} is nonsingular and \tilde{C}_{p2} is of rank m, it follows that the matrix $\tilde{C}_{p2}\Theta_{22}$ is of rank m and its null space is of dimension $n_{p2} - m$. However, since \tilde{C}_{p2} is of rank m, the equation (14) implies we must have $m \leq n_{p2} - m$ and hence we will require

$$m \le \frac{n_{p2}}{2} = \frac{n_p - \operatorname{rank}\mathcal{C}_r}{2}$$

in order for the conditions (7), (8), (9) to be satisfied.

We now introduce a change of variables

$$\tilde{x}_p = P^T x_p = \left[\begin{array}{c} \tilde{x}_{p1} \\ \tilde{x}_{p2} \end{array} \right]$$

to the system (6). It follows that

$$\dot{\tilde{x}}_{p} = \begin{bmatrix} \tilde{x}_{p1} \\ \dot{\tilde{x}}_{p2} \end{bmatrix} = P^{T} A_{p} P \tilde{x}_{p}$$

$$= 2P^{T} \Theta_{p} R_{p} P \tilde{x}_{p} = 2P^{T} \Theta_{p} P P^{T} R_{p} P \tilde{x}_{p}$$

$$= 2 \begin{bmatrix} \Theta_{11} & 0 \\ 0 & \Theta_{22} \end{bmatrix} \begin{bmatrix} R_{p11} & 0 \\ 0 & 0 \end{bmatrix} \tilde{x}_{p}$$

$$= \begin{bmatrix} 2\Theta_{11} R_{p11} \tilde{x}_{p1} \\ 0 \end{bmatrix} .$$

$$(15)$$

Also,

$$z_p = C_p P \tilde{x}_p = \begin{bmatrix} 0 & \tilde{C}_{p2} \end{bmatrix} \begin{bmatrix} \tilde{x}_{p1} \\ \tilde{x}_{p2} \end{bmatrix} = \tilde{C}_{p2} \tilde{x}_{p2}$$

using (13).

It follows from (15) that the plant variables \tilde{x}_{p2} will remain constant while the variables \tilde{x}_{p1} evolve dynamically for the plant system. Also, we have shown that the variables z_p to be estimated must be chosen to depend only on the variables \tilde{x}_{p2} and not the variables \tilde{x}_{p2} . This will mean that if the quantum plant is a closed quantum system and not coupled to the quantum observer, the variables z_p will remain constant. However, if the quantum plant is coupled to a quantum observer, this may longer apply. In the sequel, we will show that for a suitably designed quantum observer, the variables z_p will remain constant even when the quantum plant is coupled to the quantum observer.

IV. DIRECT COUPLING COHERENT QUANTUM Observers

We consider a reduced order direct coupled linear quantum observer defined by a symmetric matrix $R_o \in \mathbb{R}^{n_o \times n_o}$, and matrices $R_c \in \mathbb{R}^{n_p \times n_o}$, $C_o \in \mathbb{R}^{m_p \times n_o}$. These matrices define an observer Hamiltonian

$$\mathcal{H}_o = \frac{1}{2} x_o^T R_o x_o, \tag{16}$$

and a coupling Hamiltonian

$$\mathcal{H}_{c} = \frac{1}{2} x_{p}^{T} R_{c} x_{o} + \frac{1}{2} x_{o}^{T} R_{c}^{T} x_{p}.$$
(17)

The matrix C_o also defines the vector of output variables for the observer as $z_o(t) = C_o x_o(t)$.

The augmented quantum linear system consisting of the quantum plant and the direct coupled quantum observer is then a quantum system of the form (1) described by the total Hamiltonian

$$\mathcal{H}_{a} = \mathcal{H}_{p} + \mathcal{H}_{c} + \mathcal{H}_{o}$$

$$= \frac{1}{2} x_{a}^{T} R_{a} x_{a}$$
(18)

where $x_a = \begin{bmatrix} x_p \\ x_o \end{bmatrix}$ and $R_a = \begin{bmatrix} R_p & R_c \\ R_c^T & R_o \end{bmatrix}$. Then, using (4), it follows that the augmented quantum linear system is described by the equations

$$\begin{bmatrix} \dot{x}_{p}(t) \\ \dot{x}_{o}(t) \end{bmatrix} = A_{a} \begin{bmatrix} x_{p}(t) \\ x_{o}(t) \end{bmatrix}; x_{p}(0) = x_{0p}; x_{o}(0) = x_{0o}; z_{p}(t) = C_{p}x_{p}(t); z_{o}(t) = C_{o}x_{o}(t)$$
(19)

where $A_a = 2\Theta_a R_a$. Here

$$\Theta_a = \left[\begin{array}{cc} \Theta_p & 0\\ 0 & \Theta_o \end{array} \right].$$

We now formally define the notion of a direct coupled linear quantum observer.

