Local Distributed Decision *

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Abstract

A central theme in distributed network algorithms concerns understanding and coping with the issue of *locality*. Despite considerable progress, research efforts in this direction have not yet resulted in a solid basis in the form of a fundamental computational complexity theory. Inspired by sequential complexity theory, we focus on a complexity theory for *distributed decision problems*. In the context of locality, solving a decision problem requires the processors to independently inspect their local neighborhoods and then collectively decide whether a given global input instance belongs to some specified language.

We consider the standard \mathcal{LOCAL} model of computation and define $\mathrm{LD}(t)$ (for local decision) as the class of decision problems that can be solved in t number of communication rounds. We first study the intriguing question of whether randomization helps in local distributed computing, and to what extent. Specifically, we define the corresponding randomized class $\mathrm{BPLD}(t,p,q)$, containing languages for which there exists a randomized algorithm that runs in t rounds and accepts correct instances with probability at least p and rejects incorrect ones with probability at least q. We show that there exists a language that does not belong to $\mathrm{LD}(t)$ for any t=o(n) but which belong for $\mathrm{BPLD}(0,p,q)$ for any $p,q\in(0,1]$ such that $p^2+q\leq 1$. On the other hand, we show that, restricted to hereditary languages, $\mathrm{BPLD}(t,p,q)=\mathrm{LD}(O(t))$, for any function t and any $p,q\in(0,1]$ such that $p^2+q>1$.

In addition, we investigate the impact of non-determinism on local decision, and establish some structural results inspired by classical computational complexity theory. Specifically, we show that non-determinism does help, but that this help is limited, as there exist languages that cannot be decided non-deterministically. Perhaps surprisingly, it turns out that it is the combination of randomization with non-determinism

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that enables to decide all languages in constant time. Finally, we introduce the notion of local reduction, and establish some completeness results.

Keywords: Local distributed algorithms, local decision, nondeterminism, randomized algorithms.

1 Introduction

1.1 Motivation

Distributed computing concerns a collection of processors which collaborate in order to achieve some global task. With time, two main disciplines have evolved in the field. One discipline deals with timing issues, namely, uncertainties due to asynchrony (the fact that processors run at their own speed, and possibly crash), and the other concerns topology issues, namely, uncertainties due to locality constraints (the lack of knowledge about far away processors). Studies carried out by the distributed computing community within these two disciplines were to a large extent problem-driven. Indeed, several major problems considered in the literature concern coping with one of the two uncertainties. For instance, in the asynchrony-discipline, Fischer, Lynch and Paterson [14] proved that consensus cannot be achieved in the asynchronous model, even in the presence of a single fault, and in the locality-discipline, Linial [28] proved that $(\Delta + 1)$ -coloring cannot be achieved locally (i.e., in a constant number of communication rounds), even in the ring network.

One of the significant achievements of the asynchrony-discipline was its success in establishing unifying theories in the flavor of computational complexity theory. Some central examples of such theories are failure detectors [6, 7] and the wait-free hierarchy (including Herlihy's hierarchy) [18]. In contrast, despite considerable progress, the locality-discipline still suffers from the absence of a solid basis in the form of a fundamental computational complexity theory. Obviously, defining some common cost measures (e.g., time, message, memory, etc.) enables us to compare problems in terms of their relative cost. Still, from a computational complexity point of view, it is not clear how to relate the difficulty of problems in the locality-discipline. Specifically, if two problems have different kinds of outputs, it is not clear how to reduce one to the other, even if they cost the same.

Inspired by sequential complexity theory, we focus on decision problems, in which one is aiming at deciding whether a given global input instance belongs to some specified language. In the context of distributed computing, each processor must produce a boolean output, and the decision is defined by the conjunction of the processors' outputs, i.e., if the instance belongs to the language, then all processors must output "yes", and otherwise, at least one processor must output "no". Observe that decision problems provide a natural framework for tackling fault-tolerance: the processors have to collectively check whether the network is fault-free, and a node detecting a fault raises an alarm. In fact, many natural problems can be phrased as decision problems, like "is there a unique leader in the network?" or "is the network planar?". Moreover, decision problems occur naturally when one is aiming at

checking the validity of the output of a computational task, such as "is the produced coloring legal?", or "is the constructed subgraph an MST?". Construction tasks such as exact or approximated solutions to problems like coloring, MST, spanner, MIS, maximum matching, etc., received enormous attention in the literature (see, e.g., [5, 25, 26, 28, 30, 31, 32, 38]), yet the corresponding decision problems have hardly been considered.

The purpose of this paper is to investigate the nature of local decision problems. Decision problems seem to provide a promising approach to building up a distributed computational theory for the locality-discipline. Indeed, as we will show, one can define local reductions in the framework of decision problems, thus enabling the introduction of complexity classes and notions of completeness.

We consider the \mathcal{LOCAL} model [36], which is a standard distributed computing model capturing the essence of locality. In this model, processors are woken up simultaneously, and computation proceeds in fault-free synchronous rounds during which every processor exchanges messages of unlimited size with its neighbors, and performs arbitrary computations on its data. Informally, let us define $\mathrm{LD}(t)$ (for local decision) as the class of decision problems that can be solved in t number of communication rounds in the \mathcal{LOCAL} model. (We find special interest in the case where t represents a constant, but in general we view t as a function of the input graph. We note that in the \mathcal{LOCAL} model, every decidable decision problem can be solved in n communication rounds, where n denotes the number of nodes in the input graph.)

Some decision problems are trivially in LD(O(1)) (e.g., "is the given coloring a $(\Delta + 1)$ -coloring?", "do the selected nodes form an MIS?", etc.), while some others can easily be shown to be outside LD(t), for any t = o(n) (e.g., "is the network planar?", "is there a unique leader?", etc.). In contrast to the above examples, there are some languages for which it is not clear whether they belong to LD(t), even for t = O(1). To elaborate on this, consider the particular case where it is required to decide whether the network belongs to some specified family \mathcal{F} of graphs. If this question can be decided in a constant number of communication rounds, then this means, informally, that the family \mathcal{F} can somehow be characterized by relatively simple conditions. For example, a family \mathcal{F} of graphs that can be characterized as consisting of all graphs having no subgraph from \mathcal{C} , for some specified finite set \mathcal{C} of finite subgraphs, is obviously in LD(O(1)). However, the question of whether a family of graphs can be characterized as above is often non-trivial. For example, characterizing cographs as precisely the graphs with no induced P_4 , attributed to Seinsche [40], is not easy, and requires nontrivial usage of modular decomposition.

The first question we address is whether and to what extent randomization helps. For

 $p,q\in(0,1]$, define BPLD(t,p,q) as the class of all distributed languages that can be decided by a randomized distributed algorithm that runs in t number of communication rounds and produces correct answers on legal (respectively, illegal) instances with probability at least p (resp., q). An interesting observation is that for p and q such that $p^2 + q \le 1$, we have $\mathrm{LD}(t) \subsetneq \mathrm{BPLD}(t,p,q)$. In fact, for such p and q, there exists a language $\mathcal{L} \in \mathrm{BPLD}(0,p,q)$, such that $\mathcal{L} \notin \mathrm{LD}(t)$, for any t=o(n). To see why, consider the following Unique-Leader language. The input is a graph where each node has a bit indicating whether it is a leader or not. An input is in the language Unique-Leader if and only if there is at most one leader in the graph. Obviously, this language is not in $\mathrm{LD}(t)$, for any t < n. We claim it is in $\mathrm{BPLD}(0,p,q)$, for p and q such that $p^2 + q \le 1$. Indeed, for such p and q, we can design the following simple randomized algorithm that runs in 0 time: every node which is not a leader says "yes" with probability p. Clearly, if the graph has at most one leader then all nodes say "yes" with probability at least p. On the other hand, if there are at least p leaders, at least one node says "no", with probability at least p leaders at least p leaders, at least one node says "no", with probability at least p leaders.

It turns out that the aforementioned choice of p and q is not coincidental, and that $p^2+q=1$ is really the correct threshold. Indeed, we show that Unique-Leader $\notin \mathrm{BPLD}(t,p,q)$, for any t < n, and any p and q such that $p^2+q>1$. In fact, we show a much more general result, that is, we prove that if $p^2+q>1$, then restricted to hereditary languages, $\mathrm{BPLD}(t,p,q)$ actually collapses into $\mathrm{LD}(O(t))$, for any t.

In the second part of the paper, we investigate the impact of non-determinism on local decision, and establish some structural results inspired by classical computational complexity theory. Specifically, we show that non-determinism does help, but that this help is limited, as there exist languages that cannot be decided non-deterministically. Perhaps surprisingly, it turns out that it is the combination of randomization with non-determinism that enables to decide *all* languages in constant time. Finally, we introduce the notion of local reduction, and establish some completeness results.

1.2 Our contributions

1.2.1 Impact of randomization

We study the impact of randomization on local decision. We prove that if $p^2 + q > 1$, then restricted to hereditary languages, BPLD(t, p, q) = LD(O(t)), for any function t. This, together with the observation that $LD(t) \subseteq BPLD(t, p, q)$, for any t = o(n), may indicate that $p^2 + q = 1$ serves as a sharp threshold for distinguishing the deterministic case from the

randomized one.

