# Non-positive curvature, and the planar embedding conjecture 

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#### Abstract

The planar embedding conjecture asserts that any planar metric admits an embedding into $L_{1}$ with constant distortion. This is a well-known open problem with important algorithmic implications, and has received a lot of attention over the past two decades. Despite significant efforts, it has been verified only for some very restricted cases, while the general problem remains elusive.

In this paper we make progress towards resolving this conjecture. We show that every planar metric of non-positive curvature admits a constant-distortion embedding into $L_{1}$. This confirms the planar embedding conjecture for the case of non-positively curved metrics.


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## 1 Introduction

If $\left(X, d_{X}\right),\left(Y, d_{Y}\right)$ are metric spaces, and $f: X \rightarrow Y$ is injective, the distortion of $f$ is defined to be distortion $(f)=\|f\|_{\text {Lip }} \cdot\left\|f^{-1}\right\|_{\text {Lip }}$, where $\|f\|_{\text {Lip }}=\sup _{x \neq y \in X} \frac{d_{Y}(f(x), f(y))}{d_{X}(x, y)}$. For any metric space ( $X, d$ ), we use $c_{1}(X, d)$ to denote the $L_{1}$ distortion of $(X, d)$, i.e. the infimum over all numbers $D$ such that $X$ admits an embedding into $L_{1}$ with distortion $D$. For a graph $G=(V, E)$ we write $c_{1}(G)=\sup c_{1}(V, d)$ where $d$ ranges over all shortest-path metrics supported on $G$, and for a family $\mathcal{F}$ of graphs, we write $c_{1}(\mathcal{F})=\sup _{G \in \mathcal{F}} c_{1}(G)$. Thus for a family $\mathcal{F}$ of finite graphs, $c_{1}(\mathcal{F}) \leq D$ if and only if every geometry supported on a graph in $\mathcal{F}$ embeds into $L_{1}$ with distortion at most $D$.

In the seminal works of Linial-London-Rabinovich LLR95, and later Aumann-Rabani AR98] and Gupta-Newman-Rabinovich-Sinclair GNRS04, the geometry of graphs is related to the classical study of the relationship between flows and cuts.

A multi-commodity flow instance in $G$ is specified by a pair of non-negative mappings cap : $E \rightarrow$ $\mathbb{R}$ and dem : $V \times V \rightarrow \mathbb{R}$. We write maxflow ( $G$; cap, dem) for the value of the maximum concurrent flow in this instance, which is the maximal value $\varepsilon$ such that $\varepsilon \cdot \operatorname{dem}(u, v)$ can be simultaneously routed between every pair $u, v \in V$ while not violating the given edge capacities.

A natural upper bound on maxflow( $G$; cap, dem) is given by the sparsity of any cut $S \subseteq V$ :

$$
\begin{equation*}
\frac{\sum_{u v \in E} \operatorname{cap}(u, v)\left|\mathbf{1}_{S}(u)-\mathbf{1}_{S}(v)\right|}{\sum_{u, v \in V} \operatorname{dem}(u, v)\left|\mathbf{1}_{S}(u)-\mathbf{1}_{S}(v)\right|}, \tag{1}
\end{equation*}
$$

where $\mathbf{1}_{S}: V \rightarrow\{0,1\}$ is the indicator function for membership in $S$. We write gap $(G)$ for the maximum gap between the value of the flow and the upper bounds given by (1), over all multicommodity flow instances on $G$. This is the multi-commodity max-flow/min-cut gap for $G$. The fundamental connection between embeddings into $L_{1}$ and multi-commodity flows is captured in the following result.
Theorem 1.1 ([LLR95, GNRS04]). For every graph $G, c_{1}(G)=\operatorname{gap}(G)$.
In particular, combined with the techniques of LR99, LLR95, this implies that for any graph $G$, there exists a $c_{1}(G)$-approximation for the general Sparsest Cut problem.

### 1.1 The planar embedding conjecture

It has been shown by LLR95, LR99] that for general graphs, $c_{1}(G)=\Omega(\log n)$, and there has since been a lot of effort in trying to prove that $c_{1}(G)$ is bounded by some universal constant for interesting classes of graphs. The most well-known open case is the so-called planar embedding conjecture, summarized in the following:
Conjecture 1 (Planar embedding conjecture). For every planar graph $G$, $c_{1}(G)=O(1)$.
Despite several attempts on resolving this question, there has only been very little progress. More specifically, the work of Okamura \& Seymour OS81 implies that the metric induced on a single face of a planar graph embeds with constant distortion into $L_{1}$. In [GNRS04 it is shown that $c_{1}(G)=O(1)$ for any series-parallel, or outerplanar graph $G$. This result was extended to $O(1)$-outerplanar graphs in CGN ${ }^{+} 06$. Chakrabarti et al. CJLV08] obtained constant distortion embeddings of graphs that exclude a $\left(K_{5} \backslash e\right)$-minor. Note that even the case of planar graphs of treewidth 3 remains open. We remark that the best-known upper bound on $c_{1}(G)$ for planar graphs is $O(\sqrt{\log n})$, due to Rao Rao99], while the best-known lower bound is 2, due to Lee \& Raghavendra [LR10].

### 1.2 Generalizations: The GNRS conjecture

Gupta, Newman, Rabinovich, and Sinclair [GNRS04 posed the following generalization of the planar embedding conjecture, which seeks to characterize the graph families $\mathcal{F}$ such that $c_{1}(\mathcal{F})=$ $O(1)$, which by Theorem 1.1 also characterizes all graphs with multi-commodity gap bounded by some universal constant:

Conjecture 2 (GNRS conjecture [GNRS04]). For every family of finite graphs $\mathcal{F}$, one has $c_{1}(\mathcal{F})=$ $O(1)$ if and only if $\mathcal{F}$ forbids some minor.

We note that a strengthening of the GNRS conjecture for integral multi-commodity flows has also been considered CSW13. This is a seemingly harder problem, and progress has been even more limited in this case.

At first glance, it might appear that the GNRS conjecture is a vast generalization of the planar embedding conjecture, since planar graphs exclude $K_{5}$ as a minor. Despite this, Lee \& Sidiropoulos [LS09] have shown that the GNRS conjecture is equivalent to the conjunction of the planar embedding conjecture, with the manifestly simpler $k$-sum embedding conjecture summarized bellow. For a graph family $\mathcal{F}$, let $\oplus_{k} \mathcal{F}$ denote the closure of $\mathcal{F}$ under $k$-clique sums (see [LS09] for a more detailed exposition). We note that the case $k=1$ is folklore, while recently progress has been reported for the case $k=2$ by Lee and Poore [LP13]; even for $k=2$ however, the problem remains open.

Conjecture 3 ( $k$-sum conjecture LS09]). For any family of graphs $\mathcal{F}$, we have $c_{1}(\mathcal{F})=O(1)$ if and only if $c_{1}\left(\oplus_{k} \mathcal{F}\right)=O(1)$ for every $k \in \mathbb{N}$.

It is therefore apparent that the planar embedding conjecture is a major step towards determining the multi-commodity gap in arbitrary graphs.

### 1.3 Our results

All previous attempts on the planar embedding conjecture have been topological in nature, meaning that they seek to obtain constant-distortion embeddings by restricting the topology of the planar graph. As a consequence, all known methods are insufficient even for planar graphs of treewidth 3.

We depart from this paradigm by instead restricting the geometry of the planar metric. For any metric $(X, d)$, we have that $(X, d)$ is the shortest-path metric of a planar graph if and only if it can be realized as a set of points in a simply-connected (i.e. planar) surface. We say that a planar metric is non-positively curved if it can be realized as a set of points in a surface of non-positive curvature (see Section 1.5 for the definition of non-positively curved spaces). This leads to a natural, and very rich class of planar metrics. For instance, non-positively curved planar metrics include all trees, all regular grids (up to constant distortion), and arbitrary subsets of the hyperbolic plane $\mathbb{H}^{2}$.


Our main result is as follows.
Theorem 1.2 (Main). There exists a universal constant $\gamma>1$, such that every non-positively curved planar metric admits an embedding into $L_{1}$ with distortion at most $\gamma$.

