Forward Invariant Cuts to Simplify Proofs of Safety

Nikos Aréchiga Carnegie Mellon University Toyota InfoTechnology Center Mountain View, CA

narechig@us.toyota-itc.com

James Kapinski **Toyota Technical Center** 1630 W. 186th Gardena, CA 90248

jim.kapinski@tema.toyota.com

Jyotirmoy V. Deshmukh Toyota Technical Center 1630 W. 186th Gardena, CA 90248

jyotirmoy.deshmukh@tema.toyota.com

André Platzer Carnegie Mellon University 5000 Forbes Ave Pittsburgh, PA

aplatzer@cs.cmu.edu

Bruce Krogh Carnegie Mellon University 5000 Forbes Ave Pittsburgh, PA

krogh@rwanda.cmu.edu

ABSTRACT

The use of deductive techniques, such as theorem provers, has several advantages in safety verification of hybrid systems; however, state-of-the-art theorem provers require extensive manual intervention. Furthermore, there is often a gap between the type of assistance that a theorem prover requires to make progress on a proof task and the assistance that a system designer is able to provide. This paper presents an extension to KeYmaera, a deductive verification tool for differential dynamic logic; the new technique allows local reasoning using system designer intuition about performance within particular modes as part of a proof task. Our approach allows the theorem prover to leverage forward invariants, discovered using numerical techniques, as part of a proof of safety. We introduce a new inference rule into the proof calculus of KeYmaera, the forward invariant cut rule, and we present a methodology to discover useful forward invariants, which are then used with the new cut rule to complete verification tasks. We demonstrate how our new approach can be used to complete verification tasks that lie out of the reach of existing deductive approaches using several examples, including one involving an automotive powertrain control system.

Keywords

hybrid systems, formal verification, theorem provers

INTRODUCTION

Modern physical systems such as automobile engines, avionics, and medical devices are controlled by software running on embedded computing platforms. In the software domain, techniques such as model checking, theorem proving, and abstract interpretation have had success verifying purely software systems. For physical systems, techniques from dynamical systems theory and control theory such as Lyapunov analysis have long been used to help characterize system performance. Most cyberphysical systems, however, are hybrid, i.e., have both continuous state evolution governed by differential equations and discrete mode transitions. Most interesting analyses for such systems (e.g., reachable set estimation) are undecidable [14], and most software verification techniques are not directly applicable.

Many extant approaches to hybrid system verification focus on creating an overapproximation of the set of system states reachable over a fixed time horizon [18, 9, 10, 6]. While these approaches enjoy a high degree of automation, they are restricted in scope and scalability. Tools such as SpaceEx [10] and Flow* [6] are susceptible to approximation error that worsens when the reachable set estimation over continuous state-space interacts with discrete switching, leading to false positives. The theorem prover PVS has been used to reason about hybrid systems as composable hybrid automata in [2] [1]. However, the continuous components are modeled by the explicit solutions of the differential equations. Explicit solutions can only be obtained for restricted classes of differential equations, e.g. linear. On the other hand, $d\mathcal{L}$ allows reasoning about continuous dynamics by using only the differential equations.

An alternative approach is to employ deductive techniques that attempt to construct a symbolic proof of safety using a semi-interactive theorem prover [23]. This approach has several advantages in safety verification of hybrid systems. Unlike explicit reach-set computation techniques, theorem provers can handle nonlinear dynamics directly, without introducing approximation artifacts. Further, theorem provers can handle proof tasks that involve symbolic parameters, with only the minimal constraints required to guarantee safety. This makes the verification result reusable across systems with parameter variations. In the context of dynamical or hybrid systems verification, a human may provide insight to the theorem proving tool in the form of a safety certificate, i.e., a symbolic expression representing a set containing all reachable states from a given initial set, while excluding unsafe states [3, 23]. The tool can then use this certificate to automatically prove system safety.

In [25], the authors propose an approach that begins with a global candidate certificate (in the form of a differential *invariant*) that overapproximates the reachable set of states. Constraints are iteratively added until the overapproximation is small enough to exclude the unsafe set, at which point the invariant becomes a safety certificate. This approach has had success in verifying aircraft roundabout maneuvers using the KeYmaera theorem prover. The notable aspects of this approach are: the initial input is in the form of a global certificate of system safety, which is eagerly constructed and then (globally) refined.

Cyberphysical system designs have distinct modes of operation, with each mode corresponding to an (often) independently designed controller operating regime. Consequently,

a designer has much more nuanced information about modespecific behaviors rather than overarching knowledge about the entire system. The central thesis of this paper is that when available, such additional information can be useful for a theorem prover compared to a technique relying on construction of a global safety certificate. Our approach encourages local reasoning and lazy construction of certificates.

As an example of augmented local information, consider the scenario where a designer knows that from a given set of modes, there are no discrete transitions to unsafe system modes. This is a form of local certificate; in this case derived purely by reasoning over the finite transition structure of the discrete modes. Also consider the designer insight that a system is expected to be stable in a certain mode. This is another form of local information that makes it possible to employ Lyapunov analysis-based techniques to obtain a forward invariant set or barrier certificate that provides a local certificate for that mode.

To support local reasoning, we introduce a new proof rule that we call the forward invariant cut rule in the calculus of KeYmaera. Given a region of operation and a safe forward invariant for the behaviors of that mode, the forward invariant cut rule allows us to decompose the overall global safety proof into three proof obligations: (1) a proof of invariance of the proposed certificate, (2) a proof that the certificate guarantees safety, and (3) a proof of safety of everything but the behaviors associated with the region covered by the certificate. This makes it possible to carve out safe behavior and focus analysis only on the remaining part of the system. An advantage of the decompositional approach is that it allows us to defer the process of producing a local certificate until we reach the relevant sub-goal in the safety proof. In other words, it allows lazy construction of safe forward invariants, which is convenient as certificates for system components are often easier to obtain than certificates for the aggregate system.

We demonstrate how our methodology can be used to complete verification tasks that lie out of the reach of existing deductive techniques. The systems we consider are hybrid and contain examples with continuous dynamical behaviors that are described by nonlinear ordinary differential equations (ODEs). Deductive approaches exist for addressing this class of systems, but the existing frameworks alone are insufficient to complete the proof tasks for the examples herein. For example, the framework in [25] provides a means to address the examples we present using differential invariants, but the authors provide no general method of computing the required differential invariant candidates. Further, their technique requires reasoning about the global behavior of the system, as opposed to the local invariant property that we require (a much weaker requirement). The deductive proof system presented in [8] uses local safety certificates to reason about behaviors but applies to continuous (as opposed to hybrid) systems. Also, [8] provides no constructive means of generating the necessary local safety certificates. This is in contrast to our approach, which provides methodologies for generating the local safety certificates and including them in the proof task.

We present three examples that demonstrate the practical application of the forward invariant cut rule. The first hybrid system is a hybrid system with three stable modes and one fail mode. The second system is a non-autonomous switched system, in which a user has the freedom to switch modes at arbitrary instants. The third system is a simplified model of an automotive subsystem that is responsible for maintaining the air-to-fuel (A/F) ratio in an engine near an optimal setpoint. In the automotive context, this is one of the most important control problems with significant implications on fuel efficiency and exhaust gas emissions. We are able to prove that the A/F ratio remains within 10% of the optimal setpoint value using KeYmaera.

The paper is organized as follows. In Sec. 2, we introduce the terminology and review material on hybrid programs (the syntactic form used by KeYmaera to express hybrid systems). We introduce the forward invariant cut rule in Sec. 3, and in Sec. 4 we describe techniques for obtaining local certificates. We show how the forward invariant cut rule can be applied to specific case studies in Sec. 5. Finally, we conclude and discuss related and future work in Sec. 6.

2. HYBRID SYSTEMS AND HYBRID PRO-GRAMS

A hybrid system is a dynamical system with continuous-valued state variables \mathbf{x} that take values from a domain $X\subseteq\mathbb{R}^n$ and a discrete-valued state variable q taken from a finite set Q. The system evolves in continuous or discrete time, and the configuration of a hybrid system at time t can be described by the values of its continuous and discrete state variables. The discrete-valued states are called modes of operation. The hybrid state is given by the ordered pair $(\mathbf{x},q)\in X\times Q$. In a discrete mode q, the evolution of the continuous-valued state variables is described by ordinary differential equations (ODEs)

$$\dot{\mathbf{x}}(t) = f_q(\mathbf{x}(t)),\tag{1}$$

where f_q is a function from X to X, often called the vector field. Though hybrid systems are often described with external inputs, in this paper we consider only *autonomous systems*, i.e., systems in which all transitions depend only on the system states. The state-dependent conditions that allow the system to transition from one discrete state to another (possibly same) discrete state are called *guards*.

