PROPERTIES OF RAMANUJAN FILTER BANKS

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ABSTRACT

This paper studies a class of filter banks called the Ramanujan filter banks which are based on Ramanujan-sums. It is shown that these filter banks have some important mathematical properties which allow them to reveal localized hidden periodicities in real-time data. These are also compared with traditional comb filters which are sometimes used to identify periodicities. It is shown that non-adaptive comb filters cannot in general reveal periodic components in signals unless they are restricted to be Ramanujan filters. The paper also shows how Ramanujan filter banks can be used to generate time-period plane plots which track the presence of time varying, localized, periodic components.

***Index Terms*—** Ramanujan filter banks, Ramanujan-sum, periodicity, comb filter banks, coprime frequencies.

1. INTRODUCTION

The problem of identifying periodicities in data has been of significant interest for several decades. There have been many interesting papers in the signal processing literature to address this problem. A variety of approaches such as algebraic methods [4], [11], [12], [22], filtering-based methods [1], [6], [21], and dictionary methods [5], [16] have in the past been reported. More recently, the applicability of *Ramanujan-sums* in period estimation and related problems has been studied by a number of authors [3], [7], [8], [10], [13]–[20]. This summation was introduced nearly hundred years ago in the field of number theory by Srinivasa Ramanujan [9]. A review of the Ramanujan sum and its applications can be found in [17], [18], along with a number of new results on the representation of periodic signals using such sums. These results have since been extended in different directions, giving rise to Ramanujan dictionary based approaches [14], and multidimensional representations [20].

Ramanujan filter banks were introduced in [15] for estimation and tracking of periodicities by using a *time-period plane* approach. The purpose of this paper is to point out the suitability of such filter banks from a more fundamental point of view. Since a periodic signal has a Fourier transform made of harmonically related Dirac delta functions, the use of comb filters and their extensions have in the past been suggested for the extraction of periodic components [6], [1]. In such methods the filters are adapted according to the measured signals. Our goal is to use a fixed (non-adaptive) filter bank which offers economy of computation. In the nonadaptive case we will prove that unless the filters in the bank satisfy certain specific conditions, it is not possible to separate out the periods. We will also show that filter banks based on Ramanujan sums satisfy these conditions, and are ideally suited for this purpose. This arises from the fact that the passbands of Ramanujan filters are centered around *coprime frequencies.* In fact the only filter banks that would satisfy the required conditions are those for which the filters belong to the so-called *Ramanujan subspaces* [17].

Preliminaries. If $x(n) = x(n+P)$ for some integer P we say that $x(n)$ is periodic, and the integer P is a repetition number. P is called the period if it is the smallest positive integer with this property. Some signals have the form $x(n) = x_1(n) + x_2(n)$ where the components $x_i(n)$ are periodic, so that $x(n)$ has period equal to the lcm (or a divisor of the lcm) of the individual periods P_1 and P_2. If the data record is not longer than this period, it is not possible to see the period (say, in a plot). In fact given $x(n)$, the components $x_1(n)$ and $x_2(n)$ are not uniquely defined because there could be harmonics that are common to both, and they cannot be separated. But it is often sufficient to identify the hidden periods P_1 and P_2. This is the theme of many of the references cited at the beginning of this introduction.

Notations. (k, q) denotes the greatest common divisor (gcd) of the integers k and q, and $(k, q) = 1$ means that they are coprime. $\mathrm{lcm}(a, b)$ stands for the least common multiple of integers a, b. The notation $q|N$ means that q is a divisor (or factor) of N. $W_q = e^{-j2\pi/q}$, and $\phi(q) =$ Euler totient function (number of integers in $1 \leq n \leq q$ coprime to q).

Paper outline. Ramanujan sums are briefly reviewed in Sec. 2. Filter banks for the identification of periodicities are introduced in Sec. 3. This section also proves some unique properties of Ramanujan filter banks, which make them ideally suited for period identification and tracking. Methods to identify periodicities with such filter banks are explained in Sec. 4, along with illustrative examples. Sec. 5 concludes the paper.

