The Kesten-Stigum Reconstruction Bound Is Tight for Roughly Symmetric Binary Channels

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Abstract

We establish the exact threshold for the reconstruction problem for a binary asymmetric channel on the *b*-ary tree, provided that the asymmetry is sufficiently small. This is the first exact reconstruction threshold obtained in roughly a decade. We discuss the implications of our result for Glauber dynamics, phylogenetic reconstruction, and so-called "replica symmetry breaking" in spin glasses and random satisfiability problems.

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1 Introduction

Let

$$M = \frac{1}{2} \left[\begin{pmatrix} 1+\theta & 1-\theta\\ 1-\theta & 1+\theta \end{pmatrix} + \delta \begin{pmatrix} -1 & 1\\ -1 & 1 \end{pmatrix} \right],\tag{1}$$

be a binary asymmetric channel with second eigenvalue θ and T_b be a complete *b*-ary tree. The "reconstruction problem" is the problem of determining the state of the root, given the distribution of the Markov chain on level *n* of the tree, as *n* gets larger and larger (a precise definition is given below). For the symmetric binary channel ($\delta = 0$), it was known since 1995 that the reconstruction problem is solvable if and only if $b\theta^2 > 1$. For all other channels, it was also known and easy to prove that $b\theta^2 > 1$ implies solvability, but *exact non-solvability* results were not known. Here we show that this bound is tight provided that *M* is close enough to symmetric—i.e., we show that the reconstruction problem for *M* on *T* is not solvable if $b\theta^2 \le 1$ and $|\delta|$ is sufficiently small.

The reconstruction problem is intimately related to mixing of Glauber dynamics and to phylogenetic reconstruction. Moreover, it was recently claimed that the reconstruction problem corresponds to the "replica symmetry broken" solution of the spin glass on the tree. replica symmetry breaking is a central notion in the statistical physics theory of spin glasses and random satisfiability problems. We discuss potential applications of our results in these different areas.

1.1 Definitions and Main Result

Let $T = (V, E, \rho)$ be a tree T with nodes V, edges E and root $\rho \in V$. We direct all edges away from the root, so that if e = (x, y) then x is on the path connecting ρ to y. Let $d(\cdot, \cdot)$ denote the graph-metric distance on T, and $L_n = \{v \in V : d(\rho, v) = n\}$ be the n^{th} level of the tree. For $x \in V$ and $e = (y, z) \in E$, we denote $|x| = d(\rho, x), d(x, (y, z)) = \max\{d(x, y), d(x, z)\}$, and $|e| = d(\rho, e)$. The *b*-ary tree is the infinite rooted tree where each vertex has exactly b children.

A Markov chain on the tree T is a probability measure defined on the state space \mathcal{C}^V , where C is a finite set. Assume first that T is finite and, for each edge e of T, let $M^e = (M^e_{i,j})_{i,j\in\mathcal{C}}$ be a stochastic matrix. In this case the probability measure defined by $(M^e : e \in E)$ on T is given by

$$\overline{\mu}_{\ell}(\sigma) = \mathbf{1}_{\{\sigma(\rho)=\ell\}} \prod_{(x,y)\in E} M^{(x,y)}_{\sigma(x),\sigma(y)}.$$
(2)

In other words, the root state $\sigma(\rho)$ satisfies $\sigma(\rho) = \ell$ and then each vertex iteratively chooses its state from the one of its parent by an application of the Markov transition rule given by M^e (and all such applications are independent). We can define the measure $\overline{\mu}_{\ell}$ on an infinite tree as well, by Kolmogorov's extension theorem, but we will not need chains on infinite trees in this paper (see [8] for basic properties of Markov chains on trees).

Instead, for an infinite tree T, we let $T_n = (V_n, E_n, \rho)$, where $V_n = \{x \in V : d(x, \rho) \le n\}$, $E_n = \{e \in E : d(e, \rho) \le n\}$ and define $\overline{\mu}_{\ell}^n$ by (2) for T_n . We are particularly interested in the distribution of the states $\sigma(x)$ for $x \in L_n$, the set of leaves in T_n . This distribution, denoted by μ_k^n , is the projection of $\overline{\mu}_k^n$ on \mathcal{C}^{L_n} given by

$$\mu_k^n(\sigma) = \sum_{\overline{\sigma}} \{ \overline{\mu}_k^n(\overline{\sigma}) : \overline{\sigma} | L_n = \sigma \}.$$
(3)

Recall that for distributions μ and ν on the same space Ω the total variation distance between μ and ν is

$$D_V(\mu,\nu) = \frac{1}{2} \sum_{\sigma \in \Omega} |\mu(\sigma) - \nu(\sigma)|.$$
(4)

Definition 1 (Reconstructibility) The reconstruction problem for the infinite tree \mathcal{T} and $(M^e : e \in E)$ is **solvable** if there exist $i, j \in C$ for which

$$\liminf_{n \to \infty} D_V(\mu_i^n, \mu_j^n) > 0.$$
⁽⁵⁾

When $M^e = M$ for all e, we say that the reconstruction problem is solvable for T and M.

We will be mostly interested in binary channels, i.e., transition matrices on the state space $\{\pm\}$. In this case, the definition above says that the reconstruction problem is solvable if

$$\liminf_{n \to \infty} D_V(\mu_+^n, \mu_-^n) > 0.$$
(6)

Our main result is the following:

Theorem 1 (Main Result) For all $b \ge 2$, there exists a $\delta_0 > 0$ such that for all $|\delta| \le \delta_0$, the reconstruction problem for M on the b-ary tree T_b is not solvable if $b\theta^2 \le 1$.