Definition 1: The matrices $R_o \in \mathbb{R}^{n_o \times n_o}$, $R_c \in \mathbb{R}^{n_p \times n_o}$, $C_o \in \mathbb{R}^{m_p \times n_o}$ define a direct coupled linear quantum observer for the quantum linear plant (6) if the corresponding augmented linear quantum system (19) is such that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T (z_p(t) - z_o(t)) dt = 0.$$
 (20)

V. CONSTRUCTING A REDUCED ORDER DIRECT COUPLING COHERENT QUANTUM OBSERVER

In order to construct a reduced order direct coupled coherent observer, we assume that the quantum plant satisfies the conditions (7), (8), (9) and apply the transformation $\begin{bmatrix} \tilde{x}_{p1} \\ \tilde{x}_{p2} \end{bmatrix} = \tilde{x}_p = P^T x_p$ considered in the previous section. Also, we assume that the coupling Hamiltonian \mathcal{H}_c depends only on \tilde{x}_{p2} and x_o but not on \tilde{x}_{p1} ; i.e., we can write

$$\mathcal{H}_c = \frac{1}{2}\tilde{x}_{p2}^T\tilde{R}_c x_o + \frac{1}{2}x_o^T\tilde{R}_c^T\tilde{x}_{p2}$$
(21)

where

$$R_c = P \begin{bmatrix} 0\\ \tilde{R}_c \end{bmatrix}.$$
 (22)

Hence, we can write

$$\begin{aligned} \mathcal{H}_{a} &= \frac{1}{2} \tilde{x}_{p1}^{T} R_{p11} \tilde{x}_{p1} + \frac{1}{2} \tilde{x}_{p2}^{T} \tilde{R}_{c} x_{o} + \frac{1}{2} x_{o}^{T} \tilde{R}_{c}^{T} \tilde{x}_{p2} \\ &+ \frac{1}{2} x_{o}^{T} R_{o} x_{o} \\ &= \frac{1}{2} \tilde{x}_{a}^{T} \tilde{R}_{a} \tilde{x}_{a} \end{aligned} \\ \text{where } x_{a} &= \begin{bmatrix} \tilde{x}_{p1} \\ \tilde{x}_{p2} \\ x_{o} \end{bmatrix} \text{ and } R_{a} = \begin{bmatrix} R_{p11} & 0 & 0 \\ 0 & 0 & \tilde{R}_{c} \\ 0 & \tilde{R}_{c}^{T} & R_{o} \end{bmatrix}.$$

We now suppose that

$$n_o = \begin{cases} m \text{ if } m \text{ is even;} \\ m+1 \text{ if } m \text{ is odd.} \end{cases}$$

Thus, n_o is an even number and this corresponds to a reduced order quantum observer.

We also suppose that the matrices R_o , \dot{R}_c , C_o are such that

$$\tilde{R}_c = \alpha \beta^T, \quad \alpha = \tilde{C}_{p2}^T, \quad R_o > 0 \tag{23}$$

where $\tilde{C}_{p2}^T \in \mathbb{R}^{n_{p2} \times m}$ and $\beta \in \mathbb{R}^{n_o \times m}$ is full rank. In addition, we write $\Theta = \begin{bmatrix} \Theta_{11} & 0 & 0 \\ 0 & \Theta_{22} & 0 \\ 0 & 0 & \Theta_o \end{bmatrix}$ where Θ_{11} , Θ_{22} are defined as in (11) and $\Theta_o \in \mathbb{R}^{n_o \times n_o}$ is of the form

