### A Probability inequality using typical moments and Concentration Results

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## 1 Introduction

It is of wide interest to prove upper bounds on the probability that the sum of random variables  $X_1, X_2, \ldots, X_n$  deviates much from its mean. If the  $X_i$ are independent real-valued mean 0 random variables, then we know from the Central Limit Theorem (CLT) that their sum (in the limit) has Gaussian distribution with variance equal to the sum of the variances. More generally, Azuma's inequality deals with the case of Martingales differences, where one assumes that

$$E(X_i|X_1, X_2, \dots, X_{i-1}) = 0 \text{ and } |X_i| \le 1$$

and proves for all t > 0,

$$\Pr\left(\left|\sum_{i=1}^{n} X_{i}\right| \ge t\right) \le 2e^{-t^{2}/(2n)}.$$

The cost of replacing independence by Martingale differences is the assumption of an absolute bound  $|X_i| \leq 1$  which replaces the variance of each variable. But the absolute upper bound on  $X_i$  is a big impediment in many applications. The main aim of the current paper is to replace the upper bound by bounds on conditional moments of  $X_i$  (conditioned on  $X_1, X_2, \ldots, X_{i-1}$ ) while retaining the "sub-Gaussian" flavor, i.e., tail bounds on  $\sum_{i=1}^{n} X_i$  qualitatively similar (except for constants) to what one would get if it were normally distributed with variance equal to the sum of the variances of the  $X_i$ . To do this, we use information on "typical case" conditional moments (conditioned on "typical"  $X_1, X_2, \ldots, X_{i-1}$ ) as well as "worst-case" ones. Further, we also generalize by assuming what we call "Strong Negative Correlation" :  $EX_i(X_1 + X_2 + \ldots + X_{i-1})^l \leq 0$  for odd l up to a certain p (in place of the stronger condition of  $X_i$  being a Martingale difference sequence.)

There have been many sophisticated probability inequalities. Talagrand's elegant inequality ([22]) has numerous applications. Burkholder's inequality for Martingales and many later developments (see [11]) give bounds based on

finite moments (as opposed to the Bernstein-type exponential moment generating function method used in Azuma's and other theorems). In general, because of known lower bounds, one cannot derive sub-Gaussian bounds from the Burkholder type inequalities. Another class of inequalities are generalizations of the Efron-Stein inequality. We will briefly compare our inequality to all these later.

Here, we give an upper bound on tail probability for a more general situation than Martingale differences using conditional moments -  $E(|X_i|^l|X_1, X_2, \ldots, X_{i-1})$ . A crucial feature of our bounds is that they involve "typical" conditional moments  $(E(|X_i|^l|X_1, X_2, \ldots, X_{i-1})$  for "typical"  $X_1, X_2, \ldots, X_{i-1})$  as well as "worst-case" ones. We defer the statement of the theorem to the next section.

We demonstrate the usefulness of our bounds by applying them to several examples. We first deal with bin-packing. The random variable X here is the minimum number of capacity 1 bins into which n fractions -  $Y_1, Y_2, Y_3, \ldots, Y_n$ which are i.i.d. uniform on (0, 1) can be packed. There has been much study of this problem [8]. If  $\mu, \sigma$  are the mean and standard deviation of each  $Y_i$ , the best previous concentration result by Rhee ([23]) proves that X is concentrated in an interval of length  $O(\sqrt{n}(\mu + \sigma))$ ; Talagrand adduces a simpler proof of this (as the first of his several examples.) Here, we improve this to prove concentration in an interval of length  $O(\sqrt{n}(\mu^{3/2} + \sigma))$  with sub-Gaussian tail probabilities for the well-studied case when the items are drawn from a discrete distribution. We also show that this is the best possible interval of concentration for discrete distributions.

Then we take up the problem of proving concentration for the number of s-cliques in a random graph. For the number of triangles (s = 3), we prove the first exponential tail bounds for "small" deviations (starting with  $\Omega(1)$  standard deviations) from the mean. Extensive prior work on this (and more generally the number of copies of any fixed graph) has focussed on getting optimal results for larger deviations rather than tight concentration. Next, we take up the case of growing s for which little is known. The case when s is as high as  $O(\log n)$  is very interesting because the largest clique size in a random graph with O(1) edge probability is  $O(\log n)$ . We take a step towards this - we are able to prove for  $s \in o(\log n)$ , that the probability that the number of s-cliques is more than twice its expectation is at most  $n^{-\omega(n)}$  for a function  $\omega(n) \to \infty$  for a range of edge probabilities (roughly between two constants). These above three examples are drawn from two archetypical

categories - bin packing is combinatorial and Talagrand's inequality deals very well with such problems. The count of cliques is a polynomial and moments and the recent work of Kim and Vu [19],[20] among others deals well with these.

For the chromatic number of a random graph, (a topic on which much pioneering work using Martingale inequalities has been done), we prove (what seems to be the first) sub-Gaussian tail bounds for sparse random graphs. For the longest increasing subsequence problem, we deduce the same bound as Talagrand also by a simple proof from our inequality.

We also mention that there have been probably numerous attempts to exploit the fact that while the absolute bounds on  $X_i$  may be too large, this may happen only with very small probability. Two examples of this are [24] and [7]. For an interesting survey as well variants on Azuma's inequality, the reader is referred to McDiarmid's paper [17]. More in the spirit of what we do here are perhaps Kahn's [16] results which get around absolute bounds on  $X_i$ by using instead typical bounds in the Martingale set up (see Section 3), but only over changes of the "current" variable and later ones, not the "previous" ones; one of our main thrusts here is to leverage typical conditional moments conditioned on the previous variables. (See also Remark (2) in Section (2).)

### 2 The set-up, Main Theorem, Discussion

Suppose  $X_1, X_2, \ldots, X_n$  are real-valued random variables. p will be an even positive integer throughout. We assume

#### Strong Negative Correlation (SNC)

 $EX_i(X_1 + X_2 + \dots + X_{i-1})^l \le 0$  for  $l \le p-1$ , l odd, for  $i = 2, 3, \dots, n$ . (1)

[We use the notation that E denotes expectation of the entire expression that follows. The usual Martingale difference condition -  $E(X_i|X_1, X_2, \ldots, X_{i-1}) = 0$  clearly implies the above inequality.]

In the usual Azuma's inequality (bounded differences inequality), one assumes an absolute bound on each  $X_i$  and proves with high probability upper bounds on  $|\sum_i X_i|$ . Here instead, we will exploit upper bounds on high moments of the  $X_i$  given  $X_1 + X_2 + \ldots X_{i-1}$ . More specifically, we assume the following bounds on moments :

$$E(X_i^l | X_1 + X_2 + \dots + X_{i-1}) \le M_{il} \quad \text{for } l = 2, 4, 6, 8 \dots p \text{ and } i = 1, 2, 3, \dots n.$$
(2)

We make a further refinement. In some cases, the bound  $M_{il}$  may be as bad as (or almost as bad) as an absolute bound on  $|X_i|^l$  for "worst-case"  $X_1 + X_2 + \ldots + X_{i-1}$ . We will exploit the fact that for a "typical"  $X_1 + X_2 + \ldots + X_{i-1}$ ,  $E(X_i^l|X_1 + X_2 + \ldots + X_{i-1})$  may be much smaller. To this end, suppose

$$\mathcal{E}_{i,l}$$
,  $l = 2, 4, 6, \dots p$ ;  $i = 1, 2, \dots n$ 

are events [We assume  $\mathcal{E}_{1,l}$  is the whole sample space.]  $\mathcal{E}_{i,l}$  is to represent the "typical" case. In addition to (2), we assume that

$$E(X_i^l|X_1 + X_2 + \dots X_{i-1}, \mathcal{E}_{i,l}) \le L_{il}$$

$$\tag{3}$$

$$\Pr(\mathcal{E}_{i,l}) = 1 - \delta_{i,l} \tag{4}$$

In certain cases, the  $M_{i,l}$  may be too large to be useful. The Theorem below provides an alternative to using  $M_{i,l}$ , if one knows upper bounds on high moments of  $X_i$ , namely upper bounds on  $EX_i^{p^2}$ ; note that these are not conditional moments, so they may be easier to evaluate.

**Theorem 1.** Let  $X_1, X_2, \ldots, X_n$  be real valued random variables satisfying Strong Negative Correlation (1) and p be a positive even integer. Then with

$$\hat{M}_{il} = Min\left(M_{il}\delta_{i,l}^{2/(p-l+2)}, \left(E|X_i|^{(p-l+2)l}\delta_{i,l}\right)^{1/(p-l+2)}\right) \quad for \ even \ l,$$

$$E\left(\sum_{i=1}^{n} X_{i}\right)^{p} \leq 10(9p)^{\frac{p}{2}+1} \left(\sum_{l=1}^{p/2} \frac{p^{1-(1/l)}}{l^{2}} \left(\sum_{i=1}^{n} L_{i,2l}\right)^{1/l}\right)^{p/2} + (72p)^{p+2} \sum_{l=1}^{p/2} \frac{1}{n} \sum_{i=1}^{n} (n\hat{M}_{i,2l})^{p/2l}.$$

There are two central features of the Theorem. The first is the distinction between typical and worst case conditional moments which we have already discussed. Note that while the  $M_{il}$  may be much larger than  $L_{il}$ , the  $M_{il}$  get modulated by  $\delta_{il}^{2/(p-l+2)}$  which can be made sufficiently small. To illustrate a second feature of the Theorem, if  $\delta_{i,l} = 0$  for all i, l and  $MAX_iL_{i,2l} = L_{2l}$ , then we get an upper bound of

$$(cp)^{p/2} \left( nL_2 + \sqrt{np}L_4^{1/2} + \ldots \right)^{p/2},$$

where we note that for  $p \ll n$ , the coefficients of higher moments decline fast, so that under reasonable conditions, the  $nL_2$  term is what matters. In this case, it will not be difficult to see that we have qualitatively sub-Gaussian behavior with variance equal to the sum of the variances. We present a Corollary to quantify this in one way. [There are other more general situations where such conclusions can also be drawn. In fact, in the second term we have, with  $\hat{M}_{2l} = \text{MAX}_i \hat{M}_{i,2l}$ :

$$\sum_{l=1}^{p/2} \frac{1}{n} \sum_{i=1}^{n} (n\hat{M}_{i,2l})^{p/2l} \le n^{p/2} \hat{M}_2^{p/2} + n^{p/4} \hat{M}_4^{p/4} + \dots$$

and again under mild conditions,  $n^{p/2}M_2^{p/2}$  is the leading term. We do not pursue this line further here.] The symbol  $\sigma^2$  in the Corollary is used to suggest the variance. Note that in the Corollary, the higher moments - $E(X_i^{2l}|X_1 + X_2 + \ldots X_{i-1})$  are allowed to grow as  $\sigma^{2l} \left(\frac{n}{p}\right)^{l-1} l^{.9l}$  and since usually, we apply this with  $p \ll n$ , this allows exponential (in l) growth.