Hybrid systems are often modeled using hybrid automata. We use Fig. 1 as a running example. This example has four modes and two continuous-valued state variables, with associated ODEs. Modes are represented by nodes in the graph; each mode q has associated a unique set of ODEs (f_q) . There is a guard on the outgoing transition from q_0 to q_1 , and the transition from q_0 to q_2 is unguarded, so it can always be taken. The transition from q_0 to q_2 has a nondeterministic reset allowing a jump from current state values x_1 and x_2 to any pair of values within the circle of radius two. The set of feasible initial conditions is indicated on the default transition. Mode q_0 and q_2 have stable linear dynamics, and q_1 has stable nonlinear dynamics, as in Example 4.10 of [19].

While hybrid automata are a convenient formalism, in this paper we use the formalism of hybrid programs in order to facilitate the use of the KeYmaera theorem prover, which is the workhorse for our deductive approach. Note that any hybrid automaton can be transformed into a hybrid program [23], therefore there is no loss of generality in considering hybrid programs. KeYmaera uses the formalism of differential dynamic logic, denoted by d \mathcal{L} .

The syntax and semantics of $d\mathcal{L}$ are described in detail in [23]; we provide only a minimal overview here.

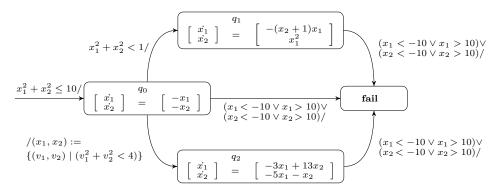


Figure 1: A running example: All the modes have stable continuous dynamics, and there is a special "fail" mode.

2.1 The logic $d\mathcal{L}$

A hybrid program is specified by the grammar

$$\alpha, \beta ::= x := \theta \mid x := * \mid \{x'_1 = \theta_1, \dots, x'_n = \theta_n \& H\}$$
 (2)
$$|?H \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$
 (3)

where α, β are hybrid programs, $\theta, \theta_1, \ldots, \theta_n$ are terms, and H is a logical formula. Intuitively, the program $x \coloneqq \theta$ means that x is assigned the value of the term θ . The program $x \coloneqq *$ means that x is nondeterministically assigned an arbitrary real value. The program $\{x_1' = \theta_1, \ldots, x_n' = \theta_n \& H\}$ means that the variables x_1, \ldots, x_n evolve continuously for some duration, with derivatives $\theta_1, \ldots, \theta_n$, subject to the constraint that x_1, \ldots, x_n satisfy H during the entire flow. The hybrid program H behaves as a skip if the logical formula H is true, and as an abort otherwise.

The nondeterministic choice $\alpha \cup \beta$ means that either α or β may be executed. The sequential composition $\alpha; \beta$ means that α is executed, then β . The nondeterministic repetition α^* means that α is executed an arbitrary (possibly zero) number of times. The logic $\mathrm{d}\mathcal{L}$ itself is a multimodal logic, in which the modalities are annotated with hybrid programs. The formulas of $\mathrm{d}\mathcal{L}$ are described by the grammar:

$$\phi, \psi ::= \theta_1 = \theta_2 \mid \theta_1 \ge \theta_2 \mid \neg \phi \mid \phi \land \psi \tag{4}$$

$$|\phi \lor \psi | \phi \to \psi | [\alpha]\phi | \langle \alpha \rangle \phi \tag{5}$$

where ϕ , ψ are formulas of $d\mathcal{L}$, θ_1 , θ_2 are terms, and α is a hybrid program. The box modality $[\alpha]\phi$ means that ϕ holds after all traces of the hybrid program α , and $\langle \alpha \rangle \phi$ means that ϕ holds after some execution of hybrid program α .

In the sequel, we will abuse notation and use a formula interchangeably with the set that it represents.

2.2 Example

Model 1 shows a hybrid program representation of the running example. Line 2 shows how the subprograms are assembled into the overall program. The system starts at a set I (Line 3), and at each iteration of the loop, one of the subprograms is nondeterministically chosen for attempted execution. If the guard of the subprogram succeeds, execution proceeds. The verification task is to show that when this loop is executed any (finite) number of times, the state remains in the set S (Line 16). Line 4 is the guard and differential equations of q_0 . Line 5 is the transition from q_0 to q_1 and the required guard. Line 7 proceeds to specify the continuous evolution of q_1 . Line 9 applies the reset of

the transition into q_2 , which indicates that the state resets anywhere in the circle of radius two. Line 10 checks the incoming guard to q_2 and Line 11 specifies the associated differential equations. Line 13 specifies the guard that allows transitions into the failure mode. Note that the guard does not check the current mode, since all of the modes may transition into the failure mode if the continuous states leave their prescribed bounds. Line 15 specifies that once the failure mode is entered, it is not possible to leave it, and states x_1, x_2 maintain their previous values and do not evolve.

Model 1: Hybrid program for the running example

```
 Ex \equiv I \to [(\mathbf{m}_0 \cup \mathbf{s}_{0 \mapsto 1} \cup \mathbf{m}_1 \cup \mathbf{s}_{0 \mapsto 2} \cup \mathbf{m}_2
 1
                                                    \cup s_{\{0,1,2\} \mapsto fail} \cup m_{fail})^*]S
 2
                            \begin{array}{c} I \equiv x_1^2 + x_2^2 \leq 10 \, \wedge \, M = q_0 \\ \mathbf{m}_0 \equiv ?(M = q_0); \; \{x_1' = -x_1, x_2' = -x_2\} \end{array}
 4
                       \mathbf{s}_{0\mapsto 1} \equiv ?(M=q_0); ?(x_1^2+x_2^2<1); (M:=q_1);
 5
                            \mathtt{m}_1 \equiv ?(M=q_1);
 6
                       \begin{cases} x_1' = -(x_2 + 1) * x_1, \ x_2' = x_1^2 \} \\ \mathbf{s}_{0 \mapsto 2} \equiv ?(M = q_0); 
 7
                                        x_1 := *; \ x_2 := *; ?(x_1^2 + x_2^2 < 4); \ (M := q_2)
  9
                            \mathbf{m}_2 \equiv ?(M = q_2);
10
                                       \{x_1' = -3x_1 + 13x_2, \ x_2' = -5x_1 - x_2\}
11
       \mathbf{s}_{\{0,1,2\}\mapsto fail} \equiv ?(-10 > x_1 \lor x_1 > 10)
12
                                        \vee -10 > x_2 \vee x_2 > 10);
13
                                       M\coloneqq fail
14
                       \mathbf{m}_{fail} \equiv ?(M = fail);
15
                              S \equiv M \neq fail
```

3. SAFETY VERIFICATION WITH THE FOR-WARD INVARIANT CUT RULE

3.1 The safety verification problem

The safety verification problem is to decide whether the state of a system is always contained within a given safe set when starting from a designated initial set, or equivalently, whether none of the behaviors enter an unsafe set.

To formalize this problem in $\mathrm{d}\mathcal{L}$, suppose α is a hybrid program representation of the system of interest. Suppose S is the safe set and I is the set of initial states. Then the behaviors of α are contained in S if the following formula is a theorem of $\mathrm{d}\mathcal{L}$.

$$I \to [\alpha^*]S$$
.

The theorem prover KeYmaera can be used to attempt to prove this.

To solve this problem, one might construct a set that contains all of the system behaviors from the initial set and is contained in the safe set. We call such a set a safety certificate. A safety certificate must contain the initial state set, exclude the unsafe set, and be invariant for system behaviors. We say that a set is initialized if it includes the initial set, safe if it excludes the unsafe set, and invariant if whenever a system behavior enters it, the behavior remains in the set for all future time. Arguments with safety certificates are captured in $\mathrm{d}\mathcal{L}$ using the invariant proof rule, where C is a safety certificate:

$$\frac{I \to C \quad C \to [\alpha]C \quad C \to S}{I \to [\alpha^*]S}$$

The general task of finding a safety certificate is difficult. In this work, we propose instead a procedure that incrementally works towards a proof. Instead of a safety certificate, we use knowledge of system structure to propose sets that are invariant and safe, but not necessarily initialized, and leverage them in the proof procedure.

In our running example, modes q_1 and q_2 have stable dynamics. If a Lyapunov function can be computed for either of these modes, its sublevel sets (i.e., sets of the form $\{x \mid V(x) \leq \ell\}$, for some $\ell \geq 0$) will be invariant. The sublevel sets will be safe if they exclude the transition to the fail mode, but they will not be initialized, since they do not contain mode zero.