Since we are motivated by the applications of metric embeddings in computer science, we will restrict our discussion to finite metrics. We remark however that our result can be extended to obtain constant-distortion embeddings of arbitrary simply-connected surfaces of non-positive curvature into $L_{1} \prod_{\square}^{1]}$

We note that embeddings of various hyperbolic spaces have been previously considered. We refer to KL06, BS, BS05b, BS05a. However, none of the previous results captures $L_{1}$ embeddings of arbitrary non-positively curved planar metrics. In fact, our approach is significantly different than all previous works.

### 1.4 A high-level overview of our approach

We now give an informal, and somewhat imprecise overview of some of the main challenges that we face when trying to embed non-positively curved planar metrics into $L_{1}$.

Distributions over monotone cuts. Let $(X, d)$ be a metric space. We will use the standard representation of $L_{1}$ as the cone of cut pseudo-metrics (see Section 1.5 for the definition). This means that in order to embed a space into $L_{1}$ with constant distortion, it suffices to find a probability distribution over cuts, such that the probability that any pair of points $x, y$ gets separated, is $\Theta(\alpha \cdot d(x, y))$, for some normalization factor $\alpha>0$.

It follows by the work of Lee and Raghavendra [LR10] (see also [CJLV08]) that when seeking a constant-distortion embedding of certain spaces into $L_{1}$ it suffices to consider distributions over a specific type of cuts, called monotone. More precisely, let $x$ be a fixed point. We say that a cut $S$ is monotone (w.r.t. $x$ ) if every shortest path starting from $x$ crosses $S$ at most once. Let us say that a metric space is a bundle if there exist two points $s, t$, such that for every point $z$, there exists an $s-t$ geodesic containing $z$. Then it is shown in [R10 that a bundle admits a constant-distortion embedding into $L_{1}$ if and only if it is a convex combination of monotone cuts (i.e. a convex combination of cut pseudo-metrics, where every indicator set is a monotone cut). It is easy to show that every finite non-positively curved metric admits an isometric embedding into a bundle. We can therefore focus our efforts into finding a good distribution over monotone cuts.

The structure of monotone cuts in non-positively curved spaces. It is convenient to demonstrate the main ideas using the following example of a "pinched square". Let $X=[0,1]^{2}$, endowed with the Euclidean distance. The space $X$ can be embedded isometrically into $L_{1}$ by taking an appropriate distribution over random half-plane cuts (e.g. by choosing a uniformly random point $p \in X$, and taking the half-plane supported by a line passing through $p$ forming a uniformly random angle with the $x$-axis). Let $A$ be one of the sides of $X$, and let $Y=X / A$ be the quotient space obtained by contracting $A$ into a single point, which we will refer to as the basepoint.

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Strictly speaking, the resulting $Y$ is not a space of non-positive curvature (in particular, there exist pairs of points in $Y$ with two distinct geodesics joining them). However, $Y$ admits a constantdistortion embedding into a planar surface of non-positive curvature, so in order to simplify the exposition, we may use $Y$ without loss of generality.

It is fairly easy to see that even though $Y$ might "look" like a triangle, its geometry is far from that of a flat Euclidean triangle. In fact, one can show that $Y$ cannot be embedded into the Euclidean plane with bounded distortion. As a consequence, embedding $Y$ into $L_{1}$ requires a significantly more involved distribution over cuts. Such a distribution can be constructed using cuts of the following form: For every $r \in[0,1]$, we have a family of cuts $S$ that are contained inside the ball of radius $r$ from the basepoint, and with boundary $\partial C$ given by a function of period $\Theta(r)$. Roughly speaking, these cuts can be obtained by random shifts along the $x$-axis of cuts from the following infinite family:


Here, the probability of a cut decreases when $r \rightarrow 0$. It is important to note that the structure of a cut depends on the distance of its boundary to the basepoint. It can be shown that this is the case for any constant distortion embedding of $Y$ into $L_{1}$. Moreover, in an constant-distortion embedding, this transition has to happen in a smooth way as $r \rightarrow 0$.

Handling multiple scales. Suppose now that we modify the space $Y$ as follows. Let $R$ be a ray in $Y$, i.e. an unbounded geodesic starting at the basepoint, and let $R^{\prime}$ be a suffix of $R$. Cutting $Y$ along $R^{\prime}$ introduces two copies $R_{1}^{\prime}, R_{2}^{\prime}$ of $R^{\prime}$ as segments of the boundary. We glue a copy of $Y$ along $R_{1}^{\prime} \cup R_{2}^{\prime}$ as follows:


The resulting space $Y^{\prime}$ again embeds with constant distortion into a planar metric of non-positive curvature. Constructing a constant-distortion embedding for $Y^{\prime}$ requires the use of even more intricate families of cuts. Intuitively, a single cut now has to "gracefully" combine information
form multiple different scales. Let $p_{1}, p_{2} \in Y^{\prime}$ be the basepoints of the two copies of $Y$ in $Y^{\prime}$. The structure of a "typical" cut $S$ has to depend on the distances between $\partial S$, and both $p_{1}$, and $p_{2}$ :


A naive way to address this problem would be to define a distribution over cuts for every single scale, and then try to combine them into a single distribution. The problem with this approach is that cuts from different scales (in our example, cuts for the two different copies of $Y$ ) might not agree on their boundary. This disagreement results in larger distortion every time we combine two different scales. Since there can be many scales, this methods leads to unbounded distortion.

We overcome this obstacle by designing a distribution over cuts that is scale-independent. This is done by starting with a distribution over cuts that handles large distances, and gradually modifying it to handle smaller scales. The main technical contribution of this paper is showing that in a nonpositively curved surface, this can be done without increasing the distortion.

### 1.5 Preliminaries

We now review some basic definitions and notions which appear throughout the paper.
Graphs. Let $G$, and let $S \subseteq V(G)$. We denote by $G[S]$ the subgraphs of $G$ induced by $S$, i.e. $G[S]=\left(S, E(G) \cap\binom{S}{2}\right)$. We will consider graphs with every edge having a non-negative length. We say that a graph is unweighted if all of its edges have unit length. Let $\operatorname{diam}(G)$ denote the diameter of $G$, i.e. $\operatorname{diam}(G)=\max _{x, y \in V(G)} d_{G}(x, y)$. We refer to a path between two vertices $x, y \in V(G)$ as a $x-y$ path.

Cuts and $L_{1}$ embeddings. A cut of a graph $G$ is a partition of $V(G)$ into $(S, \bar{S})$-we sometimes refer to a subset $S \subseteq V$ as a cut as well. A cut gives rise to a pseudometric; using indicator functions, we can write the cut pseudometric as $\rho_{S}(x, y)=\left|\mathbf{1}_{S}(x)-\mathbf{1}_{S}(y)\right|$. A central fact is that embeddings of finite metric spaces into $L_{1}$ are equivalent to sums of positively weighted cut metrics over that set (for a simple proof of this see [DL97]).

A cut measure on $G$ is a function $\mu: 2^{V} \rightarrow \mathbb{R}_{+}$for which $\mu(S)=\mu(\bar{S})$ for every $S \subseteq V$. Every cut measure gives rise to an embedding $f: V \rightarrow L_{1}$ for which

$$
\begin{equation*}
\|f(u)-f(v)\|_{1}=\int\left|\mathbf{1}_{S}(u)-\mathbf{1}_{S}(v)\right| d \mu(S) \tag{2}
\end{equation*}
$$

where the integral is over all cuts $(S, \bar{S})$. Conversely, to every embedding $f: V \rightarrow L_{1}$, we can associate a cut measure $\mu$ such that (2) holds.

Non-positively curved spaces. We will describe our proof using the definition of non-positive curvature in the sense of Busemann. We give here a brief overview of some of the relevant terminology, and we refer the reader to Pap05, TL97 for a more detailed exposition. A metric space $(X, d)$ is called geodesic if for every pair of points there exists a geodesic joining them. We say that $(X, d)$ is non-positively curved, if for any pair of affinely parameterized geodesics $\gamma:[a, b] \rightarrow X$, $\gamma^{\prime}:\left[a^{\prime}, b^{\prime}\right] \rightarrow X$, the map $D_{\gamma, \gamma^{\prime}}:[a, b] \times\left[a^{\prime}, b^{\prime}\right] \rightarrow \mathbb{R}$ defined by

$$
D_{\gamma, \gamma^{\prime}}\left(t, t^{\prime}\right)=d\left(\gamma(t), \gamma^{\prime}(t)\right)
$$

is convex. As we show, this property is sufficient to obtain constant-distortion embeddings of simply-connected surfaces into $L_{1}$.

