# Self-stabilizing Multivalued Consensus in Asynchronous Crash-prone Systems

(preliminary version)

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The problem of multivalued consensus is fundamental in the area of fault-tolerant distributed computing since it abstracts a very broad set of agreement problems in which processes have to uniformly decide on a specific value  $v \in V$ , where  $|V| \ge 2$ . Existing solutions (that tolerate process failures) reduce the multivalued consensus problem to the one of binary consensus, *e.g.*, Mostéfaoui-Raynal-Tronel and Zhang-Chen.

Our study aims at the design of an even more reliable solution. We do so through the lenses of *self-stabilization*—a very strong notion of fault-tolerance. In addition to node and communication failures, self-stabilizing algorithms can recover after the occurrence of *arbitrary transient-faults*; these faults represent any violation of the assumptions according to which the system was designed to operate (as long as the algorithm code stays intact).

This work proposes the first (to the best of our knowledge) self-stabilizing algorithm for multivalued consensus for asynchronous message-passing systems prone to process failures and arbitrary transient-faults. Our solution is also the first (to the best of our knowledge) to support wait-freedom. Moreover, using piggybacking techniques, our solution can invoke n binary consensus objects concurrently. Thus, the proposed self-stabilizing solution can terminate using fewer binary consensus objects than earlier non-self-stabilizing solutions by Mostéfaoui, Raynal, and Tronel, which uses an unbounded number of binary consensus objects, or Zhang and Chen, which is not wait-free.

# 1 Introduction

We propose, to the best of our knowledge, the first self-stabilizing, non-blocking, and memorybounded implementation of *multivalued consensus* objects for asynchronous message-passing systems whose nodes may crash.

## 1.1 Background and motivation

Fault-tolerant distributed applications span over many domains in the area of banking, transports, tourism, production, commerce, to name a few. The implementations of these applications use message-passing systems and require fault-tolerance. The task of designing and verifying these systems is known to be very hard, because the joint presence of failures and asynchrony creates uncertainties about the application state (from the process's point of view). E.g., Fischer, Lynch, and Paterson [23] demonstrated that, in any asynchronous message-passing system, it takes no more than one process crash to prevent the system from achieving consensus deterministically.

Our focal application is the emulation of finite-state machines. For the sake of consistency maintenance, all emulating processes need to apply identical sequences of state transitions. This can be done by dividing the problem into two: (i) propagation of user input to all emulating processes, and (ii) letting each emulating process execute identical sequences of state transitions. Uniform reliable broadcast [26, 41] can solve Problem (i). This work focuses on Problem (ii) since it is the core problem. *I.e.*, all processes need to agree on a common value according to which all emulating processes execute their state transitions. The consensus problem generalizes problem (ii) and requires each process to propose a value, and all non-crashed processes to reach a common decision that one of them had proposed. There is a rich literature on fault-tolerant consensus. This work advances the state of the art by offering a greater set of failures that can be tolerated.

#### 1.2 Problem definition and scope

The definition of the consensus problem appears in Definition 1.1. This work studies the multivalued version of the problem in which there are at least two values that can be proposed. Note that there is another version of the problem in which this set includes exactly two values, and referred to as binary consensus. Existing solutions for the multivalued consensus (as well as the proposed one) often use binary consensus algorithms. We present the relation among the problems mentioned above in Figure 1.

**Definition 1.1 (Consensus)** Every process  $p_i$  has to propose a value  $v_i \in V$  via an invocation of the  $propose_i(v_i)$  operation, where V is a finite set of values. Let Alg be an algorithm that solves consensus. Alg has to satisfy safety (i.e., validity, integrity, and agreement) and liveness (i.e., termination).

- Validity. Suppose that v is decided. Then some process had invoked propose(v).
- Termination. All non-faulty processes decide.
- Agreement. No two processes decide different values.
- Integrity. No process decides more than once.

# 1.3 Fault Model

We consider an asynchronous message-passing system that has no guarantees on communication delays (except that they are finite) and the algorithm cannot explicitly access the local clock. Our fault model includes (i) crashes of less than half of the processes, and (ii) communication failures, such as packet omission, duplication, and reordering.

In addition to the failures captured in our model, we also aim to recover from *arbitrary* transient-faults, *i.e.*, any temporary violation of assumptions according to which the system and network were designed to operate, *e.g.*, the corruption of control variables, such as the program counter, packet payload, and indices, *e.g.*, sequence numbers, which are responsible for the correct operation of the studied system, as well as operational assumptions, such as that

at least a majority of nodes never fail. Since the occurrence of these failures can be arbitrarily combined, it follows that these transient-faults can alter the system state in unpredictable ways. In particular, when modeling the system, we assume that these violations bring the system to an arbitrary state from which a *self-stabilizing algorithm* should recover the system after the occurrence of the last transient-fault. The system is guaranteed to satisfy the task requirements, *e.g.*, Definition 1.1, after this recovery period. Our design criteria also support wait-freedom, which requires all operations to terminate within a bounded number of algorithm steps. Waitfreedom is important since it assures starvation-freedom even in the presence of failures since all operations terminate (as long as the process that invoked them does not crash).

# 1.4 Related Work

The celebrated Paxos algorithm [28] circumvents the impossibility by Fischer, Lynch, and Paterson [23], from now on FLP, by assuming that failed computers can be detected by unreliable failure detectors [10]. Paxos has inspired many veins of research, *e.g.*, [44] and references therein. We, however, follow the family of abstractions by Raynal [41] due to its clear presentation that is easy to grasp. Also, the studied algorithm does not consider failure detectors. Instead, it assumes the availability of binary consensus objects, which uses the weakest failure detector, see Raynal [41].

#### 1.4.1 Non-self-stabilizing solutions

Mostéfaoui, Raynal, and Tronel [36], from now on MRT, reduce multivalued consensus to binary consensus via a crash-tolerant block-free algorithm. MRT uses an unbounded number of invocations of binary consensus objects and at most one uniform reliable broadcast (URB) per process. Zhang and Chen [46] proposed an algorithm for multivalued consensus that uses only x instances, where x is the number of bits it takes to represent any value in V; the domain of proposable values.

Our self-stabilizing solution is wait-free since termination is achieved within at most n invocations of binary consensus objects and at most one uniform reliable broadcast [32] (URB) operation per process, where n is the number of processes in the system. However, each such URB invocation needs to be repeated until the consensus object is deactivated by the invoking algorithm. This is due to a well-known impossibility [14, Chapter 2.3], which says that self-stabilizing systems cannot terminate and stop sending messages. Note that it is easy to trade the broadcast repetition rate with the speed of recovery from transient-faults.

Afek *et al.* [1] showed that binary and multivalued versions of the k-simultaneous consensus task are wait-free equivalent. Here, the k-simultaneous consensus is required to let each process to participate at the same time in k independent consensus instances until it decides in any one of them.

Our study focuses on deterministic solutions and does not consider probabilistic approaches, such as [4, 22, 31]. It is worth mentioning that Byzantine fault-tolerant multivalued consensus algorithms [11, 12, 35, 43] have applications to Blockchain [34]. Our fault model does not include Byzantine failures, instead, we consider arbitrary transient-faults.

## 1.4.2 Self-stabilizing solutions

We follow the design criteria of self-stabilization, which Dijkstra [13] proposed. A detailed pretension of self-stabilization was provided by Dolev [14] and Altisen *et al.* [3]. Consensus was sparsely studied in the context of self-stabilization. Blanchard *et al.* [7] presented the first

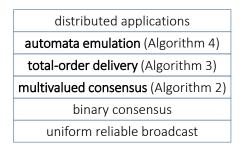


Figure 1: The studied problems of binary consensus (in **bold** font) and their context

solution in the context of self-stabilization. They presented a practically-self-stabilizing version of Paxos [28], which was the first (non-self-stabilizing) solution to the area of fault-tolerant message-passing systems. The studied solution is part of a more advanced and efficient protocol suite (Figure 1). We note that practically-self-stabilizing systems, as defined by Alon *et al.* [2] and clarified by Salem and Schiller [42], do not satisfy Dijkstra's requirements, *i.e.*, practicallyself-stabilizing systems do not guarantee recovery within a finite time after the occurrence of transient-faults. We base our self-stabilizing multivalued consensus on the self-stabilizing binary consensus by Lundström, Raynal, and Schiller [33], which is the first self-stabilizing solution to the binary consensus problem that recovers within a bounded time.

We propose, to the best of our knowledge, the first self-stabilizing solution for the multivalued version of the problem. As an application, we offer, to the best of our knowledge, the first self-stabilizing algorithm for (uniform reliable broadcast with) total order delivery. It is based on the self-stabilizing uniform reliable broadcast with FIFO delivery by Lundström, Raynal, and Schiller [32]. Our solution can facilitate the self-stabilizing emulation of state-machine replication. Dolev *et al.* [15] proposed the first practically-self-stabilizing emulation of state-machine replication, which has a similar task to one in Figure 1. However, Dolev *et al.*'s solution does not guarantee recovery within a finite time since it does not follow Dijkstra's criterion. Moreover, it is based on virtual synchrony by Birman and Joseph [6], where the one in Figure 1 considers censuses.

Georgiou, Lundström, and Schiller studied the trade-off between non-blocking and wait-free solutions for self-stabilizing atomic snapshot objects [24]. We study a similar trade-off for a different problem.

More generally, in the context of self-stabilization there are algorithms for group communications [19–21], consensus in shared-memory systems [18], wireless communications [27, 29, 30, 37–40], software defined networks [8, 9], virtual infrastructure for mobile nodes [16, 17, 45], to name a few.

# 1.5 Our contribution

We present a fundamental module for dependable distributed systems: a self-stabilizing wait-free algorithm for multivalued consensus for asynchronous message-passing systems that are prone to crash failures. To the best of our knowledge, we are the first to provide a wait-free solution for multivalued consensus that tolerates a broad fault model *i.e.*, crashes, communication failures, *e.g.*, packet omission, duplication, and reordering as well as arbitrary transient-faults using a bounded amount of resources. The latter models any violation of the assumptions according to which the system was designed to operate (as long as the algorithm code stays intact).