**Corollary 2.** Suppose  $X_1, X_2, ..., X_n$  are real-valued random variables satisfying SNC condition (1). Let t > 0. Suppose  $\sigma$  is a positive real with

$$E(X_i^{2l}|X_1 + X_2 + \dots + X_{i-1}) \le \sigma^{2l} \left(\frac{n}{p}\right)^{l-1} l^{.9l} \quad for \ l = 1, 2, \dots$$

There is a positive constant c (independent of the  $X_i, n$ ) such that

$$Pr\left(|\sum_{i=1}X_i| \ge t\right) \le 4e^{-\frac{t^2}{cn\sigma^2}}.$$

*Proof.* We apply the Theorem with all the  $\mathcal{E}_{il}$  equal to the whole sample space and p to be specified. A simple calculation (using the fact that  $\sum_{l} (1/l^{1.1})$ converges) shows that for a constant  $c_1$ ,

$$E(\sum_{i} X_i)^p \le (c_1 pn\sigma^2)^{p/2}.$$
(5)

Now we use Markov to get that

$$\Pr\left(\left|\sum_{i=1} X_i\right| \ge t\right) \le (c_1 p n \sigma^2)^{p/2} / t^p.$$

Choosing  $p = 2\lfloor t^2/6c_1n\sigma^2 \rfloor$ , the Corollary follows with  $c = 10c_1$ .

**Remark 1.** Note the "sub-Gaussian" behaviour -  $e^{-\frac{t^2}{cn\sigma^2}}$ . As is well-known, to get this behaviour, one needs that the exponent of p in the upper bound in (5) to be p/2. The well-known Burkholder type inequalities (see [11] for a discussion) assert :

For  $X_1, X_2, \ldots, X_n$  satisfying  $E(X_i | X_1, X_2, \ldots, X_{i-1}) = 0$ ,

$$E(\sum_{i=1}^{n} X_{i})^{p} \leq B^{p} E\left(\sum_{i} E(X_{i}^{2} | X_{1}, X_{2}, \dots, X_{i-1})\right)^{p/2} + B^{p} \left(E \sup(E(X_{i}^{p} | X_{1}, X_{2}, \dots, X_{i-1}))\right)$$

holds. Some effort has gone into getting the best constant B above. Burkholder's proofs only gave a B which grew exponentially with p. It is now known that the correct order of B is  $p/\ln p$ . Thus, sub-Gaussian behaviour cannot be inferred from these inequalities.

Svante Janson has pointed out that if one settles for  $\left(\frac{p}{\ln p}\right)^{cp}$  in the first term of bound in our main Theorem and also makes the stronger (than SNC (1)) assumption of  $E(X_i|X_1, X_2, \ldots, X_{i-1}) = 0$ , then we can derive our theorem from known Burkholder type inequalities. Arguably, also, our proof is more elementary.

**Remark 2.** Here we briefly compare the current result to the recent result of Boucheron, Bousquet, Lugosi and Massart [6]. They prove concentration for a real-valued function F of independent random variables  $Y_1, Y_2, \ldots, Y_n$ . Let  $Z = F(Y_1, Y_2, \ldots, Y_n)$  and suppose functions  $Z_i = Z_i(Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$ are arbitrary functions. Their main theorem is that

$$E\left((Z - EZ)_{+}\right)^{p} \le (cp)^{p/2} E\left(\sum_{i=1}^{n} (Z - Z_{i})^{2}\right)^{p/2}.$$
 (6)

Such an inequality is more in the spirit of Efron-Stein inequality, where we sum up the variations in Z caused by all the n variables and then take a high moment of it. The advantage of this would be in situations where one can show that not too many of individual  $Y_i$  cause large changes for typical  $Y_1, Y_2, \ldots Y_n$ . [See [6].] This general line of approach is also reminiscent of Talagrand's inequality; but Talagrand allows simultaneous change of variables. In a sense, [16] also combines changes caused by individual variables conditioned on (worst case values of) previous ones, but takes a product of their exponential moments. Note that (6) has an exponent of p/2 on the p which leads to the ideal sub-Gaussian behaviour. [These are the advantages of this over Burkholder type inequality.]

In contrast, our inequality (like Burkhoder's) only considers variations of one individual variable at a time which is in many cases easier to bound. We will see this in the case of Bin-Packing, coloring and other examples to follow. Even for the classical Longest Increasing Subsequence (LIS) problem, where for example, Talagrand's crucial argument is that only a small number  $O(\sqrt{n})$ of elements (namely those in the current LIS) cause a decrease in the length of the LIS by their deletion, we are able to bound individual variations (in essence arguing that EACH variable has roughly only a  $O(1/\sqrt{n})$  probability of changing the length of the LIS) sufficiently to get a concentration result.

Note that if one can only handle individual variations, then (6) essentially yields only

$$E((Z - EZ)_{+})^{p} \le (cpn)^{p/2} \max_{i} E(Z - Z_{i})^{p}.$$

In this case, arguments as in Corollary (2) as well as what we do below for Bin-Packing and LIS which is based mainly on the second moment, do not work, since the above involves a high moment.

**Remark 3.** Here we mention very briefly some contexts where the  $X_i$  are not Martingale differences, but satisfy Strong Negative Correlation (1), so our results can be applied. There is some literature on "negatively associated" random variables ([5],[9] for example). As first defined in [14], a set of random variables  $Y_1, Y_2, \ldots Y_n$  is negatively associated if for every pair of disjoint sets  $A, B \subseteq \{1, 2, \ldots n\}$  and every non-decreasing function f of  $\{y_i : i \in A\}$  and every non-decreasing function g of  $\{y_i : i \in B\}$ , we have

$$Ef(Y_i, i \in A)g(Y_i, i \in B) \le Ef(Y_i, i \in A)Eg(Y_i, i \in B).$$

It is easy to see that then  $X_i = Y_i - EY_i$  satisfy (1), since  $(X_1 + X_2 + ..., X_{i-1})^l$ is a non-decreasing function for odd l. Several interesting families of random variables, such as some arising in Occupancy (balls and bins) problems are negatively associated [14]. Another class of variables which have the Negative Association property are 0-1 variables produced by a randomized rounding algorithm of Srinivasan [21]; this is shown in [10]. [Randomized rounding is applied for combinatorial optimization problems where one is seeking a 0-1 solution, but first solves a Linear Program relaxing the integrality constraint.] Note that for (1), we really only used negative association for simple f, g, not the full power for general f, g.

We mention in passing another plausible class of variables satisfying (1). Suppose  $X_i$  is the deficit spending (by a person or institution) at time *i* and there is a "penalty/reward for overspending/underspending" (respectively) in previous periods. I.e. there are non-negative reals  $\alpha_i$  so that  $X_i$  is a random variable with mean  $-\alpha_i \sum_{i=1}^{j-1} X_j$  (and any conditional probability distribution). Then (1) holds. More generally, it suffices to have  $E(X_i|X_1 + X_2 + \dots X_{i-1})$  be of opposite sign to  $X_1 + X_2 + \dots X_{i-1}$ .

A simple third (toy) example is sampling without replacement. Suppose  $a_1, a_2, \ldots a_N$  are reals with  $\sum_{i=1}^N a_i = 0$  and we pick n < N samples -  $X_1, X_2, \ldots X_n$  from the  $a_i$  by sampling without replacement, say each sample is picked uniformly at random from what is left. Then clearly (1) is satisfied.

We do not work out here what one gets by applying the Theorem to these cases.

### 3 Proof of the Theorem (1)

*Proof.* (of the Theorem) Let  $A = X_1 + X_2 + \ldots + X_{n-1}$ .

$$E(\sum_{i=1}^{n} X_{i})^{p} \leq EA^{p} + pEX_{n}A^{p-1} + \binom{p}{2}EX_{n}^{2}A^{p-2} + \binom{p}{3}E|X_{n}|^{3}|A|^{p-3} + \dots$$
(7)

First we note that  $EX_n A^{p-1} \leq 0$  by hypothesis (1) and so the second term may be dropped. Now, for odd  $l \geq 3$ , we have

$$E|X_n|^l|A|^{p-l} = E(X_n^{l+1}A^{p-l-1})^{1/2}(X_n^{l-1}A^{p-l+1})^{1/2} \le (E(X_n^{l+1}A^{p-l-1}))^{1/2}(E(X_n^{l-1}A^{p-l+1}))^{1/2}$$

With a little calculation,

$$\binom{p}{l} \le \sqrt{4\binom{p}{l+1}\binom{p}{l-1}} \quad \text{for } l \in \{3, 4, \dots p-1\}$$

So, we get for odd  $l \geq 3$ ,

$$\binom{p}{l} E|X_n|^l |A|^{p-l} \le \sqrt{\left(2\binom{p}{l+1}EX_n^{l+1}A^{p-l-1}\right)\left(2\binom{p}{l-1}EX_n^{l-1}A^{p-l+1}\right)} \\ \le \binom{p}{l+1}EX_n^{l+1}A^{p-l-1} + \binom{p}{l-1}EX_n^{l-1}A^{p-l+1},$$

using  $2\sqrt{ab} \le a + b$  for non-negative reals a, b. Plugging this into (7), we get :

$$E(\sum_{i=1}^{n} X_{i})^{p} \le EA^{p} + 3\binom{p}{2}EX_{n}^{2}A^{p-2} + 3\binom{p}{4}EX_{n}^{4}A^{p-4} + \dots$$
(8)

Now, for  $l \geq 2$  even, we have

$$\begin{aligned} EX_{n}^{l}A^{p-l} &= \Pr(\mathcal{E}_{n,l})E(X_{n}^{l}A^{p-l}|\mathcal{E}_{n,l}) + \Pr(\neg \mathcal{E}_{n})E(X_{n}^{l}A^{p-l}|\neg \mathcal{E}_{n,l}) \\ &= \Pr(\mathcal{E}_{n,l})E_{A}\left(A^{p-l}E(X_{n}^{l}|A;\mathcal{E}_{n,l})|\mathcal{E}_{n,l}\right) \\ &+ \Pr(\neg \mathcal{E}_{n,l})E_{X_{1},X_{2},...X_{n-1}}\left((X_{1}+X_{2}+...X_{n-1})^{p-l}E_{X_{n}}(X_{n}^{l}|X_{1},X_{2},...X_{n-1};\neg \mathcal{E}_{n,l})\right) \\ &\leq L_{nl}\int_{\omega\in\mathcal{E}_{n,l}}A(\omega)^{p-l}d\omega + M_{nl}\int_{\omega\in\neg\mathcal{E}_{n,l}}A(\omega)^{p-l}d\omega \\ &\leq L_{nl}EA^{p-l} + M_{nl}\left(\int_{\neg\mathcal{E}_{n,l}}A(\omega)^{p-l+2}\right)^{(p-l)/(p-l+2)}\left(\int_{\neg\mathcal{E}_{n,l}}d\omega\right)^{2/(p-l+2)} (\text{H\"older}) \\ &\leq L_{nl}EA^{p-l} + M_{nl}(EA^{p-l+2})^{(p-l)/(p-l+2)}\delta_{n,l}^{2/(p-l+2)}. \end{aligned}$$

Alternatively, one could use Hölder's inequality differently :

$$\int_{\neg \mathcal{E}_{n,l}} X_n^l A^{p-l} \le \left( \int A^{p-l+2} \right)^{(p-l)/(p-l+2)} \left( \int X_n^{l(p-l+2)} \right)^{1/(p-l+2)} \left( \int_{\neg \mathcal{E}_{n,l}} d\omega \right)^{1/(p-l+2)}.$$

Thus we get that

$$EX_n^l A^{p-l} \le L_{n,l} EA^{p-l} + (EA^{p-l+2})^{(p-l)/(p-l+2)} \operatorname{Min} \left( M_{n,l} \delta_{n,l}^{2/(p-l+2)}, (EX_n^{l(p-l+2)} \delta_{n,l})^{1/(p-l+2)} \right)$$
$$= L_{n,l} EA^{p-l} + \hat{M}_{nl} \left( EA^{p-l+2} \right)^{(p-l)/(p-l+2)},$$

with  $\hat{M}_{n,l}$  defined as in the Theorem.