Lipschitz partitions. Let $(X, d)$ be a metric space. A distribution $\mathcal{F}$ over partitions of $X$ is called $(\beta, \Delta)$-Lipschitz if every partition in the support of $\mathcal{F}$ has only clusters of diameter at most $\Delta$, and for every $x, y \in X$,

$$
\operatorname{Pr}_{C \in \mathcal{F}}[C(x) \neq C(y)] \leq \beta \cdot \frac{d(x, y)}{\Delta} .
$$

We denote by $\beta_{(X, d)}$ the infimum $\beta$ such that for any $\Delta>0$, the metric $(X, d)$ admits a $(\Delta, \beta)$ Lipschitz random partition, and we refer to $\beta_{(X, d)}$ as the modulus of decomposability of $(X, d)$. The following theorem is due to Klein, Plotkin, and Rao KPR93, and Rao Rao99.

Theorem 1.3 (KPR93], Rao99]). For any planar graph $G$, we have $\beta_{\left(V(G), d_{G}\right)}=O(1)$.
Stochastic embeddings. A mapping $f: X \rightarrow Y$ between two metric spaces $(X, d)$ and $\left(Y, d^{\prime}\right)$ is non-contracting if $d^{\prime}(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. If $(X, d)$ is any finite metric space, and $\mathcal{Y}$ is a family of finite metric spaces, we say that $(X, d)$ admits a stochastic $D$-embedding into $\mathcal{Y}$ if there exists a random metric space $\left(Y, d^{\prime}\right) \in \mathcal{Y}$ and a random non-contracting mapping $f: X \rightarrow Y$ such that for every $x, y \in X$,

$$
\begin{equation*}
\mathbb{E}\left[d^{\prime}(f(x), f(y))\right] \leq D \cdot d(x, y) \tag{3}
\end{equation*}
$$

The infimal $D$ such that (3) holds is the distortion of the stochastic embedding. For a graph $G$ and a graph family $\mathcal{F}$ we write $G \stackrel{D}{\rightsquigarrow}$ to denote the fact that $G$ stochastically embeds into a distribution over graphs in $\mathcal{F}$, with distortion $D$. We also use the notation $G \rightsquigarrow \mathcal{F}$ to denote the fact that $G \stackrel{D}{\rightsquigarrow} \mathcal{F}$, for some universal constant $D \geq 1$. We will use the following fact.

Lemma 1.4. Let $\mathcal{F}$ be a family of graphs, such that every $H \in \mathcal{F}$ admits an embedding into $L_{1}$ with distortion at most $\alpha \geq 1$. Let $G$ be a graph, such that $G \stackrel{\beta}{\rightsquigarrow} \mathcal{F}$, for some $\beta \geq 1$. Then, $G$ admits an embedding into $L_{1}$ with distortion at most $\alpha \beta$.

Let $G$ be a graph, and let $A \subseteq V(G)$. The dilation of $A$ is defined to be

$$
\operatorname{dil}_{G}(A)=\max _{u, v \in V(G)} \frac{d_{G[A]}(u, v)}{d_{G}(u, v)}
$$

For two graphs $G, G^{\prime}$, a 1-sum of $G$ with $G^{\prime}$ is a graph obtained by taking two disjoint copies of $G$ and $G^{\prime}$, and identifying a vertex $v \in V(G)$ with a vertex $v^{\prime} \in V\left(G^{\prime}\right)$. For a graph family $\mathcal{X}$, we denote by $\oplus_{1} \mathcal{X}$ the closure of $\mathcal{X}$ under 1 -sums.

Lemma 1.5 (Peeling lemma LS09]). Let $G$ be a graph, and $A \subseteq V(G)$. Let $G^{\prime}=\left(V(G), E^{\prime}\right)$ be a graph with $E^{\prime}=E(G) \backslash E(G[A])$, and let $\beta=\beta_{\left(V, d_{G^{\prime}}\right)}$ be the corresponding modulus of decomposability. Then, there exists a graph family $\mathcal{F}$ such that $G \stackrel{D}{\mathcal{F}}$, where $D=O\left(\beta \cdot \operatorname{dil}_{G}(A)\right)$, and every graph in $\mathcal{F}$ is a 1-sum of isometric copies of the graphs $G[A]$ and $\{G[V \backslash A \cup\{a\}]\}_{a \in A}$.

### 1.6 Organization

The rest of the paper is organized as follows. In Section 2 we show how to embed an arbitrary non-positively curved planar metric into an unweighted graph of special structure, called a funnel. In Section 3 we show how to stochastically embed a funnel into a distribution over simpler graphs, called pyramids. In Section 4 we introduce some of the machinery that we will use when defining our embedding into $L_{1}$. More specifically, we describe the basic operation of cuts that will allow to gradually modify a cut when computing our embedding. Using this machinery, we describe our embedding in Section 5 . Finally, in Section 6 we prove that the constructed embedding has constant distortion.

## 2 A canonical representation of non-positively curved planar metrics

In this section we show that non-positively curved planar metrics can be embedded with constantdistortion into a certain type of unweighted planar graphs that we call funnels. Intuitively, a funnel is obtained by taking the union of a tree having all its leaves at the same level, with a collection of cycles, where every cycle spans all the vertices in a single layer of the tree.

Definition 2.1 (Funnel). Let $G$ be an unweighted planar graph, and let $v \in V(G)$. We say that $G$ is a funnel with basepoint $v$ if the following conditions are satisfied:
(1) There exists a collection of pairwise vertex-disjoint cycles $C_{1}, \ldots, C_{\Delta} \subset G$, such that $V(G)=$ $\bigcup_{i=1}^{\Delta} V\left(C_{i}\right)$. For notational convenience, we allow a cycle $C_{i}$ to consist of a single vertex, in which case it has no edges. Moreover, we have $V\left(C_{1}\right)=\{v\}$. We refer to each $C_{i}$ as a layer of $G$.
(2) For every $i \in\{2, \ldots, \Delta-1\}$, the graph $G \backslash V\left(C_{i}\right)$ has exactly two connected components, one with vertex set $\bigcup_{j=1}^{i-1} V\left(C_{j}\right)$, and another with vertex set $\bigcup_{j=i+1}^{\Delta} V\left(C_{j}\right)$.
(3) For every $i \in\{2, \ldots, \Delta\}$, every $u \in V\left(C_{i}\right)$ has exactly one neighbor $u^{\prime} \in V\left(C_{i-1}\right)$. We refer to $u^{\prime}$ as the parent of $u$. In particular, $v$ is the parent of all vertices in $V\left(C_{2}\right)$.
(4) For every $i \in\{1, \ldots, \Delta-1\}$, every $w \in V\left(C_{i}\right)$ has at least one neighbor $w^{\prime} \in V\left(C_{i+1}\right)$. We refer to every such $w^{\prime}$ as a child of $w$.

Let $R$ be a path in $G$ between $v$, and a vertex $u \in V\left(C_{\Delta}\right)$. We say that $R$ is a ray. We denote by Funnels the family of all funnel graphs. Figure 1 depicts an example of a funnel.

We will use the following two facts about metric spaces of non-positive curvature (see e.g. Pap05).
Lemma 2.2. Let $(\mathcal{S}, d)$ be a geodesic metric space of non-positive curvature. Let $x^{*}, x, y \in \mathcal{S}$, and let $\gamma:[0, d(x, y)] \rightarrow \mathcal{S}$ be a geodesic between $x$, and $y$. Then, the function $f:[0,1] \rightarrow \mathbb{R}$, with $f(t)=d\left(x^{*}, \gamma(t)\right)$ is convex.


Figure 1: A funnel.