Our wait-free solution achieves (multivalued) consensus within n invocations of binary consensus instances that can run either sequentially or concurrently, where n is the number of processes. Besides, our concurrent version can piggyback the binary consensus messages and terminate within the time that it takes to complete one uniform reliable broadcast (URB) and one binary consensus. This is also the time it takes the system to recover after the occurrence of the last transient-fault.

As an application, this technical report offers a total order extension to the self-stabilizing FIFO URB service by Lundström, Raynal, and Schiller [32]. That self-stabilizing solution uses three multivalued consensus objects and stabilizes within a constant time. The technical report also explains how to enhance this solution to a self-stabilizing emulator of a replicated state machine.

# 1.6 Organization

We state our system settings in Section 3. The task specifications and solution organization appear in Section 2. Section 4 includes a brief overview of the studied algorithm by Mostéfaoui, Raynal, and Tronel [36] that has led to the proposed solution. Our self-stabilizing algorithm for consensus multivalued object is proposed in Section 5. The correctness proof appears in Section 6. We present an application to the proposed algorithm in Section 7, which is a self-stabilizing total order uniform reliable broadcast. We conclude in Section 8 and explain how to extend the proposed application to serve as an emulator for state-machine replication.

# 2 Task Specifications and Solution Organization

The proposed solution is tailored for the protocol suite presented in Figure 1. Thus, before we specify how all these tasks are organized into one solution, we list the external building blocks and define the studied tasks.

## 2.1 External Building-Blocks: Uniform Reliable Broadcast

#### 2.1.1 Binary consensus objects

 $T_{\text{binCon}}$  denotes the task of binary and multivalued consensus, which Definition 1.1 specifies. We assume the availability of self-stabilizing binary consensus objects, such as the one by Lundström, Raynal, and Schiller [33]. As in Definition 1.1, the proposed and decided values have to be from the V domain (of proposable values). For clarity sake, we distinguish the invocation of binary and multivalued consensus. That is, for a given binary consensus object BC, the operation BC.binPropose(v) invokes the binary consensus on  $v \in V = \{\text{True}, \text{False}\}$ . (Traditionally, the result of binary consensus is either 0 or 1, but we rename them.)

#### 2.1.2 First-in first-out uniform reliable broadcast (FIFO-URB)

The task  $T_{URB}$  of Uniform reliable broadcast (URB) [26] considers an operation for URB broadcasting of message m and an event of URB delivery of message m. The requirements include URB-validity, *i.e.*, there is no spontaneous creation or alteration of URB messages, URB-integrity, *i.e.*, there is no duplication of URB messages, as well as URB-termination, *i.e.*, if the broadcasting node is non-faulty, or if at least one receiver URB-delivers a message, then all non-failing nodes URB-deliver that message. Note that the URB-termination property considers

both faulty and non-faulty receivers. This is the reason why this type of reliable broadcast is named *uniform*.

The task of FIFO-URB, denoted by  $T_{\text{TO-URB}}$ , requires, in addition to the above URB requirements, *i.e.*, URB-validity, URB-integrity, and URB-termination, that all messages that come from the same sender are delivered in the same order in which their sender has sent them; but there are no constraints regarding messages that arrive from different senders.

The proposed solution assumes the availability of a self-stabilizing uniform reliable broadcast (URB) [32]. We also assume that the operation for URB broadcasting message m returns a transmission descriptor, txDes, which is the unique message identifier. Moreover, the predicate hasTerminated(txDes) holds whenever the sender knows that all non-failing nodes in the system have delivered m. The implementation of hasTerminated(txDes) can just test that all trusted receivers have acknowledged the arrival of the message with identifier txDes. The solution in [32] can facilitate the implementation of hasTerminated() since the self-stabilizing algorithm in [32] considers such messages as 'obsolete' messages and lets the garbage collector remove them.

## 2.2 Task specifications

We specify the studied tasks.

# 2.2.1 Total order URB (TO-URB)

The task of total order URB, denoted by  $T_{\text{TO-URB}}$ , requires the total order delivery requirement, in addition to URB-validity, URB-integrity, and URB-termination. The total order delivery requirement says that if a node calls toDeliver(m) and later toDeliver(m'), then no node toDeliver(m') before toDeliver(m).

#### 2.2.2 Binary and multivalued consensus objects

 $T_{\text{mulCon}}$  denotes the task of multivalued consensus, which Definition 1.1 specifies. As in Definition 1.1, the proposed and decided values have to be from the V domain (of proposable values), where |V| > 2. The operation propose(v) invokes the multivalued consensus on  $v \in V$ .

# 2.3 Solution organization

We consider multivalued consensus objects that use an array, BC[], of n binary consensus objects, such as the one by [41, Chapter 17], where  $n = |\mathcal{P}|$  is the number of nodes in the system. The proposed algorithm considers a single multivalued consensus object, denoted by O.

The proposed application, which is a TO-URB solution, considers an array, CS[], of M multivalued consensus objects, where  $M \in \mathbb{Z}^+$  is a predefined constant. Our solution for TO-URB uses M = 3 (Section 7). Each object is uniquely identified using a single sequence number. The proposed algorithm assumes that the multivalued consensus object O is stored at  $CS[s \mod M]$ . Whenever an operation is invoked or a message is sent, the sequence number s is attached as a procedure parameter, and respectively, a message field (although the code of the proposed algorithm does not show this). We note that in case the proposed application runs out of sequence numbers, a global restart mechanism can be invoked, such as the one in [24, Section 5]. The function test(s) is used to assert consistency of the sequence number s. The function returns False whenever inconsistency is detected. Our TO-URB solution exemplifies an implementation of test(). We assume that all underlying algorithms invoke test(s) whenever

the object  $CS[s \mod M]$  or  $CS[s \mod M]$ . $BC[k] : p_k \in \mathcal{P}$  is accessed, an operation is invoked, or a message (that is associated with s) arrives (although the code of the proposed algorithm does not show this). (The term underlying algorithm refers to both the proposed algorithm for multivalued consensus as well as the one for binary consensus.) If object  $CS[s \mod M]$  is found to be inconsistent, it is simply deactivated by assigning  $\perp$  to  $CS[s \mod M]$ . Also, inconsistent operation invocations and arriving messages are simply ignored.

Definition 1.1 considers the propose(v) operation, but it does not specify how the decided value is retrieved. We clarify that it can be either via the returned value of the propose(v) (or binPropose(v)) operation (as in algorithm [25]) or via the returned value of the result() operation (as in the proposed solution). But, if  $p_i \in \mathcal{P}$  is yet to have access to the decided value,  $result_i()$ returns  $\perp$ . Otherwise, the decided value is returned. Specifically, for the case of propose(v), the parameter s should be used when calling  $result_i(s)$  and for the case of binPropose(v), also the parameter  $k : p_k \in \mathcal{P}$  should be used when calling  $result_i(s, k)$ .

We clarify that, in the absence of transient-faults,  $\operatorname{result}_i(s)$  and  $\operatorname{result}_i(s, k)$  always return either  $\perp$  or the decided value. Thus, we solve the problem specified by Definition 1.1. The studied algorithm [25] was not designed to deal with transient-faults. As we explain in Section 4.2, transient-faults can cause the studied algorithm to violate Definition 1.1's requirements without providing any indication to the invoking algorithm. After the occurrence of a transient-fault, the proposed solution allows  $\operatorname{result}_i(s)$  to provide such indication to the invoking algorithm via the return of the transient error symbol  $\Psi$ . Section 5.2.2 brings the details and Algorithm 4 exemplifies the indication handling.

# 3 System settings

We consider an asynchronous message-passing system that has no guarantees on the communication delay. Moreover, there is no notion of global (or universal) clocks and the algorithm cannot explicitly access the local clock (or timeout mechanisms). The system consists of a set,  $\mathcal{P} = \{p_0, \ldots, p_{n-1}\}$ , of *n* crash-prone nodes (or processors) with unique identifiers. Due to an impossibility [14, Chapter 3.2], we assume that any pair of nodes  $p_i, p_j \in \mathcal{P}$  have access to a bidirectional communication channel, *channel*<sub>j,i</sub>, that, at any time, has at most **channelCapacity**  $\in \mathbb{N}$ packets on transit from  $p_j$  to  $p_i$ .

In the interleaving model [14], the node's program is a sequence of (atomic) steps. Each step starts with an internal computation and finishes with a single communication operation, *i.e.*, a message send or receive. The state,  $s_i$ , of node  $p_i \in \mathcal{P}$  includes all of  $p_i$ 's variables and channel<sub>j,i</sub>. The term system state (or configuration) refers to the tuple  $c = (s_1, s_2, \dots, s_n)$ . We define an execution (or run)  $R = c[0], a[0], c[1], a[1], \dots$  as an alternating sequence of system states c[x] and (atomic) steps a[x], such that each c[x + 1], except for the starting one, c[0], is obtained from c[x] by the execution of step a[x] that some processor takes. The set of legal executions (LE) refers to all the executions in which the requirements of the task T hold.

## 3.1 The fault model and self-stabilization

Failures are environment steps rather than algorithm steps.

#### 3.1.1 Benign failures

When the occurrence of a failure cannot cause the system execution to lose legality, *i.e.*, to leave *LE*, we refer to that failure as a benign one. The system is prone to *crash failures*, in which

nodes stop taking steps forever. We assume that at most t < n/2 node may crash. We denote by *Correct* the set of indices of processors that never crash. We consider solutions that are oriented towards asynchronous message-passing systems and thus they are oblivious to the time in which the packets arrive and depart. Also, the communication channels are prone to packet failures, such as omission, duplication, reordering. However, if  $p_i$  sends a message infinitely often to  $p_j$ , node  $p_j$  receives that message infinitely often. We refer to the latter as the *fair communication* assumption. We assume that any message can reside in a communication channel only for a finite period (before it is delivered or lost). The length of that period is unbounded since we assume no bound on transmission delays. In other words, our communication model formally excludes messages that it takes an infinite time to deliver or loss them; but to say that it takes an infinite time to deliver a given message means that this message is lost.