3.2 The forward invariant cut rule

A cut in a logical proof allows introducing a lemma. The main contribution of this paper is a type of cut that simplifies the proof procedure by leveraging knowledge of local invariance properties.

The following theorem establishes that if it can be shown that a predicate (C) is locally invariant $(C \to [\alpha]C)$ and safe $(C \to S)$, then the remaining conditions $(\neg C)$ can be separately addressed to prove safety.

Theorem 1 (Forward Invariant Cut Rule). The following is a sound inference rule for the logic $d\mathcal{L}$.

$$\frac{I \wedge \neg C \to [(\alpha; ? \neg C)^*] S \quad C \to [\alpha] C \quad C \to S}{I \to [\alpha^*] S} \tag{6}$$

Proof. We first provide a sketch in natural language. Let $\nu_0, \nu_1, \ldots, \nu_n$ be any sequence of states of any length that are connected by runs of the hybrid program α .

Case a: Suppose that none of the states in this sequence satisfy C. Then this sequence is a run of the hybrid program $(\alpha; ?\neg C)^*$, and is safe by the first premise.

Case b: On the other hand, suppose $\nu_i \in C$ for some $0 \le i \le n$. Then the subsequence ν_i, \ldots, ν_n is a run of the program α starting from C. Then from the second and third premises of the rule, $\nu_j \in C \subseteq S$ for all $j \ge i$. Note that the subsequence ν_1, \ldots, ν_{i-1} is a run of program α^* such that no state satisfies C, and is therefore safe by the previous case.

The formal proof follows. Fix an interpretation \mathcal{I} and an assignment η . From semantics of the second premise, if $\nu \in C$ and $(\nu, \omega) \in \rho_{\mathcal{I},\eta}(\alpha)$, then $\omega \in C$. From the semantics of the third premise, if $\omega \in C$, then $\omega \in S$. From the semantics of the first premise, if $\nu \in I$ and $\nu \notin C$, and $\nu \in C$ are that $(\nu, \omega) \in \rho_{\mathcal{I},\eta}((\alpha; ?\neg C)^*)$, then $\nu \in C$ are the sequence of states ν_0, \dots, ν_n , with $\nu_0 = \nu \in I \land \neg C$

and $\nu_n = \omega$, $n \in \mathbb{N}$, and $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha; ?\neg C)$ for each 0 < i < n-1, it is the case that $\omega \in S$.

The proof is to show by induction that any state reachable by α^* from I in $n \geq 0$ executions of α must be contained in S. For the base case, let n = 0. Then given $\nu \in I$, the only reachable state by a sequence of length zero is ν itself. If $\nu \in C$, then $\nu \in S$ by semantics of the third premise. If $\nu \notin C$, we have that $(\nu, \nu) \in \rho_{\mathcal{I}, \eta}((\alpha; ?\neg C)^*)$ by a chain of length zero, so that by semantics of the first premise, $\nu \in S$.

As an inductive hypothesis, suppose that for every ω reachable by a chain of length $n, \omega \in S$ (i.e., there exists ν_0, \ldots, ν_n with $\nu_0 = \nu$ and $\omega = \nu_n$ such that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha)$, for $0 \le i \le n-1$. Now choose any state ξ such that there is a chain of length $n+1, \nu_0, \ldots, \nu_{n+1}$ with $\nu_0 = \nu$ and $\nu_{n+1} = \xi$, such that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha)$, for $0 \le i \le n$).

First suppose that $\nu_n \in C$. Then by semantics of the second premise, $\nu_{n+1} \in C$, and then $\nu_{n+1} \in S$ by semantics of the third premise. On the other hand, suppose $\nu_n \notin C$. We claim that for all $j \leq n$, $\nu_j \notin C$. To see this, note that if $\nu_j \in C$ for some $j \leq n$, then $\nu_n \in C$ by semantics of the second premise, which would contradict our assumption on ν_n . Then we have that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha; ?\neg C)$ for all $0 \leq i \leq n$. By semantics of the first premise, it follows that $\xi \in S$. This establishes the theorem.

3.3 Example

For the running example, mode q_1 has a Lyapunov function of the form $V_1(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - 2)^2$ as described in Example 4.10 of [19] (we discuss Lyapunov functions as sources of invariants in Section 4). The sublevel set $V_1(x_1, x_2) \leq$ 5 contains the reset into mode q_1 . We apply the forward invariant cut rule with $C_1 = V_1(x_1, x_2) \leq 5 \wedge M = M_1$, a set that is invariant and safe, but not initialized since it does not contain the initial mode q_0 of the hybrid system. The rule application causes the proof tree to split into three branches. The first branch requires showing that whenever the system begins in C_1 , it remains in C_1 . The only portions of the model that may run in this case correspond to q_1 and the transition into the fail mode (programs \mathbf{m}_1 and $\mathbf{s}_{0,1,2\mapsto fail}$. KeYmaera can readily check that since the proposed sublevel set excludes the guard into fail, C_1 will in fact be satisfied by the end of each system trace. The second branch of the proof tree requires showing $C \to S$, which is trivial, since S is simply $M \neq M_{fail}$ and C stipulates $M = M_1$. We now turn our attention to the third branch.

Mode q_2 has a Lyapunov function $V(x_1,x_2)=2x_1^2+4x_2^2$, computed using standard Lyapunov techniques for linear systems. The sublevel set $V_2(x_1,x_2) \leq 16$ contains the circle of radius 2; all incoming transitions to mode q_2 make the system state to be reset to somewhere within this circle. By applying a forward invariant cut with $C_2 = V_2(x_1,x_2) \leq 16 \land M = M_2$, we again get three branches. As before, C_2 is invariant because the only portions of the model that may run from C_2 are the programs \mathbf{m}_2 and $\mathbf{s}_{0,1,2\mapsto fail}$. Since $V_2(x_1,x_2) \leq 16$ excludes the guard to \mathbf{fail} , KeYmaera can show that C_2 represents a safe set. The next branch is to prove that C_2 implies safety, which is easy because C_2 requires $M=M_2$, which implies $M\neq M_{fail}$.

The third branch can now be easily proved with the standard tools of KeYmaera, using the loop invariant $M = M_0 \wedge x_1^2 + x_2^2 \leq 10$.

4. OBTAINING SAFE FORWARD INVARIANTS

This section describes various techniques to generate safe forward invariants, which are invariant sets that are safe but not necessarily initialized. Let $\mathbf{x}(t)$ denote any solution trajectory for a given (hybrid) dynamical system. A set S is forward invariant if for all $\mathbf{x}(0) \in S$, for all $t, \mathbf{x}(t) \in S$. The general problem of identifying safe forward invariant sets that are useful is hard, but the techniques that we present can, in some cases, automatically identify safe forward invariant sets that can be used to complete safety proofs.

4.1 Safe forward invariants based on Lyapunov analysis

Lyapunov analysis provides one way to construct forward invariant sets for hybrid systems. We briefly review the basics of Lyapunov analysis to aid our presentation. Lyapunov's direct method is a well-known method used to prove stability of dynamical systems within a region of interest. In this method, the user provides a local Lyapunov function $V: X \to \mathbb{R}$ that over the domain of interest X satisfies the following properties:

1. Positive definiteness: for all \mathbf{x} in X,

$$V(\mathbf{x}) > 0,\tag{7}$$

and V(0) = 0;

2. Derivative negative semidefiniteness: for all \mathbf{x} in X,

$$\dot{V}(\mathbf{x}) = \frac{d}{dt}V(\mathbf{x}) \le 0, \tag{8}$$

and $\dot{V}(0) = 0$.

Existing techniques from dynamical systems theory use sum-of-squares optimization [22] and semidefinite programming [4, 30] to identify Lyapunov functions [21] for systems described by polynomial differential equations. A Lyapunov function V is analogous to a ranking function for a discrete system, and it maps each continuous state \mathbf{x} to a positive real number, with the property that along any system trajectory the quantity $V(\mathbf{x})$ monotonically decreases until it reaches 0 at the equilibrium point. It is well-known that the sublevel set of a Lyapunov function, $S_{\ell} = \{\mathbf{x}|V(\mathbf{x}) \leq \ell\}$ is a forward invariant set, i.e., given any initial condition in S_{ℓ} , all future states remain in S_{ℓ} . Thus, any sublevel set of a Lyapunov function that includes the initial set and excludes the unsafe set serves as a safety certificate [16, 26].

Remark: It is well known that for stable linear systems, a quadratic Lyapunov function of the form $V = \mathbf{x}^T P \mathbf{x}$, where P is a positive definite matrix, always exists and can be computed by solving the matrix equation

$$A^T P + PA = -Q (9)$$

where Q is a positive definite matrix. Several scientific computing tools have built-in commands to solve this equation, such as lyap in MATLABand LyapunovSolve in Mathematica.