1.2.2 Impact of non-determinism

We first show that non-determinism helps local decision, i.e., we show that the class NLD(t) (cf. Section 2.3) strictly contains LD(t). More precisely, we show that there exists a language in NLD(O(1)) which is not in LD(t) for every t = o(n), where n is the size of the input graph. Nevertheless, NLD(t) does not capture all (decidable) languages, for t = o(n). Indeed we show that there exists a language not in NLD(t) for every t = o(n). Specifically, this language is $\#n = \{(G, n) \mid |V(G)| = n\}$.

Perhaps surprisingly, it turns out that it is the combination of randomization with nondeterminism that enables to decide *all* languages in constant time. Let BPNLD(O(1)) = BPNLD(O(1), p, q), for some constants p and q such that $p^2 + q \le 1$. We prove that BPNLD(O(1)) contains all languages. To sum up, LD(o(n)) \subseteq NLD(O(1)) \subset NLD(O(n)) \subseteq BPNLD(O(1)) = All.

Finally, we introduce the notion of many-one local reduction, and establish some completeness results. We show that there exits a problem, called cover, which is, in a sense, the most difficult decision problem. That is we show that cover is BPNLD(O(1))-complete. (Interestingly, a small relaxation of cover, called containment, turns out to be NLD(O(1))-complete).

1.3 Related work

Locality issues have been thoroughly studied in the literature, via the analysis of various construction problems, including $(\Delta + 1)$ -coloring and Maximal Independent Set (MIS) [1, 5, 23, 26, 28, 30, 35], Minimum Spanning Tree (MST) [12, 25, 37], Maximal Matching [19], Maximum Weighted Matching [31, 32, 41], Minimum Dominating Set [24, 27], Spanners [9, 13, 38], etc. For some problems (e.g., coloring [5, 23, 35]), there are still large gaps between the best known results on specific families of graphs (e.g., bounded degree graphs) and on arbitrary graphs.

The question of what can be computed in a constant number of communication rounds was investigated in the seminal work of Naor and Stockmeyer [34]. In particular, that paper considers a subclass of LD(O(1)), called LCL, which is essentially LD(O(1)) restricted to languages involving graphs of constant maximum degree, and involving processor inputs taken from a set of constant size, and studies the question of how to compute in O(1) rounds

the constructive versions of decision problems in LCL. The paper provides some beautiful general results. In particular, the authors show that if there exists a randomized algorithm that constructs a solution for a problem in LCL in O(1) rounds, then there is also a deterministic algorithm constructing a solution for this problem in O(1) rounds. Unfortunately, the proof of this result relies heavily on the definition of LCL. Indeed, the constant bound constraints on the degrees and input sizes allow the authors to cleverly use Ramsey theory. It is thus not clear whether it is possible to extend this result to all languages in LD(O(1)).

The question of whether randomization helps in decreasing the locality parameter of construction problems has been the focus of numerous studies. To date, there exists evidence that, for some problems at least, randomization does not help. For instance, [33] proves this for 3-coloring the ring. In fact, for low degree graphs, the gaps between the efficiencies of the best known randomized and deterministic algorithms for problems like MIS, $(\Delta+1)$ -coloring, and Maximal Matching are very small. On the other hand, for graphs of arbitrarily large degrees, there seem to be indications that randomization does help, at least in some cases. For instance, $(\Delta+1)$ -coloring can be randomly computed in expected $O(\log n)$ communication rounds on n-node graphs [1, 30], whereas the best known deterministic algorithm for this problem performs in $2^{O(\sqrt{\log n})}$ rounds [35]. $(\Delta+1)$ -coloring results whose performances are measured also with respect to the maximum degree Δ illustrate this phenomena as well. Specifically, [39] shows that $(\Delta+1)$ -coloring can be randomly computed in expected $O(\log \Delta + \sqrt{\log n})$ communication rounds whereas the best known deterministic algorithm performs in $O(\Delta + \log^* n)$ rounds [5, 23].

Recently, several results were established conserving decision problems in distributed computing. For example, [8] and [20] study specific decision problems in the $\mathcal{CONGEST}$ model. (In contrast to the \mathcal{LOCAL} model, this model assumes that the message size is bounded by $O(\log n)$ bits, hence dealing with congestion is the main issue.) Specifically, tight bounds are established in [20] for the time and message complexities of the problem of deciding whether a subgraph is an MST, and time lower bounds for many other subgraph-decision problems (e.g., spanning tree, connectivity) are established in [8]. It is interesting to note that some of these lower bounds imply strong unconditional time lower bounds on the hardness of distributed approximation for many classical construction problems in the $\mathcal{CONGEST}$ model. Decision problems have received recent attention from the asynchrony-discipline too, in the framework of wait-free computing [17]. In this framework, the focus is on task checkability. Wait-free checkable tasks have been characterized in term of covering spaces, a fundamental tool in algebraic topology.

The theory of proof-labeling schemes [21, 22] was designed to tackle the issue of locally

verifying (with the aid of a proof, i.e., a certificate, at each node) solutions to problems that cannot be decided locally (e.g., "is the given subgraph a spanning tree of the network?", or, "is it an MST?"). In fact, the model of proof-labeling schemes has some resemblance to our definition of the class NLD(O(1)). Investigations in the framework of proof-labeling schemes mostly focus on the minimum size of the certificate necessary so that verification can be performed in a single round. The notion of proof-labeling schemes also has interesting similarities with the notions of local detection [2], local checking [3], or silent stabilization [11], which were introduced in the context of self-stabilization [10].

The use of oracles that provide information to nodes was studied intensively in the context of distributed construction tasks. For instance, this framework, called *local computation with advice*, was studied in [15] for MST construction and in [15] for 3-coloring a cycle.

Finally, we note that our notion of NLD seems to be related to the theory of *lifts*, e.g., [29].

2 Decision problems and complexity classes

2.1 Model of computation

Let us first recall some basic notions in distributed computing. We consider the \mathcal{LOCAL} model [36], which is a standard model capturing the essence of locality. In this model, processors are assumed to be nodes of a network G, provided with arbitrary distinct identities, and computation proceeds in fault-free synchronous rounds. At each round, every processor $v \in V(G)$ exchanges messages of unrestricted size with its neighbors in G, and performs computations on its data. We assume that the number of steps (sequential time) used for the local computation made by the node v in some round r is bounded by some function $f_A(H(r,v))$, where H(r,v) denotes the size of the "history" seen by node v up to the beginning of round r. That is, the total number of bits encoded in the input and the identity of the node, as well as in the incoming messages from previous rounds. Here, we do not impose any restriction on the growth rate of f_A . We would like to point out, however, that imposing such restrictions, or alternatively, imposing restrictions on the memory used by a node for local computation, may lead to interesting connections between the theory of locality and classical computational complexity theory. To sum up, during the execution of a distributed algorithm A, all processors are woken up simultaneously, and, initially, a processor is solely aware of it own identity, and possibly to some local input too. Then, in each round r, every processor v

- (1) sends messages to its neighbors,
- (2) receives messages from its neighbors, and
- (3) performs at most $f_A(H(r,v))$ computations.

After a number of rounds (that may depend on the network G and may vary among the processors, simply because nodes have different identities, potentially different inputs, and are typically located at non-isomorphic positions in the network), every processor v terminates and outputs some value out(v). Consider an algorithm running in a network G with input x and identity assignment Id. The running time of a node v, denoted $T_v(G, \mathbf{x}, \mathrm{Id})$, is the maximum of the number of rounds until v outputs. The running time of the algorithm, denoted $T(G, \mathbf{x}, \mathrm{Id})$, is the maximum of the number of rounds until all processors terminate, i.e., $T(G, \mathbf{x}, \mathrm{Id}) = \max\{T_v(G, \mathbf{x}, \mathrm{Id}) \mid v \in V(G)\}$. Let t be a non-decreasing function of input configurations $(G, \mathbf{x}, \mathrm{Id})$. (By non-decreasing, we mean that if G' is an induced subgraph of G and \mathbf{x}' and Id' are the restrictions of \mathbf{x} and Id , respectively, to the nodes in G' , then $t(G', \mathbf{x}', \mathrm{Id}') \leq t(G, \mathbf{x}, \mathrm{Id})$.) We say that an algorithm A has running time at most t, if $T(G, \mathbf{x}, \mathrm{Id}) \leq t(G, \mathbf{x}, \mathrm{Id})$, for every $(G, \mathbf{x}, \mathrm{Id})$. We shall give special attention to the case that t represents a constant function. Note that in general, given $(G, \mathbf{x}, \mathrm{Id})$, the nodes may not be aware of $t(G, \mathbf{x}, \mathrm{Id})$. On the other hand, note that, if $t = t(G, \mathbf{x}, \mathrm{Id})$ is known, then w.l.o.g. one can always assume that a local algorithm running in time at most t operates at each node v in two stages: (A) collect all information available in $B_G(v,t)$, the t-neighborhood, or ball of radius t of v in G, including inputs, identities and adjacencies, and (B) compute the output based on this information.

2.2 Local decision (LD)

We now refine some of the above concepts, in order to formally define our objects of interest. Obviously, a distributed algorithm that runs on a graph G operates separately on each connected component of G, and nodes of a component G' of G cannot distinguish the underlying graph G from G'. For this reason, we consider connected graphs only.