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2. REVIEW OF RAMANUJAN-SUMS

A detailed review of Ramanujan-sums can be found in [17]. Briefly, the qth Ramanujan sum ($q \geq 1$) is a sequence in n defined as

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{-kn} \tag{1}$$

where $-\infty \leq n \leq \infty$. Thus the summation runs over only those k that are coprime to q. Clearly $c_q(n)$ is periodic:

$$c_q(n+q) = c_q(n) \tag{2}$$

and its DFT is given by

$$C_q[k] = \begin{cases} q & \text{if } (k,q) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The DFT is nonzero (equal to q) at the *coprime frequency* indices k and zero elsewhere. If $(k,q) = 1$ then it follows that $(q-k,q) = 1$ as well. Thus the DFT is real and symmetric: $C_q[k] = C_q[q-k]$ so that $c_q(n)$ is also real and symmetric ($c_q(n) = c_q(q-n)$). It is well known [9], [17] that $c_q(n)$ is *integer valued* in spite of the presence of trignometric functions in its definition. For example,

$$\begin{aligned} &c_1(n) = 1,\ c_2(n) = \{1,-1\},\ c_3(n) = \{2,-1,-1\}, \\ &c_4(n) = \{2,0,-2,0\},\ c_5(n) = \{4,-1,-1,-1,-1\}, \\ &c_6(n) = \{2,1,-1,-2,-1,1\},\ \cdots \end{aligned} \tag{4}$$

where one period is shown in each case. Finally Ramanujan sums are orthogonal in the sense that, for any integer l,

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n-l) = 0, q_1 \neq q_2 \tag{5}$$

where m is any common multiple of q_1 and q_2.

3. FILTER BANKS FOR EXTRACTION OF PERIODICITIES

Is it possible to build a linear time invariant (nonadaptive) filter such that its output is nonzero if and only if the input has a specified period P? If so, what kind of properties should such a filter have?

Adaptive comb filters have in the past been proposed for this [6], [1]. Comb filters also arise in the solution to the maximum likelihood estimation of periods, but such a solution requires further processing (such as the application of penalty functions) before a meaningful estimate can be obtained [21]. A comb filter $h_q(n)$ (or a periodic filter) has a periodic impulse response so that $H_q(e^{j\omega})$ is like a comb: it is non zero only at the harmonically related frequencies $2\pi k/q$ (see Fig. 1 (a)). We now revisit this and show that *non-adaptive* comb filters whose impulse responses are arbitrary periodic sequences will not serve this purpose. However if we build comb filters based on Ramanujan sums, then a non-adaptive filter bank based on such filters can uniquely identify periods, as well as periods of hidden periodic components.

3.1. Ramanujan filters and filter banks

If we regard the Ramanujan-sum as a digital filter with impulse response $c_q(n)$ then in view of (3) its frequency response is

$$C_q(e^{j\omega}) = 2\pi \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} \delta\Big(\omega - \frac{2\pi k}{q}\Big), \tag{6}$$

in $0 \leq \omega < 2\pi$ and repeating with period 2π outside. Thus $C_q(e^{j\omega})$ is zero everywhere except at the coprime frequencies $2\pi k_i/q$ where k_i is coprime to q (Fig. 1(b)). Now consider a periodic input $x(n)$ with period P. With $X[k]$ denoting the P-point DFT, we have

$$X(e^{j\omega}) = \frac{2\pi}{P} \sum_{l=0}^{P-1} X[l] \delta\Big(\omega - \frac{2\pi l}{P}\Big) \tag{7}$$

as demonstrated in Fig. 1(c). Suppose we apply this signal $x(n)$ as the input to the Ramanujan filter $c_q(n)$. By comparing (6) and (7) we find that the output will be zero if none of the Dirac functions in these two expressions coincide. But for each k such that

$$\frac{k}{q} = \frac{l}{P} \tag{8}$$

for some l, there can be a nonzero output. Since $(k,q) = 1$, this is possible only if P is a multiple of q (i.e., q is a divisor of P). In this case, we can always find l in the range $0 \leq l \leq P-1$ such that the above holds. Summarizing, we have proved the following:

Lemma 1. The Ramanujan filter $C_q(e^{j\omega})$ can produce a nonzero output in response to a period P signal $x(n)$ *only if* q is a divisor of P. $\diamondsuit$

Note that the "only if" cannot be replaced with "if and only if" because, it is possible that the lth harmonic component $X[l]$ is zero for every l satisfying (8), in which case the output of $c_q(n)$ will be zero.