We now show how we can use Lyapunov-like functions to construct local certificates.

Barrier Certificates. In the hybrid systems community, barrier certificates have been proposed as a Lyapunov-like analysis technique to prove that starting from an initial set

of states X_0 , no system trajectory ever enters an unsafe set U [26, 27, 28]. The main step is to identify a barrier function B from the domain X to \mathbb{R} , with the following properties:

$$\forall \mathbf{x} \in X_0 : B(\mathbf{x}) \le 0 \tag{10}$$

$$\forall \mathbf{x} \in U : B(\mathbf{x}) > 0 \tag{11}$$

$$\forall \mathbf{x} \in X \text{ s.t.} B(\mathbf{x}) = 0 : \frac{\partial B}{\partial \mathbf{x}} f(\mathbf{x}) < 0.$$
 (12)

Given a local Lyapunov function V valid in the domain X, if an ℓ can be selected such that (10) and (11) are satisfied, then $B(\mathbf{x}) = V(\mathbf{x}) - \ell$ is a barrier certificate. This follows from the definition of barrier certificates and the Lyapunov conditions (7) and (8).

Discovering Barrier Certificates. To discover barrier certificates, we employ a modification of a technique from [29], which uses concrete system executions to generate a series of candidate Lyapunov functions. Our technique, which is based on [17], uses concrete executions to generate a set of linear constraints. A candidate Lyapunov function is then generated by solving a linear program (LP) associated with the constraints. A series of candidates is iteratively improved upon, using a global optimizer to search the region of interest for executions that violate the condition (8) for the given candidate. The search is guided by a cost function that is based on the Lie derivative of the candidate Lyapunov function; if this cost function can be minimized below 0, then the minimizing argument provides a witness (which we call a counterexample) showing the candidate Lyapunov function is invalid. Once such counterexamples are obtained, we include the associated linear constraints in the LP problem and update the candidate Lyapunov function. The process terminates when the global optimizer is unable to find counterexamples to the candidate Lyapunov function. We then define the candidate barrier function $B(\mathbf{x}) = V(\mathbf{x}) - l$, where l is selected such that (10) and (11) are satisfied.

Because there are no optimality guarantees from the global optimizer used to generate the candidate barrier function, the resulting candidate may not strictly satisfy the desired constraints. To check whether the candidates satisfy (10) through (12), we rely on a satisfiability modulo theories (SMT) solver that can handle nonlinear theories over the reals. We use the dReal tool, which uses interval constraint propagation (ICP) [11]. dReal supports various nonlinear elementary functions in the framework of δ -complete decision procedures, and returns "unsat" or " δ -sat" for a given query, where δ is a precision value specified by the user. When the answer is "unsat", dReal produces a proof of unsatisfiability; when it returns " δ -SAT", it gives an interval of size δ , which contains points that may possibly satisfy the query.

When a " δ -SAT" result is returned from a query to check (10) through (12), we do the following: 1.) construct a new linear constraint based on the interval returned, 2.) add the new constraint to the existing set of linear constraints, and 3.) re-solve the LP to obtain an updated (improved) Lyapunov function candidate. If this process terminates, then the result is a barrier certificate.

Our technique attempts to use discovered barrier certificates locally, that is, for each mode we attempt to construct a certificate that proves that the system will not leave the mode. If such a local barrier certificate is found, then the forward invariant cut rule can be applied to the mode to

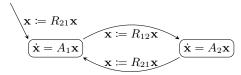


Figure 2: Hybrid automaton for the nonautonomous switched system.

simplify the safety proof for the system, which may be composed of several modes.

4.2 Other Techniques

Bounded-time Invariant Certificates. Inspired by the success of reachability analysis using bounded model checking for verifying software systems, there has been significant research in estimating the reachable set of states for hybrid and continuous-time dynamical systems. See [10] and references therein. A common theme among various approaches is to compute a *flowpipe*, or an overapproximation of the reachable states over a bounded time horizon τ . If the computed flowpipe does not intersect with the unsafe set, then it is safe, and it is invariant over bounded-time, as the initial states lie within it as wells as the set of all future states reachable within a fixed time bound also lie within it. The general form of a bounded-time invariant set is $S_{reach} = (R(\mathbf{x}) < 0) \land (t_l < t < t_u)$, where $R(\mathbf{x}) < 0$ is some compact subset of the domain, and (t_l, t_u) is the time interval over which the set $R(\mathbf{x}) < 0$ is invariant.

Discrete Transition-based Certificates. These certificates are useful to prove unreachability of certain modes because of the transition structure of an underlying hybrid automaton. Standard techniques from automata theory such as identifying strongly connected components can be used to obtain such certificates.

5. CASE STUDIES

5.1 Non-autonomous Switched System

Consider an open two-mode system, where an external input can cause the system to arbitrarily switch between the system modes. This example is significant, because neither of the two modes is invariant, so the proof cannot rely on cutting out entire modes.

The continuous dynamics are defined by matrices A_1 and A_2 , as given below:

$$A_1 = \begin{bmatrix} -1.0 & 4.0 \\ -0.25 & -1.0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.0 & -0.25 \\ 4.0 & -1.0 \end{bmatrix}.$$

Linear reset maps are applied to the state when a transition is made between Modes 1 and 2. The resets are defined by matrices R_{12} and R_{21} :

$$R_{12} = \begin{bmatrix} -0.0658 & -0.0123 \\ 0.1965 & -0.0658 \end{bmatrix},$$

$$R_{21} = \begin{bmatrix} -0.0658 & 0.1965 \\ -0.0123 & -0.0658 \end{bmatrix}.$$

Model 2: A $d\mathcal{L}$ model of the nonautonomous switched system

```
1
                 TS \equiv I \to [(s \cup m_1 \cup m_2)^*]S
 2
                    I \equiv M = 1 \land x_1^2 + x_2^2 \le 0.49
                   \mathbf{s} \equiv M \coloneqq 1 \cup \dot{M} \coloneqq \tilde{2}
 3
                  \mathbf{m}_1 \equiv (?M = 1);
 4
                            x_1 := -0.0658x_1 + 0.1965x_2;
                            x_2 := -0.0123x_1 - 0.0658x_2;

\{x'_1 = -x_1 + 4x_2, x'_2 = -(1/4)x_1 - x_2\}
 6
 7
                  m_2 \equiv (?M = 2);
 8
                            x_1 := -0.0658x_1 - 0.0123x_2
 9
                            x_2 := 0.1965x_1 - 0.0658x_2
10
                            \{x_1' = -x_1 - (1/4)x_2, x_2' = 4x_1 - x_2\}
11
                   S \equiv x_1 > -2 \land x_1 < 2 \land x_2 > -2 \land x_2 < -2
12
```

Figure 2 shows a hybrid automaton for the system, and Model 2 defines the corresponding hybrid program. For both modes, the continuous-time dynamics given by A_1 and A_2 are stable and linear. It is well known that even for switched-mode systems with stable linear continuous dynamics, switching conditions exists that lead to instability for the switched system [5]. We wish to prove that it is not possible to switch between A_1 and A_2 to create unstable behavior. The safety property for this system is that it should remain within $\|\mathbf{x}\|_{\infty} < 2.0$. We apply the forward invariant cut rule to the example to successfully prove the safety property. Below, we describe the steps of the proof.

Here, the designer provided two forward invariants of the system by independently solving the Lyapunov equation (9) for the linear dynamics of the system in each of the modes. The designer then picked level set sizes to ensure that the resulting forward invariant is contained within the safe set S. The invariants are given below:

$$C_1 = \{ \mathbf{x} \mid V_1(\mathbf{x}) < l_1 \} \tag{13}$$

$$C_2 = \{ \mathbf{x} \mid V_2(\mathbf{x}) < l_2 \}$$
 (14)

Here, $V_1(\mathbf{x}) = 0.3828x_1^2 + 0.9375x_1x_2 + 2.3750x_2^2$, and $l_1 = 1.0$, and $V_2(\mathbf{x}) = 2.3750x_1^2 + 0.9375x_1x_2 + 0.3828x_2^2$, and $l_2 = 1.0$.

We sequentially apply two forward invariant cuts in order to prove Model 2 safe. The first forward invariant cut rule uses the set C_1 as the cut. After applying C_1 , the proof tree has three branches: $I \land \neg C_1 \to [(\alpha; ? \neg C_1)^*]S$, $C_1 \to [\alpha]C_1$, and $C_1 \to S$. Of these, the third branch is trivially true as $C_1 \subseteq S$. To prove the second branch valid, KeYmaera needs to prove that C_1 is invariant for the disjuncts.