We use Young's inequality which says that for any a, b > 0 real and q, r > 0with  $\frac{1}{q} + \frac{1}{r} = 1$ , we have  $ab \le a^q + b^r$ ; we apply this below with q = (p-l+2)/2and r = (p-l+2)/(p-l) and  $\lambda_{nl}$  a positive real to be specified later :

$$\hat{M}_{nl} (EA^{p-l+2})^{(p-l)/(p-l+2)} = \left( \hat{M}_{nl}^{2p/l(p-l+2)} \lambda_{nl}^{-(p-l)/(p-l+2)} \right) \left( \hat{M}_{nl}^{(l-2)/l} \lambda_{nl} EA^{p-l+2} \right)^{(p-l)/(p-l+2)} \\
\leq \hat{M}_{nl}^{p/l} \lambda_{nl}^{-(p-l)/2} + \hat{M}_{nl}^{(l-2)/l} \lambda_{nl} EA^{p-l+2},$$

So, we get :

$$E\left(\sum_{i=1}^{n} X_i\right)^p \le \sum_{\substack{l\ge 0\\ \text{even}}}^{p} a_{nl} E A^{p-l},$$

where

$$a_{nl} = 1 + 3\lambda_{n2} \binom{p}{2}, \qquad l = 0$$
  

$$a_{nl} = 3\binom{p}{l}L_{nl} + 3\binom{p}{l+2}\hat{M}_{n,l+2}^{l/(l+2)}\lambda_{n,l+2}, \qquad 2 \le l \le p-2$$
  

$$a_{nl} = 3L_{np} + 3\sum_{\substack{l_1 \ge 2\\ \text{eVen}}}^{p} \binom{p}{l_1}\frac{\hat{M}_{nl_1}^{p/l_1}}{\lambda_{nl_1}^{(p-l_1)/2}}, \qquad l = p.$$

An exactly similar argument yields for any  $m \leq n$  and any  $q \leq p$ , even

$$E\left(\sum_{i=1}^{m} X_i\right)^q \le \sum_{\substack{l\ge0\\\text{even}}}^q a_{ml}^{(q)} E(\sum_{i=1}^{m-1} X_i)^{q-l},$$

where (since  $\delta_{m,l}^{1/(q-l+2)} \leq \delta_{m,l}^{1/(p-l+2)}$ )

$$a_{ml}^{(q)} = 1 + 3\lambda_{m2} \binom{q}{2}, \qquad l = 0$$

$$a_{ml}^{(q)} = 3 \binom{q}{l} L_{ml} + 3 \binom{q}{l+2} \hat{M}_{m,l+2}^{l/(l+2)} \lambda_{m,l+2}, \qquad 2 \le l \le q-2$$
$$a_{ml}^{(q)} = 3L_{mq} + 3 \sum_{\substack{l_1 \ge 2\\ \text{even}}}^{q} \binom{q}{l_1} \frac{\hat{M}_{ml_1}^{q/l_1}}{\lambda_{ml_1}^{(q-l_1)/2}}, \qquad l = q.$$

Now we set

$$\lambda_{ml} = \frac{1}{3p^2 n^{2/l}}$$
 for  $l = 2, 4, 6, 8, \dots$ 

Then we get

$$a_{ml}^{(q)} \le a_{ml} = 1 + \frac{1}{n}, \qquad l = 0$$

$$a_{ml}^{(q)} \le a_{ml} = 3 \binom{p}{l} \left( L_{ml} + \hat{M}_{m,l+2}^{l/(l+2)} n^{-2/(l+2)} \right), \qquad 2 \le l \le q-2$$

$$a_{mq}^{(q)} \le \hat{a}_{mq} = 3L_{mq} + 3 \sum_{\substack{l_1 \ge 2\\ e_{V} \in P}}^{q} \binom{q}{l_1} \hat{M}_{ml_1}^{q/l_1} (3p^2)^{(q-l_1)/2} n^{(q-l_1)/l_1}.$$

We note that it is important to make  $a_{m0}^{(q)}$  not be much greater than 1 because in solving the recurrence (which we do later), in this case, we do not reduce the exponent p at all, so this may be applied n times, so in essence this can be at most 1 + (1/n). For  $l \ge 2$ , an application of this would reduce p by 2, so there are at most p/2 application of these.

Note that except for l = q, the other  $a_{ml}$  do not depend upon q; we have used  $\hat{a}_{mq}$  to indicate that this extra dependence. With this, we have

$$E\left(\sum_{i=1}^{m} X_i\right)^q \le \hat{a}_{mq} + \sum_{\substack{l\ge 0\\ \text{even}}}^{q-2} a_{ml} E(\sum_{i=1}^{m-1} X_i)^{q-l}.$$

We wish to solve these recurrences by induction on m, q. Intuitively, we can imagine a tree with root marked (m, q) (since we are bounding  $E(\sum_{i=1}^{m} X_i)^q$ . The root has  $\frac{q}{2}+1$  children which are marked (m-1, q-l) for  $l = 0, 2, \ldots, q/2$ ; the node marked (m-1, q-l) is trying to bound  $E(\sum_{i=1}^{m-1} X_i)^{q-l}$ . There are also weights on the edges of  $a_{ml}$  respectively. The tree keeps going until we reach the leaves - which are marked (1, q) or (m, 0). It is intuitively easy to argue that the bound we are seeking at the root is the sum over all paths from the root to the leaves of the product of the edge weights on the path. We formalize this in a lemma.

For doing that, define for  $1 \le m \le n; 2 \le q \le p, q$  even and  $1 \le i \le m$ :  $S(m,q,i) = \{s = (s_i, s_{i+1}, s_{i+2}, \dots s_m) : s_i > 0; s_{i+1}, s_{i+2}, \dots s_m \ge 0; \sum_{j=i}^m s_j = q; s_j \text{ even } \}.$  **Lemma 1.** For any  $1 \le m \le n$  and any  $q \le p$  even, we have

$$E(\sum_{i=1}^{m} X_i)^q \le \sum_{i=1}^{m} \sum_{s \in S(m,q,i)} \hat{a}_{i,s_i} \prod_{j=i+1}^{m} a_{j,s_j}.$$

**Proof** Indeed, the statement is easy to prove for the base case of the induction -m = 1 since  $\mathcal{E}_{1l}$  is the whole sample space and  $EX_1^q \leq L_{iq}$ . For the inductive step, we proceed as follows.

$$E(\sum_{i=1}^{m} X_i)^q \le \sum_{\substack{s_m \ge 0\\ \text{eVen}}}^{q-2} a_{m,s_m} E(\sum_{i=1}^{m-1} X_i)^{q-s_m} + \hat{a}_{m,q}$$
$$\le \hat{a}_{m,q} + \sum_{i=1}^{m-1} \sum_{\substack{s_m \ge 0\\ \text{eVen}}}^{q-2} a_{m,s_m} \sum_{s \in S(m-1,q-s_m,i)} \hat{a}_{i,s_i} \prod_{j=i+1}^{m-1} a_{j,s_j}.$$

We clearly have  $S(m, q, m) = \{q\}$  and for each fixed  $i, 1 \le i \le m - 1$ , there is a 1-1 map

$$S(m-1,q,i) \cup S(m-1,q-2,i) \cup \dots S(m-1,2,i) \to S(m,q,i)$$

given by

$$s = (s_i, s_{i+1}, \dots, s_{m-1}) \to s' = (s_i, \dots, s_{m-1}, q - \sum_{j=i}^{m-1} s_j)$$

and it is easy to see from this that we have the inductive step, finishing the proof of the Lemma.

The above "sum of products" form is not so convenient to work with. We will now get this to the "sum of moments" form stated in the Theorem. This will require a series of (mainly algebraic) manipulations with ample use of Young's inequality, the inequality asserting  $(a_1 + a_2 + \ldots a_m)^q \leq m^{q-1}(a_1^q + a_2^q + \ldots a_m^q)$ for positive reals  $a_1, a_2, \ldots$  and  $q \geq 1$  and others. So far, we have (moving the l = 0 terms separately in the first step)

$$E\left(\sum_{i=1}^{n} X_{i}\right)^{p}$$

$$\leq \left(\prod_{i=1}^{n} a_{i0}\right) \sum_{i=1}^{n} \sum_{s \in S(n,p,i)} \hat{a}_{i,s_{i}} \prod_{\substack{j=i+1\\s_{j} \neq 0}}^{n} a_{j,s_{j}}$$

$$\leq 3 \sum_{i=1}^{n} \sum_{s \in S(n,p,i)} \hat{a}_{i,s_{i}} \prod_{\substack{j=i+1\\s_{j} \neq 0}}^{n} a_{j,s_{j}}$$

$$\leq 3 \sum_{t \geq 1}^{p/2} \left(\sum_{i=1}^{n} \hat{a}_{i,2t}\right) \sum_{s \in Q(p-2t)} \prod_{\substack{j=1\\s_{j} \neq 0}}^{n} a_{j,s_{j}}$$

$$(9)$$
where,  $Q(q) = \{s = (s_{1}, s_{2}, \dots s_{n}) : s_{i} \geq 0 \text{ even }; \sum_{j} s_{j} = q\}$ 

Fix q for now. For  $s \in Q(q)$ , l = 0, 1, 2, ..., p/2, let  $T_l(s) = \{j : s_j = 2l\}$  and  $t_l(s) = |T_l(s)|$ . Note that  $\sum_{l=0}^{q/2} lt_l(s) = q/2$ . Call  $t(s) = (t_0(s), t_1(s), t_2(s), ..., t_{q/2}(s))$  the "signature" of s. In the special case when  $a_{il}$  is independent of i, the signature clearly determines the "s term" in the sum (9). For the general case too, it will be useful to group terms by their signature. Let (the set of possible signatures) be

$$T = \{t = (t_0, t_1, t_2, \dots, t_{q/2}) ; t_l \ge 0 ; \sum_{l=1}^{q/2} lt_l = q/2 ; t_0 \le n; \sum_{l=0}^{q/2} t_l = n\}$$