Definition 2.1 A configuration is a pair (G, \mathbf{x}) where G is a connected graph, and every node $v \in V(G)$ is assigned as its local input a binary string $\mathbf{x}(v) \in \{0, 1\}^*$.

In some problems, the local input of every node is empty, i.e., $\mathbf{x}(v) = \epsilon$ for every $v \in V(G)$, where ϵ denotes the empty binary string. Since an undecidable collection of configurations remains undecidable in the distributed setting too, we consider only decidable collections of configurations. Formally, we define the following.

Definition 2.2 A distributed language is a decidable collection \mathcal{L} of configurations.

In general, there are several possible ways of representing a configuration of a distributed language corresponding to standard distributed computing problems. Some examples considered in this paper are the following.

Unique-Leader = $\{(G, \mathbf{x}) \mid || \mathbf{x} ||_1 \le 1\}$ consists of all configurations such that there exists at most one node with local input 1, with all the others having local input 0.

Consensus = $\{(G, (\mathbf{x}_1, \mathbf{x}_2)) \mid \exists u \in V(G), \forall v \in V(G), \mathbf{x}_2(v) = \mathbf{x}_1(u)\}$ consists of all configurations such that all nodes agree on the value proposed by some node.

Coloring = $\{(G, \mathbf{x}) \mid \forall v \in V(G), \forall w \in N(v), \mathbf{x}(v) \neq \mathbf{x}(w)\}$ where N(v) denotes the (open) neighborhood of v, that is, all nodes at distance 1 from v.

$$MIS = \{(G, \mathbf{x}) \mid S = \{v \in V(G) \mid \mathbf{x}(v) = 1\} \text{ forms a MIS}\}.$$

SpanningTree = $\{(G, (\text{name}, \text{head})) \mid T = \{e_v = (v, v^+), v \in V(G), \text{head}(v) = \text{name}(v^+)\}$ is a spanning tree of G consists of all configurations such that the set T of edges e_v between every node v and its neighbor v^+ satisfying $\text{name}(v^+) = \text{head}(v)$ forms a spanning tree of G.

(The language MST, for minimum spanning tree, can be defined similarly).

An identity assignment Id for a graph G is an assignment of distinct integers to the nodes of G. A node $v \in V(G)$ executing a distributed algorithm in a configuration (G, \mathbf{x}) initially knows only its own identity $\mathrm{Id}(v)$ and its own input $\mathbf{x}(v)$, and is unaware of the graph G. After t rounds, v acquires knowledge only of its t-neighborhood $B_G(v,t)$. In each round r of the algorithm A, a node may communicate with its neighbors by sending and receiving messages, and may perform at most $f_A(H(r,v))$ computations. Eventually, each node $v \in V(G)$ must output a local output $\mathrm{out}(v) \in \{0,1\}^*$.

Let \mathcal{L} be a distributed language. We say that a distributed algorithm \mathcal{A} decides \mathcal{L} if and only if for every configuration (G, \mathbf{x}) , and for every identity assignment Id for the nodes of G, every node of G eventually terminates and outputs "yes" or "no", satisfying the following decision rules:

- If $(G, \mathbf{x}) \in \mathcal{L}$, then $\operatorname{out}(v) = \text{"yes"}$ for every node $v \in V(G)$;
- If $(G, \mathbf{x}) \notin \mathcal{L}$, then there exists at least one node $v \in V(G)$ such that $\operatorname{out}(v) = \text{``no''}$.

We are now ready to define one of our main subjects of interest, the class LD(t), for local decision.

Definition 2.3 Let t be a non-decreasing function of triplets $(G, \mathbf{x}, \mathrm{Id})$. Define $\mathrm{LD}(t)$ as the class of all distributed languages that can be decided by a local distributed algorithm that runs in number of rounds at most t.

For instance, Coloring \in LD(1) and MIS \in LD(1). On the other hand, it is not hard to see that languages such as Unique-Leader, Consensus, and SpanningTree are not in LD(t), for any t = o(n). In what follows, we define LD(O(t)) = $\bigcup_{c>1}$ LD($c \cdot t$).

2.3 Non-deterministic local decision (NLD)

A distributed verification algorithm is a distributed algorithm A that gets as input, in addition to a configuration (G, \mathbf{x}) , a global certificate vector \mathbf{y} , i.e., every node v of a graph G gets as input a binary string $\mathbf{x}(v) \in \{0, 1\}^*$, and a certificate $\mathbf{y}(v) \in \{0, 1\}^*$. A verification algorithm A verifies \mathcal{L} if and only if for every configuration (G, \mathbf{x}) , the following hold:

- If $(G, \mathbf{x}) \in \mathcal{L}$, then there exists a certificate \mathbf{y} such that for every id-assignment Id, algorithm A applied on (G, \mathbf{x}) with certificate \mathbf{y} and id-assignment Id outputs out(v) = "yes" for all $v \in V(G)$;
- If $(G, \mathbf{x}) \notin \mathcal{L}$, then for every certificate \mathbf{y} and for every id-assignment Id, algorithm A applied on (G, \mathbf{x}) with certificate \mathbf{y} and id-assignment Id outputs out(v) = "no" for at least one node $v \in V(G)$.

One motivation for studying the nondeterministic verification framework comes from settings in which one must perform local verifications repeatedly. In such cases, one can afford to have a relatively "wasteful" preliminary step in which a certificate is computed for each node. Using these certificates, local verifications can then be performed very fast. See [21, 22] for more details regarding such applications. Indeed, the definition of a verification algorithm finds similarities with the notion of proof-labeling schemes discussed in [21, 22]. Informally, in a proof-labeling scheme, the construction of a "good" certificate \mathbf{y} for a configuration $(G, \mathbf{x}) \in \mathcal{L}$ may depend also on the given id-assignment. Since the question of whether a configuration (G, \mathbf{x}) belongs to a language \mathcal{L} is independent from the particular id-assignment, we prefer to let the "good" certificate \mathbf{y} depend only on the configuration. In other words, as defined above, a verification algorithm operating on a configuration $(G, \mathbf{x}) \in \mathcal{L}$ and a "good" certificate \mathbf{y} must say "yes" at every node regardless of the id-assignment.

We now define the class NLD(t), for nondeterministic local decision. (our terminology is by direct analogy to the class NP in sequential computational complexity).

Definition 2.4 Let t be a non-decreasing function of triplets $(G, \mathbf{x}, \mathrm{Id})$. Define $\mathrm{NLD}(t)$ as the class of all distributed languages that can be verified in at most t communication rounds.

2.4 Bounded-error probabilistic local decision (BPLD)

A randomized distributed algorithm is a distributed algorithm A that enables every node v, at any round r during the execution, to toss a number of random bits obtaining a string $r(v) \in \{0,1\}^*$. Clearly, this number cannot exceed $f_A(H(r,v))$, the bound on the number of computational steps used by node v at round r. Note however, that H(r,v) may now also depend on the random bits produced by other nodes in previous rounds. For $p, q \in (0,1]$, we say that a randomized distributed algorithm \mathcal{A} is a (p,q)-decider for \mathcal{L} , or, that it decides \mathcal{L} with "yes" success probability p and "no" success probability p and only if for every configuration (G, \mathbf{x}) , and for every identity assignment Id for the nodes of p0, every node of p1.

- If $(G, \mathbf{x}) \in \mathcal{L}$, then $\Pr[\operatorname{out}(v) = \text{"yes" for every node } v \in V(G)] \ge p$,
- If $(G, \mathbf{x}) \notin \mathcal{L}$, then $\Pr[\text{out}(v) = \text{"no" for at least one node } v \in V(G)] \ge q$,

where the probabilities in the above definition are taken over all possible coin tosses performed by nodes. We define the class BPLD(t, p, q), for "Bounded-error Probabilistic Local Decision", as follows.

Definition 2.5 For $p, q \in (0, 1]$ and a function t, BPLD(t, p, q) is the class of all distributed languages that have a local randomized distributed (p, q)-decider running in time t. (i.e., can be decided in time t by a local randomized distributed algorithm with "yes" success probability p and "no" success probability q).

3 A sharp threshold for randomization

Consider some graph G, and a subset U of the nodes of G, i.e., $U \subseteq V(G)$. Let G[U] denote the vertex-induced subgraph of G defined by the nodes in U. Given a configuration (G, \mathbf{x}) , let $\mathbf{x}[U]$ denote the input \mathbf{x} restricted to the nodes in U. For simplicity of presentation, if H is a subgraph of G, we denote $\mathbf{x}[V(H)]$ by $\mathbf{x}[H]$. A prefix of a configuration (G, \mathbf{x}) is a configuration $(G[U], \mathbf{x}[U])$, where $U \subseteq V(G)$ (note that in particular, G[U] is connected). We say that a language \mathcal{L} is hereditary if every prefix of every configuration $(G, \mathbf{x}) \in \mathcal{L}$ is also in \mathcal{L} . Coloring and Unique-Leader are clearly hereditary languages. As another example

of an hereditary language, consider a family \mathcal{G} of hereditary graphs, i.e., that is closed under vertex deletion; then the language $\{(G, \epsilon) \mid G \in \mathcal{G}\}$ is hereditary. Examples of hereditary graph families are planar graphs, interval graphs, forests, chordal graphs, cographs, perfect graphs, etc.