1.2 Previous Results

The study of the reconstruction problem began in the seventies [23, 9] when the problem was introduced in terms of the extremality of the free Gibbs measure on the tree. In [9] it is shown that the reconstruction problem for the binary symmetric channel (equation (1) where $\delta = 0$) on the binary tree is solvable when $2\theta^2 > 1$. This in fact follows from a previous work [12] which implies that for any Markov chain M, the reconstruction problem on the *b*-ary tree is solvable if $b\theta^2 > 1$ where θ is the second largest eigenvalue of M in absolute value.

Proving non-reconstructibility turned out to be harder. While coupling arguments easily yield non-reconstruction, these arguments are typically not tight. A natural way to try to prove non-reconstructibility is to analyze recursions 1) in terms of random variables each of whose values is the expectation of the chain at a vertex, given the state at the leaves of the subtree below it, 2) in terms of ratios of such probabilities, or 3) in terms of log-likelihood ratios of such probabilities. Such recursions were analyzed for a closely related model in [3]. Both the reconstruction model and the model analyzed in [3] deal with the correlation between the n^{th} -level and the root. However, while in the reconstruction problem, the two random variables are generated according to the Markov model on the tree, in [3] the nodes at level n are set to have an i.i.d. distribution and the root has the conditional distribution thus induced.

In spite of this important difference, the two models are closely related. In particular, in [3] it is shown that for the binary tree, the correlation between level n and the root decays if and only if $2\theta^2 \le 1$. Building on the techniques of [3] it was finally shown in [2] that the reconstruction problem for the binary symmetric channel is solvable if and only if $2\theta^2 > 1$. This result was later reproven in various ways [5, 10, 1, 15].

The elegance of the threshold $b\theta^2 = 1$ raised the hope that it is the threshold for reconstruction for general channels. However, previous attempts to generalize any of the proofs to other channels have failed. Moreover in [18] it was shown that for asymmetric binary channels and for symmetric channels on large alphabets the reconstruction problem is solvable in cases where $b\theta^2 < 1$. In fact [18] contains an example of a channel satisfying $\theta = 0$ for which the reconstruction problem is solvable. On the other hand, in [20, 11] it is shown that the threshold $b\theta^2 = 1$ is the threshold for two variants of the reconstruction problem, "census reconstruction" and "robust reconstruction".

The results above led some to believe that "reconstruction" unlike its siblings "census reconstruction" and "robust reconstruction" is an extremely sensitive property and that the threshold $b\theta^2 = 1$ is tight only for the binary symmetric channel. This conceptual picture was shaken by recent results in the theoretical

physics literature [17] where using variational principles developed in the context of "replica symmetry breaking" it is suggested that the bound $b\theta^2 = 1$ is tight for symmetric channels on 3 and (maybe) 4 letters.

In Theorem 1 we give the first tight threshold for the reconstruction problem for channels other than binary symmetric channels. We show that for asymmetric channels that are close to symmetric, the Kesten-Stigum bound $b\theta^2 = 1$ is tight for reconstruction. Our proof builds on ideas from [3, 2, 5, 22] and is extremely simple. In addition to giving a new result for the asymmetric channel, our proof also provides a much simpler proof of the previously known result for the binary *symmetric* channel.

1.3 The Reconstruction Problem in Mixing, Phylogeny and Replicas

Mixing of Markov Chains. One of the main themes at the intersection of statistical physics and theoretical computer science in recent years has been the study of connections between spatial and temporal mixing. It is widely accepted that spatial mixing and temporal mixing of dynamics go hand in hand though this was proven only in restricted settings.

In particular, the spatial mixing condition is usually stated in terms of uniqueness of Gibbs measures. However, as shown in [1], this spatial condition is too strong. In particular, it is shown in [1] that the spectral gap of continuous-time Glauber dynamics for the Ising model with no external field and no boundary conditions on the *b*-ary tree is $\Omega(1)$ whenever $b\theta^2 < 1$. This should be compared with the uniqueness condition on the tree given by $b\theta < 1$. In [15] this result is extended to the log Sobolev constant. In [15] it is also shown that for measures on trees, a super-linear decay of point-to-set correlations implies an $\Omega(1)$ spectral gap for the Glauber dynamics with free boundary conditions.

Thus our results not only give the exact threshold for reconstructibility. They also yield an exact threshold for mixing of Glauber dynamics on the tree for Ising models with a small external field. The details are omitted from this extended abstract.

Phylogenetic Reconstruction. Phylogenetic reconstruction is a major task of systematic biology [6]. It was recently shown in [4] that for binary symmetric channels, also called CFN models in evolutionary biology, the sampling efficiency of phylogenetic reconstruction is determined by the reconstruction threshold. Thus if for all edges of the tree it holds that $2\theta^2 > 1$ the tree can be recovered efficiently from $O(\log n)$ samples. If $2\theta^2 < 1$, then [19] implies that $n^{\Omega(1)}$ samples are needed. In fact, the proof of the lower bound in [19] implies the lower bound $n^{\Omega(1)}$ whenever the reconstruction problem is *exponentially* unsolvable. In other words, if $\lim \inf_{r\to\infty} D_V(\mu_+^r, \mu_-^r) = \exp(-\Omega(r))$ then a lower bound of $n^{\Omega(1)}$ holds for phylogenetic reconstruction.

Thus, our results here imply $n^{\Omega(1)}$ lower bounds for phylogenetic reconstruction for asymmetric channels such that $2\theta^2 < 1$ and $|\delta| < \delta_0$. The details are omitted from this extended abstract. It is natural to conjecture that this is tight and that if $2\theta^2 > 1$ then phylogenetic reconstruction may be achieved with $O(\log n)$ sequences.