Now, 
$$\sum_{s \in Q(q)} \prod_{\substack{j=1\\s_j \neq 0}}^n a_{j,sj} = \sum_{t \in T} \sum_{T_0, T_1, T_2, \dots, T_{q/2}: |T_l| = t_l}^{T_l} \prod_{l=1}^{q/2} \prod_{i \in T_l} a_{i,2l}$$
$$\leq \sum_{t \in T} \prod_{l=1}^{q/2} \frac{1}{t_l!} \left( \sum_{i=1}^n a_{i,2l} \right)^{t_l},$$

since the expansion of  $(\sum_{i=1}^{n} a_{i,2l})^{t_l}$  contains  $t_l!$  copies of  $\prod_{i \in T_l} a_{i,2l}$  (as well other terms we do not need.) Now define  $R = \{r = (r_1, r_2, \dots, r_{q/2}) : r_l \geq 0\}$ 

 $0; \sum_{l} r_{l} = q/2 \}.$  We have

$$\sum_{t \in T} \prod_{l=1}^{q/2} \frac{1}{t_l!} \left( \sum_{i=1}^n a_{i,2l} \right)^{t_l} \le \sum_{r \in R} \prod_l \frac{1}{(r_l/l)!} (\sum_{i=1}^n a_{i,2l})^{r_l/l} \\ \le \frac{1}{(q/2)!} \left( \sum_{l=1}^{q/2} p^{1-(1/l)} \left( \sum_i a_{i,2l} \right)^{1/l} \right)^{q/2}, \quad (10)$$

where the first inequality is seen by substituting  $r_l = t_l l$  and noting that the terms corresponding to the r such that  $l|r_l \forall l$  are sufficient to cover the previous expression and the other terms are non-negative. To see the second inequality, we just expand the last expression and note that the expansion contains  $\prod_l (\sum_i a_{i,2l})^{r_l/l}$  with coefficient  $\binom{q/2}{(r_1,r_2,\ldots,r_{q/2})}$  for each  $r \in R$ . Now, it only remains to see that  $p^{r_l(1-(1/l))} \geq \frac{r_l!}{(r_l/l)!}$ , which is obvious. Thus, we have plugging in (10) into (9) (and using again Young's inequality :  $ab \leq a^q + b^r$ for a, b, q, r > 0 with (1/q) + (1/r) = 1)

$$E\left(\sum_{i=1}^{n} X_{i}\right)^{p} \leq 3\sum_{t=1}^{p/2} \left(\sum_{i=1}^{n} \hat{a}_{i,2t}\right) \left(\sum_{l=1}^{(p/2)-t} p^{1-(1/l)} \left(\sum_{i} a_{i,2l}\right)^{1/l}\right)^{(p/2)-t} \frac{1}{((p/2)-t)!}$$
$$\leq 3\sum_{t=1}^{p/2} \left(\sum_{i} \hat{a}_{i,2t}\right)^{p/2t} + 3p \left(\sum_{l=1}^{(p/2)-1} p^{1-(1/l)} \left(\sum_{i} a_{i,2l}\right)^{1/l}\right)^{p/2} \frac{e^{p+2}}{p^{p/2}}$$
(11)

using

$$((p/2) - t)! \ge ((p/2) - t)^{(p/2) - t} e^{-(p/2)} \ge p^{(p/2) - t} e^{-(p/2)} \operatorname{Min}_t \left(\frac{(p/2) - t}{p}\right)^{(p/2) - t} \\ \ge p^{(p/2) - t} e^{-(p/2) - 2 - (p/2e)},$$

the last using Calculus to differentiate the log of the expression with respect to t etc.

In what follows, let  $l_1$  run over even values to p and i run from 1 to n.

$$3\sum_{t=1}^{p/2} \left(\sum_{i} \hat{a}_{i,2t}\right)^{p/2t} \leq 6^{p+1} \sum_{t} \left(\sum_{i} L_{i,2t}\right)^{p/2t} + (18p)^{p} \sum_{t} \left(\frac{1}{n} \sum_{i} \sum_{l_{1} \leq 2t} (n\hat{M}_{i,l_{1}})^{2t/l_{1}}\right)^{p/2t}$$
$$\leq 6^{p+1} \sum_{t} (\sum_{i} L_{i,2t})^{p/2t} + (18p)^{p} \sum_{t} t^{p/2t} \sum_{l_{1}} \frac{1}{n^{p/2t}} \left(\sum_{i} (n\hat{M}_{i,l_{1}})^{2t/l_{1}}\right)^{p/2t}$$
$$\leq 6^{p+1} \sum_{t} (\sum_{i} L_{i,2t})^{p/2t} + (36p)^{p} \sum_{l_{1}} (1/n) \sum_{i} (n\hat{M}_{i,l_{1}})^{p/l_{1}}.$$
(12)

$$\begin{pmatrix} {}^{(p/2)-1} \\ \sum_{l=1}^{p^{1-(1/l)}} \left(\sum_{i} a_{i,2l}\right)^{1/l} \\ {}^{p/2} \\ \leq \left(\sum_{l=1}^{(p/2)-1} p^{1-(1/l)} \left(3\frac{p^{2l}}{(2l)!}\right)^{1/l} \left(\sum_{i} L_{i,2l} + \hat{M}_{i,2l+2}^{l/(l+1)} n^{-1/(l+1)}\right)^{1/l} \right)^{p/2} \\ \leq p^{p} e^{p} \left(\sum_{l=1}^{(p/2)-1} \frac{p^{1-(1/l)}}{l^{2}} \left(\left(\sum_{i} L_{i,2l}\right)^{1/l} + \left(\sum_{i} \hat{M}_{i,2l+2}\right)^{1/(l+1)}\right)\right)^{p/2} \\ \leq 2^{p/2} e^{p} p^{p} \left(\sum_{l=1}^{p/2} \frac{p^{1-(1/l)}}{l^{2}} \left(\sum_{i} L_{i,2l}\right)^{1/l}\right)^{p/2} + 2^{p/2} e^{p} p^{p} \left(\sum_{l=2}^{p/2} \frac{p}{(l-1)^{2}} \left(\sum_{i} \hat{M}_{i,2l}\right)^{1/l}\right)^{p/2} .$$
(13)

We will further bound the last term :

$$\left(\sum_{l=2}^{p/2} \frac{p}{(l-1)^2} \left(\sum_i \hat{M}_{i,2l}\right)^{1/l}\right)^{p/2} \le p^{p/2} \left(\sum_{l=1}^{\infty} \frac{1}{l^{2p/(p-2)}}\right)^{(p-2)/2} \left(\sum_l \left(\sum_i \hat{M}_{i,2l}\right)^{p/2l}\right) \le 2^p p^{p/2} \sum_l \frac{1}{n} \sum_i (n \hat{M}_{i,2l})^{p/2l}.$$
(14)

Now plugging (13,12,14) into (11), we get the Theorem.

# 4 Concentration for functions of independent random variables

We would like to apply Theorem 1 to a real-valued function f of independent (not necessarily real-valued) random variables  $Y_1, Y_2, \ldots$  and show that the function is concentrated in a small interval. This is usually done using the Doob's Martingale construction which we recall in this section (there is no new stuff in this section.)

Let  $Y_1, Y_2, \ldots, Y_n$  be independent random variables. Denote  $Y = (Y_1, Y_2, \ldots, Y_n)$ . Let f(Y) be a real-valued function of Y. One defines the classical Doob's Martingale:

$$X_i = E(f|Y_1, Y_2, \dots, Y_i) - E(f|Y_1, Y_2, \dots, Y_{i-1}).$$

It is easy to see that

$$E(X_i|Y_1, Y_2, \dots, Y_{i-1}) = E_{Y_i}E(f|Y_1, Y_2, \dots, Y_i) - E(f|Y_1, Y_2, \dots, Y_{i-1}) = 0.$$

Since  $X_1, X_2, \ldots, X_{i-1}$  are all functions of just  $Y_1, Y_2, \ldots, Y_{i-1}$ , it follows that  $E(X_i|X_1, X_2, \ldots, X_{i-1}) = 0$ . Hence the name Doob's "martingale". So, (1) is satisfied.

Suppose now we have

$$\left(E_{Y_1,Y_2,\dots,Y_i}|X_i|^{p^2}\right)^{1/p^2} \le Q$$

We can now apply the Theorem to  $X_1, X_2, \ldots$  with

$$\mathcal{E}_{il} = \emptyset$$
;  $\delta_{il} = 1$ ;  $L_{i,l} = 0$ ;  $\hat{M}_{i,l} = Q^l$ .

**Corollary 3.** With the notation above, we get

$$E\left(\sum_{i=1}^{n} X_i\right)^p \le (100p)^{p+5} n^{p/2} Q^p.$$

**Remark 4.** Note that one would use this only if no information on  $L_{i,l}$  or  $M_{i,l}$  was available.

We introduce another standard device to deal with functions of independent random variables. Let  $Y_i, Y, f, X_i$  be as above. We now let  $Y^{(i)}$  denote the n-1-tuple of random variables  $Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n$ . Suppose also that  $f(Y^{(i)})$  is defined. Now let

$$Z_i = f(Y) - f(Y^{(i)}).$$

In some contexts, it will be easy to prove that moments of  $Z_i$  are bounded; here we argue simply that this also yields bounds on moments of  $X_i - EX_i$ : we can write  $X_i$  as

$$X_{i} = E(f(Y) - f(Y^{(i)})|Y_{1}, Y_{2}, \dots, Y_{i}) - E(f(Y) - f(Y^{(i)})|Y_{1}, Y_{2}, \dots, Y_{i-1}),$$

since  $E(f(Y^{(i)}|Y_1, Y_2, \ldots, Y_i)) = E(f(Y^{(i)}|Y_1, Y_2, \ldots, Y_{i-1}))$  simply because  $Y^{(i)}$  does not involve  $Y_i$ . Thus, we have

$$X_{i} = E_{Y_{i+1}, Y_{i+2}, \dots, Y_{n}} Z_{i} - E_{Y_{i}} \left( E_{Y_{i+1}, Y_{i+2}, \dots, Y_{n}} Z_{i} \right).$$
(15)

### 5 Bin Packing

Now we tackle bin packing. The input consists of n i.i.d. items -  $Y_1, Y_2, \ldots, Y_n \in (0, 1)$ . Suppose  $EY_1 = \mu$  and  $\operatorname{Var} Y_1 = \sigma$ . Let  $f = f(Y_1, Y_2, \ldots, Y_n)$  be the minimum number of capacity 1 bins into which the items  $Y_1, Y_2, \ldots, Y_n$  can be packed. The question we address is the best possible interval in which concentration can be proved. It was shown (after many successive developments) using non-trivial bin-packing theory ([23]) that (with c, c' > 0 fixed constants) for  $t \in (0, cn(\mu^2 + \sigma^2))$ ,

$$\Pr(|f - Ef| \ge t) \le c' e^{-ct^2/(n(\mu^2 + \sigma^2))}.$$

Talagrand [22] gives a simple proof of this from his inequality (this is the first of the six or so examples in his paper.) Colloquially, we say that this proves concentration in an interval of length  $O(\sqrt{n}(\mu + \sigma))$ . The "ideal" interval of length  $O(\sqrt{n}\sigma)$  (as for sums of independent random variables) is impossible.<sup>1</sup>

<sup>&</sup>lt;sup>1</sup>An example is when items are of size 1/k or  $(1/k) + \epsilon$  (k a positive integer) with probability 1/2 each.  $\sigma$  is  $O(\epsilon)$ . But it is easy to see that interval of concentration has to be at least  $\Omega(\sqrt{n\mu^2})$ .