 Θ_{22} are defined as in (11) and $\Theta_o \in \mathbb{R}^{n_o \times n_o}$ is of the form (3). Hence, the augmented system equations (19) describing the combined plant-observer system imply

$$\begin{aligned} \dot{\tilde{x}}_{p2}(t) &= 2\Theta_{22}\alpha\beta^T x_o(t); \\ \dot{\tilde{x}}_o(t) &= 2\Theta_o\beta\alpha^T \tilde{x}_{p2}(t) + 2\Theta_o R_o x_o(t); \\ z_p(t) &= \tilde{C}_{p2}\tilde{x}_p(t); \\ z_o(t) &= C_o x_o(t). \end{aligned}$$
(24)

We will show that the given assumptions imply that the quantity $z_p(t) = \tilde{C}_{p2}\tilde{x}_{p2}(t)$ will be constant for the augmented quantum system (24). Indeed, it follows from (24) that

$$\dot{z}_p(t) = 2\tilde{C}_{p2}\Theta_{22}\alpha\beta^T x_o(t) = 2\tilde{C}_{p2}\Theta_{22}\tilde{C}_{p2}^T\beta^T x_o(t) = 0$$

using (14). Therefore,

$$z_p(t) = z_p(0) = z_p$$
 (25)

for all $t \ge 0$.

It now follows from (24) that

$$\dot{x}_{o}(t) = 2\Theta_{o}\beta\tilde{C}_{p2}\tilde{x}_{p2}(t) + 2\Theta_{o}R_{o}x_{o}(t)$$

$$= 2\Theta_{o}R_{o}x_{o}(t) + 2\Theta_{o}\beta z_{p}.$$
 (26)

From this equation, we define the "steady state" value of the vector x_o as

$$\bar{x}_o = -R_o^{-1}\beta z_p.$$

Then we define the "error vector"

$$\tilde{x}_o(t) = x_o(t) - \bar{x}_o.$$

It follows from (26) that $\tilde{x}_o(t)$ satisfies the differential equation

$$\begin{aligned} \dot{\tilde{x}}_o(t) &= 2\Theta_o R_o x_o(t) + 2\Theta_o \beta z_p \\ &= 2\Theta_o R_o \tilde{x}_o(t) + 2\Theta_o R_o \bar{x}_o + 2\Theta_o \beta z_p \\ &= 2\Theta_o R_o \tilde{x}_o(t). \end{aligned}$$

We now show that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{x}_o(t) dt = 0$$
(27)

following the proof of a similar fact in [6]. First note that the quantity $\tilde{\mathcal{H}}_o(t) = \frac{1}{2}\tilde{x}_o(t)^T R_o \tilde{x}_o(t)$ remains constant in time. Indeed,

$$\frac{d}{dt}\tilde{\mathcal{H}}_{o}(t) = \frac{1}{2}\dot{\tilde{x}}_{o}^{T}R_{o}\tilde{x}_{o} + \frac{1}{2}\tilde{x}_{o}^{T}R_{o}\dot{\tilde{x}}_{o} \\ = -\tilde{x}_{o}^{T}R_{o}\Theta_{o}R_{o}\tilde{x}_{o} + \tilde{x}_{o}^{T}R_{o}\Theta_{o}R_{o}\tilde{x}_{o} = 0$$

since R_o is symmetric and Θ_o is skew-symmetric. That is

$$\frac{1}{2}\tilde{x}_{o}(t)^{T}R_{o}\tilde{x}_{o}(t) = \frac{1}{2}\tilde{x}_{o}(0)^{T}R_{o}\tilde{x}_{o}(0) \quad \forall t \ge 0.$$
(28)

However, $\tilde{x}_o(t) = e^{2\Theta_o R_o t} \tilde{x}_o(0)$ and $R_o > 0$. Therefore, it follows from (28) that

$$\sqrt{\lambda_{min}(R_o)} \| e^{2\Theta_o R_o t} \tilde{x}_o(0) \| \le \sqrt{\lambda_{max}(R_o)} \| \tilde{x}_o(0) \|$$

for all $\tilde{x}_o(0)$ and $t \ge 0$. Hence,

$$\|e^{2\Theta_o R_o t}\| \le \sqrt{\frac{\lambda_{max}(R_o)}{\lambda_{min}(R_o)}} \tag{29}$$

for all $t \ge 0$.