Theorem 3.1 below asserts that, for hereditary languages, randomization does not help if one imposes that $p^2 + q > 1$, i.e, the "no" success probability distribution is at least as large as one minus the square of the "yes" success probability. Somewhat more formally, we prove that for hereditary languages, we have $\bigcup_{p^2+q>1} \mathrm{BPLD}(t,p,q) = \mathrm{LD}(O(t))$. This complements the fact that for $p^2 + q \leq 1$, we have $\mathrm{LD}(t) \subsetneq \mathrm{BPLD}(t,p,q)$, for any t = o(n).

Recall that [34] investigates the question of whether randomization helps for constructing in constant time a solution for a problem in $LCL \subseteq LD(O(1))$. We stress that the technique used in [34] for tackling this question relies heavily on the definition of LCL, specifically, that only graphs of constant degree and of constant input size are considered. Hence it is not clear whether the technique of [34] can be useful for our purposes, as we impose no such assumptions on the degrees or input sizes. Also, although it seems at first glance, that Lovsz local lemma might have been helpful here, we could not effectively apply it in our proof. Instead, we use a completely different approach.

Theorem 3.1 Let \mathcal{L} be an hereditary language and let t be a function. If $\mathcal{L} \in BPLD(t, p, q)$ for constants $p, q \in (0, 1]$ such that $p^2 + q > 1$, then $\mathcal{L} \in LD(O(t))$.

Proof. Let us start with some definitions. Let \mathcal{L} be a language in BPLD(t, p, q) where $p, q \in (0, 1]$ and $p^2 + q > 1$, and t is some function. Let A be a randomized algorithm deciding \mathcal{L} , with "yes" success probability p, and "no" success probability q, whose running time is at most $t(G, \mathbf{x}, \mathrm{Id})$, for every configuration (G, \mathbf{x}) with identity assignment Id. Fix a configuration (G, \mathbf{x}) , and an id-assignment Id for the nodes of V(G). The distance $\mathrm{dist}_G(u, v)$ between two nodes of G is the minimum number of edges in a path connecting u and v in G. The distance between two subsets $U_1, U_2 \subseteq V$ is defined as

$$dist_G(U_1, U_2) = min\{dist_G(u, v) \mid u \in U_1, v \in U_2\}.$$

For a set $U \subseteq V$, let $\mathcal{E}(G, \mathbf{x}, \mathrm{Id}, U)$ denote the event that when running A on (G, \mathbf{x}) with id-assignment Id, all nodes in U output "yes". Let $v \in V(G)$. The running time of A at v may depend on the coin tosses made by the nodes. Let $t_v = t_v(G, \mathbf{x}, \mathrm{Id})$ denote the maximal running time of v over all possible coin tosses. Note that $t_v \leq t(G, \mathbf{x}, \mathrm{Id})$ (we do not assume that neither t or t_v are known to v).

The radius of a node v, denoted r_v , is the maximum value t_u such that there exists a node u, where $v \in B_G(u, t_u)$. (Observe that the radius of a node is at most t.) The radius

of a set of nodes S is $r_S := \max\{r_v \mid v \in S\}$. In what follows, fix a constant δ such that $0 < \delta < p^2 + q - 1$, and define $\lambda = 11 \lceil \log p / \log(1 - \delta) \rceil$.

A splitter of $(G, \mathbf{x}, \mathrm{Id})$ is a triplet (S, U_1, U_2) of pairwise disjoint subsets of nodes such that $S \cup U_1 \cup U_2 = V$, $\mathrm{dist}_G(U_1, U_2) \geq \lambda r_S$. (Observe that r_S may depend on the identity assignment and the input, and therefore, being a splitter is not just a topological property depending only on G). Given a splitter (S, U_1, U_2) of $(G, \mathbf{x}, \mathrm{Id})$, let $G_k = G[U_k \cup S]$, and let \mathbf{x}_k be the input \mathbf{x} restricted to nodes in G_k , for k = 1, 2.

The following structural claim does not use the fact that \mathcal{L} is hereditary.

Lemma 3.2 For every configuration (G, \mathbf{x}) with identity assignment Id, and every splitter (S, U_1, U_2) of $(G, \mathbf{x}, \mathrm{Id})$, we have

$$(G_1, \mathbf{x}_1) \in \mathcal{L} \text{ and } (G_2, \mathbf{x}_2) \in \mathcal{L} \implies (G, \mathbf{x}) \in \mathcal{L}.$$

Let (G, \mathbf{x}) be a configuration with identity assignment Id. Assume, towards contradiction, that there exists a splitter (S, U_1, U_2) of triplet $(G, \mathbf{x}, \mathrm{Id})$, such that $(G_1, \mathbf{x}_1) \in \mathcal{L}$ and $(G_2, \mathbf{x}_2) \in \mathcal{L}$, yet $(G, \mathbf{x}) \notin \mathcal{L}$. (The fact that $(G_1, \mathbf{x}_1) \in \mathcal{L}$ and $(G_2, \mathbf{x}_2) \in \mathcal{L}$ implies that both G_1 and G_2 are connected, however, we note, that for the claim to be true, it is not required that $G[U_1]$, $G[U_2]$ or G[S] are connected.) Let $d = \lambda r_S$.

Given a vertex $u \in S$, we define the *level* of u by $\ell(u) = \operatorname{dist}_G(U_1, \{u\})$. For an integer $i \in [1, d]$, let L_i denote the set of nodes in S of level i. For an integer $i \in (r_S, d - r_S)$, let $S_i = \bigcup_{j=i-r_S}^{i+r_S} L_j$, and finally, for a set $J \subseteq (r_S, d - r_S)$ of integers, let $S_J = \bigcup_{i \in J} S_i$.

Define

$$I = \{i \in (2r_S, d - 2r_S) \mid \Pr[\mathcal{E}(G, \mathbf{x}, \operatorname{Id}, S_i)] < 1 - \delta\}.$$

Claim 3.3 There exists $i \in (2r_S, d - 2r_S)$ such that $i \notin I$.

Proof. For proving Claim 3.3, we upper bound the size of I by $d-4r_S-2$. This is done by covering the integers in $(2r_S, d-2r_S)$ by at most $4r_S+1$ sets, such that each one is $(4r_S+1)$ -independent, that is, for every two integers in the same set, they are at least $4r_S+1$ apart. Specifically, for $s \in [1, 4r_S+1]$ and $m(S) = \lceil (d-8r_S)/(4r_S+1) \rceil$, we define $J_s = \{s+2r_S+j(4r_S+1) \mid j \in [0, m(S)]\}$. Observe that, as desired, $(2r_S, d-2r_S) \subset \bigcup_{s \in [1, 4r_S+1]} J_s$, and for each $s \in [1, 4r_S+1]$, J_s is $(4r_S+1)$ -independent. In what follows, fix $s \in [1, 4r_S+1]$ and let $J = J_s$. Since $(G_1, \mathbf{x}_1) \in \mathcal{L}$, we know that,

$$\Pr[\mathcal{E}(G_1, \mathbf{x}_1, \operatorname{Id}, S_{J \cap I})] \geq p$$
.

Observe that for $i \in (2r_S, d-2r_S)$, $t_v \leq r_v \leq r_S$, and hence, the t_v -neighborhood in G of

every node $v \in S_i$ is contained in $S \subseteq G_1$, i.e., $B_G(v, t_v) \subseteq G_1$. It therefore follows that:

$$\Pr[\mathcal{E}(G, \mathbf{x}, \operatorname{Id}, S_{J \cap I})] = \Pr[\mathcal{E}(G_1, \mathbf{x}_1, \operatorname{Id}, S_{J \cap I})] \ge p. \tag{1}$$

Consider two integers a and b in J. We know that $|a-b| \ge 4r_S + 1$. Hence, the distance in G between any two nodes $u \in S_a$ and $v \in S_b$ is at least $2r_S + 1$. Thus, the events $\mathcal{E}(G, \mathbf{x}, \mathrm{Id}, S_a)$ and $\mathcal{E}(G, \mathbf{x}, \mathrm{Id}, S_b)$ are independent. It follows by the definition of I, that

$$\Pr[\mathcal{E}(G, \mathbf{x}, \mathrm{Id}, S_{J \cap I})] < (1 - \delta)^{|J \cap I|}$$
(2)

By (1) and (2), we have that $p < (1 - \delta)^{|J \cap I|}$ and thus $|J \cap I| < \log p / \log(1 - \delta)$. Since $(2r_S, d - 2r_S)$ can be covered by the sets J_s , $s = 1, \ldots, 4r_S + 1$, each of which is $(4r_S + 1)$ -independent, we get that

$$|I| = \sum_{s=1}^{4r_S+1} |J_s \cap I| < (4r_S+1)(\log p/\log(1-r)).$$

Combining this bound with the fact that $d = \lambda r_S$, we get that $d - 4r_S - 1 > |I|$. It follows by the pigeonhole principle that there exists some $i \in (2r_S, d - 2r_S)$ such that $i \notin I$, as desired. This completes the proof of Claim 3.3.