Replica Symmetry Breaking. The replica and cavity methods were invented in the theoretical physics literature to solve Ising spin glass problems on the complete graph—the so-called Sherrington-Kirkpatrick model. These methods, while not mathematically rigorous, led to numerous predictions on the spin glass and other models on dense graphs, a few of which were proved many years later. When applied to random satisfiability problems, which turn out to be equivalent to dilute spin glasses—i.e., spin glasses on sparse random graphs—these methods led to the empirically best algorithms for solving random satisfiability problems [16, 21].

A central concept in this theory is the notion of a "glassy phase" of the spin glass measure. In the glassy

phase, the distribution on the random graphs decomposes into an exponential number of "lumps". One of the standard techniques for determining the glassy phase is via "replica symmetry breaking". Moreover, there are certain glassy phases for which the replica symmetry breaking is relatively simple—those which are said to have "one-step replica symmetry breaking"; and others in which the replica symmetry breaking is more complicated—those with so-called "full replica symmetry breaking".

In a recent paper [17] it is claimed that the parameters for which a "glassy phase occurs" are exactly the same as the parameters for which the reconstruction problem is not solvable. More formally, for determining if the glassy phase occurs for random (b+1)-regular graphs and Gibbs measures with some parameters, one needs to check if the reconstruction problem for the *b*-ary tree and associated parameters is solvable or not.

Furthermore, it is claimed in [17] that the reconstruction problem determines the type of glassy phase as follows. Mezard and Montanari predict that one-step replica symmetry breaking occurs exactly when when the Kesten-Stigum bound is not equal to the reconstruction bound; otherwise full replica symmetry breaking occurs.

Thus our results proved here, in conjunction with the theoretical physics predictions of [17], suggest the existence of two types of glassy phases for spin systems on random graphs. It is an interesting challenge to state these predictions in a rigorous mathematical way and to prove or disprove them.

2 Preliminaries and General Result

For convenience, we sometimes write the channel

$$M = \left(\begin{array}{cc} 1 - \varepsilon^+ & \varepsilon^+ \\ 1 - \varepsilon^- & \varepsilon^- \end{array}\right).$$

Note first that the stationary distribution $\pi = (\pi_+, \pi_-)$ of M is given by

$$\pi_{+} = \frac{1 - \varepsilon^{-}}{1 - \theta} = \frac{1}{2} - \frac{\delta}{2(1 - \theta)}, \qquad \pi_{-} = \frac{\varepsilon^{+}}{1 - \theta} = \frac{1}{2} + \frac{\delta}{2(1 - \theta)}.$$

In particular, this expression implies that the stationary distribution depends only on the ratio $\delta/(1-\theta)$. Or put differently, each two of the parameters π_+ , δ and θ determine the third one uniquely. Note also that

$$\theta = \varepsilon^{-} - \varepsilon^{+}, \qquad \pi_{-} - \pi_{+} = \frac{\delta}{1 - \theta}$$

Without loss of generality, we assume throughout that $\pi_{-} \ge \pi_{+}$ or equivalently that $\delta \ge 0$. (Note that δ can be made negative by inverting the role of + and -.) Below, we will use the notation

$$\pi_{-/+} \equiv \pi_{-}\pi_{+}^{-1}, \qquad \Delta \equiv \pi_{-/+} - 1.$$

2.1 General Trees

In this section, we state our Theorem in a more general setting. Namely, we consider general rooted trees where different edges are equipped with different transition matrices—all having the same stationary distribution $\pi = (\pi_+, \pi_-)$. In other words, we consider a general infinite rooted tree $\mathcal{T} = (V, E)$ equipped with a function $\theta : E \to [-1, 1]$ such that the edge *e* of the tree is equipped with the matrix M^e with $\theta(M^e) = \theta(e)$ and the stationary distribution of M^e is (π_+, π_-) .

In this general setting the notion of degree is extended to the notion of *branching number*. In [7], Furstenberg introduced the Hausdorff dimension of a tree. Later, Lyons [13, 14] showed that many probabilistic properties of the tree are determined by this number which he named the branching number. For our purposes it is best to define the branching number via cutsets.

Definition 2 (Cutsets) A cutset S for a tree T rooted at ρ , is a finite set of vertices separating ρ from ∞ . In other words, a finite set S is a cutset if every infinite self avoiding path from ρ intersects S. An antichain or minimal cutset is a cutset that does not have any proper subset which is also a cutset.

Definition 3 (Branching Number) Consider a rooted tree $\mathcal{T} = (V, E, \rho)$ equipped with an edge function $\theta : E \to [-1, 1]$. For each vertex $v \in V$ we define

$$\eta(x) = \prod_{e \in \texttt{path}(\rho, x)} \theta^2(e),$$

where $path(\rho, x)$ is the set of edges on the unique path between ρ and x in \mathcal{T} . The branching number $br(\mathcal{T}, \theta)$ of (\mathcal{T}, θ) is defined as

$$\operatorname{br}(\mathcal{T},\theta) = \inf\left\{\lambda > 0 : \inf_{\operatorname{cutsets}} \sum_{X \in S} \eta(X) \lambda^{-|X|} = 0\right\}.$$

In our main result we show

Theorem 2 (Reconstructibility on General Trees) Let $0 \le \theta_0 < 1$. Then there exists $\delta_0 > 0$ such that, for all distributions $\pi = (\pi_+, \pi_-)$ with $\max\{|\delta(\pi, \theta_0)|, |\delta(\pi, -\theta_0)|\} < \delta_0$ and for all trees (\mathcal{T}, θ) with $\sup_e |\theta(e)| \le \theta_0$ and $\operatorname{br}(\mathcal{T}, \theta) \le 1$, the reconstruction problem is not solvable.