Lemma 2.3. Let $(\mathcal{S}, d)$ be a geodesic metric space of non-positive curvature, and let $x^{*}, x, y \in \mathcal{S}$. Let $\gamma_{x}:\left[0, d\left(x^{*}, x\right)\right] \rightarrow \mathcal{S}$ be a geodesic between $x^{*}$, and $x$, and let $\gamma_{y}:\left[0, d\left(x^{*}, y\right)\right] \rightarrow \mathcal{S}$ be a geodesic between $x^{*}$, and $y$. Then, the function $f:[0,1] \rightarrow \mathbb{R}$, with $f(t)=d\left(\gamma_{x}(t), \gamma_{y}(t)\right)$ is non-decreasing.

Recall that for a metric space $(X, d)$, and some $r>0$, an $r$-net in $(X, d)$ is a maximal subset $X^{\prime} \subseteq X$ such that for any $x, y \in X^{\prime}$, we have $d(x, y) \geq r$.

Lemma 2.4 (Funnel representation). Let $\mathcal{S}$ be a simply-connected surface, and let d be a nonpositively curved metric on $\mathcal{S}$. Let $X \subset \mathcal{S}$ be a finite set of points. Then, $(X, d)$ admits an embedding into a funnel with constant distortion.

Proof. By scaling $d$, we may assume w.l.o.g. that the minimum distance in $(X, d)$ is at least 1 (note that scaling $d$ results into a metric which is still of non-positive curvature). Let $x^{*} \in \mathcal{S}$ be an arbitrary point. For any $x \in \mathcal{S}$, let $\gamma(x)$ denote the unique geodesic between $x$, and $x^{*}$. Let $r=1 / 8$. For any integer $i \geq 0$, let

$$
D_{i}=\left\{x \in \mathcal{S}: d\left(x^{*}, x\right) \leq i r\right\} .
$$

Since $d$ is non-positively curved, we have that for every $i$, the set $D_{i}$ is a disk (see e.g. Pap05). Let $\Gamma_{i}$ be the cycle in $\mathcal{S}$ bounding $D_{i}$. Let $\Delta=\min \left\{i \in \mathbb{N}: X \subset D_{i}\right\}$. Let $N_{\Delta}$ be an $r$-net in $\Gamma_{\Delta}$. Note that since $(\mathcal{S}, d)$ is non-positively curved, there exists a unique geodesic between any pair of points. This implies that the subspace

$$
T=\bigcup_{x \in N_{\delta}} \gamma(x)
$$

is a (simplicial) tree. For every $i \in\{0, \ldots, \Delta-1\}$, we define an $r$-net $N_{i}$ of $\Gamma_{i}$ as follows. Suppose that $N_{i+1}$ is already defined. Let $Y_{i}^{\prime}$ be the set of points $p \in N_{\Delta}$ such that $\gamma(p)$ intersects $N_{i+1}$. Let $N_{i}^{\prime}=\Gamma_{i} \cap\left(\bigcup_{p \in Y_{i}^{\prime}} \gamma(p)\right)$. Note that for any $x \in \Gamma_{i}$, there exists $y \in N_{i}^{\prime}$ such that $d(x, y)<r$. Therefore, we can set to be a maximal subset $N_{i} \subseteq N_{i}^{\prime}$, such that $N_{i}$ is an $r$-net. This concludes the definition of the sequence of subsets $N_{0}, \ldots, N_{\Delta}$. Note that $N_{0}=\left\{x^{*}\right\}$.

We define a graph $G$, with $V(G)=\bigcup_{i=0}^{\Delta} N_{i}$. The set of edges $E(G)$ is defined as follows. For every $i \in\{1, \ldots, \Delta\}$, we add a unit-length edge $\{x, y\} \in E(G)$ for any two points $x, y \in N_{i}$, such that $x$, and $y$ appear consecutively in a clockwise traversal of $\Gamma_{i}$. Moreover, for every $z \in N_{i}$, let $z^{\prime} \in N_{\Delta}$ be such that $z \in \gamma\left(z^{\prime}\right)$. Let $z^{\prime \prime}$ be the point in the intersection of $\gamma\left(z^{\prime}\right)$ with $\Gamma_{i-1}$. If $z^{\prime \prime} \in N_{i-1}$, then we add the unit-length edge $\left\{z, z^{\prime \prime}\right\}$. Otherwise, let $w$ be the first point in $N_{i-1}$ that we visit in a clockwise traversal of $\Gamma_{i-1}$ starting from $z^{\prime \prime}$. We add the unit-length edge edge
$\{z, w\}$. This concludes the definition of the graph $G$. It is straightforward to check that $G$ is a funnel with basepoint $x^{*}$.

We can now define an embedding $f: X \rightarrow V(G)$, by mapping every points $x \in X$ to its nearest neighbor in $V(G)$. It remains to verify that $f$ has constant distortion. Observe that the set $V(G)$ contains a $2 r$-net in $D_{\Delta}$, and therefore for any $x \in X$, we have $d(x, f(x))<2 r$. Since the minimum distance in $X$ is at least 1 , this implies that $f$ is an injection, and for any $x, y \in X$, we have $d(x, y)=\Theta(d(f(x), f(y)))$. It therefore suffices to show that for any $x, y \in V(G)$, we have $d_{G}(x, y)=\Theta(d(x, y))$.

We first show that for any $x, y \in V(G)$, we have $d_{G}(x, y)=\Omega(d(x, y))$. To that end, it suffices to show that for any edge $\{x, y\} \in E(G)$, we have $d(x, y)=O\left(d_{G}(x, y)\right)=O(1)$.

We consider first case where there exists $i \in\{1, \ldots, \Delta\}$ such that $x, y \in N_{i}$, and $x, y$ are consecutive in $\Gamma_{i}$. Let $\alpha$ be the arc of $\Gamma_{i}$ between $x$, and $y$, that does not contain any other points in $N_{i}$. By the triangle inequality, there exists $z \in \alpha$, such that $d(x, z) \geq d(x, y) \geq 2$, and $d(y, z) \geq d(x, y) \geq 2$. Since $N_{i}$ is an $r$-net in $\Gamma_{i}$, it follows that there exists $z^{\prime} \in N_{i}$, such that $d\left(z, z^{\prime}\right)<r$. Let $\beta$ be the geodesic between $z$, and $z^{\prime}$. The arc $\beta$ intersects either $\gamma(x)$, or $\gamma(y)$. Assume w.l.o.g. that it intersects $\gamma(x)$ at some points $z^{\prime \prime}$. By lemma 2.2 we have that as we travel along $\beta$, the distance to $x^{*}$ is a convex function. This implies that $d\left(x, z^{\prime \prime}\right) \leq d\left(z, z^{\prime}\right)$. We conclude that $d(x, y) \leq 2 d(x, z) \leq 2\left(d\left(x, z^{\prime \prime}\right)+d\left(z^{\prime \prime}, z\right)\right) \leq 2\left(d\left(x, z^{\prime \prime}\right)+d\left(z^{\prime}, z\right)\right) \leq 4 d\left(z, z^{\prime}\right) \leq 4 r=O(1)$.

Next, we consider the case where $x \in N_{i}$, and $y \in N_{i+1}$, for some $i \in\{0, \ldots, \Delta\}$. Let $y^{\prime}$ be the point where $\gamma(y)$ intersects $\Gamma_{i}$. Arguing as above, we have that $d\left(y^{\prime}, x\right)=O(1)$. Therefore, $d(x, y) \leq d\left(y, y^{\prime}\right)+d\left(y^{\prime}, x\right) \leq r+O(1)=O(1)$. This concludes that proof that for any edge $\{x, y\} \in$ $E(G)$, we have $d(x, y)=O(1)$, and therefore for any $x, y \in V(G)$, we have $d_{G}(x, y)=\Omega(d(x, y))$.