Fix $i \in (2r_S, d - 2r_S)$ such that $i \notin I$, and let $\mathcal{F} = \mathcal{E}(G, \mathbf{x}, \mathrm{Id}, S_i)$. By definition,

$$\Pr[\overline{\mathcal{F}}] \le \delta < p^2 + q - 1. \tag{3}$$

Let H_1 denote the subgraph of G induced by the nodes in $(\bigcup_{j=1}^{i-r_S-1} L_j) \cup U_1$. We similarly define H_2 as the subgraph of G induced by the nodes in $(\bigcup_{j>i+r_S} L_j) \cup U_2$. Note that $S_i \cup V(H_1) \cup V(H_2) = V$, and for any two nodes $u \in V(H_1)$ and $v \in V(H_2)$, we have $d_G(u,v) > 2r_S$. It follows that, for k=1,2, the t_u -neighborhood in G of each node $u \in V(H_k)$ equals the t_u -neighborhood in G_k of u, that is, $B_G(u,t_u) \subseteq G_k$. (To see why, consider, for example, the case k=2. Given $u \in V(H_2)$, it is sufficient to show that $\nexists v \in V(H_1)$, such that $v \in B_G(u,t_u)$. Indeed, if such a vertex v exists then $d_G(u,v) > 2r_S$, and hence $t_u > 2r_S$. Since there must exists a vertex $w \in S_i$ such that $w \in B(u,t_u)$, we get that $r_w > 2r_S$, in contradiction to the fact that $w \in S$.) Thus, for k=1,2, since $(G_i,\mathbf{x}_i) \in \mathcal{L}$, we get $\Pr[\mathcal{E}(G,\mathbf{x},\mathrm{Id},V(H_i))] = \Pr[\mathcal{E}(G_i,\mathbf{x}_i,\mathrm{Id},V(H_i))] \geq p$.

Let $\mathcal{F}' = \mathcal{E}(G, \mathbf{x}, \operatorname{Id}, V(H_1) \cup V(H_2))$. As the events $\mathcal{E}(G, \mathbf{x}, \operatorname{Id}, V(H_1))$ and $\mathcal{E}(G, \mathbf{x}, \operatorname{Id}, V(H_2))$ are independent, it follows that $\Pr[\mathcal{F}'] > p^2$, that is

$$\Pr[\overline{\mathcal{F}'}] \le 1 - p^2 \tag{4}$$

By Eqs. (3) and (4), and using union bound, it follows that $\Pr[\overline{\mathcal{F}} \vee \overline{\mathcal{F}'}] < q$. Thus

$$\Pr[\mathcal{E}(G, \mathbf{x}, \operatorname{Id}, V(G))] = \Pr[\mathcal{E}(G, \mathbf{x}, \operatorname{Id}, S_i \cup V(H_1) \cup V(H_2))] = \Pr[\mathcal{F} \wedge \mathcal{F}'] > 1 - q.$$

This is in contradiction to the assumption that $(G, \mathbf{x}) \notin \mathcal{L}$. This concludes the proof of Lemma 3.2.

Our goal now is to show that $\mathcal{L} \in \mathrm{LD}(O(t))$ by proving the existence of a deterministic local algorithm D that runs in time O(t) and recognizes \mathcal{L} . (No attempt is made here to minimize the constant factor hidden in the O(t) notation.) Recall that both $t = t(G, \mathbf{x}, \mathrm{Id})$ and $t_v = t_v(G, \mathbf{x}, \mathrm{Id})$ may not be known to v. Nevertheless, by inspecting the balls $B_G(v, 2^i)$ for increasing $i = 1, 2, \dots$, each node v can compute an upper bound on t_v as given by the following claim.

Claim 3.4 Fix a a configuration (G, \mathbf{x}) , an id-assignment Id, and a constant c. In O(t) time, each node v can compute a value $t_v^* = t_v^*(c)$ such that (1) $c \cdot t_v \leq t_v^* = O(t)$ and (2) for every $u \in B_G(v, c \cdot t_v^*)$, we have $t_u \leq t_v^*$.

To establish the claim, observe first that in O(t) time, each node v can compute a value t'_v satisfying $t_v \leq t'_v \leq 2t$. Indeed, given the ball $B_G(v, 2^i)$, for some integer i, and using the upper bound on number of (sequential) local computations, node v can simulate all its possible executions up to round $r = 2^i$. The desired value t'_v is the smallest $r = 2^i$ for which all executions of v up to round r conclude with an output at v. Once t'_v is computed, node v aims at computing t^*_v . For this purpose, it starts again to inspect the balls $B_G(v, 2^i)$ for increasing $i = 1, 2, \cdots$, to obtain t'_u from each $u \in B_G(v, 2^i)$. (For this purpose, it may need to wait until u computes t'_u , but this delays the whole computation by at most O(t) time.) Now, node v outputs $t^*_v = 2^i$ for the smallest i satisfying (1) $c \cdot t'_v \leq 2^i$ and (2) for every $u \in B_G(v, c \cdot 2^i)$, we have $t'_u \leq t^*_v$. It is easy to see that for this i, we have $2^i = O(t)$, hence $t^*_v = O(t)$.

Given a configuration (G, \mathbf{x}) , and an id-assignment Id, Algorithm D, applied at a node u first calculates $t_u^* = t_u^*(6\lambda)$, and then outputs "yes" if and only if the $2\lambda t_u^*$ -neighborhood of u in (G, \mathbf{x}) belongs to \mathcal{L} . That is,

$$\operatorname{out}(u) = \text{"yes"} \iff (B_G(u, 2\lambda t_u^*), \mathbf{x}[B_G(u, 2\lambda t_u^*)]) \in \mathcal{L}.$$

Obviously, Algorithm D is a deterministic algorithm that runs in time O(t). We claim that Algorithm D decides \mathcal{L} . Indeed, since \mathcal{L} is hereditary, if $(G, \mathbf{x}) \in \mathcal{L}$, then every prefix of (G, \mathbf{x}) is also in \mathcal{L} , and thus, every node u outputs out(u) ="yes". Now consider the case where $(G, \mathbf{x}) \notin \mathcal{L}$, and assume by contradiction that by applying D on (G, \mathbf{x}) with

id-assignment Id, every node u outputs out(u) = "yes". Let $U \subseteq V(G)$ be maximal by inclusion, such that G[U] is connected and $(G[U], \mathbf{x}[U]) \in \mathcal{L}$. Obviously, U is not empty, as $(B_G(u, 2\lambda t_v^*), \mathbf{x}[B_G(u, 2\lambda t_v^*)]) \in \mathcal{L}$ for every node u. On the other hand, we have |U| < |V(G)|, because $(G, \mathbf{x}) \notin \mathcal{L}$.

Let $u \in U$ be a node with maximal t_u such that $B_G(u, 2t_u)$ contains a node outside U. Define G' as the subgraph of G induced by $U \cup V(B_G(u, 2t_u))$. Observe that G' is connected and that G' strictly contains U. Towards contradiction, our goal is to show that $(G', \mathbf{x}[G']) \in \mathcal{L}$.

Let H denote the graph which is maximal by inclusion such that H is connected and

$$B_G(u, 2t_u) \subset H \subseteq B_G(u, 2t_u) \cup (U \cap B_G(u, 2\lambda t_u^*))$$
.

Let W^1, W^2, \dots, W^ℓ be the ℓ connected components of $G[U] \setminus B_G(u, 2t_u)$, ordered arbitrarily. Let W^0 be the empty graph, and for $k = 0, 1, 2, \dots, \ell$, define the graph $Z^k = H \cup W^0 \cup W^1 \cup W^2 \cup \dots \cup W^k$. Observe that Z^k is connected for each $k = 0, 1, 2, \dots, \ell$. We prove by induction on k that $(Z^k, \mathbf{x}[Z^k]) \in \mathcal{L}$ for every $k = 0, 1, 2, \dots, \ell$. This will establish the contradiction since $Z^\ell = G'$. For the basis of the induction, the case k = 0, we need to show that $(H, \mathbf{x}[H]) \in \mathcal{L}$. However, this is immediate by the facts that H is a connected subgraph of $B_G(u, 2\lambda t_u^*)$, the configuration $(B_G(u, 2\lambda t_u^*), \mathbf{x}[B_G(u, 2\lambda t_u^*)]) \in \mathcal{L}$, and \mathcal{L} is hereditary. Assume now that we have $(Z^k, \mathbf{x}[Z^k]) \in \mathcal{L}$ for $0 \leq k < \ell$, and consider the graph $Z^{k+1} = Z^k \cup W^{k+1}$. Define the sets of nodes

$$S = V(Z^k) \cap V(W^{k+1}), \quad U_1 = V(Z^k) \setminus S, \quad \text{and} \quad U_2 = V(W^{k+1}) \setminus S.$$

A crucial observation is that (S, U_1, U_2) is a splitter of Z^{k+1} . This follows from the following arguments. Let us first show that $r_S \leq t_u^*$. By definition, we have $t_v \leq t_u^*$, for every $v \in B_G(u, 6\lambda t_u^*)$. Hence, in order to bound the radius of S (in Z^{k+1}) by t_u^* it is sufficient to prove that there is no node $w \in U \setminus B_G(u, 6\lambda t_u^*)$ such that $B_G(w, t_w) \cap S \neq \emptyset$. Indeed, if such a node w exists then $t_w > 4\lambda t_u^*$ and hence $B_G(w, 2t_w)$ contains a node outside U, in contradiction to the choice of u. It follows that $r_S \leq t_u^*$.