#### 3.1.2 Arbitrary transient-faults

We consider any violation of the assumptions according to which the system was designed to operate. We refer to these violations and deviations as *arbitrary transient-faults* and assume that they can corrupt the system state arbitrarily (while keeping the program code intact). The occurrence of an arbitrary transient-fault is rare. Thus, our model assumes that the last arbitrary transient-fault occurs before the system execution starts [14]. Also, it leaves the system to start in an arbitrary state.

#### 3.1.3 Dijkstra's self-stabilization criterion

An algorithm is *self-stabilizing* with respect to the task of LE, when every (unbounded) execution R of the algorithm reaches within a finite period a suffix  $R_{legal} \in LE$  that is legal. That is, Dijkstra [13] requires that  $\forall R : \exists R' : R = R' \circ R_{legal} \land R_{legal} \in LE \land |R'| \in \mathbb{Z}^+$ , where the operator  $\circ$  denotes that  $R = R' \circ R''$  concatenates R' with R''. The complexity measure of self-stabilizing systems, called *stabilization time*, is the time it takes the system to recover after the occurrence of the last transient-fault, *i.e.*, |R'|. The studied and proposed solutions allow nodes to interact and share information via binary consensus objects and uniform reliable broadcast (URB). Thus, we measure the stabilization time as the number of accesses to these primitives plus the number of URB accesses.

# 4 Background: Non-self-stabilizing Non-blocking Multivalued Consensus

We review in sections 4.1 and 4.2 a non-self-stabilizing non-blocking algorithm for multivalued consensus by Mostéfaoui, Raynal, and Tronel [36], which uses an unbounded number of binary consensus objects.

# 4.1 Algorithm 1: non-self-stabilizing multivalued consensus

The non-self-stabilizing solution in Algorithm 1 is the basis for its self-stabilizing variation in Algorithm 2. For the sake of a simple presentation, the line numbers of Algorithm 2 continues the ones of Algorithm 1. The operation propose(v) (line 5) invokes an instance of a multivalued consensus object. Algorithm 1 uses a uniform reliable broadcast (URB) [41] for letting any  $p_i \in \mathcal{P}$  disseminate its proposed value  $v_i$  (line 7). Each  $p_j \in \mathcal{P}$  that delivers this proposal, stores this value in  $proposal_j[i]$ . Also,  $p_k \in \mathcal{P}$  can concurrently broadcast its proposal,  $v_k$ , which  $p_j$  Algorithm 1: Non-self-stabilizing non-blocking multivalued consensus using an unbounded number of binary consensus instances; code for  $p_i$ 

```
1 local variables:
```

```
2 proposals[0, ., n-1];
                                                   /* array of the received proposals */
 3 k ;
                                                                      /* the round counter */
 4 BC[0, ., n-1];
                                      /* binary consensus objects (unbounded list) */
 5 operation propose(v) begin
       (proposals, BC) \leftarrow ([\bot, \dots, \bot], [\bot, \dots, \bot]);
 6
       urbBroadcast PROPOSAL(v);
 \mathbf{7}
       while (k \leftarrow 0; \text{True}; k \leftarrow k+1) do
 8
           if BC[k].binPropose((proposals[k \mod n] \neq \bot)) then
 9
               wait(proposals[k mod n] \neq \perp);
10
               return (proposals[k \mod n]);
\mathbf{11}
12 upon urbDelivered PROPOSAL(v) from p_j do {proposals[j] \leftarrow v;}
```

stores in  $proposal_j[k]$ . Therefore, Algorithm 1 needs to decide which entry in  $proposal_j[]$  the propose(v) operation should return. This decision is coordinated via an unbounded global array  $BC[0], BC[1], \ldots$ , of binary consensus objects.

Algorithm 1 starts by URB-broadcasting  $p_i$ 's proposed value,  $v_i$  (line 7). This broadcast assures that all correct nodes receive identical sets of messages (Section 2.1.2). Also, the set of delivered messages must include every message URB-broadcast by any correct node. The arrival of PROPOSAL(v) from  $p_j$ , informs  $p_i$  about  $p_j$ 's proposal, and thus,  $p_i$  stores  $v_j$  in proposals<sub>i</sub>[j] (line 12).

Following the proposal broadcast, Algorithm 1 proceeds in asynchronous rounds. The variable k stores the round counter (lines 3 and 8). Once  $p_i$  decides, it leaves the loop by returning the value of  $proposals_i[x_i]$  (line 11), where  $x_i = k_i \mod n$ . In other words,  $x_i \in \{0, \ldots, n-1\}$  is the identifier of the node that has broadcast the proposal stored in  $proposals_i[x_i]$ .

As mentioned, the selection of  $x_i$  is facilitated via the unbounded array, BC[], of binary consensus objects. Since all correct nodes eventually receive the same set of broadcasts,  $p_i$  proposes  $proposals_i[x] \neq \bot$  to the  $k_i$ -th object,  $BC[k_i]$  (line 9). *I.e.*,  $p_i$  proposes True on the  $k_i$ -th round if, and only if, it received  $p_{x_i}$ 's proposal.

Algorithm 1 continues to the next round whenever  $BC[k_i]$  decides False. Otherwise,  $p_i$  decides the value,  $proposals_i[x_i]$ , proposed by  $p_{x_i}$ . Due to asynchrony,  $p_i$  might need to wait until  $p_{x_i}$ 's broadcast was URB-delivers (line 10). However, if any node proposed to decide  $v_{x_i}$ , it must be the case that  $proposals_i[x_i]$  was delivered to the node that has proposed True at  $BC[k_i]$ . Therefore, eventually,  $p_i$  is guaranteed to URB-deliver  $v_{x_i}$  and stores it at  $proposals_i[x_i]$ . For this reason, Algorithm 1 does not block forever in line 10 and the decided value is eventually returned in line 11.

#### 4.2 Executing Algorithm 1 in the presence of transient-faults

Before describing Algorithm 2, we review the main challenges that one faces when transferring Algorithm 1 to an algorithm that can recover after the concurrence of transient-faults.

(a) Upon propose(v), uniform reliable broadcast ⟨v⟩.
(b) By URB-termination, eventually, there is p<sub>j</sub> ∈ P and round k', such that p<sub>j</sub>'s message arrived at all non-faulty processors, *i.e.*, ∀ℓ ∈ Correct ⇒ proposals<sub>ℓ</sub>[j] ≠ ⊥.
(c) For k ∈ {0, 1, 2, ...}, p<sub>i</sub> invokes BC[k].binPropose(proposals[k mod n] ≠ ⊥).
(d) By BC-termination and stage (b), eventually, the k<sub>min</sub>-th binary consensus objects is the first to decide True while all x-th objects decide False, where x ∈ {0, 1, 2, k<sub>min</sub>-1}.
(e) Due to URB-termination, eventually, proposals[k<sub>min</sub> mod n] includes a non-⊥ value.
(f) Then, return proposals[k<sub>min</sub> mod n] as the decided value.

Figure 2: High-level stages in the execution of Algorithm 1; code for  $p_i$ 

#### 4.2.1 Use of an unbounded number of binary objects

Self-stabilizing systems can only use a bounded amount of memory [14]. This is because, in practice, computer systems can use only a finite amount of memory. However, a single transient-fault can set every counter (or data-structure) to its maximum value (respectively, exhaust the memory capacity of the data-structure).

#### 4.2.2 Corrupted round number counter

In the context of self-stabilization, one cannot simply rely on counter k (line 3) to count the number of asynchronous rounds. This is because a single transient-fault can set the value of k to zero. It can also alter the state of every  $BC[k]_{k \in \{0,...,z\} \land z \in \mathbb{Z}^+}$ , such that a call to propose() returns False, where z can be practically infinite, say  $z = 2^{64} - 1$ . In this case, the system will have to iterate for  $2^{64}$  times before a fresh binary consensus object is reached.

#### 4.2.3 Corrupted program counter

A transient-fault can set the program counter of every  $p_j \in \mathcal{P}$  to skip over the broadcast in line 7 and to point to line 8. If this happens, then validity or termination can be violated. Therefore, there is a need to repeat the transmission of  $v_i$  in order to make sure that at least one proposal is known to all correct processors.

#### 4.2.4 A corrupted array of binary objects

Transient faults can corrupt binary objects in the array BC[]. Specifically, since the array BC[] should include only a bounded number of binary consensus objects, a transient-fault can change the state of all objects in BC[] to encode 'decide False'. In this case, Algorithm 1 cannot finish the multivalued consensus.

# 5 The Proposed Solution: Self-stabilizing Wait-free Multivalued Consensus

This section presents a new self-stabilizing algorithm for multivalued consensus that is waitfree and uses n binary consensus objects and n self-stabilizing uniform reliable broadcasts (URBs) [32]. The correctness proof appears in Section 6. (a) Upon propose(v), uniform reliable broadcast ⟨v⟩.
(b) Wait until hasTerminated() says that ⟨v⟩ arrived at all non-faulty processors.
(c) For k ∈ {0,...,n-1}, p<sub>i</sub> invokes BC[k].binPropose(proposals[k mod n] ≠ ⊥).
(d) By BC-termination and stage (b), eventually, the k<sub>min</sub>-th binary consensus objects is the first to decide True while all x-th objects decide False, where x ∈ {0, 1, 2, k<sub>min</sub>-1}.
(e) Due to URB-termination, eventually, proposals[k<sub>min</sub> mod n] includes a non-⊥ value.
(f) Then, return proposals[k<sub>min</sub> mod n] as the decided value.

Figure 3: A bounded alternative to Figure 2; code for  $p_i$ 

#### 5.1 The algorithm idea

We sketch the key notions that are needed for Algorithm 2 by addressing the challenges raised in Section 4.2.