The above bound can also be derived easily from our theorem as we outline first before going to the main result of this section : let  $g(Y_1, Y_2, \ldots, Y_n; u) =$ g be the minimum overflow (sum of items not packed) when we pack the  $Y_1, Y_2, \ldots, Y_n$  into u bins.

$$Z_i = g(Y_1, Y_2, \dots, Y_n; u) - g(Y_1, Y_2, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n; u).$$

As discussed in section 4, for the Doob's Martingale -

$$X_i = E(g|Y_1, Y_2, \dots, Y_i) - E(g|Y_1, Y_2, \dots, Y_{i-1}),$$

we have

$$X_i = E(Z_i | Y_1, Y_2, \dots, Y_i) - E(Z_i | Y_1, Y_2, \dots, Y_{i-1}).$$

Clearly,  $0 \leq Z_i \leq Y_i$ , thus for any  $l \geq 1$ ,

$$E(X_i^{2l}|Y_1, Y_2, \dots, Y_{i-1}) \le EY_i^{2l} \le \mu^2 + \sigma^2.$$

Take u to be the median number of bins and use the fact that if we have overflow of a, the overflow items can be packed into 2a + 1 bins by the first-fit algorithm. We also let  $p = cn(\sigma^2 + \mu^2)$  and note that  $\mu^2 + \sigma^2 \leq (n/p)^{l-1}(\sigma^2 + \mu^2)$ . We can now get the required sub-Gaussian behavior using Corollary (2).

Our main aim here is to prove that the best length of the interval of concentration is  $O(\sqrt{n}(\mu^{3/2} + \sigma))$  when the items take on only one of a fixed finite set of values (discrete distributions - a case which has received much attention in the literature for example [18] and references therein) by proving that indeed whp, the number of bins is concentrated in an interval of this length and then giving a simple example in which the interval has to be at least this long.

**Theorem 4.** Suppose  $Y_1, Y_2, \ldots, Y_n$  are *i.i.d.* drawn from a discrete distribution with r atoms each with probability at least  $\frac{1}{\log n}$ . Let  $EY_1 = \mu \leq \frac{1}{r^2 \log n}$ and  $\operatorname{Var} Y_i = \sigma^2$ . Then for any  $t \in (0, n(\mu^3 + \sigma^2))$ , we have

$$Pr(|f - Ef| \ge t + r) \le c_1 e^{-ct^2/(n(\mu^3 + \sigma^2))}.$$

**Proof** Let item sizes be  $\zeta_1, \zeta_2, \ldots, \zeta_j \ldots, \zeta_r$  and the probability of picking type j be  $p_j$ . We have : mean  $\mu = \sum_j p_j \zeta_j$  and standard deviation  $\sigma = (\sum_j p_j (\zeta_j - \mu)^2)^{1/2}$ .

[While our proof of the upper bound here is only for problems with a fixed finite number of types, it would be nice to extend this to continuous distributions.]

Note that if  $\mu \leq r/\sqrt{n}$ , then earlier results already give concentration in an interval of length  $O(\sqrt{n}(\mu + \sigma))$  which is then  $O(r + \sigma)$ , so there is nothing to prove. So assume that  $\mu \geq r/\sqrt{n}$ .

Define a "bin Type" as an r- vector of non-negative integers specifying number of items of each type which are together packabale into one bin. If bin type i packs  $a_{ij}$  items of type j for  $j = 1, 2, \ldots r$  we have  $\sum_j a_{ij} \zeta_j \leq 1$ . Say there are s bin types. Note that s depends only on  $\zeta_j$ , not on n.

For any set of given items, we may write a Linear Programming relaxation of the bin packing problem whose answers are within additive error r of the integer solution. If there are  $n_j$  items of size  $\zeta_j$  in the set, the Linear program is :

Primal :  $(x_i \text{ number of bins of type } i.)$ 

$$\operatorname{Min} \sum_{i=1}^{s} x_{i} \quad \text{ subject to } \sum_{i=1}^{s} x_{i} a_{ij} \geq n_{j} \forall j \ ; x_{i} \geq 0.$$

Since an optimal basic feasible solution has at most r non-zero variables, we may just round these r up to integers to get an integer solution; thus the additive error is at most r as claimed. In what follows, we prove concentration not for the integer program's value, but for the value of the Linear Program.

The Linear Program has the following

Dual :  $(y_j \text{ "imputed" size of item } j)$ :

MAX 
$$\sum_{j=1}^{r} n_j y_j$$
 subject to  $\sum_j a_{ij} y_j \le 1$  for  $i = 1, 2, \dots s; y_j \ge 0$ .

Suppose now, we have already chosen all but  $Y_i$ . Now, we pick  $Y_i$  at random; say  $Y_i = \zeta_k$ . Let  $Y = (Y_1, Y_2, \ldots, Y_n)$  and  $Y' = (Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$ We denote by f(Y) the value of the Linear Program for the set of items Y. Let

$$Z_i = f(Y) - f(Y').$$

Suppose we have the optimal solution of the LP for Y'. Let  $i_0$  be the index of the bin type which packs  $\lfloor 1/\zeta_k \rfloor$  copies of item of type k. Clearly if we increase  $x_{i_0}$  by  $\frac{1}{\lfloor 1/\zeta_k \rfloor}$ , we get a feasible solution to the new primal LP. So

$$Z_i \le \frac{1}{\lfloor 1/\zeta_k \rfloor} \le \zeta_k + 2\zeta_k^2.$$

Now, we lower bound  $Z_i$  by looking at the dual. For this, let y be the dual optimal solution for Y'. (Note : Thus, y = y(Y') is a function of Y'.) y is feasible to the new dual LP (after adding in  $Y_i$ ) since the constraints do not change. So, we get:  $Z_i \ge y_k$  and also  $y_k \le \zeta_k + 2\zeta_k^2$ . We will use these lower and upper bounds on  $Z_i$  in what follows.  $0 \le Z_i \le \zeta_k + 2\zeta_k^2$  gives us

$$E(Z_i^2|Y') \le \sum_j p_j(\zeta_j + 2\zeta_j^2)^2 \le \sum_j p_j\zeta_j^2 + 4\sum_j p_j\zeta_j^4 + 4\sum_j p_j\zeta_j^3$$
  
$$\le \mu^2 + \sigma^2 + 64\sum_j p_j|\zeta_j - \mu|^3 + 64\mu^3 \le \mu^2 + 65\sigma^2 + 64\mu^3.$$
(16)

$$E(Z_i|Y') \ge \sum_j p_j y_j(Y') = \mu - \delta(Y') \text{ (say)}.$$
(17)

Say the number of items of type j in Y' is  $(n-1)p_j + \Delta_j$ . [Ideally,  $\Delta_j = 0$ .] Now we wish to argue that  $\delta(Y')$  is not too large. Indeed, we have (recalling that  $\zeta$  is a feasible dual solution)

$$\sum_{j} ((n-1)p_j + \Delta_j)y_j \ge \sum_{j} ((n-1)p_j + \Delta_j)\zeta_j \text{ since } y \text{ is optimal dual soln.}$$
  
So, 
$$\sum_{j} \Delta_j (y_j - \zeta_j) \ge (n-1)\delta(Y')$$
$$\delta(Y') \le \frac{1}{n-1} (\sum_{j} (\Delta_j^2/p_j))^{1/2} (\sum_{j} p_j (y_j - \zeta_j)^2)^{1/2} \le \frac{32(\mu + \sigma)r}{n} \text{MAX}_j |\Delta_j/\sqrt{p_j}|$$
(18)

where we have used the fact that  $-\zeta_j \leq y_j - \zeta_j \leq 2\zeta_j^2 \leq 2\zeta_j$ . Let  $(i-1)p_j + \Delta'_j$ and  $(n-i)p_j + \Delta''_j$  respectively be the number of items of size  $\zeta_j$  among  $Y_1, Y_2, \ldots, Y_{i-1}$  and  $Y_{i+1}, \ldots, Y_n$ . Since  $\Delta''_j$  is the sum of n-i i.i.d. random variables, each taking on value  $-p_j$  with probability  $1 - p_j$  and  $1 - p_j$  with probability  $p_j$ , we have  $E(\Delta''_j)^2 = \operatorname{Var}(\Delta''_j) \leq np_j$ . Consider the event

$$\mathcal{E}_i: |\Delta'_j| \le 100\sqrt{p\ln(10p/\mu)p_j(i-1)} \quad \forall j.$$

p is to be specified later, but will satisfy  $p \leq \frac{1}{10}n(\mu^3 + \sigma^2)$ . The expected number of "successes" in the i - 1 Bernoulli trials is  $p_j(i - 1)$ . To get an upper bound on  $\delta_i = \Pr(\neg \mathcal{E}_i)$ , note that

$$\Delta'_j = \sum_{k=1}^{i-1} W_k \quad \text{where } W_k = \chi(Y_k = \zeta_j) - p_j.$$

We will use Corollary (2).  $|W_k| \leq 1$  and  $\operatorname{Var}(W_k) \leq p_j$ . Let  $\sigma = \sqrt{p_j}$ . For  $l \geq 2$ ,  $E(W_k^{2l}) \leq p_j \leq \sigma^{2l} n^{l-1} / p^{l-1}$  assuming  $p \leq np_j$ . Applying the Corollary, we get with some manipulation that

$$\delta_i \le \mu^{4p} p^{-4p}$$

Using (17) and (18), we get

$$E(Z_i|Y_1, Y_2, \dots, Y_{i-1}; \mathcal{E}_i) \ge \mu - E(\delta|Y_1, Y_2, \dots, Y_{i-1}; \mathcal{E}_i)$$
  
$$\ge \mu - \frac{32\mu r}{n} E(\max_j \frac{1}{\sqrt{p_j}} (100\sqrt{p \ln(10p/\mu)p_j(i-1)} + (E(\Delta'')^2)^{1/2}))$$
  
$$\ge \mu - c\mu^{5/2} r \sqrt{\ln(10p/\mu)} - \frac{c\mu r}{\sqrt{n}}$$

So, we get recalling (16) and using  $\operatorname{Var} Z_i = E Z_i^2 + (E Z_i)^2$ 

$$\operatorname{Var}(Z_i|Y_1, Y_2, \dots, Y_{i-1}; \mathcal{E}_i) \le c(\mu^3 + \sigma^2) + c\mu^{7/2}r\sqrt{\ln(10p/\mu)} + \frac{c\mu^2 r}{\sqrt{n}} = c(\mu^3 + \sigma^2),$$

using  $\frac{r}{\sqrt{n}} \le \mu \le \frac{1}{r^2 \log n}$ . Also, we have

$$\operatorname{Var}(Z_i|Y_1, Y_2, \dots, Y_{i-1}) \le E(Z_i^2|Y_1, Y_2, \dots, Y_{i-1}) \le c\mu^2$$

We now appeal to (15) to see that these also give upper bounds on  $\operatorname{Var}(X_i)$ . Note that  $|Z_i| \leq 1$  implies that  $L_{i,2l} \leq L_{i,2}$ . Now to apply the Theorem, we have

$$L_{i,2l} \le c(\mu^3 + \sigma^2).$$

So the "L terms" are bounded as follows :

$$\sum_{l=1}^{p/2} \frac{p^{1-(1/l)}}{l^2} \left(\sum_{i=1}^n L_{i,2l}\right)^{1/l} \le \sum_{l=1}^{p/2} \frac{p}{l^2} \left(\frac{cn(\mu^3 + \sigma^2)}{p}\right)^{1/l} \le cn\left(\mu^3 + \sigma^2\right)$$

noting that  $p \leq n(\mu^3 + \sigma^2)$  implies that the maximum of  $((n/p)(\mu^3 + \sigma^2)^{1/l})$  is attained at l = 1 and also that  $\sum_l (1/l^2) \leq 2$ .