Now consider Fig. 2 which shows an analysis filter bank where each filter is a Ramanujan filter with impulse response $\{c_q(n)\}$, for $1 \leq q \leq N$. We call this the *Ramanujan filter bank.* Its usefulness arises from the following theorem which follows immediately from the preceding lemma:

Theorem 1 (Ramanujan filter banks): Consider a Ramanujan analysis filter bank $\{c_q(n)\}$ with $1 \leq q \leq N$ and let $x(n)$ be a period-P input signal with $1 \leq P \leq N$. Then nonzero outputs can only be produced by those filters $c_q(n)$ such that the filter index q is a divisor of P, that is, $q|P$. $\diamondsuit$

In practice, one would replace the filters with FIR versions, and the signal $x(n)$ would have finite duration (with localized periodicities). So the Dirac passbands will spread out in frequency, and the above result will only be approximate.

3.2. Arbitrary comb filter banks

Suppose we replace the Ramanujan filters $c_q(n)$ in the filter bank with arbitrary comb filters $h_q(n)$. Say, the filter $h_q(n)$ has period q, but is otherwise arbitrary, so the DFT coefficients $H_q[k]$ can in general be nonzero for all k. Then the frequency responses are

$$H_q(e^{j\omega}) = \frac{2\pi}{q}\sum_{k=0}^{q-1} H_q[k]\delta\Big(\omega - \frac{2\pi k}{q}\Big) \tag{9}$$

which is similar to Fig. 1(c) except that the uniform spacing is $2\pi/q$ rather than $2\pi/P$ (see Fig. 1(a)). Assume

$$H_q[0] = 0, \quad \text{unless } q = 1. \tag{10}$$

Thus, the DC or zero-frequency component $X[0]$ of $x(n)$ only passes through $c_1(n)$ and not other filters. We will now argue that a result like Theorem 1 does not hold anymore. We know that for an arbitrary period-P input (7) the output of $h_q(n)$ is nonzero if (8) holds for some k, l pair such that $H_q[k] \neq 0$ and $X[l] \neq 0$. If the coprimality $(k, q) = 1$ is true, then (8) would hold only for filter index q a divisor of P. If coprimality $(k, q) = 1$ is not true, it is possible that (8) holds for q not a divisor of P. In fact, we can rewrite (8) as

$$\frac{P}{q} = \frac{l}{k} \tag{11}$$

where $1 \leq k \leq q-1$ and $1 \leq l \leq P-1$. If there exist l and k such that the above is true then the lth harmonic of the input passes through the kth passband of $h_q(n)$ to produce a nonzero output. This of course is possible if and only if P and q are *not coprime*. We summarize this important conclusion as follows:

Theorem 2 (Comb filter banks): Consider a comb analysis filter bank $\{h_q(n)\}$ with $1 \leq q \leq N$ and let $x(n)$ be a period-P input signal with $1 \leq P \leq N$. Assume that all P-point DFT coefficients $X[l]$ of $x(n)$ are nonzero, and that $H_q[0] = 0$ for $q > 1$. Then the period-P input produces nonzero output if and only if the filter index q is *not* coprime to P. $\diamondsuit$

Summarizing, two things are true:

1. None of the filters whose index q is coprime to P can produce a nonzero output.

2. If q is not coprime to P then the output of $h_q(n)$ can be nonzero. In particular if q is *either a divisor or a multiple* of P, then the output of $h_q(n)$ can be nonzero. To be more precise, it is nonzero if and only if the lth harmonic of $x(n)$ is nonzero for some l such that $P/q = l/k$.