We now claim that $\operatorname{dist}_{Z^{k+1}}(U_1, U_2) \geq \lambda t_u^*$. Consider a simple directed path P in Z^{k+1} going from a node $x \in U_1$ to a node $y \in U_2$. Since $x \notin V(W^{k+1})$ and $y \in V(W^{k+1})$, we get that P must pass through a vertex in $B_G(u, 2t_u)$. Let z be the last vertex in P such that $z \in B_G(u, 2t_u)$, and consider the directed subpath $P_{[z,y]}$ of P going from z to y. Now, let $P' = P_{[z,y]} \setminus \{z\}$. The first $d' = \min\{(2\lambda - 2)t_u^*, |P'|\}$ vertices in the directed subpath P' must belong to $V(H) \subseteq V(Z^k)$. In addition, observe that all nodes in P' must be in $V(W^{k+1})$. It follows that the first d' nodes of P' are in S. Since $y \notin S$, we get that $|P'| \geq d' = (2\lambda - 2)t_u^*$,

and thus $|P| > \lambda t_u^*$. Consequently, $\operatorname{dist}_{Z^{k+1}}(U_1, U_2) \ge \lambda t_u^*$, as desired. This completes the proof that (S, U_1, U_2) is a splitter of Z^{k+1} .

Now, by the induction hypothesis, we have $(G_1, \mathbf{x}[G_1]) \in \mathcal{L}$, because $G_1 = G[U_1 \cup S] = Z^k$. In addition, we have $(G_2, \mathbf{x}[G_2]) \in \mathcal{L}$, because $G_2 = G[U_2 \cup S] = W^{k+1}$, and W^{k+1} is a prefix of G[U]. We can now apply Lemma 3.2 and conclude that $(Z^{k+1}, \mathbf{x}[Z^{k+1}]) \in \mathcal{L}$. This concludes the induction proof. The theorem follows.

Let Planar = $\{(G, \epsilon) : G \text{ is planar}\}$, Interval = $\{(G, \epsilon) : G \text{ is an interval graph}\}$ and CycleFree = $\{(G, \epsilon) : G \text{ has no cycle}\}$. One can easily check that neither of these three languages is in LD(t), for any t = o(n). Hence, Theorem 3.1 yields the following.

Corollary 3.5 Let $p, q \in (0, 1]$ such that $p^2+q > 1$, then Unique-Leader, Planar, Interval and CycleFree are not in BPLD(t, p, q), for any t = o(n).

4 Nondeterminism and complete problems

4.1 Separation results

Our first separation result indicates that non-determinism helps for local decision. Indeed, we show that there exists a language, specifically, $tree = \{(G, \epsilon) \mid G \text{ is a tree}\}$, which belongs to NLD(1) but not to LD(t), for any t = o(n). The proof follows by rather standard arguments.

Theorem 4.1 $LD(t) \subsetneq NLD(t)$, for any t = o(n).

Proof. To establish the theorem it is sufficient to show that there exists a language \mathcal{L} such that $\mathcal{L} \notin \mathrm{LD}(o(n))$ and $\mathcal{L} \in \mathrm{NLD}(1)$. Let $\mathsf{tree} = \{(G, \epsilon) \mid G \text{ is a tree}\}$. We have $\mathsf{tree} \notin \mathrm{LD}(o(n))$. To see why, consider a cycle C with nodes labeled consecutively from 1 to 4n, and the path P_1 (resp., P_2) with nodes labeled consecutively $1, \ldots, 4n$ (resp., $2n+1,\ldots,4n,1,\ldots,2n$), from one extremity to the other. For any algorithm A deciding tree , all nodes $n+1,\ldots,3n$ output "yes" in configuration (P_1,ϵ) for any identity assignment for the nodes in P_1 , while all nodes $3n+1,\ldots,4n,1,\ldots,n$ output "yes" in configuration (P_2,ϵ) for any identity assignment or the nodes in P_2 . Thus if A is local, then all nodes output "yes" in configuration (C,ϵ) , a contradiction. In contrast, we next show that $\mathsf{tree} \in \mathrm{NLD}$. The (nondeterministic) local algorithm A verifying tree operates as follows. Given a configuration (G,ϵ) , the certificate given at node v is $\mathsf{y}(v) = \mathrm{dist}_G(v,r)$ where $r \in V(G)$ is an arbitrary fixed node. The verification procedure is then as follows. At each node v, A

inspects every neighbor (with its certificates), and verifies the following:

- $\mathbf{y}(v)$ is a non-negative integer,
- if $\mathbf{y}(v) = 0$, then $\mathbf{y}(w) = 1$ for every neighbor w of v, and
- if $\mathbf{y}(v) > 0$, then there exists a neighbor w of v such that $\mathbf{y}(w) = \mathbf{y}(v) 1$, and, for all other neighbors w' of v, we have $\mathbf{y}(w') = \mathbf{y}(v) + 1$.

If G is a tree, then applying Algorithm A on G with the certificate yields the answer "yes" at all nodes regardless of the given id-assignment. On the other hand, if G is not a tree, then we claim that for every certificate, and every id-assignment Id, Algorithm A outputs "no" at some node. Indeed, consider some certificate \mathbf{y} given to the nodes of G, and let G be a simple cycle in G. Assume, for the sake of contradiction, that all nodes in G output "yes". In this case, each node in G has at least one neighbor in G with a larger certificate. This creates an infinite sequence of strictly increasing certificates, in contradiction with the finiteness of G.

Theorem 4.2 There exists a language \mathcal{L} such that $\mathcal{L} \notin \text{NLD}(t)$, for any t = o(n).

Proof. Let InpEqSize = $\{(G, \mathbf{x}) \mid \forall v \in V(G), \ \mathbf{x}(v) = |V(G)|\}$. We show that InpEqSize \notin NLD(t), for any t = o(n). Assume, for the sake of contradiction, that there exists a local nondeterministic algorithm A deciding InpEqSize. Let t < n/4 be the running time of A. Consider the cycle C with 2t + 1 nodes $u_1, u_2, \cdots, u_{2t+1}$, enumerated clockwise. Assume that the input at each node u_i of C satisfies $\mathbf{x}(u_i) = 2t + 1$. Then, there exists a certificate \mathbf{y} such that, for any identity assignment Id, algorithm A outputs "yes" at each node of C. Now, consider the configuration (C', \mathbf{x}') where the cycle C' has 4t + 2 nodes, and for each node v_i of C', $\mathbf{x}'(v_i) = 2t + 1$. We have $(C', \mathbf{x}') \notin$ InpEqSize. To fool Algorithm A, we enumerate the nodes in C' clockwise, i.e., $C = (v_1, v_2, \cdots, v_{4t+2})$. We then define the certificate \mathbf{y}' as follows:

$$\mathbf{y}'(v_i) = \mathbf{y}'(v_{i+2t+1}) = \mathbf{y}(u_i) \text{ for } i = 1, 2, \dots 2t + 1.$$

Fix an id-assignment Id' for the nodes in V(C'), and fix $i \in \{1, 2, \dots 2t + 1\}$. There exists an id-assignment Id₁ for the nodes in V(C), such that the output of A at node v_i in (C', \mathbf{x}') with certificate \mathbf{y}' and id-assignment Id' is identical to the output of A at node u_i in (C, \mathbf{x}) with certificate \mathbf{y} and id-assignment Id₁. Similarly, there exists an id-assignment Id₂ for the nodes in V(C) such that the output of A at node v_{i+2t+1} in (C', \mathbf{x}') with certificate \mathbf{y}' and id-assignment Id' is identical to the output of A at node u_i in (C, \mathbf{x}) with with

certificate \mathbf{y} and id-assignment Id₂. Thus, Algorithm A at both v_i and v_{i+2t+1} outputs "yes" in (C', \mathbf{x}') with certificate \mathbf{y}' and id-assignment Id'. Hence, since i was arbitrary, all nodes output "yes" for this configuration, certificate and id-assignment, contradicting the fact that $(C', \mathbf{x}') \notin \text{InpEqSize}$.

For $p, q \in (0, 1]$ and a function t, let us define BPNLD(t, p, q) as the class of all distributed languages that have a local randomized non-deterministic distributed (p, q)-decider running in time t.

Theorem 4.3 Let $p, q \in (0, 1]$ such that $p^2 + q \leq 1$. For every language \mathcal{L} , we have $\mathcal{L} \in BPNLD(1, p, q)$.

Proof. Let \mathcal{L} be a language. The certificate of a configuration $(G, \mathbf{x}) \in \mathcal{L}$ is a map of G, with nodes labeled with distinct integers in $\{1, ..., n\}$, where n = |V(G)|, together with the inputs of all nodes in G. In addition, every node v receives the label $\lambda(v)$ of the corresponding vertex in the map. Precisely, the certificate at node v is $\mathbf{y}(v) = (G', \mathbf{x}', i)$ where G' is an isomorphic copy of G with nodes labeled from 1 to n, \mathbf{x}' is an n-dimensional vector such that $\mathbf{x}'[\lambda(u)] = \mathbf{x}(u)$ for every node u, and $i = \lambda(v)$. The verification algorithm involves checking that the configuration (G', \mathbf{x}') is identical to (G, \mathbf{x}) . This is sufficient because distributed languages are sequentially decidable, hence every node can individually decide whether (G', \mathbf{x}') belongs to \mathcal{L} or not, once it has secured the fact that (G', \mathbf{x}') is the actual configuration. It remains to show that there exists a local randomized non-deterministic distributed (p,q)-decider for verifying that the configuration (G', \mathbf{x}') is identical to (G, \mathbf{x}) , and running in time 1.