Now, we work on the *M* terms in the Theorem.  $\max_i \delta_i \leq \mu^{4p} p^{-4p} = \delta^*$  (say).

$$\sum_{l=1}^{p/2} (1/n) \sum_{i=1}^{n} (n \hat{M}_{i,2l})^{p/2l} = \sum_{l=1}^{p/2} e^{h(l)},$$

where  $h(l) = \frac{p}{2l} \log n + \frac{p}{l(p-2l+2)} \log \delta^*$ . We have  $h'(l) = -\frac{p}{2l^2} \log n - \log \delta^* \frac{p(p-4l+2)}{l^2(p-2l+2)^2}$ . Thus for  $l \ge (p/4) + (1/2)$ ,  $h'(l) \le 0$  and so h(l) is decreasing. Now for l < (p/4) + (1/2), we have  $\frac{p}{2l^2} \log n \ge -(\log \delta^*) \frac{p(p-4l+2)}{l^2(p-2l+2)^2}$ , so again  $h'(l) \le 0$ . Thus, h(l) attains its maximum at l = 1, so  $(36p)^{p+2} \sum_{l=1}^{p/2} e^{h(l)} \le p(36p)^{p+3} n^{p/2} \delta^*$  giving us

$$(36p)^{p+2} \sum_{l=1}^{p/2} (n\hat{M}_{2l}^*)^{p/2l} \le (cnp(\mu^3 + \sigma^2))^{p/2}$$

Thus we get from the Main Theorem that

$$E(f - Ef)^p \le (cpn(\mu^3 + \sigma^2))^{p/2},$$

from which Theorem (4) follows by the choice of  $p = \lfloor \frac{t^2}{c_5 n(\mu^3 + \sigma^2)} \rfloor$ .

### 5.1 Lower Bound on Spread for Bin Packing

Suppose again  $Y_1, Y_2, \ldots, Y_n$  are the i.i.d. items. Suppose the distribution is :

$$\Pr\left(Y_1 = \frac{k-1}{k(k-2)}\right) = \frac{k-2}{k-1}$$
$$\Pr\left(Y_1 = \frac{1}{k}\right) = \frac{1}{k-1}$$

This is a "perfectly packable distribution" (well-studied class of special distributions) (k-2 of the large items and 1 of the small one pack.) Also,  $\sigma$  is small. But we can have number of 1/k items equal to

$$\frac{n}{k-1} - c\sqrt{\frac{n}{k}}$$

Number of bins required  $\geq \sum_{i} X_{i} = \frac{n}{k} + \frac{n}{k(k-1)} + c\sqrt{\frac{n}{k}} \left(\frac{1}{k} \left(\frac{k-1}{k-2} - 1\right)\right) \geq \frac{n}{k-1}$ . So at least  $c\sqrt{\frac{n}{k}}$  bins contain only (k-1)/k(k-2) sized items (the big items). The gap in each such bin is at least 1/k for a total gap of  $\Omega(\sqrt{n}/k^{3/2})$ . On the other hand, if the number of small items is at least n/(k-1), then each bin except two is perfectly fillable.

### 6 Chromatic Number

Martingale inequalities have been used in two different ways on the chromatic number  $\chi$  of a random graph  $G_{n,\nu}$ , where each edge is chosen independently to be in with probability  $\nu$ . The first is to prove concentration of  $\chi$  following Shamir and Spencer's work, [24], see [3] for details. The second was to actually pin down the value of  $\chi$  following Bollobás [4]. For  $\nu \in \Omega(1)$ ,  $\chi$ is known to be a.s. concentrated in an interval of length  $\omega(n)\sqrt{n}$  ( $\omega(n)$ is any function going to infinity). For  $\nu \in O(n^{-\alpha})$ , and  $\alpha \in [0, 1/2)$ , the a.s. concentration interval is of length  $O(\nu\omega(n)\sqrt{n}\log n)$  and for  $\alpha < 1/2$  of length O(1). We stress that these are all a.s. concentration results. There seem to be very few results giving sub-Gaussian (or even exponential) tail probabilities. One simple observation in this regard is that since changing one vertex changes the chromatic number by at most 1, classical Martingale inequalities imply bounds of the form

$$\Pr(|\chi - E\chi| \ge t) \le c_1 e^{-c_2 t^2/n}.$$
(19)

Talagrand generalizes this in certain directions, but his result (which has a complicated term in the conclusion) does not seem to yield better tail bounds in general for the sparse case (when  $\nu \in o(1)$ ).

Here, we prove (what seems to be the first) sub-Gaussian tail bounds for all values of  $\nu$ . In essence, our result replaces the n in (19) by  $n\nu$ . We conjecture that this is the best possible.

**Theorem 5.** For any  $t \in (0, cn\nu)$ , we get

$$Pr(|\chi - E\chi| \ge t) \le c_1 e^{-ct^2/(n\nu(\log(1/\nu))^2)}.$$

Proof. (c denotes a generic constant.) Let  $s = \lceil 1/\nu \rceil$ ; divide the *n* vertices into r = (n/s) groups -  $G_1, G_2, \ldots, G_r$  of *s* vertices each. Define  $Y_i$  for  $i = 1, 2, \ldots, n\nu$  as the set of edges in  $G_i \times (G_1 \cup G_2 \cup \ldots, G_{i-1})$ . We can define the Doob's Martingale  $X_i = E(\chi | Y_1, Y_2, \ldots, Y_i) - E(\chi | Y_1, Y_2, \ldots, Y_{i-1})$ . Also define  $Z_i$  as in section 4 by

$$Z_i = \chi(G, Y_1, Y_2, \dots, Y_n) - \chi(G, Y_1, Y_2, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n)$$

Let  $d_j$  be the degree of vertex j in  $G_i$  in the graph induced on  $G_i$  alone. It is easy to see that  $Z_i$  is at most  $\max_{j \in G_i} d_j$ , since we can always color  $G_i$ with this many additional colors. By traditional Chernoff, it follows that for a fixed  $j \in G_i$ ,

$$\Pr(d_i \ge \lambda) \le e^{-\lambda/2} \text{ for } \lambda \ge 4s\nu_i$$

Thus

$$\Pr(Z_i \ge \lambda) \le se^{-\lambda/2} \text{ for } \lambda \ge 4s\nu.$$

Hence, (after a suitable integration), we get for any l,

$$E(Z_i^{2l}|Y_1, Y_2, \dots, Y_{i-1}) \le (c_1 l)^{2l} + (c_2 \ln s)^{2l}.$$
(20)

Let  $t \in (0, n\nu)$ . We take p = the even integer nearest to  $t^2/(cn\nu(\ln s)^2)$ . It is easy to check that n > sp. From (20), we may apply the Theorem with  $L_{i,2l} = (cl)^{2l} + (c\ln s)^{2l}$ , whence we get

$$\frac{1}{l^2} \left(\frac{\sum_i L_{i,2l}}{p}\right)^{1/l} \le (c \ln s)^2 \left(\frac{n}{sp}\right)^{1/l} \le (c \ln s)^2 n/(sp),$$

and hence we get

$$E(\chi - E\chi)^p \le (cp)^{p/2} (n(\ln s)^2/s)^p.$$

Now Theorem (5) follows by Markov.

### 7 Number of Triangles in a random graph

Again consider the random graph  $G_{n,\nu}$  and let  $f(G_{n,\nu})$  be the number of triangles in the graph. The concentration of f is well-studied. It has been pointed out that (see [19], [20] for a discussion) Talagrand's inequality cannot prove good concentration when  $\nu$  the edge probability is small. [But we assume that  $n\nu \geq 1$ , so that  $Ef = O(n^3\nu^3)$  is  $\Omega(1)$ .] For  $\nu \leq 1/\sqrt{n}$ , it is known (by a simple calculation) that

$$\mathbf{Var}f = O(n^3\nu^3).$$

One would like sub-Gaussian tail bounds on f, i.e. :

**Question** Assuming  $\nu \leq 1/\sqrt{n}$ , can we prove  $\Pr\left(|f - Ef| \geq t(n\nu)^{3/2}\right) \leq ce^{-c't^2}$  for  $t \in (0, a)$  for a suitable a?

This question (raised in a slightly different form for example by Vu [25]) is still open. But by now there are sharp results for larger deviations. The most popular question on these lines has been to prove upper bounds on  $\Pr(f \ge (1 + \epsilon)Ef)$  for essentially  $\epsilon \in \Omega(1)$  (see [19],[20]). In a culmination of this line of work, [13] have proved that

$$\Pr\left(f \ge (1+\epsilon)Ef\right) \le ce^{-c\epsilon^2 n^2 \nu^2}.$$

(This is a special case of their theorem on the number of copies of any fixed graph in  $G_{n,\nu}$ .) The dependence on  $\epsilon$  here has been calculated from their proof. It is not spelt out; instead their focus is on  $\epsilon \in \Omega(1)$ , whence they show that their result is best possible within log factors. But in the Question above,  $\epsilon$  would be  $O(1/(n\nu)^{3/2})$  whence the result of [13] does not help us at all. The situation here is in total contrast to that concerning the chromatic number; for the number of triangles problem, (and more generally, number of copies of any fixed graph in  $G_{n,\nu}$ ) much attention has been paid to getting the correct tail probabilities at the cost of looking only at larger deviations rather than proving tight intervals of concentration.

Here we provide a tail bound which is exponential in the number of standard deviations; it is operative even for deviations as small as a constant number of standard deviations from the mean. But it falls only as  $e^{-O(t)}$  and not  $e^{-O(t^2)}$ ; so it is not as effective as the question demands for larger than constant standard deviations from the mean. It also assumes  $\nu \leq n^{-3/5}$ .