It remains to show that for any $x, y \in V(G)$, we have $d_{G}(x, y)=O(d(x, y))$. We consider first the case where there exists $i \in\{1, \ldots, \Delta\}$, such that $x, y \in N_{i}$ (the case $i=0$ is trivial since $N_{0}$ contains only $x^{*}$ ). Let $\beta$ be a geodesic between $x$, and $y$. By lemma 2.2 , we have $\beta \subset D_{i}$. Let $x^{\prime}$ be the unique point in $\gamma(x) \cap \Gamma_{i-1}$, and let $y^{\prime}$ be the unique point in $\gamma(y) \cap \Gamma_{i-1}$. By lemma 2.3 we have $d\left(x^{\prime}, y^{\prime}\right) \leq d(x, y)$. Let $x^{\prime \prime}$ be the parent of $x$, and let $y^{\prime \prime}$ be the parent of $y$ in $G$. Let $x^{\prime}=z_{1}, \ldots, z_{k}=y^{\prime}$ be the points in $N_{i-1}$ that appear between $x^{\prime}$, and $y^{\prime}$ along $\Gamma_{i-1}$. For any $i \in\{1, \ldots, k\}$, pick a child $w_{i}$ of $z_{i}$, with $w_{1}=x$, and $w_{k}=y$. For any $i \in\{1, \ldots, k\}$, the curve $\beta$ intersects $\gamma\left(w_{i}\right)$. By the above discussion we have that the distance between any two such consecutive intersection points is $\Omega(1)$. Therefore, $d(x, y)=\operatorname{len}(\beta)=\Omega(k)$. The $x-y$ path in $G$ that visits the vertices $x z_{1} \ldots z_{k} y$ in this order has length $k+2$, and therefore $d_{G}(x, y)=O(d(x, y))$.

Next, we consider the case where there exists $i \in\{1, \ldots, \Delta\}$, such that $x \in N_{i}$, and $y \in N_{i-1}$. This case is identical to the case above, by replacing $y$ with $y^{\prime \prime}$. We therefore also obtain $d_{G}(x, y)=$ $O(d(x, y))$ in this case.

Finally, we consider the case of arbitrary points $x, y \in V(G)$. Let $\beta$ be the geodesic between $x$, and $y$. The curve $\beta$ can be decomposed into consecutive segments $\beta_{1}, \ldots, \beta_{k}$, such that every such segment is contained in (the closure of) $D_{i} \backslash D_{i-1}$, for some $i \in\{1, \ldots, \Delta\}$. Consider such a segment $\beta_{i}$. There exists $j, \ell \in\{0, \ldots, \Delta\}$, with $|j-\ell| \leq 1$, and such that $x_{i} \in \Gamma_{j}$, and $y_{i} \in \Gamma_{\ell}$. Let $x_{i}^{\prime}$ be the nearest neighbor of $x_{i}$ in $N_{j}$, and let $y_{i}^{\prime}$ be the nearest neighbor of $y_{i}$ in $N_{\ell}$. Since $N_{j}$ is a $O(1)$-net for $\Gamma_{j}$, and $N_{\ell}$ is a $O(1)$-net for $\Gamma_{\ell}$, we have $d\left(x_{i}^{\prime}, y_{i}^{\prime}\right) \leq d\left(x_{i}, y_{i}\right)+O(1)=O\left(d\left(x_{i}, y_{i}\right)\right)$. By the above analysis we have $d_{G}\left(x_{i}^{\prime}, y_{i}^{\prime}\right)=O\left(d\left(x_{i}^{\prime}, y_{i}^{\prime}\right)\right)$. Therefore, we obtain $d_{G}\left(x_{i}, y_{i}\right)=O\left(d\left(x_{i}, y_{i}\right)\right)$. We conclude that $d_{G}(x, y) \leq \sum_{i} d_{G}\left(x_{i}, y_{i}\right)=O\left(\sum_{i} d\left(x_{i}, y_{i}\right)\right)=O(d(x, y))$, as required.

## 3 Cutting along a ray

We now show that every funnel admits a constant-distortion stochastic embedding into a distribution over simpler graphs, that we call pyramids. Intuitively, a pyramid is obtained by "cutting" a funnel along a ray. The structure of pyramids will simplify the exposition of the embedding into $L_{1}$ that we describe in the subsequent sections.

Definition 3.1 (Pyramid). Let $G$ be an unweighted planar graph, let $v \in V(G)$, and let $\Delta \geq 1$ be an integer. We say that $G$ is a pyramid with basepoint $v$, and of depth $\Delta$ if the following conditions are satisfied:
(1) There exists a collection of pairwise vertex-disjoint paths $P_{1}, \ldots, P_{\Delta} \subset G$, with $P_{i}=u_{i, 1} \ldots u_{n_{i}, i}$, such that $V(G)=\bigcup_{i=1}^{\Delta} V\left(P_{i}\right)$. For notational convenience, we allow a path $P_{i}$ to consist of a single vertex, in which case it has no edges. Moreover, we have $V\left(P_{1}\right)=\{v\}$. We refer to each $P_{i}$ as a layer.
(2) For every $i \in\{2, \ldots, \Delta-1\}$, the graph $G \backslash V\left(P_{i}\right)$ has exactly two connected components, one with vertex set $\bigcup_{j=1}^{i-1} V\left(P_{j}\right)$, and another with vertex set $\bigcup_{j=i+1}^{\Delta} V\left(P_{j}\right)$.
(3) For every $i \in\{2, \ldots, \Delta\}$, every $u \in V\left(P_{i}\right)$ has exactly one neighbor $u$ in $V\left(P_{i-1}\right)$. We refer to this neighbor as the parent of $u$. In particular, $v$ is the parent of all vertices in $V\left(P_{2}\right)$.
(4) For every $i \in\{1, \ldots, \Delta-1\}$, every $w \in V\left(P_{i}\right)$ at least one neighbor $w^{\prime}$ in $V\left(P_{i-1}\right)$. We refer to every such $w^{\prime}$ as a child of $w$.
(5) For any $i \in\{1, \ldots, \Delta-1\}$, and for any $\left\{u_{i, j}, u_{i+1, j^{\prime}}\right\},\left\{u_{i, t}, u_{i+1, t^{\prime}}\right\} \in E(G)$, we have $j \leq$ $t \Longleftrightarrow j^{\prime} \leq t^{\prime}$. In other words, the ordering of the vertices in $P_{i+1}$ agrees with the ordering of their parents in $P_{i}$.

We say that a path $R$ in $G$ between $v$, and a vertex $u \in V\left(P_{\Delta}\right)$, is a ray. We denote by Pyramids the family of all pyramid graphs. Figure 2 depicts an example of a pyramid.

Definition 3.2 (Skeleton of a pyramid). Let $G$ be a pyramid with basepoint $v \in V(G)$. We define the skeleton of $G$ to be a tree $T$, with $V(T)=V(G)$, with root $v$, and with

$$
E(T)=\left\{\{x, y\} \in\binom{V(G)}{2}: x \text { is the parent of } y\right\} .
$$

For any $x, y \in V(G)$, we denote by nca the nearest common ancestor of $x$, and $y$ in $T$. We also define for any $x \in V(G)$,

$$
\operatorname{depth}(x)=d_{T}(v, x)+1 .
$$

Figure 2 depicts an example of a skeleton.
Definition $3.3(\prec)$. For any $i \in\{1, \ldots, \Delta\}$, for any $u_{i, j}, u_{i, j^{\prime}} \in V\left(P_{i}\right)$, with $j<j^{\prime}$, we write $u_{i, j} \prec u_{i, j^{\prime}}$. Moreover, for any $x, y \in V(G)$, such that $x$, and $y$ do not lie on the same ray, let $z=\mathrm{nca}(x, y)$, and let $x^{\prime}$ (resp. $y^{\prime}$ ) be the child of $z$ in the $z-x$ (resp. $z-y$ ) path in T. Then, we write $x \prec y$ if and only if $x^{\prime} \prec y^{\prime}$. Finally, for any $x^{\prime \prime}, y^{\prime \prime} \in V(G)$, we write $x^{\prime \prime} \preceq y^{\prime \prime}$ if and only if either $x^{\prime \prime} \preceq y^{\prime \prime}$, or $x^{\prime \prime}$, and $y^{\prime \prime}$ lie on the same ray.


Figure 2: A pyramid (left), and its skeleton (right).