#### 5.1.1 Using a bounded number of binary objects

We explain how Algorithm 2 can use only at most n binary consensus objects. Figure 2 is a high-level description of Algorithm 2's execution and Figure 3 shows how this process can be revised. The key differences between figures 2 and 3 appear in the boxed text of Figure 3. Specifically, Figure 3 waits until  $p_i$ 's broadcast has terminated in line (b). At that point in time,  $p_i$  knows that all non-faulty processors have received its message. Only then does  $p_i$  allow itself to propose values via the array of binary objects. This means that no processor starts proposing any binary value before there is at least one index  $k \in \{0, \ldots, n-1\}$  for which  $proposals_j[k] \neq \bot$ , where  $p_j$  is any node that has not failed. This means that, regardless of who is going to invoke the k-th binary consensus object, only the value True can be proposed. For this reason, there is no need to use more than n binary consensus values until at least one of them decides True, cf. line (c) in Figure 3.

## 5.1.2 Dealing with corrupted round number counter

Using the object values in BC[], Algorithm 2 calculates k(), which returns the current round number. This way, a transient-fault cannot create inconsistencies between k()'s value and BC[].

In detail, for an active multivalued consensus object O, *i.e.*,  $O \neq \bot$ , we say that the binary consensus object O.BC[k] is active when  $O.BC[k] \neq \bot$ . Algorithm 2 calculates k() (line 19) by counting the number of active binary consensus objects that have terminated and the decided value is False. We restrict this counting to consider only the entries BC[k], such that k = 0 or  $\forall k' < k : BC[k']$  is an active binary consensus objects that have terminated and the decided False. This is defined by the set  $\mathsf{K} = (\{k \in S(n-1) : O.BC[k] \neq \bot \land O.BC[k].\mathsf{result}(k) = \mathsf{False}\})$ , where  $S(x) = \{0, \ldots, x\}$  is the set of all integers between zero and x. This way, the value of  $\mathsf{k}()$  is simply  $\max(\{\{-1\} \cup \{x \in S(n-1) : (S(x) \cap \mathsf{K}) = S(x)\})$ . Note that the value of -1 is used to indicate that there are no active binary objects in BC[] that have terminated with a decided value of False, *i.e.*,  $\mathsf{K} = \emptyset$ .

# 5.1.3 Dealing with a corrupted program counter

As explained in Section 4.2.3, there is a need to repeat the transmission of  $v_i$  in order to make sure that at least one proposal is known to all correct processors. Specifically, after  $\operatorname{propose}_i(v_i)$ 's invocation,  $p_i \in \mathcal{P}$  need to store  $v_i$  and broadcasts  $v_i$  repeatedly due to a well-known impossibility [14, Chapter 2.3]. Note that there is an easy way to trade the broadcast repetition rate with the recovery speed from transient-faults. Also, once the first broadcast has terminated, all correct processors  $p_i \in \mathcal{P}$  are ready to decide by proposing  $\operatorname{binPropose}_i(k, \operatorname{proposals}_i[k] \neq \bot$ ) for any  $p_k \in \mathcal{P}$ , see steps (b) and (c) in Figure 3.

#### 5.1.4 Dealing with a corrupted array of binary objects

Algorithm 2 uses only *n* binary consensus objects. Due to the challenge in Section 4.2.4, we explain how to deal with the case in which a transient-fault changes the state of all objects in BC[] to encode 'decide False'. In this case, the algorithm cannot satisfy the requirements of the multivalued consensus task (Definition 1.1). Therefore, our solution identifies such situations and informs the invoking algorithm via the return of the *transient error* symbol  $\Psi$ .

#### 5.2 Algorithm description

## **5.2.1** The propose(v) operation and variables

The operation propose(v) activates a multivalued object by initializing its fields (line 20). These are the proposed value, v, the array, proposals[], of received proposals, where proposals[j]stores the value received from  $p_j \in \mathcal{P}$ . Moreover, BC[] is the array of binary consensus objects, where the active object BC[j] determines whether the value in proposals[j] should be the decided value. Also, txDes is the transmission descriptor (initialized with  $\bot$ ), and oneTerm is a boolean that indicates that at least one transmission has completed, which is initialized with False. Note that only v has its (immutable) value initialized in line 20 to its final value. The other fields are initialized to  $\bot$  or an array of  $\bot$  values; their values can change later on.

#### 5.2.2 The result() operation

Algorithm 2 allows retrieving the decided value via result() (line 21). As long as the multivalued consensus object is not active (line 22), or there is no decision yet (line 24), the operation returns  $\perp$ . As explained in Section 5.1.3, Algorithm 2 might enter an error state. In this case, result() returns  $\Psi$  (lines 23 and 25). The only case that is left (the else clause of line 25) is when there is a binary consensus object O.BC[k] and a matching  $O.proposals[k] \neq \perp$ , where k = k(). Here, due to the definition of k() (line 19), for any  $k' \in \{0, \ldots, k\text{-}1\}$  the decided value of O.BC[k'] is False and O.BC[k] decides True. Thus, result<sub>i</sub>() returns the value of O.proposals[k].

#### 5.2.3 The do-forever loop

As explained above, Algorithm 2 has to make sure that the proposed value, v, arrives at all processors and records in *oneTerm* the fact that at least once transmission has arrived. To that end, in line 27,  $p_i$  tests the predicate  $(txDes \neq \bot \land hasTerminated(txDes))$  and makes sure that the transmission descriptor, txDes, refers to an active broadcast, *i.e.*, txDes stores a descriptor that has not terminated (cf. hasTerminated()'s definition in Section 2.1.2). In detail, whenever  $x.txDes \neq \bot$  holds, hasTerminated<sub>i</sub>(txDes) holds eventually (URB-termination). Thus, the if-statement condition in line 27 holds eventually and  $p_i$  URB-broadcast PROPOSAL(v) (line 29)

**Algorithm 2:** Self-stabilizing non-blocking multivalued consensus;  $p_i$ 's code

13 variables: /\* initialization is optional in the context of self-stabilization \*/ /\* local decision estimates \*/ 14 v; **15** proposals[0, ., n-1]; /\* array of arriving proposals \*/ **16** BC[0, ., n-1]; /\* array of n binary consensus objects \*/ 17 txDes; /\* URB transmission descriptor for decision sharing \*/ 18 oneTerm ; /\* true once at least one broadcast termination occured \*/ 19 macro  $k() = max(\{\{-1\} \cup \{x \in S(n-1) : (S(x) \cap K) = S(x)\})$ : where S(x) = $\{0,\ldots,x\}$  and  $\mathsf{K} = (\{k \in S(n-1) : O.BC[k] \neq \bot \land O.BC[k].\mathsf{result}(k) = \mathsf{False}\});$ /\* k() is the max consecutive BC[] entry index with the decision False \*/ 20 operation propose(v) do {if  $v \neq \bot \land O = \bot$  then  $O.(v, proposals, BC, txDes, oneTerm) \leftarrow (v, [\bot, ..., \bot], [\bot, ..., \bot], \bot, \mathsf{False}) \};$ 21 operation result() begin if  $O = \bot$  then return  $\bot$ ;  $\mathbf{22}$ else if  $O.v = \bot \lor k \ge n-1$  then return  $\Psi$  where k = k(); 23 else if  $BC[k+1] = \bot \lor BC[k+1]$ .result $(k+1) \neq$  True then return  $\bot$ ;  $\mathbf{24}$ else if  $x = \bot$  then return  $\Psi$  else return x where x = O.proposals[k+1]; $\mathbf{25}$ **26** do forever foreach  $O \neq \bot$  with O's fields v, proposals, BC, and txDes do if  $(v \neq \bot \land (txDes = \bot \lor hasTerminated(txDes))$  then 27  $oneTerm \leftarrow oneTerm \lor (txDes \neq \bot \land hasTerminated(txDes));$  $\mathbf{28}$  $txDes \leftarrow urbBroadcast PROPOSAL(v)$ 29 /\* use either lines 30 to 31 or lines 32 to 33 \*/ if  $oneTerm \land k < n-1 \land BC[k+1] = \bot \land (k=-1 \lor BC[k].result(k) \neq \bot)$  then  $\mathbf{30}$ binPropose(k+1, proposals[k+1]  $\neq \perp$ ) where k = k()31 if  $|oneTerm \land \exists \ell : BC[\ell] = \bot |$  then /\* invoke BC objects concurrently \*/ 32 for each  $k \in \{0, \dots, n-1\}$ :  $BC[k] = \bot$  do binPropose $(k, proposals[k+1] \neq \bot)$ 33 upon PROPOSAL(vJ) urbDelivered from  $p_j$  begin  $\mathbf{34}$ if  $vJ \neq \bot$  then 35 if  $O \neq \bot \land O.proposals[j] = \bot$  then  $O.proposals[j] \leftarrow vJ;$ 36 37 else if  $O = \bot$  then (O.(v, proposals, BC, txDes)),  $O.proposals[j]) \leftarrow ((sJ, vJ, [\bot, ..., \bot], [\bot, ..., \bot], \bot), vJ);$ 

after checking that  $v \neq \perp$  (line 27). Note that  $p_i$  records the fact that at least one transmission was completed by assigning True to *oneTerm* (line 28).

Upon the URB-delivery of  $p_i$ 's PROPOSAL(vJ) at  $p_j \in \mathcal{P}$ , processor  $p_j$  considers the following two cases. If O is an active object,  $p_j$  merely checks whether O.proposals[i] needs to be updated with vJ (line 36). Otherwise, O is initialized with vJ as the proposed value (line 37) similarly to line 20.

Going back to the sender side, Algorithm 2 uses either lines 30 to 31, which sequentially access the array, BC[], of binary consensus objects, or lines 32 to 33, which simply access all binary objects concurrently. In both methods, processor  $p_i$  makes sure that at least one broadcast was completed, *i.e.*, oneTerm = True (lines 28, 30 and 32). When following the sequential method (lines 30 to 31), the aim is to invoke binary consensus by calling  $\text{binPropose}(k+1, proposals[k+1] \neq \bot$ ) (line 31), where  $k = k_i$ (). This can only happen when the (k+1)-th object in BC[] is not active, *i.e.*,  $BC[k+1] = \bot$  and BC[k+1] is either the first in BC[], *i.e.*, k = -1 or BC[k] has terminated, *i.e.*, BC[k].result<sub>i</sub>(k)  $\neq \bot$  (line 30).