The non-deterministic (p,q)-decider operates as follows. First, every node v checks that it has received the input as specified by \mathbf{x}' , i.e., v checks wether $\mathbf{x}'[\lambda(v)] = \mathbf{x}(v)$, and outputs "no" if this does not hold. Second, each node v communicates with its neighbors to check that (1) they all got the same map G' and the same input vector \mathbf{x}' , and (2) they are labeled the way they should be according to the map G'. If some inconsistency is detected by a node, then this node outputs "no". Finally, consider a node v that passed the aforementioned two phases without outputting "no". If $\lambda(v) \neq 1$ then v outputs "yes" (with probability 1), and if $\lambda(v) = 1$ then v outputs "yes" with probability p.

We claim that the above implements a non-deterministic distributed (p,q)-decider for verifying that the configuration (G', \mathbf{x}') is identical to (G, \mathbf{x}) . Indeed, if all nodes pass the two phases without outputting "no", then they all agree on the map G' and on the input vector \mathbf{x}' , and they know that their respective neighborhood fits with what is indicated on the map. Hence, (G', \mathbf{x}') is a lift of (G, \mathbf{x}) . If follows that $(G', \mathbf{x}') = (G, \mathbf{x})$ if and only

if there exists at most one node $v \in G$, whose label satisfies $\lambda(v) = 1$. Consequently, if $(G', \mathbf{x}') = (G, \mathbf{x})$ then all nodes say "yes" with probability at least p. On the other hand, if $(G', \mathbf{x}') \neq (G, \mathbf{x})$ then there are at least two nodes in G whose label is "1". These two nodes say "yes" with probability p^2 , hence, the probability that at least one of them says "no" is at least $1 - p^2 \geq q$. This completes the proof of Theorem 4.3.

The above theorem guarantees that the following definition is well defined. Let BPNLD = BPNLD(1, p, q), for some $p, q \in (0, 1]$ such that $p^2 + q \leq 1$. The following follows from Theorems 4.1, 4.2 and 4.3.

Corollary 4.4 $LD(o(n)) \subsetneq NLD(O(1)) \subset NLD(o(n)) \subsetneq BPNLD = All.$

4.2 Completeness results

Let us first define a notion of reduction that fits the class LD. For two languages $\mathcal{L}_1, \mathcal{L}_2$, we say that \mathcal{L}_1 is locally reducible to \mathcal{L}_2 , denoted by $\mathcal{L}_1 \leq \mathcal{L}_2$, if there exists a constant time local algorithm A such that, for every configuration (G, \mathbf{x}) and every id-assignment Id, A produces $\operatorname{out}(v) \in \{0, 1\}^*$ as output at every node $v \in V(G)$ so that

$$(G, \mathbf{x}) \in \mathcal{L}_1 \iff (G, \text{out}) \in \mathcal{L}_2$$
.

By definition, LD(O(t)) is closed under local reductions, that is, for every two languages $\mathcal{L}_1, \mathcal{L}_2$ satisfying $\mathcal{L}_1 \leq \mathcal{L}_2$, if $\mathcal{L}_2 \in LD(O(t))$ then $\mathcal{L}_1 \in LD(O(t))$.

We now show that there exists a natural problem, called cover, which is in some sense the "most difficult" decision problem; that is, we show that cover is BPNLD-complete. Language cover is defined as follows. Every node v is given as input an element $\mathcal{E}(v)$, and a finite collection of sets $\mathcal{S}(v)$. The union of these inputs is in the language if there exists a node v such that one set in $\mathcal{S}(v)$ equals the union of all the elements given to the nodes. Formally, we define cover v = v

Theorem 4.5 cover is BPNLD-complete.

Proof. The fact that $\mathsf{cover} \in \mathsf{BPNLD}$ follows from Theorem 4.3. To prove that cover is BPNLD -hard, we consider some $\mathcal{L} \in \mathsf{BPNLD}$ and show that $\mathcal{L} \preceq \mathsf{cover}$. For this purpose, we describe a local distributed algorithm A transforming any configuration for \mathcal{L} to a configuration for cover preserving the memberships to these languages. Let (G, \mathbf{x}) be a configuration for \mathcal{L} and let Id be an identity assignment. Algorithm A operating at a node v outputs a pair $(\mathcal{E}(v), \mathcal{S}(v))$, where $\mathcal{E}(v)$ is the "local view" at v in (G, \mathbf{x}) , i.e., the star subgraph of G consisting of v and its neighbors, together with the inputs of these nodes and their identities,

and S(v) is the collection of sets S defined as follows. For a binary string x, let |x| denote the length of x, i.e., the number of bits in x. For every vertex v, let $\psi(v) = 2^{|\operatorname{Id}(v)| + |\mathbf{X}(v)|}$. Node v first generates all configurations (G', \mathbf{x}') where G' is a graph with $k \leq \psi(v)$ vertices, and \mathbf{x}' is a collection of k input strings of length at most $\psi(v)$, such that $(G', \mathbf{x}') \in \mathcal{L}$. For each such configuration (G', \mathbf{x}') , node v generates all possible Id' assignments to V(G') such that for every node $u \in V(G')$, $|\operatorname{Id}(u)| \leq \psi(v)$. Now, for each such pair of a graph (G', \mathbf{x}') and an Id' assignment, algorithm A associates a set $S \in S(v)$ consisting of the k = |V(G')| local views of the nodes of G' in (G', \mathbf{x}') . We show that $(G, \mathbf{x}) \in \mathcal{L} \iff A(G, \mathbf{x}) \in \mathsf{cover}$.

If $(G, \mathbf{x}) \in \mathcal{L}$, then by the construction of Algorithm A, there exists a set $S \in \mathcal{S}(v)$ such that S covers the collection of local views for (G, \mathbf{x}) , i.e., $S = \{\mathcal{E}(u) \mid u \in G\}$. Indeed, the node v maximizing $\psi(v)$ satisfies $\psi(v) \geq \max\{\mathrm{Id}(u) \mid u \in V(G)\} \geq n$ and $\psi(v) \geq \max\{\mathbf{x}(u) \mid u \in V(G)\}$. Therefore, that specific node has constructed a set S which contains all local views of the given configuration (G, \mathbf{x}) and Id assignemnt. Thus $A(G, \mathbf{x}) \in \mathsf{cover}$.

Now consider the case that $A(G, \mathbf{x}) \in \text{cover}$. In this case, there exists a node v and a set $S \in \mathcal{S}(v)$ such that $S = \{\mathcal{E}(u) \mid u \in G\}$. Such a set S is the collection of local views of nodes of some configuration $(G', \mathbf{x}') \in \mathcal{L}$ and some Id' assignment. On the other hand, S is also the collection of local views of nodes of the given configuration $(G, \mathbf{x}) \in \mathcal{L}$ and Id assignment. It follows that $(G, \mathbf{x}) = (G', \mathbf{x}') \in \mathcal{L}$.

We now define a natural problem, called containment, which is NLD(O(1))-complete. Somewhat surprisingly, the definition of containment is quite similar to the definition of cover. Specifically, as in cover, every node v is given as input an element $\mathcal{E}(v)$, and a finite collection of sets $\mathcal{S}(v)$. However, in contrast to cover, the union of these inputs is in the containment language if there exists a node v such that one set in $\mathcal{S}(v)$ contains the union of all the elements given to the nodes. Formally, we define containment $= \{(G, (\mathcal{E}, \mathcal{S})) \mid \exists v \in V(G), \exists S \in \mathcal{S}(v) \text{ s.t. } S \supseteq \{\mathcal{E}(v) \mid v \in V(G)\}.$

Theorem 4.6 containment is NLD(O(1))-complete.

Proof. We first prove that containment is NLD(O(1))-hard. Consider some $\mathcal{L} \in NLD(O(1))$; we show that $\mathcal{L} \leq$ containment. For this purpose, we describe a local distributed algorithm D transforming any configuration for \mathcal{L} to a configuration for containment preserving the memberships to these languages.