Lemma 3.4 (Pyramid representation). For every funnel $G$, we have $G \rightsquigarrow \oplus_{1}\{$ Pyramids $\}$.
Proof. Let $G$ be a funnel with basepoint $x^{*} \in V(G)$, and depth $\Delta$. Let $R$ be a ray in $G$. Replace $R \backslash x^{*}$ by a $\Delta \times 4$ grid $H$. Clearly, this results into an embedding of $G$ into a funnel $G^{\prime}$ with distortion $O(1)$. Let $R^{\prime}$ be the union of the two central columns of $H$, and let $A=R^{\prime} \cup\left\{x^{*}\right\}$. Observe that $\operatorname{dil}_{G^{\prime}}(A)=1$. Applying lemma 1.5 on $G^{\prime}$ and the set $A$, we obtain a stochastic embedding of $G^{\prime}$ into a distribution of graphs $\mathcal{D}$. Since $G^{\prime}$ is planar, and $\operatorname{dil}_{G^{\prime}}(A)=1$, it follows by Theorem 1.3 that the distortion of the resulting stochastic embedding is $O(1)$. Every graph in the support of $\mathcal{D}$ is obtained via 1 -sums of $G^{\prime}[A]$, with $G^{\prime}[V \backslash A \cup\{a\}]$, for some $a \in A$. The graph $G^{\prime}[A]$ is a $\Delta \times 2$ grid, with the basepoint $x^{*}$ connected to the two vertices in the top row, and is therefore a pyramid. For any $a \in A$, the graph $G^{\prime}[V \backslash A \cup\{a\}]$ is obtained from $G^{\prime}$ by cutting along a ray, and is therefore also a pyramid. This concludes the proof.

## 4 Monotone cuts

In this section we describe the family of cuts, that we will use when defining our embedding into $L_{1}$. These are cuts that we call monotone, and intuitively correspond to sets that only cross every ray at most once. We also describe a specific "shifting" operation that will allow us to modify a cut in order to adapt to the finer geometry of a given space.
Definition 4.1 (Monotone cut). Let $G$ be a pyramid with basepoint $v \in V(G)$, and let $S \subseteq V(G)$. We say that $S$ is $v$-monotone (or monotone when $v$ is clear from the context) if $v \in S$, and for any ray $R$ in $G, R \cap S$ is a prefix of $R$. In particular, this implies that $G[S]$ is a connected subgraph (see Figure 3).

Definition 4.2 (Boundary of a monotone cut). Let $S \subseteq V(G)$ be a monotone cut. We define the vertex boundary of $S$, denoted by $\partial_{V} S$, to be the set of all $u \in S$, such that all children of $u$ are not in $S$. We also define the edge boundary of $S$, denoted by $\partial_{E} S$, to be

$$
\partial_{E} S=\left\{\{x, y\} \in E(G): x, y \in \partial_{V} S \text { and } \operatorname{depth}(x)=\operatorname{depth}(y)\right\} .
$$

Finally, we define the graph $\partial S=\left(\partial_{V} S, \partial_{E} S\right)$ (see Figure 3).
Definition 4.3. Let $G$ be a pyramid, let $T$ be the skeleton of $G$. Let $u \in V(G)$, and $r \geq 0$. Then, we denote by $\widetilde{N}(u, r)$ the set of all vertices $w \in V(G)$, such that $u$ is an ancestor of $w$ in $T$, and $d_{T}(u, w) \leq r$.


Figure 3: A monotone cut (top-left), its boundary (top-right), its odd ( $2,\left\{e_{1}, e_{2}\right\}$ )-shift (bottomleft), and its even ( $2,\left\{e_{1}, e_{2}\right\}$ )-shift (bottom-right).

Definition 4.4 (Odd/even shift of a monotone cut). Let $S \subseteq V(G)$ be a monotone cut, let $r>0$, and $Z \subseteq \partial_{E} S$. Let $Z=\left\{\left\{x_{i}, y_{i}\right\}\right\}_{i=1}^{k}$, with

$$
x_{1} \prec y_{1} \preceq x_{2} \prec y_{2} \preceq \ldots \preceq x_{k} \prec y_{k} .
$$

Let $\left\{V_{i}\right\}_{i=1}^{k+1}$ be a decomposition of $\partial_{V} S$, with $V_{1}=\left\{u \in \partial_{V} S: u \preceq x_{1}\right\}, V_{k+1}=\left\{u \in \partial_{V} S\right.$ : $\left.y_{k} \preceq u\right\}$, and for any $i \in\{2, \ldots, k\}, V_{i}=\left\{u \in \partial_{V} S: y_{i} \preceq u \preceq x_{i+1}\right\}$. We define a partition $\partial_{V} S=V_{\text {odd }} \cup V_{\text {even }}$, by setting $V_{\text {odd }}=\bigcup_{i=1}^{[t / 2\rceil} V\left(Q_{2 i-1}\right)$, and $V_{\text {even }}=\bigcup_{i=1}^{\lfloor t / 2\rfloor} V\left(Q_{2 i}\right)$. We define the odd $(r, Z)$-shift of $S$ to be the cut $S_{\text {odd }}$ given by

$$
S_{o d d}=S \cup \bigcup_{u \in V_{\text {odd }}} \tilde{N}(u, r) .
$$

Similarly, we define the even $(r, Z)$-shift of $S$ to be the cut $S_{\text {even }}$ given by

$$
S_{\text {even }}=S \cup \bigcup_{u \in V_{\text {even }}} \widetilde{N}(u, r) .
$$

We say that a cut $S^{\prime}$ is a $(r, Z)$-shift of $S$, if it is either the odd, or the even $(r, Z)$-shift of $S$ (see Figure 3 for an example).

## 5 The embedding

In this section we present a constant-distortion embedding of pyramids into $L_{1}$. Combining with lemmas 1.4, 2.4, \& 3.4, this implies that every planar metric of non-positive curvature embeds into
$L_{1}$ with constant distortion.
Let $G$ be a pyramid, with basepoint $v \in V(G)$. Let $\Delta \geq 1$ be the depth of $G$, and let $\delta=\lceil\log \Delta\rceil$. It will be convenient for our exposition to isometrically embed $G$ into a larger pyramid $G^{\prime}$, with depth $\Delta^{\prime}=O(\Delta)$, as follows. The pyramid $G^{\prime}$ contains a copy of $G$, and a new basepoint $v^{\prime}$, that is connected to the basepoint $v$ of $G$ via a path of length $2 \Delta$, resulting into a pyramid of depth $\Delta^{\prime}=3 \Delta$.


We will then compute an embedding for $G^{\prime}$, and prove that its restriction on $G$ has small distortion. We remark that our embedding will have unbounded distortion for points in $G^{\prime}$ close to $v^{\prime}$ (more precisely, pairs of vertices at distance $\varepsilon$ from $v^{\prime}$, will be distorted by a factor of $\left.O(1 / \varepsilon)\right)$. However, this does not affect our result, since we only case about distances in $G$, which lies far from $v^{\prime}$.

Definition 5.1 (Evolution of a monotone cut). Let $r>0$, and let $S \subseteq V\left(G^{\prime}\right)$ be a monotone cut. The r-evolution of $S$ is a probability distribution $\mathcal{D}$ over monotone cuts, defined by the following random process. Let

$$
Y=\left\{\{x, y\} \in \partial_{E} S: \operatorname{depth}(x)-\operatorname{depth}(\operatorname{nca}(x, y)) \in[r, 6 r)\right\}
$$

Pick a random subset $Y^{\prime} \subseteq Y$, by choosing every $e \in Y$ independently, with probability $1 / r$. We probability $1 / 2$, let $S^{\prime}$ be the odd $\left(r, Y^{\prime}\right)$-shift of $S$, and otherwise let $S^{\prime}$ be the even $\left(r, Y^{\prime}\right)$-shift of $S$. The resulting random cut $S^{\prime}$ defines the distribution $\mathcal{D}$.

Let $\mathcal{M}$ be the set of all $v$-monotone cuts in $G^{\prime}$. We inductively define a sequence $\left\{\mu_{i}\right\}_{i=0}^{\delta+1}$, where each $\mu_{i}$ is a probability distribution over $\mathcal{M}$. We define $\mu_{0}$ as follows. Let $P_{1}, \ldots, P_{\Delta^{\prime}}$ be the layers of $G^{\prime}$. For any $j \in\left\{1, \ldots, \Delta^{\prime}\right\}$, let $X_{i}=\bigcup_{t=1}^{j} V\left(P_{t}\right)=\operatorname{ball}\left(v^{\prime}, i\right)$. Let $\mu_{0}$ be the uniform distribution over the collection of cuts $X_{1}, \ldots, X_{\Delta^{\prime}}$.