The advantage of the sequential access method over the concurrent one is that it is more conservative with respect to the number of consensus objects that are being used since once the decision is True, there is no need to use more objects. The concurrent access method, marked in the boxed lines, encourages to piggyback of the messages related to binary concurrent objects. This is most relevant when every message (of binary consensus) can carry the data-loads of n proposals. In this case, the concurrent access method is both simpler and faster than the sequential one.

# 6 Correctness of Algorithm 2

Theorems 6.1 and 6.6 show that Algorithm 2 implements a self-stabilizing multivalued consensus. Definition 6.1 is used by Theorem 6.1. As explained in Section 2.3, for the sake of a simple presentation, we make the following assumptions. Let R be an Algorithm 2's execution,  $p_i \in \mathcal{P}$ , and  $O_i$  a multivalued consensus object.

**Definition 6.1 (Consistent multivalued consensus object)** Let R be an Algorithm 2's execution and  $O_i$  a multivalued consensus object, where  $p_i \in \mathcal{P}$ . Suppose either (i)  $O_i = \bot$  is inactive or that (ii)  $O_i \neq \bot$  is active,  $O_i \cdot v \neq \bot \land (k < n - 1) \land ((BC[k+1] = \bot \lor BC[k+1].result(k+1) = \bot \lor (BC[k+1].result(k+1) = True \land O_i.proposals[k+1] \neq \bot)))$ , where  $k = k_i()$ . In either case, we say that  $O_i$  is consistent in c.

Theorem 6.1 shows recovery from arbitrary transient-faults.

**Theorem 6.1 (Convergence)** Let R be an Algorithm 2's execution. Suppose that there exists a correct processor  $p_j \in \mathcal{P} : j \in Correct$ , such that throughout R it holds that  $O_j \neq \bot$  is an active multivalued consensus object. Moreover, suppose that any correct processor  $p_i \in \mathcal{P} : i \in Correct$ calls result<sub>i</sub>() infinitely often in R. Within n invocations of binary consensus, (i) the system reaches a state  $c \in R$  after which result<sub>i</sub>()  $\neq \bot$  holds. Specifically, (ii)  $O_i$  is either consistent (Definition of 6.1) or eventually reports the occurrence of a transient-fault, i.e., result<sub>i</sub>() =  $\Psi$ .

**Proof of Theorem 6.1** Lemmas 6.2 and 6.5 implies the proof.

**Lemma 6.2** Invariant (i) holds, i.e.,  $\operatorname{result}_i() \neq \bot$  holds in c.

**Proof of Lemma 6.1** Suppose, towards a contradiction, that c does not exist. Specifically, let R' be the longest prefix of R that includes no more than n invocations of binary consensus. The proof of Invariant (i) needs to show that the system reaches a contradiction by showing that  $c \in R'$ . To that end, arguments (1) to (3), as well as claims 6.3 to 6.4, show the needed contradiction.

Argument (1) implies that it is enough to show that the if-statement in line 24 cannot hold eventually.

Argument (1) The if-statement conditions in lines 22, 23, and 25 do not hold for  $p_j$  throughout R. By the theorem assumption that  $O_j \neq \bot$  is an active multivalued consensus object throughout R, we know that the if-statement condition in line 22 cannot hold. Moreover, by the assumption that c does not exist, we know that the if-statement conditions in lines 23 and 25 do not hold for any (correct)  $p_i$  throughout R.

**Argument (2)** The invariant  $O_j . v \neq \bot$  holds throughout R. Since the if-statement condition in line 23 does not hold,  $O_j . v \neq \bot$  holds in R's starting system state. Moreover, only lines 20, 36, and 37 change the value of  $O_j . v$  but this happens only after testing that the assigned value is not  $\bot$  (lines 20 and 35).

**Argument (3)** R has a suffix in which all correct processors  $p_i \in \mathcal{P}$  are active. Since  $p_j$  is active and  $O_j . v \neq \bot$  holds throughout R, the if-statement condition in line 27 holds eventually since either  $txDes = \bot$  or hasTerminated(txDes) holds eventually due to the URB-termination property. By line 29,  $p_j$  broadcasts the  $\langle v \rangle$  message to all correct processors  $p_i$ . By the URB-termination property,  $p_i$  receives  $\langle v \rangle$  and by lines 36 to 37, processor  $p_i$  is active.

Argument (4)  $\forall i, j \in Correct : O_i.proposals[j] \neq \bot \land O_i.txDes \neq \bot \land O_i.oneTerm = \mathsf{True}$ holds eventually. By URB-termination, hasTerminated( $O_i.txDes$ ) holds eventually. Once that happens, the if-statement condition in line 27 holds (due to arguments (2) and (3)) and  $O_i.txDes = \bot$  cannot hold (line 29). By Argument (2),  $O_i.v \neq \bot$ . Thus,  $p_i$  eventually URB-broadcasts PROPOSAL( $O_i.v$ ). Once  $p_i$  self-delivers this message, line 36 assigns v to  $O_i.proposals[i]$  due to the assumption that  $O_i \neq \bot$  throughout R. We can now repeat the reasoning that hasTerminated( $O_i.txDes$ ) holds eventually and thus the if-statement condition in line 27 hold. Thus,  $O_i.oneTerm = \mathsf{False}$  does not hold eventually (line 28). By Argument (3), the same holds for  $p_j$ . Specifically,  $p_j$  eventually URB-broadcasts PROPOSAL( $O_j.v$ ). Once  $p_i$ URB-delivers this message from  $p_j, p_i$ 's state can possibly change, even in the case that  $O_i \neq \bot$ , cf. lines 36 and 37.

**Claim 6.3** The *if-statement condition in lines 30 and 32 can only hold at most n times for any*  $p_i \in \mathcal{P} : i \in Correct.$ 

**Proof of Claim 6.1** The if-statement condition in line 32 can only hold at most once due to line 33. Thus, the rest of the proof focuses on line 31.

By the proof of Argument (4), eventually, the system reaches a state,  $c' \in R$ , in which  $O_i.txDes \neq \perp \land O_i.proposals[j] \neq \perp \land O_i.oneTerm =$ True holds. Note that the if-statement condition in line 30 holds whenever k = -1. Arguments (5) and (6) assumes that k > -1 and consider the cases in which  $O_i.BC[k+1] \neq \perp$  holds and does not hold, respectively, where k = k(). Argument (7) shows that the if-statement condition in line 30 can hold at most n times.

Argument (5) Suppose that  $k > -1 \land O_i.BC[k+1] \neq \bot$  holds. Eventually, either  $k_i() < n-1$ does not hold or the if-statement condition in line 30 holds.  $BC[k+1].\text{result}_i(k+1) \neq \bot$  holds eventually due to the termination property of binary consensus objects.

In case BC[k+1].result<sub>i</sub>(k+1) = True, we know that result<sub>i</sub> $() \neq \bot$  holds due to the definition of k() (line 19). However, this implies a contradiction with the assumption made at the start of this lemma's proof.

In case BC[k+1].result<sub>i</sub>(k+1) = False holds, the if-statement condition in line 30 holds in c' if k+1 < n-1 and  $O_i.BC[k+1] = \bot$ . In case the former predicate holds and the latter does not, we can repeat the reasoning above for at most n times until either the former does not hold or both predicates hold. In either case, the proof of the argument is done.

**Argument (6)** Suppose that  $k > -1 \land O_i.BC[k+1] = \bot$  holds. Eventually, either  $k_i() < n-1$  does not hold or the if-statement condition in line 30 holds. The if-statement condition in line 30 holds. The if-statement condition in line 30 holds. The reasoning in the proof of

Argument (5), which shows that BC[k+1].result<sub>i</sub> $(k+1) \neq \bot$  holds, can be used for showing that BC[k].result<sub>i</sub> $(k) \neq = \bot$  holds eventually.

Argument (7) Within n invocations of  $binPropose_i()$ , the if-statement condition in line 30 does not hold. Suppose that the if-statement condition in line 30 holds. In line 31,  $p_i$  invokes the operation  $binPropose_i(k+1, O_i.proposals[k+1] \neq \bot)$  of the (k+1)-th binary consensus object. This invocation changes  $p_i$ 's state, such that  $O_i.BC[k+1] = \bot$  does not hold any longer (because the  $binPropose_i()$  operation initializes the state of  $O_i.BC[k+1]$ ). Since BC[] has n entries, there could be at most n such invocations until the system reaches  $c'' \in R$ , after which the if-statement condition in line 30 cannot hold.

**Claim 6.4** Once the if-statement condition in line 30 (or 32) does not hold, also the if-statement condition in line 24 does not hold.

**Proof of Claim 6.1** Since if-statement condition in line 30 does not hold, we know that  $BC_i[k+1] = \bot$  does not hold, see Argument (5) of Claim 6.3. In the case of line 32, the same holds in a straight forward manner. By BC-termination, BC[k+1].result<sub>i</sub> $(k+1) \neq \bot$  holds eventually. Since  $\forall p_x \in \mathcal{P} : O_i.BC[x] \neq \bot \land BC[x]$ .result<sub>i</sub>(x) = False implies a contradiction with Argument (1), we know that  $BC_i[k+1]$ .result $(k+1) \neq$  True cannot hold.  $\Box_{Claim 6.4}$ 

**Lemma 6.5** Invariant (ii) holds, i.e.,  $O_i$  is either consistent or result<sub>i</sub>() =  $\Psi$ .

**Proof of Lemma 6.1** Recall that the theorem assumes that  $O_i$  is an active object throughout R. The argument is implied by Definition 6.1 and lines 22 to 25.