For the hybrid program \mathbf{m}_1 , KeYmaera computes the forward image of the set C_1 when transformed by the linear transformation R_{21} , i.e., the set $F = \{\mathbf{y} \mid \mathbf{y} = R_{21}\mathbf{x} \land V_1(\mathbf{x}) < l_1\}$. Note that this step requires performing quantifier elimination, and KeYmaera utilizes Mathematica for this purpose. It then uses C_1 as a differential invariant to prove that $F \to [\{\mathbf{x}' = A_1\mathbf{x}\}]C_1$. This is facilitated by the fact that C_1 is in fact invariant for the linear system $\dot{\mathbf{x}} = A_1\mathbf{x}$.

The difficult branch is the one requiring us to prove that C_1 is invariant for mode \mathbf{m}_2 . To do so, we assist KeYmaera with certain lemmas; the intuition for these lemmas is as follows: Any state in set C_1 upon executing the program \mathbf{m}_2 is linearly transformed by R_{12} . Let $\hat{C}_1 = \{\hat{\mathbf{x}} \mid \mathbf{x} \in C_1 \land \hat{\mathbf{x}} = R_{12}\mathbf{x}\}$ represent the forward image of C_1 under R_{12} . Next,

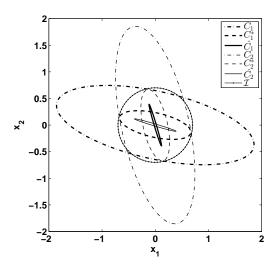


Figure 3: Illustration of forward invariant sets for Example 5.1.

we show that the set \hat{C}_1 is a subset of a specific sublevel set C_2^* of $V_2(\mathbf{x})$. As C_2^* is a sublevel set of $V_2(\mathbf{x})$, it is invariant under the dynamics $\dot{\mathbf{x}} = A_2 \mathbf{x}$; thus, any state beginning in C_2^* will remain in C_2^* . Finally, we choose C_2^* in such a way that $C_2^* \subseteq C_1$. This essentially proves that any state starting in the set C_1 will be contained in set \hat{C}_1 , of which any state will under the dynamics $\dot{\mathbf{x}} = A_2 \mathbf{x}$ remain in the set C_2^* , i.e., in the state C_1 .

Formally, we establish the following:

$$C_1 \to [\mathbf{x} := R_{12}\mathbf{x}]\hat{C}_1 \tag{15}$$

$$\hat{C}_1 \subseteq C_2^* \tag{16}$$

$$\hat{C}_1 \subseteq C_2^*$$

$$C_2^* \to [\{\mathbf{x}' = A_2 \mathbf{x}\}]C_2^*$$
(16)

$$C_2^* \subseteq C_1 \tag{18}$$

We can combine these to infer that $C_1 \to [m_2]C_1$.

Finally, the first branch of the proof considers $I \wedge \neg C_1$; this contains the set of initial states not in C_1 . These can now be addressed by the second forward invariant cut (set C_2) following a symmetric argument as above. After applying the second cut C_2 , the first branch has an empty antecedent $(I \wedge \neg C_1 \wedge \neg C_2 \text{ is empty})$, i.e., the proof has accounted for all initial states, which closes the proof. The sets we have discussed are shown in Figure 3.

5.2 **Engine fuel control**

Model. Our second case study is a hybrid system representing an automotive fuel control application. Environmental concerns and government legislation require that the fuel economy be maximized and the exhaust gas emissions (e.g., hydrocarbons, carbon monoxide, and nitrogen oxides) be minimized. At the ideal air-to-fuel (A/F) ratio, also known as the stoichiometric value, both these quantities are optimized. We present an automotive control system whose purpose is to accurately regulate the A/F ratio.

The system dynamics and parameters were derived from a published model [15] and then simplified, as in [17]. The model consists of a simplified version of the physics of engine subsystems responsible for air intake and A/F ratio measurement, along with a computer control system tasked with regulating the A/F ratio. The objective of the controller is to maintain the A/F ratio within 10% of the nominal operating conditions. The experiment that we model involves an engine connected to a dynamometer – a device that can control the speed of the engine and measure the output torque. In our setting, the dynamometer maintains the engine at a constant rotational velocity. The controller has two modes of operation: (1) a recovery mode, which controls fuel in an open-loop manner, i.e., with only feedforward control action, where the system runs for at most 8ms, and (2) a normal run mode, which uses feedback control to regulate the A/F ratio.

The controller measures both the air flow through the airintake manifold, which it uses to estimate the air pressure in the manifold, and the oxygen content of the exhaust gas, which it uses to compute the A/F ratio. The recovery mode represents the behavior of the controller when recovering from a sensor fault (e.g., aberrant sensor readings, environmental conditions that cause suspicion of the sensor readings). During the recovery mode, the controller has no access to oxygen sensor measurements and so must operate in a feedforward manner (i.e., using only the manifold air flow rate). The normal mode is the typical mode of operation, where the oxygen sensor measurements are used to do feedback control.

Model 3 is a hybrid program representing this system. The ODEs representing the continuous dynamics in each mode and the model parameters are presented in the Appendix. The state variables $\hat{p}, \hat{r}, \hat{p}_{est}$, and \hat{i} represent the manifold pressure, the ratio between actual air-fuel ratio and the stoichiometric value, the controller estimate of the manifold pressure, and the internal state of the PI controller; these variables have all been translated so that the equilibrium point coincides with the origin. In the recovery mode, the continuous-time state **x** is the tuple $(\hat{p}, \hat{r}, \hat{p}_{est}, i, \tau)$. The additional state variable in the recovery mode represents the state of a timer that evolves according to the ODE $\dot{\tau} = 1$. In the normal mode, the state is given by $(\hat{p}, \hat{r}, \hat{p}_{est}, \hat{i})$.

We assume the system is within 1.0% of the nominal value at the initialization of the recovery mode. This represents the case where the system was previously in a mode of operation that accurately regulated the A/F ratio to the desired setpoint. A domain of interest for the state variables is given by $\|\mathbf{x}\|_{\infty} < 0.2$.

Safety proof using forward invariant cut. The verification goal is to ensure that in the given experimental setting, the system always remains within 10% of the nominal A/F ratio after a fixed recovery time of 0.8 ms has passed. In other words, we wish to show that the system begins in the recovery mode, with the initial set of continuous states defined by init = $\{\mathbf{x} \mid ||\mathbf{x}||_{\infty} < 0.01\}$; the system transitions to the normal mode after at most 8.0 ms; and the system never transitions to the unsafe set, where |r| > 0.1, within the domain of interest $\|\mathbf{x}\|_{\infty} < 0.2$.

In previous work [17], the authors had established a forward invariant set for the *normal* mode of operation using a barrier certificate formulation. The authors formulated the barrier certificate using simulation-guided techniques to obtain a candidate Lyapunov function V and a number ℓ to propose a barrier function of the form $B(\mathbf{x}) = V(\mathbf{x}) - \ell$. Here, $V(\mathbf{x}) = \mathbf{z}^T P \mathbf{z}$, and \mathbf{z} is a vector of all monomials of

Model 3: A $d\mathcal{L}$ model of a closed-loop fuel control system

```
EFC \equiv
 1
     I \to [(\mathbf{m}_1 \cup \mathbf{m}_2 \cup \mathbf{s}_{1 \mapsto 2} \cup \mathbf{s}_{\{1,2\} \mapsto fail} \cup \mathbf{m}_{fail})^*]S
                      I \equiv (-0.001 \le p \le 0.001)
 2
 3
                             \wedge \ (-0.001 \le r \le 0.001) \ \wedge
                             \wedge (p_{est} = 0 \land i = 0 \land M = 1)
                    \mathbf{m}_1 \equiv (?M = recovery; ?\tau \leq 0.008;
 5
 6
                             \{\exists \ell_1. \exists \ell_2. \exists \ell_3
                                 (-0.86 \le \ell_1 \le 0.74)
                                 (-0.17 \le \ell_2 \le 0.18) 
 (-0.81 \le \ell_3 \le 0.68)
 8
 9
                                10
11
               s_{1\mapsto 2} \equiv (?M = recovery; ?\tau \geq 0.008;
12
                               M := normal;
13
                    \mathbf{m}_2 \equiv (?M = normal;
14
                               \begin{cases} p' = f_p, \\ r' = f_r, \end{cases}
15
16
                               p'_{est} = f_{p_{est}}, 
i' = f_i,
17
18
                               \& -0.02 \le p \le 0.02 \land -0.02 \le r \le 0.02
19
                                \land -0.02 \le p_{est} \le 0.02 \land -0.02 \le \overline{i} \le 0.02 \}
20
     s_{\{1,2\}\mapsto fail} \equiv (?(r < -0.1 \lor r > 0.1);
21
                                 M := fail
                \mathbf{m}_{fail} \equiv (?(r < -0.1 \lor r > 0.1);
23
                                 M \coloneqq fail
24
25
                         \equiv M \neq fail
```

degree ≤ 2 of the state variables \hat{p} , \hat{r} , \hat{p}_{est} and \hat{i} . Note that **z** thus contains 14 monomials, and P is a 14x14 matrix. We omit the resulting P matrix for brevity.