It is easy to see that the conditions of Theorem 2 hold for T_b if $\theta(e) = \theta$ for all e and $b\theta^2 \leq 1$.

2.2 Magnetization

Let T be a finite tree rooted at x with edge function θ . Let σ be the leaf states generated by the Markov chain on (T, θ) with stationary distribution (π_+, π_-) . We denote by $\mathbb{P}_T^+, \mathbb{E}_T^+$ (resp. $\mathbb{P}_T^-, \mathbb{E}_T^-$, and $\mathbb{P}_T, \mathbb{E}_T$) the probability/expectation operators with respect to the measure on the leaves of T obtained by conditioning the root to be + (resp. -, and stationary). With a slight abuse of notation, we also write $\mathbb{P}_T[+ | \sigma]$ for the probability that the state at the the root of T is + given state σ at the leaves. The main random variable we consider is the *weighted magnetization of the root*

$$X = \pi_{-}^{-1} \left[\pi_{-} \mathbb{P}_{T}[+ | \sigma] - \pi_{+} \mathbb{P}_{T}[- | \sigma] \right].$$

Note that the weights are chosen to guarantee

$$\mathbb{E}_T[X] = \pi_-^{-1} \left[\pi_- \pi_+ - \pi_+ \pi_- \right] = 0,$$

while the factor π_{-}^{-1} is such that $|X| \leq 1$ with probability 1.

Note that for any random variable depending only on the leaf states, $f = f(\sigma)$, we have $\pi_+ \mathbb{E}_T^+[f] + \pi_- \mathbb{E}_T^-[f] = \mathbb{E}_T[f]$, so that in particular

$$\pi_{+}\mathbb{E}_{T}^{+}[X] + \pi_{-}\mathbb{E}_{T}^{-}[X] = \mathbb{E}_{T}[X] = 0, \qquad \pi_{+}\mathbb{E}_{T}^{+}[X^{2}] + \pi_{-}\mathbb{E}_{T}^{-}[X^{2}] = \mathbb{E}_{T}[X^{2}].$$

We define the following analogues of the Edwards-Anderson order parameter for spin glasses on trees rooted at x

$$\bar{x} = \mathbb{E}_T[X^2], \qquad \bar{x}_+ = \mathbb{E}_T^+[X^2], \qquad \bar{x}_- = \mathbb{E}_T^-[X^2].$$



Figure 1: A finite tree T.

Now suppose \mathcal{T} is an infinite tree rooted at ρ with edge function θ . Let $T_n = (V_n, E_n, x_n)$, where $V_n = \{u \in V : d(u, \rho) \le n\}$, $E_n = \{e \in E : d(e, \rho) \le n\}$, and x_n is identified with ρ . It is not hard to see that non-reconstructibility on (T, θ) is equivalent in our notation to

$$\limsup_{n \to \infty} \bar{x}_n = 0.$$

(Note that the total variation distance is monotone in the cutsets. Therefore the limit goes to 0 with the levels if and only if there exists a sequence of cutsets for which it goes to 0.)

2.3 Expectations

Fix a stationary distribution $\pi = (\pi_+, \pi_-)$. Let T = (V, E) be a finite tree rooted at x with edge function $\{\theta(f), f \in E\}$ and weighted magnetization at the root X. Let y be a child of x and T' be the subtree of T rooted at y. Let Y be the weighted magnetization at the root of T'. See Figure 1. Denote by σ the leaf states of T and let σ' be the restriction of σ to the leaves of T'. Assume the channel on e = (x, y) is given by

$$M^{e} = \begin{pmatrix} 1 - \varepsilon^{+} & \varepsilon^{+} \\ 1 - \varepsilon^{-} & \varepsilon^{-} \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \begin{pmatrix} 1 + \theta & 1 - \theta \\ 1 - \theta & 1 + \theta \end{pmatrix} + \delta \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{bmatrix}.$$

We collect in the next lemmas a number of useful identities.

Lemma 1 (Radon-Nikodym Derivative) The following hold:

$$\frac{\mathrm{d}\mathbb{P}_T^+}{\mathrm{d}\mathbb{P}_T} = 1 + \pi_{-/+}X, \qquad \frac{\mathrm{d}\mathbb{P}_T^-}{\mathrm{d}\mathbb{P}_T} = 1 - X,$$

$$\mathbb{E}_{T}^{+}[X] = \pi_{-/+}\mathbb{E}_{T}[X^{2}], \qquad \mathbb{E}_{T}^{-}[X] = -\mathbb{E}_{T}[X^{2}].$$

Proof: Note that

$$X = \pi_{-}^{-1} \left[\pi_{-} \mathbb{P}_{T}[+ | \sigma] - \pi_{+} \mathbb{P}_{T}[- | \sigma] \right] = \pi_{-}^{-1} \left[\mathbb{P}_{T}[+ | \sigma] - \pi_{+} \right] = \pi_{-/+}^{-1} \left[\frac{\mathbb{P}_{T}[+ | \sigma]}{\pi_{+}} - 1 \right],$$

so that

$$\frac{\mathrm{d}\mathbb{P}_T^+}{\mathrm{d}\mathbb{P}_T} = \frac{\mathbb{P}_T[+\mid\sigma]}{\pi_+} = 1 + \pi_{-/+}X.$$

Likewise,

$$\frac{\mathrm{d}\mathbb{P}_T^-}{\mathrm{d}\mathbb{P}_T} = \frac{\mathbb{P}_T[-\mid\sigma]}{\pi_-} = 1 - X.$$

Then, it follows that

$$\mathbb{E}_T^+[X] = \mathbb{E}_T\left[X\left(1 + \pi_{-/+}X\right)\right] = \pi_{-/+}\mathbb{E}_T[X^2],$$

and similarly for $\mathbb{E}_T^-[X]$.