In detail, line 22 handles the case in which  $O_i = \bot$ . Suppose that  $O_i \neq \bot$  is active, which indicates that an inconsistent was detected. Line 23 handles the case in which  $O_i \cdot v \neq \bot \land k < n-1$  does not hold by returning  $\Psi$ , where  $k = \mathsf{k}_i()$ , which indicates that an inconsistent was detected. Line 24 allows the case in which  $BC[k+1] = \bot \lor BC[k+1]$ .result $(k+1) = \bot$  (note that the case of BC[k+1].result $(k+1) = \mathsf{False}$  does not exist due to the definition of k() in line 19). This case is allowed since it is consistent, see Definition 6.1. Line 25 handles the case in which  $(BC[k+1].\mathsf{result}(k+1) = \mathsf{True} \land O_i.proposals[k+1] \neq \bot)$  does not holds by returning  $\Psi$ , which indicates that an inconsistent was detected.  $\Box_{Lemma \ 6.5}$ 

Definition 6.2 is used by Theorem 6.6.

**Definition 6.2 (Complete execution with respect to propose() invocations)** Let R be an execution of Algorithm 2 that starts in  $c \in R$ . We say that c is completely free of PROPOSAL(-) messages if (i) the communication channels do not include PROPOSAL(-) messages, and (ii) for any non-failing  $p_i \in \mathcal{P}$ , there is no active multivalued consensus object  $O_i = \bot$  in c. Let  $c_s \in R$  be the system state that is: (a) completely free of PROPOSAL(-), and (b) it appears in R immediately before a step that includes  $p_i$ 's invocation of propose(-) (lines 20) in which  $O_i$  becomes active (rather than due to the arrival of a PROPOSAL(-) message in lines 35 to 37). In this case, we say that  $p_i$ 's invocation is authentic. Suppose that  $p_i$  sends a PROPOSAL(-) message after  $c_s$ . In this case, we say that PROPOSAL(-) is an authentic message transmission. An arrival of PROPOSAL(-) to  $p_j \in \mathcal{P}$  (lines 20) is said to be authentic if it is due to an authentic arrival (rather than an invocation of the propose(-) operation). In this case, we also say that  $p_j$ 's invocation is authentic. Suppose that any invocation is arrival, and invocations by applying the transitive closures of them. Suppose that any invocation in R

of  $propose_k(-): p_k \in \mathcal{P}$  is authentic as well as the transmission and reception of PROPOSAL(-)messages from or to  $p_k$ . In this case, we say that R is authentic.

Theorem 6.6 shows that Algorithm 2 satisfies the task requirements (Section 2.2).

**Theorem 6.6 (Closure)** Let R be an authentic execution of Algorithm 2. The system demonstrates in R the construction of a multivalued consensus object.

**Proof of Theorem 6.6** Validity holds since only the user input is stored in the field v (line 20), which is then URB-broadcast (line 29), stored in the relevant entry of *proposals* (lines 36 to 37), and returned as the decided value (line 25). Moreover, any value in v can be traced back to an invocation of propose(v) since R is authentic.

Lemma 6.7 demonstrates termination and agreement.

**Lemma 6.7** Let  $a_i \in R$  be the first step in R that includes an invocation, say, by  $p_i \in \mathcal{P}$  of  $\operatorname{propose}_i(v_i)$ . Suppose that  $v_i \neq \bot$  holds in any system state of R. There exists  $v \notin \{\bot, \Psi\}$ , such that for every correct  $p_j \in \mathcal{P}$  it holds that  $\operatorname{result}_j()$  returns v within n invocations of binary consensus.

**Proof of Lemma 6.6** Arguments (1) to (7) imply the proof.

**Argument (1)**  $O_i.(v, proposals, BC, txDes, oneTerm) = (v, [\bot, ..., \bot], [\bot, ..., \bot], \bot, False)$ holds immediately after  $a_i$ . We show that the if-statement condition in line 20 holds immediately before  $a_i$ . Recall the theorem assumption that  $v_i \neq \bot$  holds in R. By the assumption that R is authentic, we know that  $O_i = \bot$  holds immediately before  $a_i$ . Therefore,  $p_i$  assigns  $(v_i, [\bot, ..., \bot], [\bot, ..., \bot], \bot, False)$  to  $O_i.(v_i, proposals_i, BC_i, txDes_i, oneTerm_i)$  (line 20).

Argument (2) Eventually PROPOSAL $(v_i)$  messages are URB broadcast and oneTerm<sub>i</sub> holds. By URB-termination, hasTerminated $(O_i.txDes)$  does not hold eventually. Since  $O_i.v \neq \bot$  (by the lemma assumption), the if-statement condition in line 27 holds and  $p_i$  URB-broadcasts PROPOSAL $(O_i.v)$ . By applying again the same argument, the assignment in line 28 makes sure that  $oneTerm_i = \text{True}$ .

**Argument (3)** For any  $p_x \in \mathcal{P}$ :  $x \in Correct$ , eventually  $O_x.proposals[i] \neq \bot$  and  $O_x.proposals[x] \neq \bot$  hold. By URB-termination, every correct processor,  $p_x$ , eventually URB-delivers Argument (2)'s PROPOSAL $(v_i)$  message. By the assumption that  $v_i \neq \bot$  holds in any system state of R, the if-statement condition in line 35 holds (even if  $p_x$  has invoked propose<sub>x</sub> $(v_x)$  before this URB delivery).

In case there was no earlier invocation of  $propose_x(v_x)$ , the assignment  $O_x.v \leftarrow v_i$  occurs due to line 37 (otherwise, a similar assignment occurs due to line 36). Moreover, due to the reasons that cause  $p_i$  URB broadcasts in Argument (2), also  $p_x$  URB broadcasts PROPOSAL $(v' \neq \bot)$  messages. Upon the URB delivery of  $p_x$  message to itself, the  $O_x.proposals[x] \leftarrow v' \neq \bot$  assignment occurs (line 36). (Note that this time, the if-statement condition in line 36 must hold since  $O_x \neq \bot$ .)

Argument (4) The if-statement condition in lines 30 and 32 hold eventually. Since  $oneTerm_i$  holds eventually (Argument (2)), the if-statement condition in line 32 holds eventually. Also, the fact that  $O_x.proposals[x] \neq \bot$  (Argument (3)) and URB-termination imply that eventually, in  $p_x$ 's do-forever loop, the if-statement condition in line 30 holds. In detail, since R is an authentic execution,  $k = -1 \land BC[k+1] = \bot$  holds in R's second state, where  $k = k_x()$ .

Let  $S(z) = \{0, ..., z\}$ . The proof of Argument (5) shows  $\exists y \in S(n-1), \forall x \in Correct, p_x$ invokes  $binPropose_x()$  at most y times and it observes that  $O_x.BC[k] \neq \bot : k \in S(y-1)$ . Argument (5) The Termination property holds. By line 33, the if-statement condition in line 32 can hold at most once. The if-statement condition in line 30 cannot hold for more than n times due to the (k < n-1) clause. Thus, the termination property is implied.

Let  $r(j) = [x(0), \ldots, x(n-1)]$ :  $x(k) = BC_j[k]$ .result(k) and  $S = \{[\bot, \ldots, \bot], [\ldots, \mathsf{False}, \bot, \ldots, \bot], [\ldots, \mathsf{False}, \mathsf{True}, \bot, \ldots, \bot], [\ldots, \mathsf{False}, \mathsf{True}]\}$ . For the case of using lines 30 to 31, the proof of Argument (6) shows  $\forall p_j \in \mathcal{P} : r(j) \in S$ , *i.e.*, sequential invocation of binPropose().

Argument (6) For the case of using lines 30 to 31,  $r(j) \in S$  holds. Due to lines 19 and 30 as well as the agreement property of binary consensus objects and the fact that R is authentic, we know that eventually, all non-failing nodes must observe the same results from their consensus objects. Specifically for the case of lines 30 to 31, it holds that  $\forall p_j \in \mathcal{P} : r(j) = r_s$ . Also, at any time, r(j) can only include a finite number (perhaps empty but with no more than n-1) of False values that are followed by at most one True value and the only  $\perp$ -values (if space is left), *i.e.*,  $r(j) \in S$ .

Argument (7) The Agreement property holds.

Since no  $p_j \in \mathcal{P}$  invokes  $\operatorname{binPropose}_j(k_j+1, O_x.proposals[k_j+1] \neq \bot$ ) (lines 30 and 32), before it had assured the safe URB delivery of  $O_x.txDes$ 's transmission, we know that eventually, at least one element of r(j) is True. Thus, by the agreement property of binary consensus, every  $p_x$ eventually calculates the same value of  $k_j()$ , such that  $BC[k_j()]$ .result $_x(k_j() + 1) =$  True. This implies the agreement property since result $_x()$  returns  $O_x.proposals[k_j()+1]$  for any non-failing  $p_x \in \mathcal{P}$  (line 25).  $\Box_{Lemma} 6.7$ 

Lemma 6.8 demonstrates the property of integrity.

**Lemma 6.8** Suppose that  $\exists v' \notin \{\bot, \Psi\}$ :  $\exists p_j \in \mathcal{P} : \exists c' \in R : \mathsf{result}_j() = v' \text{ in } c'. \notin c'' \in R : \mathsf{result}_j() = v'' \text{ in } c'', \text{ such that } v' \neq v''.$ 

**Proof of Lemma 6.6** The proof is by contradiction. Suppose that  $c'' \in R$  exists and, without the loss of generality, c' appears before c'' in R. Since R is authentic and  $c' \in R$  exists, then there is a  $p_k \in \mathcal{P} : k \in S(n-1)$ , such that for any  $p_j \in \mathcal{P}$ , there is a system state  $c'_j$  that appears in R not after c' in which for any  $k' \in S(n-1)$  it holds that BC[k'].result<sub>j</sub>(k') = False for the case of k' < k and BC[k].result<sub>j</sub>(k') = True for the case of k' = k. This is due the definition of k() (line 19). Note that in any system state that follows  $c'_j$ , the value of  $k = k_j()$  does not change due to the integrity of binary consensus objects. Therefore, result<sub>j</sub>() must return the value of  $O_j.proposal[k]$  in any system state that follows  $c'_j$ . Since line 36 does not allow any change in the value of  $O_j.proposal[k]$  between c' and c'', it holds that v' = v''. Thus, the proof reached a contradiction and the lemma is true.  $\Box_{Lemma 6.8}$ 

# 7 Application: Self-stabilizing Total-order Message Delivery using Multivalued Consensus

We exemplify the use of Algorithm 2 by implementing the task total order uniform reliable broadcast (TO-URB), which we specified in Section 2.2.1. We describe our implementation before bringing the correctness proof.