Now since Θ_o and R_o are non-singular,

$$\int_{0}^{T} e^{2\Theta_{o}R_{o}t} dt = \frac{1}{2} e^{2\Theta_{o}R_{o}T} R_{o}^{-1} \Theta_{o}^{-1} - \frac{1}{2} R_{o}^{-1} \Theta_{o}^{-1}$$

and therefore, it follows from (29) that

$$\begin{split} &\frac{1}{T} \| \int_{0}^{T} e^{2\Theta_{o}R_{o}t} dt \| \\ &= \frac{1}{T} \| \frac{1}{2} e^{2\Theta_{o}R_{o}T} R_{o}^{-1} \Theta_{o}^{-1} - \frac{1}{2} R_{o}^{-1} \Theta_{o}^{-1} \| \\ &\leq \frac{1}{2T} \| e^{2\Theta_{o}R_{o}T} \| \| R_{o}^{-1} \Theta_{o}^{-1} \| \\ &\quad + \frac{1}{2T} \| R_{o}^{-1} \Theta_{o}^{-1} \| \\ &\leq \frac{1}{2T} \sqrt{\frac{\lambda_{max}(R_{o})}{\lambda_{min}(R_{o})}} \| R_{o}^{-1} \Theta_{o}^{-1} \| \\ &\quad + \frac{1}{2T} \| R_{o}^{-1} \Theta_{o}^{-1} \| \\ &\quad \to 0 \end{split}$$

as $T \to \infty$. Hence,

$$\begin{split} \lim_{T \to \infty} \frac{1}{T} \| \int_0^T \tilde{x}_o(t) dt \| \\ &= \lim_{T \to \infty} \frac{1}{T} \| \int_0^T e^{2\Theta_o R_o t} \tilde{x}_o(0) dt \| \\ &\leq \lim_{T \to \infty} \frac{1}{T} \| \int_0^T e^{2\Theta_o R_o t} dt \| \| \tilde{x}_o(0) \| \\ &= 0. \end{split}$$

This implies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{x}_o(t) dt = 0.$$

Now we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T z_o(t) dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T C_o x_o(t) dt$$
$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T C_o(\tilde{x}_o(t) + \bar{x}_o) dt$$
$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T C_o \bar{x}_o dt$$
$$= C_o \bar{x}_o = -C_o R_o^{-1} \beta z_p.$$

We now choose the matrices $C_o \in \mathbb{R}^{m \times n_o}$ and $\beta \in \mathbb{R}^{n_o \times m}$ so that

$$-C_o R_o^{-1} \beta = I. \tag{30}$$

This is always possible since $n_o \ge m$. It follows that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T z_o(t) dt = z_p$$

and hence, the condition (20) is satisfied. Thus, we have proved the following theorem.

Theorem 1: Consider a quantum plant of the form (6) satisfying the conditions (7), (8), (9). Then the matrices R_o , \tilde{R}_c , C_o constructed as in (23), (30) will define a reduced order direct coupled quantum observer achieving time-averaged consensus convergence for this quantum plant.

VI. ILLUSTRATIVE EXAMPLE

We now present some numerical simulations to illustrate the reduced order direct coupled quantum observer described in the previous section. We choose the quantum plant to have

Then, the corresponding matrix C_r defined in (10) is given by

$C_r =$	F I	1	1	1	1	1	1	1	1	1	1	1 -	1
	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	
	1	1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	·
	1	1	1	1	1	1	1	1	1	1	1	1	
	L 1	1	1	1	1	1	-1	-1	-1	-1	-1	-1.	

This matrix has rank 2. From this, the orthogonal matrix P is calculated by finding the svd of C_r . This yields

P =		0.0000	0.5825	-0.5722	0.0000	-0.00001	٦.
	0	-0.5774	0.5722	0.5825	-0.0000	0.00001	
	-0.5774	-0.0000	-0.2912	0.2861	0.6938	-0.13651	
	0	-0.5774	-0.2861	-0.2912	-0.1365	-0.69381	· ·
	-0.5774	-0.0000	-0.2912	0.2861	-0.6938	0.13651	
	L o	-0.5774	-0.2861	-0.2912	0.1365	0.6938	