Lemma 2 (Child Magnetization) We have,

$$\mathbb{E}_T^+[Y] = \theta \mathbb{E}_{T'}^+[Y], \qquad \mathbb{E}_T^-[Y] = \theta \mathbb{E}_{T'}^-[Y],$$

and

$$\mathbb{E}_{T}^{+}[Y^{2}] = (1-\theta)\mathbb{E}_{T'}[Y^{2}] + \theta\mathbb{E}_{T'}^{+}[Y^{2}], \qquad \mathbb{E}_{T}^{-}[Y^{2}] = (1-\theta)\mathbb{E}_{T'}[Y^{2}] + \theta\mathbb{E}_{T'}^{-}[Y^{2}].$$

Proof: By the Markov property, we have

$$\mathbb{E}_{T}^{+}[Y] = (1-\varepsilon^{+})\mathbb{E}_{T'}^{+}[Y] + \varepsilon^{+}\mathbb{E}_{T'}^{-}[Y] = \left[(1-\varepsilon^{+}) - \varepsilon^{+}\frac{\pi_{+}}{\pi_{-}}\right]\mathbb{E}_{T'}^{+}[Y] = \left[(1-\varepsilon^{+}) - (1-\varepsilon^{-})\right]\mathbb{E}_{T'}^{+}[Y]$$
$$= \theta\mathbb{E}_{T'}^{+}[Y],$$

and similarly for $\mathbb{E}_T^-[Y]$.

Also,

$$\mathbb{E}_{T}^{+}[Y^{2}] = (1 - \varepsilon^{+})\mathbb{E}_{T'}^{+}[Y^{2}] + \varepsilon^{+}\mathbb{E}_{T'}^{-}[Y^{2}] = (1 - \varepsilon^{+})\mathbb{E}_{T'}^{+}[Y^{2}] + \frac{\varepsilon^{+}}{\pi_{-}}(\mathbb{E}_{T'}[Y^{2}] - \pi_{+}\mathbb{E}_{T'}^{+}[Y^{2}])$$
$$= \theta\mathbb{E}_{T'}^{+}[Y^{2}] + (1 - \theta)\mathbb{E}_{T'}[Y^{2}],$$

where we have used the calculation above. A similar expression holds for $\mathbb{E}_T^-[Y^2]$.

3 Tree Operations

To derive moment recursions, the basic graph operation we perform is the following *Add-Merge* operation. Fix a stationary distribution $\pi = (\pi_+, \pi_-)$. Let T' (resp. T'') be a finite tree rooted at y (resp. z) with edge function θ' (resp. θ''), leaf state σ' (resp. σ''), and weighted magnetization at the root Y (resp. Z). Now add an edge $e = (\hat{y}, z)$ with edge value $\theta(e) = \theta$ to T'' to obtain a new tree \hat{T} . Then merge \hat{T} with T' by identifying $y = \hat{y}$ to obtain a new tree T. To avoid ambiguities, we denote by x the root of T and X the magnetization of the root of T (where we identify the edge function on T with those on T', T'', and e). We let $\sigma = (\sigma', \sigma'')$ be the leaf state of T. See Figure 2. Let also \hat{Y} be the magnetization of the root on \hat{T} . Assume

$$M^e = \left(\begin{array}{cc} 1 - \varepsilon^+ & \varepsilon^+ \\ 1 - \varepsilon^- & \varepsilon^- \end{array}\right)$$

We first analyze the effect of adding an edge and merging subtrees on the magnetization variable.

Lemma 3 (Adding an Edge) With the notation above, we have

$$\widehat{Y} = \theta Z.$$



Figure 2: Tree T after the Add-Merge of T' and T''. The dashed subtree is \hat{T} .

Proof: Note that by Bayes' rule, the Markov property, and Lemma 1,

$$\begin{split} \widehat{Y} &= \pi_{+} \sum_{\gamma=+,-} \gamma \, \frac{\mathbb{P}_{\widehat{T}}[\gamma \mid \sigma'']}{\pi^{\gamma}} = \pi_{+} \sum_{\gamma=+,-} \gamma \, \frac{\mathbb{P}_{\widehat{T}}[\sigma'' \mid \gamma]}{\mathbb{P}_{\widehat{T}}[\sigma'']} \\ &= \pi_{+} \frac{\mathbb{P}_{T''}[\sigma'']}{\mathbb{P}_{\widehat{T}}[\sigma'']} \sum_{\gamma=+,-} \gamma \, \left[(1 - \varepsilon^{\gamma}) \frac{\mathbb{P}_{T''}[\sigma''] + 1}{\mathbb{P}_{T''}[\sigma'']} + \varepsilon^{\gamma} \frac{\mathbb{P}_{T''}[\sigma'']}{\mathbb{P}_{T''}[\sigma'']} \right] \\ &= \pi_{+} \sum_{\gamma=+,-} \gamma \, \left[(1 - \varepsilon^{\gamma}) \left(1 + \pi_{-/+}Z \right) + \varepsilon^{\gamma} \left(1 - Z \right) \right], \end{split}$$

where we have used $\mathbb{P}_{\widehat{T}}[\sigma''] = \mathbb{P}_{T''}[\sigma'']$. We now compute the expression in square brackets. We have

$$(1 - \varepsilon^{\gamma})\left(1 + \pi_{-/+}Z\right) + \varepsilon^{\gamma}\left(1 - Z\right) = 1 + \pi_{-}Z\left[\frac{1 - \varepsilon^{\gamma}}{\pi_{+}} - \frac{\varepsilon^{\gamma}}{\pi_{-}}\right]$$