#### 7.1 Refinement of the system settings

Our self-stabilizing total order message delivery implementation (Algorithm 3) provides the toBroadcast(m) operation (line 44). It uses a self-stabilizing URB with FIFO-order delivery,

Algorithm 3: Self-stabilizing TO-URB via consensus; code for  $p_i \in \mathcal{P}$ 

**38 notations:**  $x \operatorname{opr}_3 y \equiv (x \operatorname{opr} y) \mod 3 : \operatorname{opr} \in \{-,+\};$ **39 constants and variables:**  $\delta \in \mathbb{Z}^+$  max number of messages after which is delivery is enforced;  $CS[0..2] = [\bot, ..., \bot]$  array of multivalued consensus objects; obsS = 0highest obsolete sequence number 40 macro  $S() = \{CS[k].seq : CS[k] \neq \bot\}_{k \in \{0,...,2\}};$ 41 macro getSeq() do return  $max({obsS} \cup S());$ 42 macro test(s) do return ( $s \in (S() \cup {getSeq() + 1});$ 43 macro  $\Delta()$  do return ((allHaveTerminated()  $\land 0 < \ell) \lor \delta \leq \ell$ ) where  $(x, y, \ell) = (minReady(), maxReady(), \sum_{p_k \in \mathcal{P}} (y[k] - x[k]));$ 44 **operation** toBroadcast(m) **do** fifoURB(toURB(m)); 45 do forever begin if 46  $(\exists k \in \{0, \dots, 2\} : CS[k] \neq \bot \land CS[k].seq \mod 3 \neq k) \lor (\mathsf{S}() \neq \emptyset \land (obsS > \max \mathsf{S}()) \lor$  $\max \mathsf{S}() - \min \mathsf{S}() > 1)$  then  $CS \leftarrow [\bot, \ldots, \bot]$ ;  $sn \leftarrow sn + 1;$  $\mathbf{47}$ repeat  $\mathbf{48}$ for each  $p_i \in \mathcal{P}$  do send SYNC(*sn*) to  $p_i$ ; 49 until SYNCack $(sn, \bullet)$  received from all  $p_j : j \in trusted;$  $\mathbf{50}$ let  $(allReady, maxSeq, allSeq) = (entrywise-min\{x\}_{(\bullet,x)\in X},$  $\mathbf{51}$  $\max\{x\}_{(-,x,\bullet)\in X}, \cup_{(-,x,y,-)\in X}\{x,y\}$  where X is the set of messages received in line 50; let (x, y, z) = (obsS, getSeq(), maxSeq);52if  $\neg (x+1=y=z \lor x=y=z \lor x=y=z-1)$  then 53  $obsS \leftarrow \max\{x, y, z\}$  $\mathbf{54}$ foreach  $k \in \{0, \dots, 2\} \setminus (\{x \mod 3 : x < y\} \cup \{y \mod 3\} \cup \{z_{+3}1 : |allSeq| = 1\})$ 55do  $CS[k] \leftarrow \bot;$ if  $(|allSeq| = 1 \land \Delta())$  then 56  $| CS[maxSeq +_3 1].propose(maxSeq + 1, allReady)$  $\mathbf{57}$ if  $obsS + 1 = getSeq() \land x \neq \bot \land x.result() \neq \bot$  where x = CS[(obsS + 1)] then 58 if x.result()  $\neq \Psi$  then 59 **foreach**  $m \in \mathsf{bulkRead}(x.\mathsf{result}())$  **do** toDeliver(m)60  $obsS \leftarrow obsS + 1$ 61 62 upon SYNC(snJ) arrival from  $p_j$  do send SYNCack(snJ, getSeq(), obsS, maxReady()) to  $p_i$ ;

such as the one by Lundström, Raynal, and Schiller [32], to broadcast the protocol message, toURB(msg). As before, the line numbers of Algorithm 3 continues the ones of Algorithm 2.

The proposed solution assumes that the FIFO-URB module has interface functions that facilitate the aggregation of protocol messages before their delivery. For example, we assume that the interface function allHaveTerminated<sub>i</sub>() returns True whenever there are no active URB transmissions sent by  $p_i \in \mathcal{P}$ . Also, given  $p_i \in \mathcal{P}$ , the functions readyMin<sub>i</sub>() and readyMax<sub>i</sub>() return each a vector,  $r_i[0, ..., n-1]$ , such that for any  $p_j \in \mathcal{P}$ , the entry  $r_i[j]$  holds the lowest, and respectively, highest FIFO-delivery sequence number that is ready to be FIFO-delivered. These FIFO-delivery sequence numbers are the unique indices that the senders attach to the URB messages. Also, the function  $\mathsf{bulkRead}_i(r_{\max})$  returns immediately after system state c a determinately ordered sequence,  $sqnc_i$ , that includes all the messages between  $r_{\min}$  and  $r_{\max}$ , such that  $r_{\min} = \mathsf{readyMin}_i()$  in c, as well as  $r_{\max}$ , is a vector that is entry-wise greater or equal to  $r_{\min}$  and entry-wise smaller or equal  $r = \mathsf{readyMax}_i()$  in c.

Algorithm 3 assumes access to a self-stabilizing perfect failure detector, such as the one by Beauquier and Kekkonen-Moneta [5]. The local set,  $trusted_i$ , includes the indices of the nodes that  $p_i$ 's failure detector trusts. We follow Assumption 7.1 for the sake of a simple presentation.

Assumption 7.1 Any sent message arrives or is lost within  $\mathcal{O}(1)$  asynchronous cycles. Any URB message arrives within  $\mathcal{O}(1)$  asynchronous cycles [32]. Each active multivalued consensus object decides within  $\mathcal{O}(1)$  asynchronous cycles [33].

# 7.2 Algorithm description

The algorithm idea uses the fact that  $sqnc_i$  is deterministically ordered. Namely, if all nodes  $p_j \in \mathcal{P}$  share the same sequence  $r_1, r_2, \ldots$  when calling  $\mathsf{bulkRead}_j(r_x) : x \in \mathbb{Z}^+$ , the studied task is reduced to invoking the event of  $\mathsf{toDeliver}(m)$  for every  $m \in \mathsf{bulkRead}(r_x)$ . To that end, Algorithm 3 queries all nodes about the messages that are ready to be delivered (lines 47 to 51), validates the consistency of the control variables (line 46 and lines 54 to 55), agree on the current value of  $r_x$  (lines 56 to 57), and delivers the ready messages (lines 58 to 61). We discuss in detail each part after describing the local constant, variables, and macros; sn the query number.

#### 7.2.1 Constant, variables, and macros

The constant M defines the number of multivalued consensus objects that Algorithm 3 needs to use. The proof shows that, at any time during a legal execution, Algorithm 3 uses at most two active objects at a time and one more object that is always non-active. The array CS[0..2] holds all the multivalued consensus objects that Algorithm 3 uses. Algorithm 3 uses CS[] cyclically.

Algorithm 3 aims at aggregating URB messages and delivering them only when all transmission activities have terminated. To that end, it uses the allHaveTerminated() function (Section 7.1). Since the number of such transmissions is unbounded, there is a need to stop aggregating after some predefined number of transmissions that we call  $\delta$ . The variable *obsS* points to the highest obsolete sequence number; the one that was already delivered locally. The variable *sn* stores the number of the next query. As mentioned in Section 2.2, whenever Algorithm 3 runs out of query numbers, a global restart mechanism is invoked, such as the one in [24, Section 5]. Thus, it is possible to have bounded query numbers.

The macro S() returns the set of sequence numbers used by the active multivalued consensus objects. The macro getSeq() returns the locally maximum sequence number. The macro test(s)returns True whenever the sequence number s is used by an active or is greater by one than getSeq(). The macro  $\Delta()$  facilitates the decision about whether to invoke a new consensus. It returns True if there are non-delivered messages but no on-going transmissions. It also returns True when the number of ready to be delivered messages exceeds  $\delta$  (regardless of the presence of active URB transmissions).

#### 7.2.2 Querying (lines 47 to 51)

Algorithm 3 uses a simple synchronization query mechanism. Each query instance is associated with a unique sequence number that is stored in the variable sn and incremented in line 47. Line 49 broadcasts the synchronization query repeatedly until a reply is received from every trusted node. The query response (line 62) includes the correspondent's (local) maximum sequence number stored by any multivalued consensus object (that the macro getSeq() retrieves), the maximum obsolete sequence number (that its respective multivalued consensus object is no longer needed), and the latest value returned from readyMax<sub>i</sub>(). Using these responses, line 50 aggregates the query results and store them in minReady, maxSeq, and allSeq. The vector minReady includes the entry-wise minimum (per sender) FIFO-URB messages that are ready to be delivered at all nodes. Also, maxSeq is the maximum known sequence number. And, the set allSeq includes all the collected maximum sequence numbers and obsolete sequence numbers.

#### 7.2.3 Consistency assertion and stale information removal (line 46 and lines 54 to 55)

Line 46 makes sure that CS[k].seq, when taken its reminder from the division by M, equals to k. It also tests that the local obsolete sequence number, obsS, is not greater than the largest sequence number. Besides, the gap between the maximum and the minimum sequence number cannot be greater than one. Line 54 verifies that obsS, getSeq(), and maxSeq follow a consistent pattern. Line 55 removes stale information by deactivating any obsolete multivalued consensus object.

#### 7.2.4 Repeated agreement (lines 56 to 57)

The if-statement condition in line 56 tests whether all trusted nodes in the system share the same sequence number. This happens when all trusted nodes  $p_j \in \mathcal{P}$  have  $obsS_j = \mathsf{getSeq}_j()$ . Line 56 also checks whether  $\Delta()$  indicates that it is the time to deliver a batch of messages. If this is the case, then line 57 proposes to agree on the value of *allReady*.