For any $i \geq 0$, given $\mu_{i}$, we inductively define $\mu_{i+1}$ via the following random process: We first pick a random cut $S_{i}$ according to $\mu_{i}$. Let $\mathcal{D}=\mathcal{D}\left(S_{i}\right)$ be the $\Delta / 3^{i}$-evolution of $S_{i}$. We pick a random cut $S_{i+1}$ according to $\mathcal{D}$. The resulting random variable $S_{i+1}$ defines the probability distribution $\mu_{i+1}$.

We define the embedding $f$ induced by the probability distribution $\mu_{\delta}$, and the embedding $f_{0}$ induced by the probability distribution $\mu_{0}$. Finally, we set the resulting embedding to be

$$
g=f \oplus f_{0}
$$

i.e. the concatenation of the embeddings $f$, and $f_{0}$. In the next section we show that the distortion of $g$ restricted on $G$ is bounded by some universal constant.

## 6 Distortion analysis

We now analyze the distortion of the embedding $g$ constructed in the previous section.

### 6.1 Distortion of vertical pairs of points

Lemma 6.1. Let $u \in V(G)$, with depth $(u)<\Delta^{\prime}$, and let $i \in\{1, \ldots, \delta\}$. Then, $\operatorname{Pr}\left[u \in \partial_{V} S_{i}\right]=$ $1 / \Delta^{\prime}$.

Proof. The proof is by induction on $i$. For $i=0$, the assertion holds since $\mu_{0}$ is the uniform distribution over the cuts $X_{1}, \ldots, X_{\Delta^{\prime}}$. Suppose next that $i>0$. Let $r=\Delta / 3^{i-1}$, and let $u^{\prime}$ be the ancestor of $u$ in $T$, with $d_{T}(u, u)=r$. Fix some $S_{i-1}$ in the support of $\mu_{i-1}$, and suppose that $S_{i}$ is sampled from the $r$-evolution of $S_{i-1}$. This means that we first sample a set of edges $Y$, and for any such $Y$ we set $S_{i}$ to be the odd $(r, Y)$-shift of $S_{i-1}$ with probability $1 / 2$, or otherwise we set $S_{i}$ to be the even $(r, Y)$-shift of $S_{i-1}$. Therefore, we have have $u \in \partial_{V} S_{i}$, only if either $u \in \partial_{V} S_{i-1}$, or $u^{\prime} \in \partial_{V} S_{i-1}$. Conditioned on either of these two events, and for any $Y$, exactly one of the odd/even shifts of $S_{i-1}$ has $u$ in its boundary. This implies that

$$
\begin{aligned}
\operatorname{Pr}\left[u \in \partial_{V} S_{i}\right]= & \operatorname{Pr}\left[u \in \partial_{V} S_{i} \mid u \in \partial_{V} S_{i-1}\right] \cdot \operatorname{Pr}\left[u \in \partial_{V} S_{i-1}\right] \\
& +\operatorname{Pr}\left[u \in \partial_{V} S_{i} \mid u^{\prime} \in \partial_{V} S_{i-1}\right] \cdot \operatorname{Pr}\left[u^{\prime} \in \partial_{V} S_{i-1}\right] \\
= & \frac{1}{\Delta^{\prime}} \cdot \frac{1}{2}+\frac{1}{\Delta^{\prime}} \cdot \frac{1}{2} \\
= & 1 / \Delta^{\prime},
\end{aligned}
$$

as required.
Lemma 6.2. Let $x, y \in V(G)$, such that $x$, $y$ lie on the same ray. Then, $\|f(x)-f(y)\|_{1}=$ $d_{G}(x, y) / \Delta$.

Proof. Let $R$ be the ray containing both $x$, and $y$. Let $R^{\prime}$ be the subpath of $R$ between $x$, and $y$, including $x$, and excluding $y$. By the monotonicity of $\mathcal{S}_{\delta}$, it follows that $\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(y)$, if and only if there exists $z \in V\left(R^{\prime}\right)$, such that $z \in \partial_{V} S_{\delta}$. Since these events are disjoint for different $z$, we obtain by lemma 6.1 that $\|f(x)-f(y)\|_{1}=\operatorname{Pr}\left[\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(y)\right]=\left|V\left(R^{\prime}\right)\right| / \Delta^{\prime}=d_{G}(x, y) / \Delta^{\prime}$, as required.

### 6.2 Distortion of horizontal pairs of points

We now bound the distortion on pairs of vertices $x, y \in V(G)$ that lie on the same layer of $G^{\prime}$, i.e. such that depth $(x)=\operatorname{depth}(y)=h$. Let $d_{G}(x, y)=L$. Let also $h^{\prime}=\operatorname{depth}(\operatorname{nca}(x, y))$. We assume w.l.o.g. that $x \preceq y$. Let $P$ be the subpath of $P_{h}$ between $x$, and $y$.

Let

$$
E_{\mathrm{top}}=\{\{z, w\} \in E(P): \operatorname{depth}(\operatorname{nca}(z, w)) \leq h-L / 2\},
$$

and

$$
E_{\mathrm{bottom}}=E(P) \backslash E_{\mathrm{top}} .
$$

Lemma 6.3. $\left|E_{\text {top }}\right| \leq L$.

Proof. Suppose, to the contrary, that $\left|E_{\mathrm{top}}\right|>L$. For any $i \in\{h-L / 2, \ldots, h\}$, let $Z_{i}$ be the subpath of $P_{i}$ between the ancestor of $x$, and the ancestor of $y$ in $P_{i}$. For any $e=\{z, w\} \in E_{\text {top }}$, with $z \prec w$, let $R_{z}$ be a ray containing $z$, and let $W_{e}$ be the subpath of $R_{e}$ contained between $P_{h-L / 2}$, and $P_{h}$.


The union of all these paths $\left(\bigcup_{i} Z_{i}\right) \cup\left(\bigcup_{e} W_{e}\right)$ forms a $(L / 2+1) \times L$ grid minor in $G^{\prime}$, with $x$, and $y$ being the bottom-left, and bottom-right vertices respectively. Since the $x-y$ shortest path in $G^{\prime}$ is contained in ball $\left(v^{\prime}, h\right)$, this implies that $d_{G}(x, y)>L$, which is a contradiction.

Let $H$ be the subgraph of $G$ induced on the set of vertices

$$
V(H)=\left\{u \in V(G): h^{\prime} \leq \operatorname{depth}(u) \leq h \text { and } x \preceq u \preceq y\right\} .
$$

Definition 6.4 (Straight cut). Let $i \in\{1, \ldots, \delta\}$, and $j \in\left\{1, \ldots, \Delta^{\prime}\right\}$. We say that $S_{i}$ is $j$-straight if $\partial S_{i} \cap H \subseteq P_{j}$.

Let $e=\{z, w\} \in E(P)$. We say that an edge $e^{\prime}=\left\{z^{\prime}, w^{\prime}\right\} \in E(G)$ is an ancestor of $e$, if $z^{\prime}$ is an ancestor of $z$ in $T, w^{\prime}$ is an ancestor of $w$ in $T$, and depth $\left(z^{\prime}\right)=\operatorname{depth}\left(w^{\prime}\right)$.
Definition 6.5 (Bend). Let $e \in E(P)$. We say that $e$ bends $S_{i}$, if the following events happen.
(1) There exists $j \in\left\{1, \ldots, \Delta^{\prime}\right\}$, such that $S_{i}$ is $j$-straight.
(2) Let $Y \subseteq \partial_{E} S_{i}$, such that $S_{i+1}$ is the $(r, Y)$-shift of $S_{i}$, for some $r>0$. Then, there exists an ancestor of e in $Y$.