For $\gamma = +$, we get

$$\frac{1-\varepsilon^+}{\pi_+} - \frac{\varepsilon^+}{\pi_-} = (1-\theta) \left[\frac{1-\varepsilon^+}{1-\varepsilon^-} - 1 \right] = (1-\theta) \left[\frac{\varepsilon^- - \varepsilon^+}{1-\varepsilon^-} \right] = \frac{\theta}{\pi_+}.$$

A similar calculation for the - case gives for $\gamma=+,-$

$$(1-\varepsilon^{\gamma})\left(1+\pi_{-/+}Z\right)+\varepsilon^{\gamma}\left(1-Z\right)=1+\gamma\theta\pi_{-}\pi_{\gamma}^{-1}Z.$$

Plugging above gives $\widehat{Y} = \theta Z$.

Lemma 4 (Merging Subtrees) With the notation above, we have

$$X = \frac{Y + \widehat{Y} + \Delta Y \widehat{Y}}{1 + \pi_{-/+} Y \widehat{Y}}.$$

The same expression holds for a general \hat{T} .

Proof: By Bayes' rule, the Markov property, and Lemma 1, we have

$$\begin{aligned} X &= \pi_{+} \sum_{\gamma=+,-} \gamma \, \frac{\mathbb{P}_{T}[\gamma \mid \sigma]}{\pi^{\gamma}} = \pi_{+} \sum_{\gamma=+,-} \gamma \, \frac{\mathbb{P}_{T}[\sigma \mid \gamma]}{\mathbb{P}_{T}[\sigma]} = \pi_{+} \frac{\mathbb{P}_{T'}[\sigma'] \mathbb{P}_{\widehat{T}}[\sigma'']}{\mathbb{P}_{T}[\sigma]} \sum_{\gamma=+,-} \gamma \, \frac{\mathbb{P}_{T'}[\sigma' \mid \gamma]}{\mathbb{P}_{T'}[\sigma']} \frac{\mathbb{P}_{\widehat{T}}[\sigma'']}{\mathbb{P}_{\widehat{T}}[\sigma'']} \\ &= \pi_{+} \frac{\mathbb{P}_{T'}[\sigma'] \mathbb{P}_{\widehat{T}}[\sigma'']}{\mathbb{P}_{T}[\sigma]} \sum_{\gamma=+,-} \gamma \, \left[1 + \gamma \pi_{-} \pi_{\gamma}^{-1} (Y + \widehat{Y}) + (\pi_{-} \pi_{\gamma}^{-1})^{2} Y \widehat{Y} \right]. \end{aligned}$$

Similarly, we have

$$\frac{\mathbb{P}_{T}[\sigma]}{\mathbb{P}_{T'}[\sigma']\mathbb{P}_{\widehat{T}}[\sigma'']} = \frac{1}{\mathbb{P}_{T'}[\sigma']\mathbb{P}_{\widehat{T}}[\sigma'']} \sum_{\gamma=+,-} \pi^{\gamma} \mathbb{P}_{T}[\sigma \mid \gamma] = \sum_{\gamma=+,-} \pi_{\gamma} \left[1 + \gamma \pi_{-} \pi_{\gamma}^{-1} (Y + \widehat{Y}) + (\pi_{-} \pi_{\gamma}^{-1})^{2} Y \widehat{Y} \right].$$

Note that

$$\sum_{\gamma=+,-} \gamma \left[1 + \gamma \pi_{-} \pi_{\gamma}^{-1} (Y + \widehat{Y}) + (\pi_{-} \pi_{\gamma}^{-1})^{2} Y \widehat{Y} \right] = \pi_{+}^{-1} (Y + \widehat{Y}) + \pi_{+}^{-2} (\pi_{-} - \pi_{+}) Y \widehat{Y},$$

where we have used

$$\pi_{-}^{2} - \pi_{+}^{2} = (\pi_{-} - \pi_{+})(\pi_{-} + \pi_{+}) = \pi_{-} - \pi_{+}.$$

Similarly,

$$\sum_{\gamma=+,-} \pi_{\gamma} \left[1 + \gamma \pi_{-} \pi_{\gamma}^{-1} (Y + \widehat{Y}) + (\pi_{-} \pi_{\gamma}^{-1})^{2} Y \widehat{Y} \right] = 1 + \pi_{-} \pi_{+}^{-1} Y \widehat{Y}.$$

The result follows.

4 Symmetric Channels On Regular Trees

As a warm-up, we start by analyzing the binary symmetric channel on the infinite *b*-ary tree. Our proof is arguably the simplest proof to date of this result. The same proof structure will be used in the general case.

Theorem 3 (Symmetric Channel. See [2, 5, 10, 20, 11, 1, 15].) Let M be a transition matrix with $\delta = 0$ and $b\theta^2 \leq 1$. Let T be the infinite b-ary tree. Then, the reconstruction problem on (T, M) is not solvable.