We use the set enclosed by the barrier function to formulate the forward invariant cut

$$C \equiv (M = normal) \land (B(\mathbf{x}) \le 0). \tag{19}$$

Application of the forward invariant cut inference rule (6), generates three proof obligations that KeYmaera has to discharge.

Obligation 1. $C \rightarrow [\alpha]C$

Note that once we define C, the hybrid programs \mathfrak{m}_1 , $s_{1\mapsto 2}$ can be excised by KeYmaera, as both have the hybrid program ?M = recovery as their first item, which is inconsistent with C. Thus, KeYmaera can then focus on proving this obligation only for the programs \mathfrak{m}_2 , $\mathfrak{s}_{\{1,2\}\mapsto fail}$ and \mathfrak{m}_{fail} .

In order to discharge the obligation for the program m₂, we first perform some trivial simplifications with KeYmaera that leaves us with the following proof goal:

$$(B(\mathbf{x}) \le 0) \to [\{\mathbf{x}' = f(\mathbf{x}) \& H\}](B(\mathbf{x}) \le 0) \land (M = normal)$$
(20)

To discharge (20), we can use the barrier certificate rule shown in (21) that we have added to KeYmaera's proof calculus.

$$\frac{init \to B(\mathbf{x}) \le 0 \quad B(\mathbf{x}) = 0 \to \frac{\partial B}{\partial \mathbf{x}} \cdot f(\mathbf{x}) < 0 \quad B(\mathbf{x}) \le 0 \to safe}{init \to [\{x' = f(\mathbf{x})\}] safe}$$
(21)

where H is the domain of evolution of the continuous dynamics. In our application of the barrier certificate rule, we substitute init with $(B(\mathbf{x}) \leq 0)$ and safe with $(B(\mathbf{x}) \leq 0) \wedge (M = normal)$. The first and the third proof obligations in the barrier certificate rule are then trivially satisfied. For

the remaining (middle) proof obligation KeYmaera uses the SMT solver dReal [11]. In particular, it asks dReal if the query $(B(\mathbf{x})=0) \wedge (\frac{\partial B}{\partial \mathbf{x}} \cdot f_{normal}(\mathbf{x}) > -\epsilon)$ is unsatisfiable, where ϵ is a small positive number.

In order to discharge the proof obligation for \mathtt{m}_2 , $\mathtt{s}_{\{1,2\}\mapsto fail}$, KeYmaera needs to show that if $B(\mathbf{x})<0$ holds, either of these programs cannot invalidate C by transitioning to mode fail. It proves this by showing that the set $B(\mathbf{x})<0$ is a subset of the safe set using dReal.

Obligation 2. $C \rightarrow S$

This obligation is trivial as S requires the mode to be fail, while C says that the mode is normal mode.

Obligation 3.
$$I \wedge \neg C \rightarrow [(\alpha; ? \neg C)^*]S$$

To prove this obligation, we use the lemma that the set C1 is an invariant for all states remaining in $I \land \neg C$. This is a bounded-time invariant certificate.

$$C1 \equiv (M \neq m_{fail}) \land (0 \le \tau \le 0.008) \land (x \in S_{reach}) \quad (22)$$

Here S_{reach} is an overapproximation of reachable sets by using upper and lower bounds on \dot{p} and \dot{r} computed using dReal. The proof for this branch continues using standard KeYmaera deduction procedures. There is one additional barrier certificate application to show that the normal mode, when starting from this set, lands within the barrier certificate and therefore also respects this invariant. This requires a derivative negativity argument, which KeYmaera again handles via an external dReal query.

6. RELATED WORK AND CONCLUSIONS

Lazy abstraction. In software verification using conservative abstractions, an abstract program can be viewed as a proof of program correctness if it satisfies the correctness property of interest. A popular paradigm is that of lazy abstraction [13], where the abstract program is not derived from a global set of predicates, but is an abstract model in which predicates change from state to state. Such an abstraction is obtained through the process of lazy refinement, where abstraction is done on-the-fly with a goal of eliminating local spurious counterexamples. While the exact mechanics of our technique are different, our technique also generates correctness proofs consisting of lazily generated local invariants.

Logical cuts. In classical logic, a cut serves the role of a lemma. In Gentzen's sequent calculus [12], the cut rule splits the proof tree into two branches, one in which the lemma can be used as an assumption, and another in which it must be proved. The cut-elimination theorem, states that any proof of the sequent calculus that uses the cut rule has another proof that does not use the cut rule. Ideas similar to Gentzen's cut rule have been developed for other reasoning frameworks. Craig interpolants [7] have been used to compute cuts in frameworks that leverage first-order logic, and they have been used successfully in a model checking framework [20]. The differential cut rule of $d\mathcal{L}$ makes it possible to introduce lemmas about the continuous evolution of differential equations. It has been shown that there are theorems that cannot be proved without differential cuts, i.e., the differential cut strictly adds deductive power [24]. Overall, the approach provides an iterative method to find a

safety certificate, by proposing sets that are initialized and invariant, and repeating the differential cut procedure until safety can be proved. This work proposes a forward invariant cut rule, in which a lemma is proved about the evolution of a hybrid system model. The proof rule requires showing that a certain set is safe and invariant, and allows the proof to continue for the behaviors that are not initialized within the set. The forward invariant cut may be repeated, until a proof of overall system safety is attained. Most crucially, the proposed cuts allow the verification process to leverage a designer's knowledge of local system properties.

Deductive Proof System for Temporal Logic. In [8], the authors present a deductive proof system for proving alternating-time temporal logic assertions on a continuous dynamical system. Some of the proof rules presented require the user to provide auxiliary predicates to establish proof-subgoals. These predicates are essentially logical cuts, and in particular can be barrier certificates. The key feature of our approach is that we provide an automated mechanism to leverage user insight about parts of the system to obtain localized forward invariant cuts. It would be interesting to see if the automation that we develop in this paper could be used to mechanize the proof system presented in [8].

Conclusions. This paper presents a method to leverage knowledge of local system behavior within a deductive framework. In this framework, designer knowledge of system behavior can be leveraged lazily as part of a proof of global system safety. The designer proposes sets that are invariant and safe, which allows certifying the safety of some region of state space. In future work, we would like to investigate the use of sets that are safe, but not initialized or invariant, as part of a proof effort. An example of this is when a collection of modes have continuous barriers that the differential equations may not cross, but the set is not invariant because there are outgoing transitions that are not excluded by the set.

7. REFERENCES

- Abraham-Mumm, E., Hannemann, U., Steffen, M.: Verification of hybrid systems: formalization and proof rules in pvs. In: Engineering of Complex Computer Systems, 2001. Proceedings. Seventh IEEE International Conference on. pp. 48–57 (2001)
- [2] Abraham-Mumm, E., Hannemann, U., Steffen, M.: Assertion-based analysis of hybrid systems with pvs. In: Moreno-Diaz, R., Buchberger, B., Luis Freire, J. (eds.) Computer Aided Systems Theory, EUROCAST 2001, Lecture Notes in Computer Science, vol. 2178, pp. 94–109. Springer Berlin Heidelberg (2001), http://dx.doi.org/10.1007/3-540-45654-6_8
- [3] Beckert, B., Hähnle, R., Schmitt, P.H.: Verification of object-oriented software: The KeY approach. Springer-Verlag (2007)
- [4] Boyd, S., Ghaoui, L.E., Feron, E., Balakrishnan, V.: Linear Matrix Inequalities in System and Control Theory, vol. 15. SIAM (1994)
- [5] Branicky, M.S.: Stability of switched and hybrid systems. In: Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on. vol. 4, pp. 3498–3503. IEEE (1994)
- [6] Chen, X., Abraham, E., Sankaranarayanan, S.: Flow*: An Analyzer for Non-Linear Hybrid Systems. In: CAV