Let $t = t_{\mathcal{L}} \geq 0$ be some (constant) integer such that there exists a local nondeterministic algorithm $A_{\mathcal{L}}$ deciding \mathcal{L} in time at most t. Let (G, \mathbf{x}) be a configuration for \mathcal{L} and let Id be an identity assignment. Algorithm D operating at a node v outputs a pair $(\mathcal{E}(v), \mathcal{S}(v))$, where $\mathcal{E}(v)$ is the "t-local view" at v in (G, \mathbf{x}) , i.e., the ball of radius t around v, $B_G(v, t)$, together

with the inputs of these nodes and their identities, and $\mathcal{S}(v)$ is the collection of sets S defined as follows. For a binary string x, let |x| denote the length of x, i.e., the number of bits in x. For every vertex v, let $\psi(v) = 2^{|\mathrm{Id}(v)| + |\mathbf{x}(v)|}$. Node v first generates all configurations (G', \mathbf{x}') where G' is a graph with $m \leq \psi(v)$ vertices, and \mathbf{x}' is a collection of m input strings of length at most $\psi(v)$, such that $(G', \mathbf{x}') \in \mathcal{L}$. For each such configuration (G', \mathbf{x}') , node v generates all possible Id' assignments to V(G') such that for every node $u \in V(G')$, $|\mathrm{Id}(u)| \leq \psi(v)$. Now, for each such pair of a graph (G', \mathbf{x}') and an Id' assignment, algorithm D associates a set $S \in \mathcal{S}(v)$ consisting of the m = |V(G')| t-local views of the nodes of G' in (G', \mathbf{x}') . We show that $(G, \mathbf{x}) \in \mathcal{L} \iff D(G, \mathbf{x}) \in \mathtt{containment}$.

If $(G, \mathbf{x}) \in \mathcal{L}$, then by the construction of Algorithm D, there exists a set $S \in \mathcal{S}(v)$ such that S covers the collection of t-local views for (G, \mathbf{x}) , i.e., $S = \{\mathcal{E}(u) \mid u \in G\}$. Indeed, the node v maximizing $\psi(v)$ satisfies $\psi(v) \geq \max\{\mathrm{Id}(u) \mid u \in V(G)\} \geq n$ and $\psi(v) \geq \max\{\mathbf{x}(u) \mid u \in V(G)\}$. Therefore, that specific node has constructed a set S that precisely corresponds to (G, \mathbf{x}) and its given Id assignment; hence, S contains all corresponding t-local views. Thus, $D(G, \mathbf{x}) \in \mathtt{containment}$.

Now consider the case that $D(G, \mathbf{x}) \in \text{containment}$. In this case, there exists a node v and a set $S \in \mathcal{S}(v)$ such that $S \supseteq \{\mathcal{E}(u) \mid u \in G\}$. Such a set S is the collection of t-local views of nodes of some configuration $(G', \mathbf{x}') \in \mathcal{L}$ and some Id' assignment. Since $(G', \mathbf{x}') \in \mathcal{L}$, there exists a certificate \mathbf{y}' for the nodes of G', such that when algorithm $A_{\mathcal{L}}$ operates on $(G', \mathbf{x}', \mathbf{y}')$, all nodes say "yes". Now, since S contains the t-local views of nodes (G, \mathbf{x}) , with the corresponding identities, there exists a mapping $\phi : (G, \mathbf{x}, \mathrm{Id}) \to (G', \mathbf{x}', \mathrm{Id}')$ that preserves inputs and identities. Moreover, when restricted to a ball of radius t around a vertex $v \in G$, ϕ is actually an isomorphism between this ball and its image. We assign a certificate \mathbf{y} to the nodes of G: for each $v \in V(G)$, $\mathbf{y}(v) = \mathbf{y}'(\phi(v))$. Now, Algorithm $A_{\mathcal{L}}$ when operating on $(G, \mathbf{x}, \mathbf{y})$ outputs "yes" at each node of G. By the correctness of $A_{\mathcal{L}}$, we obtain $(G, \mathbf{x}) \in \mathcal{L}$.

We now show that containment $\in \text{NLD}(O(1))$. For this purpose, we design a nondeterministic local algorithm A that decides whether a configuration (G, \mathbf{x}) is in containment. Such an algorithm A is designed to operate on $(G, \mathbf{x}, \mathbf{y})$, where \mathbf{y} is a certificate. The configuration (G, \mathbf{x}) satisfies that $\mathbf{x}(v) = (\mathcal{E}(v), \mathcal{S}(v))$. Algorithm A aims at verifying whether there exists a node v^* with a set $S^* \in \mathcal{S}(v^*)$ such that $S^* \supseteq \{\mathcal{E}(v) \mid v \in V(G)\}$.

Given a correct instance, i.e., a configuration (G, \mathbf{x}) , we define the certificate \mathbf{y} as follows. For each node v, the certificate $\mathbf{y}(v)$ at v consists of several fields, specifically, $\mathbf{y}(v) = (\mathbf{y}_c(v), \mathbf{y}_s(v), \mathbf{y}_{id}(v), \mathbf{y}_l(v))$. The candidate configuration field $\mathbf{y}_c(v)$ is a triplet $\mathbf{y}_c(v) = (G', \mathbf{x}', \mathrm{Id}')$, where (G', \mathbf{x}') is an isomorphic copy (G', \mathbf{x}') of (G, \mathbf{x}) and Id' is an

identity assignment for the nodes of G'. The candidate set field $\mathbf{y}_s(v)$ is a copy of S^* , i.e., $\mathbf{y}_s(v) = S^*$. In addition, let u and u^* be the nodes in (G', \mathbf{x}') corresponding to v and v^* , respectively. The candidate identity field $\mathbf{y}_{id}(v)$ is $\mathbf{y}_{id}(v) = \mathrm{Id}'(u)$, and the candidate leader field $\mathbf{y}_l(v)$ is $\mathbf{y}_l(v) = \mathrm{Id}'(u^*)$.

We describe the operation of Algorithm A on some triplet $(G, \mathbf{x}, \mathbf{y})$. First, each node v verifies that it agrees with its neighbors on the candidate configuration and candidate set fields in their certificates. That way, if all nodes say "yes" then we know that all nodes hold the same candidate configuration which is some triplet $(G', \mathbf{x}', \mathrm{Id}')$, and the same candidate set S'. Second, each node verifies that $\mathcal{E}(v) \in S'$. Also, each node checks that it agrees with its neighbors on the candidate leader field in their certificates. I.e, that there exists some integer k such that for all nodes v we have $\mathbf{y}_l(v) = k$. Each node v checks that there exists a node $v \in V(G')$ such that $\mathrm{Id}'(u^*) = k$, and that there exists a node $v' \in V(G')$ such that $\mathbf{y}_{id}(v) = \mathrm{Id}'(v')$. Moreover, node v verifies that the input \mathbf{x}' at v' contains a collection of sets $\mathcal{S}'(v')$ that contains S', that is, $S' \in \mathcal{S}'(v')$. Finally, each node v verifies that its immediate neighborhood $B_G(v,1)$ agrees with the corresponding neighborhood of v' in G', and that the candidate identities $\mathbf{y}_{id}(w)$ of its neighbors $w \in B_G(v,1)$ are compatible with the corresponding identities $\mathrm{Id}'(w')$ in G'. We term this verification the neighborhood check of v.

It is easy to see that when applying Algorithm A on a correct instance, together with the certificate described above, each node outputs "yes". We now show the other direction. Assume that Algorithm A applied on some triplet $(G, \mathbf{x}, \mathbf{y})$ outputs "yes" at each node, our goal is to show that $(G, \mathbf{x}) \in \mathcal{L}$. Since all nodes say "yes" on $(G, \mathbf{x}, \mathbf{y})$, it follows that the certificate $\mathbf{y}(v)$ at every node $v \in V(G)$ contains the same candidate configuration field $(G', \mathbf{x}', \mathrm{Id}')$, the same candidate set S' and the same pointer $\mathrm{Id}'(v')$ to a vertex $v' \in G'$, such that $S' \in \mathcal{S}(v')$. Since each node $v \in V(G)$ verifies that $\mathcal{E}(v) \in S'$, it follows that $S' \supseteq \{\mathcal{E}(v) \mid v \in V(G)\}$. It remains to show that there exists a node $v^* \in V(G)$ such that $S' \in \mathcal{S}(v^*)$. Indeed, this follows by the neighborhood checks of all nodes.

5 Future work

This paper aims to make a first step in the direction of establishing a complexity theory for the locality discipline. Many interesting questions are left open. For example, it would be interesting to investigate the connections between BPLD(t, p, q) for different p and q such that $p^2 + q \le 1$. (A simple observation shows that BPLD $(t, p, q) \subseteq BPLD(t, p^k, 1 - (1-q)^k)$, for every integer k. Indeed, given an algorithm with a "yes" and "no" success probabilities

p and q, one can modify the success probabilities by performing k runs and requiring each node to individually output "no" if it decided "no" on at least one of the runs. In this case, the "no" success probability increases from q to at least $1 - (1 - q)^k$, and the "yes" success probability then decreases from p to p^k .) Another interesting question is whether the phenomena we observed regarding randomization occurs also in the non-deterministic setting, that is, whether BPNLD(t, p, q) collapses into NLD(O(t)), for $p^2 + q > 1$.

Our model of computation, namely, the \mathcal{LOCAL} model, focuses on difficulties arising from purely locality issues, and abstracts away other complexity measures. Naturally, it would be very interesting to come up with a rigorous complexity framework taking into account also other complexity measures. For example, it would be interesting to investigate the connections between classical computational complexity theory and the local complexity one. The bound on the (centralized) running time in each round (given by the function f, see Section 2) may serve a bridge for connecting the two theories, by putting constrains on this bound (i.e., f must be polynomial, exponential, etc). Also, one could restrict the memory used by a node, in addition to, or instead of, bounding the sequential time. Finally, it would be interesting to come up with a complexity framework taking also congestion into account.

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