Proof: Consider again the setup of Section 3. Note first that, by Lemmas 1, 2 and 3, we have

$$\mathbb{E}_{\widehat{T}}[\widehat{Y}^2] = \mathbb{E}_{\widehat{T}}^+[\widehat{Y}] = \theta \mathbb{E}_{\widehat{T}}^+[Z] = \theta^2 \mathbb{E}_{T''}^+[Z] = \theta^2 \mathbb{E}_{T''}[Z^2],\tag{7}$$

where we have used the fact that $\pi_{-/+} = 1$ when $\delta = 0$ (although note that it is not needed). In other words, adding an edge to the root of a tree and re-rooting at the new vertex has the effect of multiplying the second moment of the magnetization by θ^2 . Now consider the *Add-Merge* operation defined in Section 3. Using the expansion

$$\frac{1}{1+r} = 1 - r + \frac{r^2}{1+r},\tag{8}$$

the inequality $|X| \leq 1$, and Lemma 4, we get

$$X = Y + \hat{Y} - Y\hat{Y}(Y + \hat{Y}) + Y^{2}\hat{Y}^{2}X \le Y + \hat{Y} - Y\hat{Y}(Y + \hat{Y}) + Y^{2}\hat{Y}^{2}.$$
(9)

Note that from Lemmas 1 and 2, we have

$$\mathbb{E}_{T}^{+}[X] = \bar{x}, \quad \mathbb{E}_{T}^{+}[Y] = \mathbb{E}_{T}^{+}[Y^{2}] = \bar{y}, \quad \mathbb{E}_{T}^{+}[\widehat{Y}] = \mathbb{E}_{T}^{+}[\widehat{Y}^{2}] = \theta^{2}\bar{z},$$

where we have used that $\bar{y}_+ = \bar{y}_- = \bar{y}$ and $\bar{z}_+ = \bar{z}_- = \bar{z}$ by symmetry. Taking \mathbb{E}_T^+ on both sides of (9), we get

$$\bar{x} \leq \bar{y} + \theta^2 \bar{z} - \theta^2 \bar{y} \bar{z} - \theta^2 \bar{y} \bar{z} + \theta^2 \bar{y} \bar{z} = \bar{y} + \theta^2 \bar{z} - \theta^2 \bar{y} \bar{z}.$$

Now, let $T_n = (V_n, E_n, x_n)$ be as in Section 2.2. Repeating the Add-Merge operation (b-1) times, we finally have by induction

$$\bar{x}_n \le b\theta^2 \bar{x}_{n-1} - (b-1)\theta^4 \bar{x}_{n-1}^2.$$

Indeed, note that for 0 < a < b,

$$(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{4}\bar{x}_{n-1}^{2}) + \theta^{2}\bar{x}_{n-1} - \theta^{2}(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{4}\bar{x}_{n-1}^{2})\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} - a\theta^{4}\bar{x}_{n-1}^{2} + \theta^{2}\bar{x}_{n-1} - \theta^{2}(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{4}\bar{x}_{n-1}^{2})\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} - a\theta^{4}\bar{x}_{n-1}^{2} + \theta^{2}\bar{x}_{n-1} - \theta^{2}(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{4}\bar{x}_{n-1}^{2})\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} - \theta^{2}(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{2}\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} - \theta^{2}(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{2}\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} - \theta^{2}(a\theta^{2}\bar{x}_{n-1} - (a-1)\theta^{2}\bar{x}_{n-1} \le (a+1)\theta^{2}\bar{x}_{n-1} \le ($$

and the first step of the induction is given by (7). This concludes the proof. \blacksquare

5 Roughly Symmetric Channels on General Trees

We now tackle the general case. We start by analyzing the Add-Merge operation.

Proposition 1 (Basic Inequality) Consider the setup of Section 3. Assume $|\theta| < 1$. Then, there is a $\delta_0(|\theta|) > 0$ depending only on $|\theta|$ such that

$$\bar{x} \le \bar{y} + \theta^2 \bar{z},$$

whenever δ (on e) is less than $\delta_0(|\theta|)$.

Proof: The proof is similar to that in the symmetric case. By expansion (8), inequality $|X| \leq 1$, and Lemma 4, we have

$$X \le Y + \hat{Y} + \Delta Y \hat{Y} - \pi_{-/+} Y \hat{Y} (Y + \hat{Y} + \Delta Y \hat{Y}) + \pi_{-/+}^2 Y^2 \hat{Y}^2.$$
(10)

Let $\rho' = (\bar{y})^{-1} \bar{y}_+$ and $\rho'' = (\bar{z})^{-1} \bar{z}_+$. Then, by Lemmas 1 and 2, we have

$$\begin{split} \mathbb{E}_{T}^{+}[X] &= \pi_{-/+}\bar{x}, \quad \mathbb{E}_{T}^{+}[Y] = \pi_{-/+}\bar{y}, \quad \mathbb{E}_{T}^{+}[Y^{2}] = \bar{y}\rho', \\ \mathbb{E}_{T}^{+}[\widehat{Y}] &= \pi_{-/+}\theta^{2}\bar{z}, \quad \mathbb{E}_{T}^{+}[\widehat{Y}^{2}] = \theta^{2}\bar{z}[(1-\theta) + \theta\rho'']. \end{split}$$

Taking $\pi_{-/+}^{-1} \mathbb{E}_T^+$ on both sides of (10), we get

$$\begin{split} \bar{x} &\leq \bar{y} + \theta^2 \bar{z} + \Delta \pi_{-/+} \theta^2 \bar{y} \bar{z} \\ &- \pi_{-/+} \theta^2 \bar{y} \bar{z} \rho' - \pi_{-/+} \theta^2 \bar{y} \bar{z} [(1-\theta) + \theta \rho''] - \Delta \theta^2 \bar{y} \bar{z} \rho' [(1-\theta) + \theta \rho''] \\ &+ \pi_{-/+} \theta^2 \bar{y} \bar{z} \rho' [(1-\theta) + \theta \rho''] \\ &\leq \bar{y} + \theta^2 \bar{z} - \pi_{-/+} \theta^2 \bar{y} \bar{z} [\mathcal{A} - \Delta \mathcal{B}], \end{split}$$