Let us consider some examples. Let $P = 6$. Then the filters which can have nonzero outputs correspond to $q = 2, 3, 4, 6, 8, 9, 10, 12, \ldots$ Again, for clearer understanding let us examine how filter $h_9(n)$ can produce a nonzero output. We have $P = 6, q = 9$. Since $6/9 = 2/3 = l/k$ it follows that the 2nd harmonic ($l = 2$) of the input signal passes through the 3rd passband ($k = 3$).

The main consequence of Theorem 2 is that by knowing the filter indices $\{q_i\}$ corresponding to nonzero outputs, we *cannot in general identify the period* P of the input signal $x(n)$, as demonstrated by the following examples: suppose $P = 2$. Then all the even numbered filters $h_q(n)$ can have nonzero output. If $P = 4$ or more generally $P = 2^m$ then the same is true. Thus the filter bank cannot distinguish between input periods $P = 2$ and $P = 2^m, m > 1$. Similarly, since P and q are coprime if and only if P^m and q are coprime, it follows that an arbitrary comb filter bank cannot distinguish between a period P signal and a period P^m signal ($m > 1$). More generally if $x(n)$ is a sum of periodic components, it is not possible to identify the hidden periods.

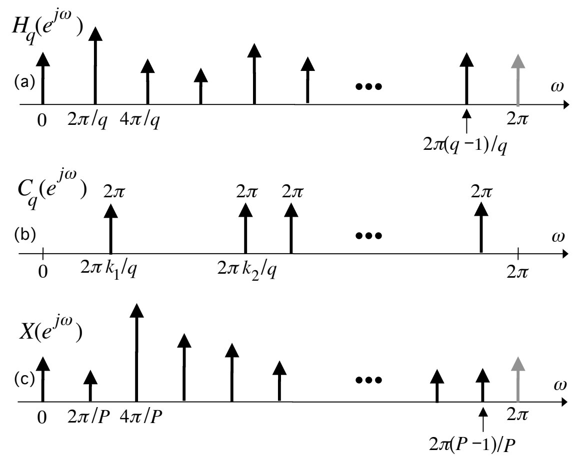

Fig. 1. (a) Frequency response of an arbitrary period-q filter $h_q(n)$, (b) the frequency response of the ideal (infinite duration) Ramanujan filter $c_q(n)$, and (c) the Fourier transform of a typical period-P signal.

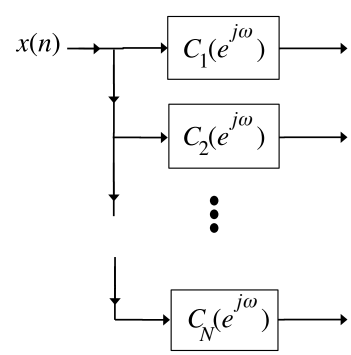

Fig. 2. The Ramanujan analysis filter bank.

4. IDENTIFYING PERIODS WITH RAMANUJAN FILTER BANKS

In view of Theorems 1 and 2 Ramanujan filter banks have special properties when compared with arbitrary comb filter banks. To show how this can be utilized to identify periods, we now prove the following:

Theorem 3 (The lcm property of Ramanujan filter banks): Consider a Ramanujan analysis filter bank $\{c_q(n)\}$ with $1 \leq q \leq N$ and let $x(n)$ be a period-P input signal with $1 \leq P \leq N$. Let nonzero outputs be produced by the subset of filters $c_{q_i}(n)$ with periods $q_1, q_2, \cdots, q_K$. Then the period P is given by

$$P = \text{lcm}\,\{q_1, q_2, \cdots, q_K\} \tag{12}$$

So the least common multiple of these filter indices $\{q_k\}$ reveals the period. $\diamondsuit$