#### 7.2.5 Message delivery (lines 58 to 61)

The delivery of the next message batch becomes possible the multivalued consensus object has terminated (line 58). Before the actual delivery (line 60), there is a need to check that no error was reported (line 60) due to conditions that appear at line 23 of Algorithm 2. In any case, obsS is incremented (line 61) so that even if an error occurred, the object is ready for recycling.

# 7.3 Correctness of Algorithm 3

Theorem 7.2 uses Definition 7.1.

**Definition 7.1 (Consistent states and legal executions)** Let c be a system state and  $p_i \in \mathcal{P}$  be any processor in the system. Suppose that in c, it holds that (i)  $\forall k \in \{0, \ldots, 2\} : CS_i[k] = \perp \lor CS_i[k]$ .seq mod 3 = k and either  $S = \emptyset \land obsS_i \in \mathbb{Z}^+$  or  $S \neq \emptyset \land (obsS_i \leq \max S \land \max S - \min S \leq 1)$ , where  $S = \{CS_i[k].seq : CS_i[k] \neq \bot\}_{k \in \{0,\ldots,2\}}$  and getSeq<sub>i</sub>() returns seq = max( $\{obsS_i\} \cup S$ ). Moreover, (ii.a)  $sn_i$ 's value is greater equal to any snJ field in the message SYNC(snJ) in a communication channel from  $p_i$  as well as SYNCack(snJ,  $\bullet$ ) message in a communication channel to  $p_i$ . And (ii.b)  $obsS_i \leq getSeq_i$ ()  $\leq obsS_i + 1$ . In this case, we say that c is consistent concerning Algorithm 3.

Suppose that R is an execution of Algorithm 3, such that every  $c \in R$  is consistent. In addition, (iii.a) suppose that if toBroadcast() is not invoked during R nor do any FIFO-broadcast becomes available for delivery, then pred holds throughout R, where  $pred \equiv \exists z \in \mathbb{Z}^+ : \forall k \in$  $Correct : getSeq_k() = z \land maxSeq_k = z \land obsS_k = z \land allSeq_k = \{z\}$ . Furthermore, (iii.b) suppose that if toBroadcast() is invoked during R infinitely often, then pred holds infinitely often. In this case, we say that R is legal.

**Theorem 7.2** Within  $\mathcal{O}(1)$  asynchronous cycles, Algorithm 3's execution is legal.

**Proof of Theorem 7.2** Due to line 46, Definition 7.1's Invariant (i) holds after  $p_i$  first complete iteration of Algorithm 3's do-forever loop (lines 45 to 61). Lemma 7.3 shows Invariant (ii.a). Line 54 implies Invariant (ii.b). Lemma 7.5 shows invariant (iii).

Lemma 7.3 Invariant (ii.a) holds.

**Proof of Lemma 7.2** Only line 47 modifies  $sn_i$ 's value, *i.e.*, by increasing the value of  $sn_i$ . The rest of the proof is implied by Assumption 7.1, which says that all messages that appear in the communication channels in R's starting system state are either delivered or lost within  $\mathcal{O}(1)$  asynchronous cycles.  $\Box_{Lemma 7.3}$ 

We observe from the code of Algorithm 2 that once invariants (i) and (ii) hold, they are not violated. Thus, the rest of the proof assumes that invariants (i) and (ii) hold in every system state of R. Lemma 7.4 is needed for the proof of Lemma 7.5.

**Lemma 7.4** Every complete iteration of the do-forever loop (lines 45 to 61) allows the collection of  $M_{sn_i} = \{(sn_i, s_k, o_k, r_k)\}_{k \in trusted_i}$ , such that  $s_k = seq_k, o_k = obsS_k$ , and  $r_k = readyMax_k()$ in the system state  $c_k$ , where  $c_i^{line \ \ell} \in \mathbb{R} : \ell \in \{47, 50\}$  is the system state when  $p_i$  executed line  $\ell$  with  $sn_i$  and  $c_k$  appears between  $c_i^{line \ 47}$  and  $c_i^{line \ 50}$  when  $p_k$  executed line 62 on the arrival of SYNC( $snJ = sn_i$ ). Moreover, minReady<sub>i</sub> (line 50) is entry-wise smaller equal to every readyMax<sub>k</sub>() in  $c_i^{line \ 50}$ , maxSeq<sub>i</sub> is greater equal than every  $seq_k$  in  $c_i^{line \ 47}$ , and allSeq<sub>i</sub> includes the union  $\cup_{k \in trusted_i} aS_k$ , where  $aS_k = \{obsS_k, seq_k\}$  in  $c_k$ .

**Proof of Lemma 7.2** Since invariants (ii.a) holds, the increment of  $sn_i$  (line 47) creates a sequence number that is (associated with  $p_i$  and) greater than all other associated sequence numbers in the system. With this unique sequence number, the repeat-until loop (lines 49 to 50) gets a fresh collection of  $M_{sn_i} = \{(sn_i, s_k, o_k, r_k)\}_{k \in trusted_i}$ . Note that this loop cannot block due to the end-condition (line 50), which considers only the trusted nodes in the system. The rest of the proof is implied directly by lines 47, 50, and 62.

**Lemma 7.5** Within  $\mathcal{O}(1)$  asynchronous cycles,  $R = R' \circ R''$  reaches a suffix, R'', in which invariants (iii.a) and (iii.b) hold.

**Proof of Lemma 7.2 Argument (1)** Invariant (iii.a) holds. By the assumption that no FIFO-broadcast becomes ready during R, it holds that the if-statement condition in line 56 does not hold during R. By Assumption 7.1, all active multivalued consensus objects have terminated with  $\mathcal{O}(1)$  asynchronous cycles. Therefore, within  $\mathcal{O}(1)$  asynchronous cycles, the if-statement condition in line 60 cannot hold. Due to Lemma 7.4,  $maxSeq_i$  is greater equal to  $getSeq_k(): k \in Correct_i$ . Due to the if-statement line 53 and line 55, within  $\mathcal{O}(1)$  asynchronous cycles, line 55 deactivates any multivalued consensus object,  $O_{i:p_i \in \mathcal{P}, x \in \{0, ..., 2\}} = CS_i[x]$  for which  $O_{i,x}.seq < maxSeq_i - 1$ . By using Assumption 7.1 again, any re-activated multivalued

**Algorithm 4:** Self-stabilizing emulation of a replicated state-machine; code for  $p_i \in \mathcal{P}$ 

Same code as in lines 38 to 44. 52 do forever begin Same code as in lines 46 to 55. if  $(|allSeg| = 1 \land \Delta())$  then 62  $CS[maxSeq +_3 1].propose(maxSeq + 1, (state = getState(), msg = maxReady()))$ 63 if  $obsS + 1 = getSeq() \land x \neq \bot \land x.result() \neq \bot$  where x = CS[(obsS + 1)] then 64 if x.result()  $\neq \Psi$  then 65 setState(x.result().state); foreach  $m \in \mathsf{bulkRead}(x.\mathsf{result}().msg)$  do 66 toDeliver(m)Same code as in lines 61 to 62.

consensus object has to terminate with  $\mathcal{O}(1)$  asynchronous cycles. Thus, the above implies that the state of all multivalued consensus objects, active or not, do not change and that  $maxSeq_i = getSeq_k() : k \in Correct_i.$ 

We show that  $obsS_i = getSeq_i()$  holds within  $\mathcal{O}(1)$  asynchronous cycles. By Invariant (ii.b), we know that either  $obsS_i + 1 = getSeq_i()$  or  $obsS_i = getSeq_i()$ . Suppose that  $obsS_i + 1 = getSeq_i()$  holds. Due to the definition of getSeq() as well as lines 46 and 55,  $x_i \neq \bot$ , where  $x_i = CS_i[(obsS_i + 3 1)]$  (line 58). By Assumption 7.1, within  $\mathcal{O}(1)$  asynchronous cycles, the multivalued consensus object  $x_i$  terminates. Thus, the if-statement condition in line 58 holds and line 61 increments  $obsS_i$  once. Therefore,  $obsS_i = getSeq_i()$  within  $\mathcal{O}(1)$  asynchronous cycles.

Since  $maxSeq_i = getSeq_k() = obsS_k : k \in Correct_i$ , then  $allSeq_k = \{z\}$ , where  $z = maxSeq_i = getSeq_k() = obsS_k : k \in Correct_i$ . Thus, pred holds.

Argument (2) Invariant (iii.b) holds. We note that  $\Delta_i()$  holds infinitely often by the assumption that toBroadcast() is invoked during R infinitely often and the URB-termination property. We show that the if-statement condition in line 56 holds within  $\mathcal{O}(1)$  asynchronous cycles once  $\Delta_i()$  holds. Suppose, towards a contradiction, that |allSeq| = 1 does not hold for a period longer than  $\mathcal{O}(1)$  asynchronous cycles. Then, the then-statement in line 57 is not executed for a period longer than  $\mathcal{O}(1)$  asynchronous cycles. By the proof of Argument (1), pred holds within  $\mathcal{O}(1)$  asynchronous cycles. Thus, the if-statement condition in line 56 holds within  $\mathcal{O}(1)$  asynchronous cycles. Thus, the if-statement condition in line 56 holds within  $\mathcal{O}(1)$  asynchronous cycles. In other words, Invariant (iii.b) holds.  $\Box_{Lemma 7.5}$ 

# 8 Discussion

We showed how a non-self-stabilizing algorithm for multivalued consensus by Mostéfaoui, Raynal, and Tronel [36] can become one that recovers from transient-faults. Interestingly, our solution is both wait-free and incurs a bounded number of binary consensus invocations whereas earlier work either uses an unbounded number of binary consensus objects [36] or is blocking [46]. Therefore, we present a more attractive transformation technique than the studied algorithm (regardless of the presence or absence of transient-faults).

As an application, we showed a self-stabilizing total-order message delivery. As an enhancement to this application, Algorithm 4 explains how to construct a self-stabilizing emulator for state-machine replication. Line 63 of Algorithm 4 proposes to agree on both on the automaton state, which is retrieved by getState(), and the bulk of FIFO-URB messages, as in line 57 of Algorithm 3. Line 66 of Algorithm 4 uses setState() for updating the automaton state using the agreed state.

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