where

$$\mathcal{A} = \rho' + (1 - \rho')[(1 - \theta) + \theta\rho''],$$

and

$$\mathcal{B} = 1 - \pi_{-/+}^{-1} \rho' [(1 - \theta) + \theta \rho''].$$

Note that $[(1 - \theta) + \theta \rho''] \ge 0$ by Lemma 2. So $\mathcal{B} \le 1$ and it suffices to have $\mathcal{A} \ge \Delta$. Note also that \mathcal{A} is multilinear in (ρ', ρ'') . Therefore, to minimize \mathcal{A} , we only need to consider extreme cases in (ρ', ρ'') . By $\pi_+ y^+ + \pi_- y^- = y$ it follows that $0 \le \rho' \le \pi_+^{-1}$. The same holds for ρ'' . At $\rho' = 0$, we have

$$\mathcal{A} = 1 - \theta [1 - \rho''] \ge \begin{cases} 1 - \theta, & \text{if } \theta \ge 0, \\ 1 - \pi_{-/+} |\theta|, & \text{if } \theta \le 0, \end{cases}$$

where we have used

$$1 - \pi_+^{-1} = -\pi_{-/+}.$$

At $\rho' = \pi_+^{-1}$, we have

$$\mathcal{A} = \pi_{+}^{-1} + (1 - \pi_{+}^{-1})[1 - \theta[1 - \rho'']] = 1 + \theta \pi_{-/+}[1 - \rho''] \ge \begin{cases} 1 - \pi_{-/+}^{2}\theta, & \text{if } \theta \ge 0, \\ 1 - \pi_{-/+}|\theta|, & \text{if } \theta \le 0. \end{cases}$$

Since $\pi_{-/+} \ge 1$ by assumption, it follows that

$$\mathcal{A} \ge 1 - \pi_{-/+}^2 |\theta|.$$

At $\delta = 0$, this bound is strictly positive and moreover $\Delta = 0$. Therefore, by continuity in δ of Δ and the bound above, the result follows.

Proposition 2 (Induction Step) Let T be a finite tree rooted at x with edge function θ . Let w_1, \ldots, w_α be the children of x in T and denote by e_a the edge connecting x to w_a . Let $\theta_0 = \max\{|\theta(e_1)|, \ldots, |\theta(e_\alpha)|\}$ and assume that on each edge $e_a, \delta \leq \delta_0(\theta_0)$, where δ_0 is defined in Proposition 1. Then

$$\bar{x} \le \sum_{a=1}^{\alpha} \theta(e_a)^2 \bar{w}_a$$

Proof: As noted in the proof of Theorem 3, adding an edge e to the root of a tree and re-rooting at the new vertex has the effect of multiplying the second moment of the magnetization by $\theta^2(e)$. The result follows by applying Proposition 1 ($\alpha - 1$) times.

Proof of Theorem 2: It suffices to show that for all $\varepsilon > 0$ there is an N large enough so that $\bar{x}_n \leq \varepsilon$, $\forall n \geq N$. Fix $\varepsilon > 0$. By definition of the branching number, there exists a cutset S of T such that

$$\sum_{u \in S} \eta(u) \le \varepsilon$$

Assume w.l.o.g. that S is actually an antichain and let N be such that S is in T_N . It is enough to show that

$$\bar{x}_n \le \sum_{u \in S} \eta(u), \quad \forall \ n \ge N.$$

Fix $n \ge N$. Applying Proposition 2 repeatedly from the root of T_n down to S, it is clear that

$$\bar{x}_n \le \sum_{u \in S} \eta(u) \mathbb{E}_{T_n(u)}[U^2] \le \sum_{u \in S} \eta(u),$$

where $T_n(u)$ is the subtree of T_n rooted at u and U is the magnetization at u on $T_n(u)$ (with $|U| \le 1$). This concludes the proof.

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A Lower bound on δ_0

Lemma 5 (Bound on δ_0) Let δ_0 be as in Proposition 1. Let $0 \le \theta_0 < 1$. Then, $\delta_0(\theta_0)$ can be set as large as $\bar{\delta} = (1 - \theta_0)\beta(\theta_0)$, where $\beta(\theta_0)$ is the smallest root of

$$(1 - \theta_0) - (4 + 2\theta_0)\beta + (3 - \theta_0)\beta^2 = 0.$$

In particular, if $\theta_0 = 1/\sqrt{b}$ (as in the b-ary case), $\bar{\delta} \approx 0.016$ when b = 2 and $\bar{\delta} \approx 1/3$ when b is large.

Proof: Let

$$\phi = \frac{\delta}{1 - \theta_0}.$$

Then (letting $|\theta| = \theta_0$)

$$\pi_{-/+} = \frac{1+\phi}{1-\phi}.$$

From the proof of Proposition 1, we seek the largest value of $\delta \ge 0$ such that

$$(1 - \pi_{-/+}^2 \theta_0) - (\pi_{-/+} - 1) \ge 0.$$

Multiplying by $(1-\phi)^2$ and rearranging, we get

$$2(1-\phi)^2 - (1+\phi)(1-\phi) - \theta_0(1+\phi)^2 = (1-\theta_0) - (4+2\theta_0)\phi + (3-\theta_0)\phi^2.$$

This expression is positive at $\phi = 0$ and remains positive until it reaches its smallest root in ϕ .

When $\theta_0 = 0$, the polynomial above reduces to

$$1 - 4\phi + 3\phi^2 = (1 - 3\phi)(1 - \phi),$$

which has its smallest root at 1/3. The special case b = 2 in the statement of the lemma can be computed numerically.