Proof. We already know that nonzero outputs can only be produced by Ramanujan filters with periods $q|P$. So clearly P is a common multiple of $\{q_i\}$. Suppose it is not the lcm. Then let $L = \text{lcm}\,\{q_1, q_2, \cdots, q_K\}$ so that $P = LM$ for some integer $M > 1$. Now, $x(n)$ has the frequency components $2\pi l/P$ for $0 \leq l \leq P-1$ where $1 \leq P \leq N$. The filter $c_q(n)$ on the other hand passes the components $2\pi k/q$ with $1 \leq k \leq q$ with $(k, q) = 1$. But the set of rationals l/P where $1 \leq l \leq P$ and $1 \leq P \leq N$ is the same as the set of rationals k/q satisfying

$$(k, q) = 1, \ 1 \leq k \leq q, \quad 1 \leq q \leq N. \tag{13}$$

Thus, *every nonzero frequency component in* $x(n)$ *will be seen by one or other of the filters,* and will produce a nonzero output in that filter. So $x(n)$ can be represented as a linear combination of $W_{q_i}^{k_i n}$ where $(k_i, q_i) = 1$ and $1 \leq k_i \leq q_i$.

Now, since $L = \text{lcm}\,\{q_1, q_2, \cdots, q_K\}$ it follows that $L = q_i m_i$ for some integer m_i so that $W_{q_i} = W_L^{m_i}$. This shows that $x(n)$ is a linear combination of the form

$$x(n) = \sum_{m=0}^{L-1} \beta_m W_L^{mn} \tag{14}$$

That is, $x(n)$ has a period L (or a divisor of L) which is smaller than P. This a contradiction. $\bigtriangledown\bigtriangledown\bigtriangledown$

Some important remarks are in order:

1. *Large periods.* While any input period P in the range $1 \leq P \leq N$ can be identified by the Ramanujan filter bank, larger periods cannot in general be identified. For example if $x(n) = e^{j2\pi n/(N+1)}$ then its frequency $2\pi/(N+1)$ is not seen by any of the N filters, and we cannot identify it.

2. *Superposition of periodic components.* Consider the case where a signal is a sum of periodic components:

$$x(n) = x_1(n) + x_2(n) \tag{15}$$

where $x_1(n)$ and $x_2(n)$ have periods $P_i \leq N$. In this case the period of $x(n)$ can be as large as $\text{lcm}\,(P_1, P_2)$ and is generally larger than N. However, we can still identify the components P_1 and P_2 because the output of the filter bank is the sum of the outputs due to $x_1(n)$ and the outputs due to $x_2(n)$. In this case we can partition the set of filter indices $\{q_i\}$ which produce nonzero outputs into two classes: one class has lcm $= P_1$ and the other has lcm $= P_2$. The periods can therefore be identified. However, the individual component signals $x_1(n)$ and $x_2(n)$ cannot be uniquely identified because there may exist common harmonic frequencies. Thus, if $2\pi k_1/P_1 = 2\pi k_2/P_2$ for some k_1 and k_2, then this frequency component of $x(n)$ can be distributed arbitrarily between $x_1(n)$ and $x_2(n)$. So these components are not uniquely defined.

In practice the filters $c_q(n)$ in the Ramanujan filter bank can be replaced with causal FIR filters

$$C_q^{(l)}(z) = \sum_{n=0}^{lq-1} c_q(n) z^{-n} \tag{16}$$

Thus l consecutive periods of $c_q(n)$ are retained to form the impulse responses. The qth filter's duration is then ql samples, and is proportional to the period q. If the input signal $x(n)$ has periodic components that are localized in time, we can track this time-varying periodicity by observing the filter bank output. A large value of l implies that the filter responses approximate the Dirac functions of Fig. 1(a) more accurately. On the other hand, in order to obtain good time domain resolution for tracking varying periods, l has to be small. So, l is a tradeoff parameter between time and frequency resolutions. Since the filter duration ql is proportional to the period q, smaller periods can be localized accurately. It is possible to obtain variations of the FIR Ramanujan filter bank by incorporating *Hamming* or *Kaiser* windows in the time domain instead of rectangular windows as in (16), but we keep it simple here.