**Theorem 6.** Suppose  $c \log n/n \le \nu \le cn^{-3/5}$ . Then, for  $t \in [10, O(n\nu)]$ , we have

$$Pr(|f - Ef| \ge t(n\nu)^{3/2}) \le c_1 e^{-ct}.$$

*Proof.* Let  $Y_i$  be the set of neighbours of vertex i among [i-1] and imagine adding the  $Y_i$  in order. [This is often called the vertex-exposure Martingale.] We will also let  $Y_{ij}$  be the 0-1 variable denoting whether there is an edge between i and j for j < i. The number of triangles f can be written as  $f = \sum_{i>j>k} Y_{ij}Y_{jk}Y_{ik}$ . We will first need a simple fact which we state without proof.

**Claim 1.** If Z is the sum of m Bernoulli random variables, each with expectation  $\nu$ , then for any q,  $EZ^q \leq q^{q+1} + q(4m\nu)^q$ .

As usual consider the Doob Martingale difference sequence

$$X_i = E(f|Y_1, Y_2, \dots, Y_i) - E(f|Y_1, Y_2, \dots, Y_{i-1}).$$

It is easy to see that

$$X_{i} = \sum_{j < k \in [i-1]} Y_{jk} (Y_{ij} Y_{ik} - \nu^{2}) + (n-i)\nu^{2} \sum_{j < i} (Y_{ij} - \nu) = X_{i,1} + X_{i,2}$$
(say).

Let  $\tilde{E}$  denote  $E(\cdot|Y_1, Y_2, \ldots, Y_{i-1})$ . Let  $q = O(n\nu)^{3/2}$ . [We will avoid giving exact constants in this section. So,  $A \leq cB$  or  $A \in O(B)$  is shorthand for "there exists a constant c such that  $A \leq cB$ " holds, with possibly different constants each time the statement is made.]  $\tilde{E}(X_i^q) \leq 2^q \tilde{E}(X_{i,1}^q) + 2^q \tilde{E}(X_{i,2}^q)$ . Of the two, it is much easier to deal with  $X_{i,2}$ . Indeed we have using the claim

$$\tilde{E}(X_{i,2}^q) \le c^q n^q \nu^{2q} E(\sum_{j < i} Y_{ij})^q \le (cn\nu^2)^q (q(n\nu)^q + q^{q+1}) \le (O(n^{2.5}\nu^{3.5}))^q.$$
(21)

Now we deal with  $X_{i,1}$ . Let  $N(i) = \{j < i : Y_{ij} = 1\}$ . Let  $Z = \sum_{j < k \in N(i)} Y_{jk}$  be the number edges in  $N(i) \times N(i)$ . Define

$$\mathcal{E}_i : |N(k)| \le c n^{3/2} \nu^{3/2} \ \forall k < i \ ; \ \sum_{j,k < i} Y_{jk} \le 10 n^2 \nu \ ; \ \text{and} \ Z \le c n^{3/2} \nu^{3/2}.$$

Let  $\delta_i = \Pr(\neg \mathcal{E}_i)$ . Using the Claim, we have

$$EZ^{q} = E_{Y_{i}}E(Z^{q}|N(i)) \le qE_{Y_{i}}(|N(i)|^{2}\nu)^{q} + q^{q+1}$$
  
$$\le q\nu^{q}E|N(i)|^{2q} + q^{q+1} \le q^{2}\nu^{q}((n\nu)^{2q} + (2q)^{2q+1}) + q^{q+1} \le (O(n\nu))^{3q/2},$$

using  $\nu < n^{-3/5}$ . Hence,  $\Pr(Z > cn^{3/2}\nu^{3/2}) \leq c^{-q}$ . It is not difficult to see using the claim that the probability of the other events in  $\mathcal{E}_i$  are also bounded so as to get  $\delta_i \leq c^{-q}$ . Now we bound moments under  $\mathcal{E}_i$  starting with the second moment.

$$\tilde{E}(X_{i,1}^2) \le \sum_{j_1, j_2, k_1, k_2} Y_{j_1 k_1} Y_{j_2 k_2} E(Y_{i j_1} Y_{i k_1} - \nu^2) (Y_{i j_2} Y_{i k_2} - \nu^2)$$
(22)

If  $|\{j_1, j_2, k_1, k_2\}| = 4$ , then the term is 0. Terms with  $|\{j_1, j_2, k_1, k_2\}| = 3$  sum to at most

$$4\sum_{j_{1},j_{2},k}\nu^{3}Y_{j_{1},k}Y_{j_{2},k} \le 8\nu^{3}\sum_{k< i}|N(k)|^{2}.$$

Under  $\mathcal{E}_i$ ,  $\sum_{k < i} |N(k)|^2 \leq 2 \text{MAX}_k |N(k)| \sum_{j,k} Y_{jk} \leq n^{3.5} \nu^{2.5}$ . We thus get that these terms contribute at most  $n^{3.5} \nu^{5.5}$ . It is easier to see that the terms with  $|\{j_1, j_2, k_1, k_2\}| = 2$  contribute at most  $\nu^2 \sum_{j,k} Y_{jk} \leq n^2 \nu^3$ . Thus,

$$\tilde{E}(X_{i,1}^2|\mathcal{E}_i) \le O(n^2\nu^3).$$

Now we tackle higher moments. Noting that  $|X_{i,1}| \leq Z + \nu^2 \sum_{j,k} Y_{jk} \in O(n^{3/2}\nu^{3/2})$  under  $\mathcal{E}_i$ , we have for l,  $\tilde{E}(X_{i,1}^{2l}|\mathcal{E}_i) \leq \tilde{E}(X_{i,1}^2|\mathcal{E}_i)(n\nu)^{3(l-1)} \leq (n\nu)^{3l}/n$ . Now set

$$p = ct$$

and note that  $p \leq q$ . We set

$$\mathcal{E}_{i,l} = \mathcal{E}_i \text{ and } \delta_{i,l} = \delta_i \text{ for } l = 1, 2, \dots l_0 = \min(p/2, \sqrt{n\nu})$$
  
$$\mathcal{E}_{i,l} = \text{ whole space and } \delta_{i,l} = 0 \text{ for } l = l_0 + 1, \dots p/2.$$

Note that always

$$\tilde{E}(X_i^{2l}) \le E|N(i)|^{4l} \le 2l(n\nu)^{4l} + 2l(4l)^{4l},$$
(23)

using again the claim. We have (with  $L_{i,2l} = \tilde{E}(X_i^{2l}|\mathcal{E}_i))$ ,

$$\sum_{l=1}^{l_0} \left( (p/l^2) \left( \frac{\sum_i L_{i,2l}}{p} \right)^{1/l} \right) \le p(n\nu)^3.$$
 (24)

Also,

$$\sum_{l=l_0+1}^{p/2} \left( (p/l^2) \left( \frac{\sum_i L_{i,2l}}{p} \right)^{1/l} \right) \le 2 \sum_{l=l_0}^{p/2} cp \left( l^2 + \frac{(n\nu)^4}{l^2} \right) \le p(n\nu)^3, \quad (25)$$

the last since  $(l^2 + ((n\nu)^4/l^2))$  is a convex function of  $l^2$  and its maximum over the range is attained at one of the extremes. Now, we have to bound the *M* terms in the Theorem. To this end, note that for  $l \leq l_0/2$ , we get from (23),  $\hat{M}_{i,2l} \leq (cn\nu)^{4l} c^{-q/(p-l+2)}$ , so

$$(n\hat{M}_{i,2})^{p/2l} \in O(n\nu)^{3p/2}.$$
 (26)

Putting (24,25 and 26) into the Theorem, we get

$$E|f - Ef|^p \le (cp)^p (n\nu)^{3p/2}.$$

Now the theorem follows by Markov inequality.

### 8 Number of s- cliques in $G(n, \nu)$ for $s \to \infty$

Much is known about the number of copies a FIXED graph H in  $G(n, \nu)$ (Janson and Rucinski's [15],[12] and Kim and Vu's [19],[20] results pertain to this case.) However, in G(n, 1/2) for example, the largest clique size in  $\Omega(\log n)$  None of the previous results apply to such cases. In Kim and Vu's results, the clique size dictates the degree of the polynomial; as they point out, their results do not seem to extend to degrees which are more than  $O(\log n/\log \log n)$ . Note that the degree of the polynomial is  $\binom{s}{2}$  for clique size s. Here we make a beginning towards handling cliques of size  $O(\log n)$ - We are able to handle cliques of size almost upto  $O(\log n)$ , namely we handle here cliques of size  $\log n/\omega(n)$  for any  $\omega(n) \to \infty$ , but so far only for a restricted range of parameters. We show that the probability that there are more than 3/2 times the expected number of such cliques in a random graph is  $n^{-\omega'(n)}$  for some  $\omega' \to \infty$ . Let f(G) denote the number of s cliques in  $G(n, \nu)$ . To clarify the definition of f, we point out that

$$E(f(G(n,\nu))) = \binom{n}{s} \nu^{s(s-1)/2} \approx \frac{n^s}{s!} \nu^{s(s-1)/2}.$$

We just write Ef for  $E(f(G(n, \nu)))$ 

Theorem 7. Let

$$s = \frac{\ln n}{\omega(n)} \quad ; \quad \nu = \exp\left(-2\omega(n) + \frac{\lambda\omega(n)^2}{\ln n} + \frac{2\omega(n)\ln\ln n}{\ln n}\right)$$

with  $1 + \frac{1}{20\omega^{1/4}} < \lambda < 2$ .

$$Pr(f \ge 2Ef) \le n^{-(\lambda - 1)\omega^{1/4}/10}$$

First note that the condition on  $\nu$  is sensible; it essentially says that  $\nu$  is at most roughly a constant and at least roughly another constant, if  $\omega(n)$  grows slowly enough. Also, by direct calculation,

$$Ef = n^{\frac{\lambda}{2} + o(1)},$$

(which also says this range of parameters is sensible; much smaller  $\nu$  will make Ef go below 1.)