The corresponding transformed plant Hamiltonian matrix $\tilde{R}_p = P^T R_p P$ is in the form (12) where

$$R_{p11} = \left[\begin{array}{cc} 3 & 3\\ 3 & 3 \end{array} \right].$$

Also, the transformed commutation matrix $\tilde{\Theta}_p = P^T \Theta_p P$ is in the form (11) where

$$\Theta_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Theta_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In order to choose a suitable value of the matrix C_p so that condition (7) is satisfied, we choose $\tilde{C}_p = C_p P$ of the form (13) where $\tilde{C}_{p2} \in \mathbb{R}^{2\times 4}$. Also, we require that the condition (14) is satisfied. It is straightforward to verify that this condition is satisfied by the matrix

$$\tilde{C}_{p2} = \left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{array} \right].$$

This corresponds to the matrix

$$C_p = \left[\begin{array}{ccccc} 0.0103 & 1.1547 & 0.5522 & -1.4076 & -0.5625 & 0.2530 \\ 0.0103 & 1.1547 & -0.5625 & 0.2530 & 0.5522 & -1.4076 \end{array} \right]$$

which is such that conditions (7), (8), (9) are satisfied.

The quantum plant defined by the matrices R_p and C_p given above is a plant of the form considered in Section III where $n_p = 6$, $n_{p1} = 2$, $n_{p2} = 4$, and m = 2. Hence, we will construct a reduced order observer as described in Section V with $n_o = 2$. In order to construct the observer, we need to choose matrices $R_o > 0$, β and C_o such that (30) is satisfied. In this example, we will choose

$$R_o = I, \quad C_o = I, \quad \beta = -I$$

Then the matrix \tilde{R}_c is constructed according to (23) as

$$\tilde{R}_c = \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

From this, the matrix R_c is constructed according to (22) as

$$R_c = \begin{bmatrix} -0.0103 & -0.0103 \\ -1.1547 & -1.1547 \\ -0.5522 & 0.5625 \\ 1.4076 & -0.2530 \\ 0.5625 & -0.5522 \\ -0.2530 & 1.4076 \end{bmatrix}$$

The augmented plant-observer system is described by the equations (19). To simulate these equations we can write

$$x_a(t) = \Phi(t)x_a(0)$$

where $\Phi(t) = e^{2\Theta_a R_a t}$. Furthermore, the plant variables to be estimated are given by

$$z_p(t) = \begin{bmatrix} C_p & 0 \end{bmatrix} \Phi(t) x_a(0)$$

and the observer output variables are given by

$$z_o(t) = \begin{bmatrix} 0 & C_o \end{bmatrix} \Phi(t) x_a(0).$$

Although the quantities $z_p(t)$ and $z_o(t)$ are operators which cannot be plotted directly, we can plot the coefficients in the above equations which define the components of $z_p(t)$ or $z_o(t)$ with respect to the initial condition operators in $x_a(0)$.

In Figure 1, we plot these coefficients corresponding to the first plant variable to be estimated. In Figure 2, we plot these coefficients corresponding to the second plant variable to be estimated. These figures verify that the quantity $z_p(t)$ remains constant at its initial value.

In Figure 3, we plot these coefficients corresponding to the first observer output variable, which is designed to provide an



Fig. 1. Coefficient functions defining the first component of $z_p(t)$.



Fig. 2. Coefficient functions defining the second component of $z_p(t)$.



Fig. 3. Coefficient functions defining the first component of $z_o(t)$.

estimate of the first plant variable to be estimated. In Figure 4, we plot these coefficients corresponding to the second observer output variable, which is designed to provide an estimate of the second plant variable to be estimated. From



Fig. 4. Coefficient functions defining the second component of $z_o(t)$.

Figures 3 and 4, we can see that $z_o(t)$ evolves in a timevarying and oscillatory way.

To illustrate the time average convergence property of the quantum observer (20), we now plot the time averaged quantities corresponding to Figures 3 and 4. In Figure 5, we plot the time averaged coefficients corresponding to the first observer output variable. Comparing this figure with Figure



Fig. 5. Time averaged coefficient functions defining the first component of $\boldsymbol{z}_o(t).$

1, we can see that the time average of the first component of $z_o(t)$ converges to the first component of z_p .

In Figure 6, we plot the time averaged coefficients corresponding to the second observer output variable. Comparing this figure with Figure 2, we can see that the time average of the second component of $z_o(t)$ converges to the second component of z_p .

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Fig. 6. Time averaged coefficient functions defining the second component of $z_o(t)$.

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