- (2013)
- [7] Craig, W.: Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. J. Sym. Logic 3, 269–285 (1957)
- [8] Dimitrova, R., Majumdar, R.: Deductive control synthesis for alternating-time logics. In: Proc. of the International Conference on Embedded Software. p. 14 (2014)
- [9] Frehse, G.: PHAVer: Algorithmic verification of hybrid systems past HyTech. STTT 10(3), 263–279 (2008)
- [10] Frehse, G., Le Guernic, C., Donzé, A., Cotton, S., Ray, R., Lebeltel, O., Ripado, R., Girard, A., Dang, T., Maler, O.: SpaceEx: Scalable verification of hybrid systems. In: CAV. pp. 379–395 (2011)
- [11] Gao, S., Avigad, J., Clarke, E.M.: δ -complete decision procedures for satisfiability over the reals. In: J. Automated Reasoning. pp. 286–300 (2012)
- [12] Gentzen, G.: Untersuchungen Äijber das logische Schließen I,II. Mathematische Zeitschrift 39(2), 39(3): 176–210, 405–431 (1935)
- [13] Henzinger, T.A., Jhala, R., Majumdar, R., Sutre, G.: Lazy abstraction. In: ACM SIGPLAN Notices. vol. 37, pp. 58–70 (2002)
- [14] Henzinger, T.A., Kopke, P.W., Puri, A., Varaiya, P.: What's Decidable about Hybrid Automata? JCSS 57(1), 94 – 124 (1998)
- [15] Jin, X., Deshmukh, J.V., Kapinski, J., Ueda, K., Butts, K.: Powertrain Control Verification Benchmark. In: Hybrid Systems: Computation and Control (2014)
- [16] Johansson, M., Rantzer, A.: Computation of piecewise quadratic Lyapunov functions for hybrid systems. IEEE Tran. on Automatic Control 43(4), 555–559 (1998)
- [17] Kapinski, J., Deshmukh, J.V., Sankaranarayanan, S., Aréchiga, N.: Simulation-guided Lyapunov analysis for hybrid dynamical systems. In: Hybrid Systems: Computation and Control (2014)
- [18] Kapinski, J., Krogh, B.H.: Verifying asymptotic bounds for discrete-time sliding mode systems with disturbance inputs. In: ACC. vol. 3, pp. 2852–2857 (2004)
- [19] Khalil, H.K.: Nonlinear Systems. Prentice Hall (2002)
- [20] McMillan, K.L.: Lazy abstraction with interpolants. In: Computer Aided Verification. pp. 123–136 (2006)
- [21] Meiss, J.D.: Differential Dynamical Systems (Monographs on Mathematical Modeling and Computation). SIAM (2007)
- [22] Parrilo, P.A.: Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization. Ph.D. thesis, California Institute of Technology (2000)
- [23] Platzer, A.: Logical Analysis of Hybrid Systems. Springer (2010)
- [24] Platzer, A.: The structure of differential invariants and differential cut elimination. Logical Methods in Computer Science 8(4), 1–38 (2012)
- [25] Platzer, A., Clarke, E.M.: Computing differential invariants of hybrid systems as fixedpoints. In: Computer Aided Verification (2008)

- [26] Prajna, S.: Optimization-based methods for nonlinear and hybrid systems verification. Ph.D. thesis, California Institute of Technology, Caltech, Pasadena, CA, USA (2005)
- [27] Prajna, S.: Barrier certificates for nonlinear model validation. Automatica 42(1), 117–126 (2006)
- [28] Prajna, S., Jadbabaie, A.: Safety Verification of Hybrid Systems Using Barrier Certificates. In: Hybrid Systems: Computation and Control. pp. 477–492 (2004)
- [29] Topcu, U., Seiler, P., Packard, A.: Local stability analysis using simulations and sum-of-squares programming. Automatica 44, 2669–2675 (2008)
- [30] Vandenberghe, L., Boyd, S.: Semidefinite Programming. SIAM Review 38(1), 49–95 (March 1996)

APPENDIX

Appendix

A Semantics of $d\mathcal{L}$

We follow the development of [23], Chapter 2. Symbols in $d\mathcal{L}$ are classified into three different syntactic categories, depending on their role.

- 1. Σ_r represents a set of *rigid* symbols that cannot change their value, such as $0, 1, +, \cdot$;
- 2. Σ_{fl} represents a set of *flexible* symbols, also called *state* variables, which change their value as the system evolves;
- 3. V represents a set of *logical variables*, which do not change as the system evolves, but can be quantified over universally and existentially; they often serve the role of parameters.

An interpretation is a function \mathcal{I} that associates functions and relations over the reals to function and relation symbols in Σ_r . The standard arithmetic operators and relations symbols, such as $+,\cdot,\geq$, are interpreted as usual. A state is a map $\nu:\Sigma_{fl}\mapsto\mathbb{R}$, which maps a real value to each state variable. An assignment $\eta:V\mapsto\mathbb{R}$ is a map that prescribes the value of the logical variables. Note that the value of the logical variables does not depend on the state.

A state variable is a term, and a logical variable is also a term. The result of applying a function of arity n to n terms is also a term. Nothing else is a term.

Definition 1 (Valuation of terms ([23], Defn. 2.5)). The valuation of terms with respect to interpretation \mathcal{I} , assignment η , and state ν is defined as

- 1. $val_{\mathcal{I},\eta}(\nu,p) = \eta(p)$ if p is a logical variable.
- 2. $val_{\mathcal{I},\eta}(\nu,x) = \nu(x)$ if x is a state variable.
- 3. $val_{\mathcal{I},\eta}(\nu, f(\theta_1, \dots, \theta_n)) = \mathcal{I}(f)(val_{\mathcal{I},\eta}(\theta_1), \dots, val_{\mathcal{I},\eta}(\theta_2))$ if f is a function of arity $n \geq 0$ and $\theta_1, \dots, \theta_n$ are terms.

The notation $\eta[x\mapsto d]$ represents the function that agrees with η except for the interpretation of x, where it takes the value d. The notation $\nu[x\mapsto d]$ denotes the modification of a state ν , that agrees with ν everywhere except the interpretation of the state variable x, where it takes the value d

Definition 2 (Valuation of $d\mathcal{L}$ formulas ([23], Defn. 2.6). The valuation $val_{\mathcal{I},\eta}(\nu,\cdot)$ of formulas with respect to interpretation \mathcal{I} , assignment η , and state ν is defined as

- 1. $val_{\mathcal{I},\eta}(\nu, p(\theta_1, \dots, \theta_n) = I(p)(val_{\mathcal{I},\eta}(\nu, \theta_1), \dots, val_{\mathcal{I},\eta}(\nu, \theta_n))$
- 2. $val_{\mathcal{I},\eta}(\nu,\phi \wedge \psi) = \text{true } \textit{iff } val_{\mathcal{I},\eta}(\nu,\phi) = \text{true } \textit{and } val_{\mathcal{I},\eta}(\nu,\psi) = \text{true}.$
- 3. $val_{\mathcal{I},\eta}(\nu,\phi \lor \psi) = \text{true } iff \ val_{\mathcal{I},\eta}(\nu,\phi) = \text{true } or \ val_{\mathcal{I},\eta}(\nu,\psi) = \text{true}.$
- 4. $val_{\mathcal{I},\eta}(\nu,\neg\phi) = \text{true } iff \ val_{\mathcal{I},\eta}(\nu,\phi) \neq \text{true}.$
- 5. $val_{\mathcal{I},\eta}(\nu,\phi \to \psi) = \text{true } iff \ val_{\mathcal{I},\eta}(\nu,\phi) \neq \text{true } or \ val_{\mathcal{I},\eta}(\nu,\psi) = \text{true}.$
- 6. $val_{\mathcal{I},\eta}(\nu, \forall x\phi) = \text{true iff } val_{\mathcal{I},\eta_{[x\mapsto d]}}(\nu,\phi) = \text{true for all } d \in \mathbb{R}.$
- 7. $val_{\mathcal{I},\eta}(\nu, \exists x\phi) = \text{true iff } val_{\mathcal{I},\eta_{[x\mapsto d]}}(\nu,\phi) = \text{true for } some \ d \in \mathbb{R}.$

- 8. $val_{\mathcal{I},\eta}(\nu, [\alpha]\phi) = \text{true iff } val_{\mathcal{I},\eta}(\omega,\phi) = \text{true for all } states \omega \text{ for which the transition relation (defined below)} satisfies <math>(\nu,\omega) \in \rho_{\mathcal{I},\eta}(\alpha)$.
- 9. $val_{\mathcal{I},\eta}(\nu,\langle\alpha\rangle\phi) = \text{true iff } val_{\mathcal{I},\eta}(\omega,\phi) \text{ for some state } \omega \text{ such that the transition relation satisfies } (\nu,\omega) \in \rho_{\mathcal{I},\eta}(\alpha).$

We now define the transition semantics of hybrid programs. We already saw a glimpse of it in the definition of valuation of formulas, since the formulas and programs of $\mathrm{d}\mathcal{L}$ are constructed coinductively.