*Proof.* Consider the independent random variables  $Y_{ij}$ ,  $1 \le j < i \le n$  which are Bernoulli (with  $Y_{ij} = 1$  iff edge  $(i, j) \in G(n, \nu)$ ). We will now consider the vertex-exposure Martingale. For this, we let  $Y_i$  denote the set of edges from vertex *i* to previous vertices. The  $Y_i$  are of course independent. We again use Doob's Martingale :

$$X_i = E(f|Y_1, Y_2, \dots, Y_i) - E(f|Y_1, Y_2, \dots, Y_{i-1}).$$

We may expand

$$f = \sum_{S; |S|=s} \prod_{j \neq k \in S} Y_{jk}.$$

It is easy to see that if  $i \notin S$ , then the term contributes 0 to  $X_i$ . Thus,

$$X_{i} = \sum_{|S|=s-1; i \notin S} \left( E \prod_{j,k \in S; j > i,k} Y_{jk} \right) \prod_{k < j \in S \cap [i-1]} Y_{jk} \left( \prod_{j \in S \cap [i-1]} Y_{ij} - \nu^{|S \cap [i-1]|} \right)$$

Let  $q = \sqrt{\omega(n)}$ . Consider  $X_i^q$  for q an even positive integer. We just expand this below. [The sum below is taken over  $S_1, S_2, \ldots, S_q$  all of cardinality s - 1 and not containing i and possibly intersecting.]

$$EX_{i}^{q} = \sum_{S_{1},S_{2},\dots,S_{q}} \prod_{r=1}^{q} \left( E \prod_{j,k\in S_{r};j>i,k} Y_{jk} \right) E \prod_{r=1}^{q} \prod_{k< j\in S_{r},j  
$$\leq 2^{q} \sum_{S_{1},S_{2},\dots,S_{q}} E \prod_{r=1}^{q} \prod_{k< j\in S_{r}\cup\{i\}} Y_{jk}.$$$$

We will upper bound  $\sum_{S_1,S_2,\ldots,S_q} E \prod_{r=1}^q \prod_{k < j \in S_r \cup \{i\}} Y_{jk}$  by considering the effect of each  $S_r$ , one at a time. First, for  $r = 2, 3, \ldots, q$ , let

$$s_r = |S_r \setminus \bigcup_{t=1}^{r-1} S_t|$$

be the number of "new" elements in  $S_r$ . The number of possible  $S_r \setminus \bigcup_{t=1}^{r-1} S_t$ satisfying  $s_r = |S_r \setminus \bigcup_{t=1}^{r-1} S_t|$  is at most  $\binom{n}{s_r} \leq n^{s_r}/s_r!$ . For each such  $S_r \setminus \bigcup_{t=1}^{r-1} S_t$ , there are at most  $2^{qs_r}$  ways of these  $s_r$  new vertices belonging to the later  $S_t$  (for t > r). [Note that since we will be paying this factor of  $2^{qs_r}$ for each  $r \geq 2$ , for  $r \geq 3$ , the vertices of  $S_r$  which are in  $S_2, S_3, \ldots, S_{r-1}$  are already fixed by the earlier  $S_t$ .] For the s - 1 vertices of  $S_1$ , we follow a different procedure : for each  $j \in S_1$ , we write down the list of  $S_r, r \geq 2$  to which j **does not** belong; this is written in "sparse" representation - i.e., we write down only the  $\log q$  bit address of each  $S_r$  to which j does not belong. We argue that the total number of such  $\log q$  bit strings written down for all  $j \in S_1$  is at most  $(q-1) \sum_{r\geq 2} s_r$  : to see this, let

$$R = \{(j,t) : j \in S_r \setminus \bigcup_{r'=1}^{r-1} S_{r'}; r \ge 2; t \ge r; j \in S_t\}$$
  

$$R_1 = \{(j,r) : j \in S_1; r \ge 2; j \notin S_r\}$$
  

$$R'_1 = \{(j,r) : j \in S_1; r \ge 2; j \in S_r\}.$$

It is easy to see that

$$|R_1| + |R'_1| = (q-1)(s-1) = |R'_1| + |R|$$
; Thus,  $|R| = |R_1| \le (q-1)\sum_{r\ge 2} s_r$ ,

as claimed. So, the number of bits describing  $R_1$  is at most  $q \log q \sum_{r \ge 2} s_r$ which implies that the number of possible  $R_1$  is at most  $q^{q \sum_{r \ge 2} s_r}$ . Now, it is easy to see that there are at least  $(1/2)s(s-1) - (1/2)(s-1-s_r)(s-s_r) = ss_r - (s_r/2) - (s_r^2/2)$  "new" edges in  $S_r \times S_r$  (i.e., edges not in any previous  $S_t \times S_t$ ) since at most  $(1/2)(s-1-s_r)(s-s_r)$  edges could be between two "old" vertices. Thus  $S_r$  contributes at most  $\nu^{ss_r-(s_r/2)-(s_r^2/2)}$  extra factor to  $E \prod_{r=1}^q \prod_{k < j \in S_r} Y_{jk}$ . Thus, with

$$A_r = \frac{n^{s_r}}{s_r!} 2^{qs_r} q^{qs_r} \nu^{ss_r - (s_r/2) - (s_r^2/2)}, \text{ for } r \ge 2$$

and

$$A_1 = \frac{n^{s-1}}{(s-1)!} \nu^{(1/2)s(s-1)} = \frac{s}{n} Ef,$$

we see that

$$EX_i^q \le \sum_{s_2, s_3, \dots s_q} \prod_{r \ge 1} A_r.$$

We will argue that each  $A_r, r \ge 2$  is maximized when  $s_r = 0$  or  $s_r = s - 1$  (at the extremes) by showing that the  $\ln A_r$ , namely

$$g(a) = a \ln n + qa \ln(2q) + (la - (a/2)) \ln \nu - (1/2)a^2 \ln \nu - \ln a!$$

is essentially a convex function of a. Indeed, it is easy to see by direct calculation (assuming  $\nu < 1/2$ ) that :

$$g(a+1) - 2g(a) + g(a-1) \ge 0$$
 for  $1 \le a \le s - 1$ ,

from which it follows that  $g(a) \leq MAX(g(a-1), g(a+1))$  for  $0 \leq a \leq s-1$ and so we have

$$A_r \le MAX(1, \frac{s}{n}(Ef)(2q)^{qs}) \quad \forall r.$$

Also we see that  $A_1 = \frac{s}{n}(Ef)$ . Now, it is easy to see that for the parameter settings, (noting that  $\lambda < 2$ ,) we have  $\frac{s}{n}(2q)^{qs}Ef \leq 1$ . Thus, we have

$$EX_i^q \le 1 \forall q.$$

So we may apply Corollary (3) with Q = 1, to get with  $p = (\omega(n))^{1/4}$ ,

$$E(f - Ef)^p \le (cp)^p n^{p/2}$$
  
 $\Pr(f \ge 2Ef) \le \frac{(cp)^p n^{p/2}}{(Ef)^p} \le n^{-(\lambda - 1)\omega^{1/4}/3}$ 

since  $\lambda > 1 + \frac{1}{20\omega^{1/4}}$ .

**Remark 5.** Note that  $X_i$  is at most the number of s - 1 cliques among the neighbours of vertex *i*. We are bounding this. A question which arises is : why not directly bound the moments of the number of *s* cliques instead ? But note that we have  $Ef^q \ge (Ef)^q$  and so such a bound (via Markov) would at best only yield the trivial inequality  $Pr(f \ge 2Ef) \le (Ef)^p/(Ef)^p = 1$  ! What we are really doing is getting "crude" bounds on moments of numbers of s - 1 cliques which give us good bounds on "central moments" (about the mean) of the number of *s* cliques via our Theorem.

### 9 Longest Increasing Subsequence

Let  $Y_1, Y_2, \ldots, Y_n$  be i.i.d., each distributed uniformly in [0, 1]. An increasing subsequence (IS) in  $Y = (Y_1, Y_2, \ldots, Y_n)$  consists of  $1 \leq i_1 < i_2 < i_3 \ldots < i_k \leq$ n with  $Y_{i_1} < Y_{i_2} < \ldots < Y_{i_k}$ ; k is the length of this sequence. We consider here f(Y) = the length of the longest increasing subsequence (LIS) of Y. This is a well-studied problem. It is known that  $Ef = (2 + o(1))\sqrt{n}$ . Here we supply a (fairly simple) proof from Theorem (1) that f is concentrated in an interval of length  $O(n^{1/4})$  with sub-Gaussian tails. A similar result is also obtained by Talagrand [22]. [But by now better intervals of concentration, namely  $O(n^{1/6})$  are known, but using detailed arguments specific to this problem [2].] Our argument follows from two claims below. Call  $Y_i$  essential for Y if  $Y_i$  belongs to every LIS of Y (equivalently,  $f(Y \setminus Y_i) = f(Y) - 1$ .) Fix  $Y_1, Y_2, \ldots, Y_{i-1}$  and for  $j \geq i$ , let  $a_j = \Pr(Y_j$  is essential for  $Y | Y_1, Y_2, \ldots, Y_{i-1})$ 

Claim 2.  $a_i, a_{i+1}, \ldots, a_n$  form a non-increasing sequence.

Proof. Let  $j \geq i$ . Consider a point  $\omega$  in the sample space where  $Y_j$  is essential, but  $Y_{j+1}$  is not. Map  $\omega$  onto  $\omega'$  by swapping the values of  $Y_j$  and  $Y_{j+1}$ ; this is clearly a 1-1 measure preserving map. If  $\theta$  is a LIS of  $\omega$  with  $j \in \theta, j+1 \notin \theta$ , then  $\theta \setminus j \cup j + 1$  is an increasing sequence in  $\omega'$ ; so  $f(\omega') \geq f(\omega)$ . If  $f(\omega') = f(\omega) + 1$ , then an LIS  $\alpha$  of  $\omega'$  must contain both j and j + 1 and so contains no k such that  $Y_k$  is between  $Y_j, Y_{j+1}$ . Now  $\alpha \setminus j$  is an LIS of  $\omega$ contradicting the assumption that j is essential for  $\omega$ . So  $f(\omega') = f(\omega)$ . So, j + 1 is essential for  $\omega'$  and j is not. So,  $a_j \leq a_{j+1}$ .

**Remark** In fact, one can show that  $a_i = a_{i+1} = \ldots a_n$ ; we do not need this.

Claim 3.  $a_i \le c/\sqrt{n-i+1}$ .

*Proof.*  $a_i \leq \frac{1}{n-i+1} \sum_{j\geq i} a_j$ . Now  $\sum_{j\geq i} a_j = a$  (say) is the expected number of essential elements among  $Y_i, \ldots, Y_n$  which is clearly at most  $Ef(Y_i, Y_{i+1}, \ldots, Y_n) \leq 4\sqrt{n-i+1}$ , so the claim follows.

Now we will apply Theorem (1) to the Doob's Martingale  $X_i = E(f|Y_1, Y_2, \ldots, Y_i) - E(f|Y_1, Y_2, \ldots, Y_{i-1})$ . Define  $Z_i$  to be  $f(Y) - f(Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$ .  $Z_i$  is a 0-1 random variable with  $E(Z_i|Y_1, Y_2, \ldots, Y_{i-1}) \leq c/\sqrt{n-i+1}$ . Thus it follows (using (15) of section (4)) that

$$E(X_i^2|Y_1, Y_2, \dots, Y_{i-1}) \le c/\sqrt{n-i+1}.$$

Clearly,  $E(X_i^l|Y_1, Y_2, \ldots, Y_{i-1}) \leq E(X_i^2|Y_1, Y_2, \ldots, Y_{i-1})$  for  $l \geq 2$ , even. Thus we may apply the Theorem with  $\mathcal{E}_{il}$  equal to the whole sample space. Assuming  $p \leq \sqrt{n}$ , we see that (using  $\sum_l (1/l^2) = O(1)$ )

$$E(f - Ef)^p \le (c_1 p)^{(p/2)+2} n^{p/4},$$

from which one can derive the asserted sub-Gaussian bounds.

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