We now consider an example with $l = 5$ and consider an inverse chirp signal $x_c(t) = \sin(1/at)$. With $a = 0.01/\pi$, we sample this signal at the spacing $T = 0.005s$ to obtain $x(n) = x_c(nT)$ in the interval $1 \leq t \leq 6$. The signal $x(n)$ is shown in Fig. 3(a). A time-period plane plot is generated in Fig. 3(b) by using $x(n)$ as the input to the Ramanujan filter bank. The vertical axis is the period or filter index q and the horizontal axis is the integer time-index at the filter output. The intensity represents the average energy at the output of $C_q^{(l)}(z)$, within a sliding window (the details are similar to the examples in [15]). Each $c_q^{(l)}(n)$ is a linear phase filter with group delay $(ql-1)/2$. Since these delays are different for different q, the outputs are appropriately shifted so that all filters have identical group delays, before obtaining the plot of Fig. 3(b).

It is clear that the filter bank tracks the chirp beautifully along time. Similar examples can be found in [15] where a comparison is made with spectrograms based on the short-time Fourier transform. Since the STFT has a fixed time window it cannot resolve different periodicities uniformly. The Ramanujan filter bank, on the other hand, resolves the various periods in the time domain and gives localization information because the qth filter length is proportional to the period q.

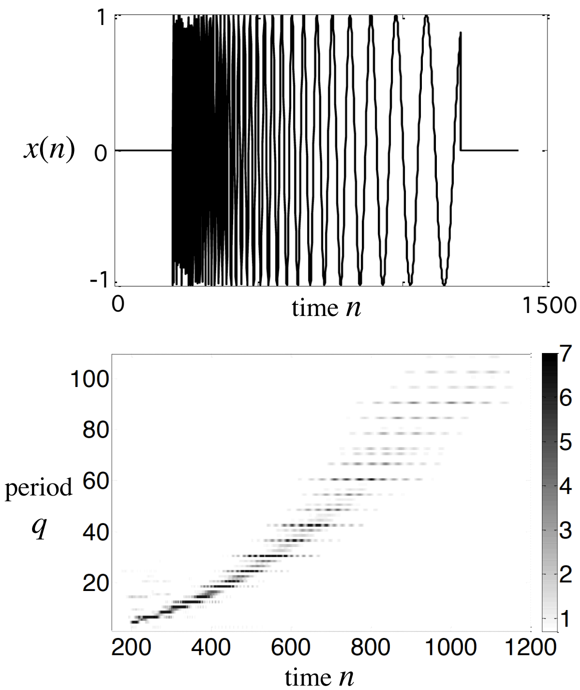

Fig. 3. A time-period plane plot obtained from the Ramanujan filter bank, for a inverse-chirp signal.

5. CONCLUDING REMARKS

We showed that arbitrary comb filter banks do not enjoy the period discrimination property of Ramanujan filter banks, as shown in Theorems 1, 2, and 3. The question which arises is: what is the most general class of filter banks $\{s_q(n)\}$ satisfying properties similar to Theorem 1 (hence Theorem 3)? It can be shown that the necessary and sufficient condition is that each filter $s_q(n)$ have a q-point DFT such that

$$S_q[k] = \begin{cases} \neq 0 & \text{for } (k,q)=1 \\ 0 & \text{for } (k,q)\neq 1 \end{cases} \tag{17}$$

Clearly $c_q(n)$ is a special case of such filters because of (3). A sequence $s_q(n)$ satisfying the above property belongs to a space called the *Ramanujan-subspace* $\mathcal{S}_q$, defined in [17], and elaborated further in [19]. Thus, as long as the filter $s_q(n)$ belongs to $\mathcal{S}_q$ and has nonzero DFT for $(k,q)=1$, the filter bank is suitable for identifying and separating periods. The fact that the nonzero DFT coefficients $S_q[k]$ can have values other than q (unlike $C_q[k]$) gives more freedom in the design of these filter banks especially in the presence of noise.

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