Definition 3 (Transition semantics of hybrid programs ([23], Defn. 2.7)). The valuation of a hybrid program α , denoted $\rho_{\mathcal{I},\eta}(\alpha)$ is a transition relation on states that specifies which states are reachable from a state ν under the program α , and is defined inductively as follows.

- 1. $(\nu, \omega) \in \rho_{\mathcal{I},\eta}(x_1 := \theta_1, \dots, x_n := \theta_n)$ iff the state ω equals the state obtained by modification of ν as $\nu[x_1 \mapsto val_{\mathcal{I},\eta}(\nu,\theta_1)], \dots, \nu[x_n \mapsto val_{\mathcal{I},\eta}(\nu,\theta_n)].$
- 2. $(\nu,\omega) \in \rho_{\mathcal{I},\eta}(\{x_1'=\theta_1,\ldots,x_n'=\theta_n\&H\})$ iff there is a flow f of some duration $r \geq 0$ from ν to ω along the differential equations $x_1'=\theta_1,\ldots,x_n'=\theta_n$ that always respects the invariant H.
- 3. $\rho_{\mathcal{I},\eta}(?\chi) = \{(\nu,\nu) \mid val_{\mathcal{I},\eta}(\nu,\chi) = \text{true}\}$
- 4. $\rho_{\mathcal{I},\eta}(\alpha \cup \beta) = \rho_{\mathcal{I},\eta}(\alpha) \cup \rho_{\mathcal{I},\eta}(\beta)$
- 5. $\rho_{\mathcal{I},\eta}(\alpha;\beta) = \rho_{\mathcal{I},\eta}(\alpha) \circ \rho_{\mathcal{I},\eta}(\beta)$
- 6. $(\nu, \omega) \in \rho_{\mathcal{I},\eta}(\alpha^*)$ iff there is a sequence of states states ν_0, \ldots, ν_n with $n \geq 0$, $\nu = \nu_0$, and $\nu_n = \omega$ such that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I},\eta}(\alpha)$ for each $0 \leq i \leq n-1$.

B Soundness proof for forward invariant cut

Fix an interpretation \mathcal{I} and an assignment η . From semantics of the first premise, if $\nu \in C$ and $(\nu,\omega) \in \rho_{\mathcal{I},\eta}(\alpha)$, then $\omega \in C$. From the semantics of the second premise, if $\omega \in C$, then $\omega \in S$ From the semantics of the third premise, if $\nu \in I$ and $\nu \notin C$, and ω is such that $(\nu,\omega) \in \rho_{\mathcal{I},\eta}((\alpha;?\neg C)^*)$, then $\omega \in S$. This is equivalent to saying that for any ω such that there is a sequence of states ν_0,\ldots,ν_n , with $\nu_0=\nu \in I$ and $\nu_n=\omega, n\in \mathbb{N}$, and $(\nu_i,\nu_{i+1})\in \rho_{\mathcal{I},\eta}(\alpha;?\neg C)$ for each $0\leq i\leq n-1$, it is the case that $\omega \in S$.

The proof is to show by induction that any state reachable by α^* from I in $n\geq 0$ executions of α must be contained in

For the base case, let n=0. Then given $\nu \in I$, the only reachable state by a sequence of length zero is ν itself. If $\nu \in C$, then νinS by semantics of the second premise. If $\nu \notin C$, we have that $(\nu, \nu) \in \rho_{\mathcal{I}, \eta}((\alpha; ?\neg C)^*)$ by a chain of length zero, so that by semantics of the third premise, $\nu \in S$.

As an inductive hypothesis, suppose that for every ω reachable by a chain of length $n, \omega \in S$ (i.e., there exists ν_0, \ldots, ν_n with $\nu_0 = \nu$ and $\omega = \nu_n$ such that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha)$, for $0 \le i \le n-1$. Now choose any state ξ such that there is a chain of length $n+1, \nu_0, \ldots, \nu_{n+1}$ with $\nu_0 = \nu$ and $\nu_{n+1} = \xi$, such that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha)$, for $0 \le i \le n$).

First suppose that $\nu_n \in C$. Then by semantics of the first premise, $\nu_{n+1} \in C$, and then $\nu_{n+1} \in S$ by semantics of the second premise. On the other hand, suppose $\nu_n \notin C$. We claim that for all $j \leq n$, $\nu_j \notin C$. To see this, note that if $\nu_j \in C$ for some $j \leq n$, then $\nu_n \in C$ by semantics of the first premise, which would contradict our assumption on ν_n . Then we have that $(\nu_i, \nu_{i+1}) \in \rho_{\mathcal{I}, \eta}(\alpha; ?\neg C)$ for all

Table 1: Model Parameters for the Engine Fuel Control System.

Parameter	Value
c_1	0.41328
c_2	200.0
c_3	-0.366
c_4	0.08979
c_5	-0.0337
c_6	0.0001
c_7	2.821
c_8	-0.05231
<i>C</i> 9	0.10299
c_{10}	-0.00063
c_{11}	1.0
c_{12}	14.7
c_{13}	0.9
c_{14}	0.4
c_{15}	0.4
c_{16}	1.0
$\hat{u_1}$	23.0829

 $0 \leq i \leq n.$ By semantics of the third premise, it follows that $\xi \in S$. This establishes the theorem.

System dynamics for the Engine Fuel Control Model

We now present the model parameters and the ODEs for the Engine Fuel Control model. Figure 4 details the equathe Engine Fuel Control model. Figure 4 details the equations for the recovery mode, and Fig. 5 provides the dynamic equations for the normal mode. In the figures, $\frac{dp}{dt} = f_p$, $\frac{dr}{dt} = f_r$, $\frac{dp_{est}}{dt} = f_{pest}$, and $\frac{di}{dt} = f_i$.

We translate the system so that the origin coincides with the normal equilibrium point $p \approx 0.8987$, r = 1.0, $p_{est} \approx 1.077$, is also and solutions the translated parishbes \hat{n} , \hat{n} , \hat{n}

1.077, $i \approx 0.0$ and call the translated variables \hat{p} , \hat{r} , \hat{p}_{est} , and, \hat{i} , respectively.

$$f_{p} = c_{1} \left(2\hat{u_{1}} \sqrt{\frac{p}{c_{11}} - \left(\frac{p}{c_{11}}\right)^{2}} - \left(c_{3} + c_{4}c_{2}p + c_{5}c_{2}p^{2} + c_{6}c_{2}^{2}p\right) \right)$$

$$f_{r} = 4 \left(\frac{c_{3} + c_{4}c_{2}p + c_{5}c_{2}p^{2} + c_{6}c_{2}^{2}p}{c_{13}(c_{3} + c_{4}c_{2}p_{est}^{2} + c_{5}c_{2}p_{est}^{2} + c_{6}c_{2}^{2}p_{est})} - r \right)$$

$$f_{p_{est}} = c_{1} \left(2\hat{u_{1}} \sqrt{\frac{p}{c_{11}} - \left(\frac{p}{c_{11}}\right)^{2}} - c_{13} \left(c_{3} + c_{4}c_{2}p_{est} + c_{5}c_{2}p_{est}^{2} + c_{6}c_{2}^{2}p_{est}\right) \right)$$

$$f_{i} = 0$$

Figure 4: System dynamics for the Engine Fuel Control System in the recovery mode.

$$f_{p} = c_{1} \left(2\hat{u_{1}} \sqrt{\frac{p}{c_{11}} - \left(\frac{p}{c_{11}}\right)^{2}} - \left(c_{3} + c_{4}c_{2}p + c_{5}c_{2}p^{2} + c_{6}c_{2}^{2}p\right) \right)$$

$$f_{r} = 4 \left(\frac{c_{3} + c_{4}c_{2}p + c_{5}c_{2}p^{2} + c_{6}c_{2}^{2}p}{c_{13}(c_{3} + c_{4}c_{2}p_{est}^{2} + c_{5}c_{2}p_{est}^{2} + c_{6}c_{2}^{2}p_{est})(1 + i + c_{14}(r - c_{16}))} - r \right)$$

$$f_{pest} = c_{1} \left(2\hat{u_{1}} \sqrt{\frac{p}{c_{11}} - \left(\frac{p}{c_{11}}\right)^{2}} - c_{13} \left(c_{3} + c_{4}c_{2}p_{est} + c_{5}c_{2}p_{est}^{2} + c_{6}c_{2}^{2}p_{est} \right) \right)$$

$$f_{i} = c_{15}(r - c_{16})$$

Figure 5: System dynamics for the Engine Fuel Control System in the normal mode.