Lemma 6.6. Let $j \in\left\{h^{\prime}, \ldots, h\right\}$, and let $i \in\{1, \ldots, \delta\}$. Then, $\operatorname{Pr}\left[S_{i}\right.$ is $j$-straight $] \leq 1 / \Delta^{\prime}$.
Proof. Let $z$ be an arbitrary vertex in $V\left(P_{j}\right) \cap V(H)$. Clearly, $S_{i}$ can only be $j$-straight if $z \in \partial_{V} S_{i}$. Therefore, by lemma 6.1 we obtain $\operatorname{Pr}\left[S_{i}\right.$ is $j$-straight $] \leq \operatorname{Pr}\left[z \in \partial_{V} S_{i}\right]=1 / \Delta^{\prime}$, as required.

For any edge $e=\{z, w\} \in E(P)$, and for any $i \in\{1, \ldots, \delta\}$, let $\mathcal{E}(e, i)$ be the conjunction of the following two events:

- $\mathcal{E}_{1}(e, i)$ : There exists $j$, such that the following event, denoted by $\mathcal{E}_{1}(e, i, j)$, holds: Intuitively, the event $\mathcal{E}_{1}(e, i, j)$ describes a necessary condition such that a bend of $S_{i}$ can potentially lead to a cut $S_{\delta}$ that separates $x$, and $y$. Formally, we have that $S_{i}$ is $j$-straight, with

$$
\begin{equation*}
\Delta / 3^{i} \leq j-\operatorname{depth}(\operatorname{nca}(z, w))<6 \Delta / 3^{i} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
h-j \leq 2 \Delta / 3^{i} . \tag{5}
\end{equation*}
$$

- $\mathcal{E}_{2}(e, i): e$ bends $S_{i}$.

Lemma 6.7. Suppose that $\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(y)$. Then, there exists $e \in E(P)$, and $i \in\{1, \ldots, \delta\}$, such that the event $\mathcal{E}(e, i)$ occurs.

Proof. Recall that by the definition of $\mu_{0}$, the cut $S_{0}$ is $j_{0}$-straight, for some $j_{0} \in\left\{1, \ldots, \Delta^{\prime}\right\}$. Since $1_{S_{\delta}}(x) \neq 1_{S_{\delta}}(y)$, it follows that for all $j_{\delta} \in\left\{1, \ldots, \Delta^{\prime}\right\}$, the cut $S_{\delta}$ is not $j_{\delta}$-straight. Let $i^{*} \in\{0, \ldots, \delta-1\}$ be the smallest integer such that for all $j \in\left\{1, \ldots, \Delta^{\prime}\right\}$, the cut $S_{i^{*}+1}$ is not $j$-straight. This means that $S_{i^{*}}$ is $j^{*}$-straight, for some $j^{*} \in\left\{1, \ldots, \Delta^{\prime}\right\}$. Therefore, there exists $e=\{z, w\} \in E(P)$, such that $e$ bends $S_{i^{*}}$, which means that the event $\mathcal{E}_{2}\left(e, i^{*}\right)$ occurs. It suffices to show that the event $\mathcal{E}_{1}\left(e, i^{*}, j^{*}\right)$ also occurs. We have established that $S_{i^{*}}$ is $j^{*}$-straight, so its remains to show that (4) \& (5) hold. Condition (4) follows immediately form the fact that $e$ bends $S_{i^{*}}$, and $S_{i^{*}+1}$ is the $\left(Y, \Delta / 3^{i^{*}}\right.$ )-shift of $S_{i^{*}}$, with $e \in Y$. Since $S_{i^{*}}$ is $j^{*}$-straight, we have $S_{i^{*}} \subseteq$ ball $\left(v^{\prime}, j^{*}\right)$. The cut $S_{\delta}$ is obtained from $S_{i^{*}}$ via a sequence of $(Y, r)$-shifts, with exponentially decreasing values of $r$. This implies $S_{\delta} \subseteq$ ball $\left(v^{\prime}, t\right)$, for some $t \leq j^{*}+\sum_{i=i^{*}}^{\delta} \Delta / 3^{i}<j^{*}+2 \Delta / 3^{*}$. Since $\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(y)$, we have $t>h$, and therefore $h-j^{*}<2 \Delta / 3^{i^{*}}$, which implies (5), and concludes the proof.

Lemma 6.8 (Expansion of horizontal pairs). Let $x, y \in V(G)$, such that $\operatorname{depth}(x)=\operatorname{depth}(y)$. Then, $\|f(x)-f(y)\|_{1}=O\left(d(x, y) / \Delta^{\prime}\right)$.

Proof. Let $\mathcal{E}_{\text {top }}$ denote the event that there exists $e \in E_{\text {top }}$, and $i \in\{1, \ldots, \delta\}$, such that $\mathcal{E}(e, i)$ occurs. Similarly, let $\mathcal{E}_{\text {bottom }}$ denote the event that there exists $e \in E_{\mathrm{bottom}}$, and $i \in\{1, \ldots, \delta\}$, such that $\mathcal{E}(e, i)$ occurs. By lemma 6.7 we have

$$
\begin{aligned}
\|f(x)-f(y)\|_{1} & =\operatorname{Pr}\left[\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(x)\right] \\
& \leq \operatorname{Pr}\left[\mathcal{E}_{\text {top }}\right]+\operatorname{Pr}\left[\mathcal{E}_{\text {bottom }}\right] .
\end{aligned}
$$

Let us bound the two latter quantities separately.
We first bound $\operatorname{Pr}\left[\mathcal{E}_{\text {top }}\right]$. Let $e=\{z, w\} \in E_{\text {top }}$, and $i \in\{1, \ldots, \delta\}$. Let $h^{\prime}=\operatorname{depth}(\mathrm{nca}(\{z, w\}))$. Recall that by the definition of $\mathcal{E}_{1}(e, i, j)$, in order for $\mathcal{E}_{1}(e, i, j)$ to occur for some $j$, we must have by (4) that $j-h^{\prime} \leq 6 \Delta / 3^{i}$, and by (5) that $h-j \leq 2 \Delta / 3^{i}$. We therefore obtain that

$$
h-h^{\prime}=h-j+j-h^{\prime}=O\left(\Delta / 3^{i}\right) .
$$

Note that $S_{i+1}$ is the $(Y, r)$-shift of $S_{i}$, for some $r=\Delta / 3^{i}$, and for random some $Y \subseteq \partial_{E} S_{i}$. In order for the edge $e$ to bend $S_{i}$, its must be the case that its unique ancestor (if it exists) in $\partial_{E} S_{i}$ is chosen in $Y$. Every edge in chosen in $Y$ with probability at most $1 / r$. Therefore, for any $j$, and for any $i$, we have

$$
\operatorname{Pr}\left[\mathcal{E}_{2}(e, i) \mid \mathcal{E}_{1}(e, i, j)\right] \leq 3^{i} / \Delta .
$$

Moreover, $\mathcal{E}_{1}(e, i, j)$ can occur only if $j \in\left\{h^{\prime}, \ldots, h\right\}$. For each such value $j \in\left\{h^{\prime}, \ldots, h\right\}$, and for any $i$, we have by lemma 6.6 that

$$
\operatorname{Pr}\left[\mathcal{E}_{1}(e, i, j)\right]=O(1 / \Delta)
$$

To summarize, we have

$$
\begin{align*}
\operatorname{Pr}\left[\mathcal{E}_{\text {top }}\right] & \leq \sum_{e \in E_{\text {top }}} \sum_{i \in\{1, \ldots, \delta\}} \operatorname{Pr}[\mathcal{E}(e, i)] \\
& \leq \sum_{e \in E_{\text {top }}} \sum_{j \in\left\{h^{\prime}, \ldots, h\right\}} \sum_{i \in\{1, \ldots, \delta\}} \operatorname{Pr}\left[\mathcal{E}_{2}(e, i) \mid \mathcal{E}_{1}(e, i, j)\right] \cdot \operatorname{Pr}\left[\mathcal{E}_{1}(e, i, j)\right] \\
& \leq \sum_{e \in E_{\text {top }}} \sum_{j \in\left\{h^{\prime}, \ldots, h\right\}} \sum_{i \in\{1, \ldots, \delta\}} \frac{3^{i}}{\Delta} O(1 / \Delta) \\
& \leq \sum_{e \in E_{\text {top }}} \sum_{j \in\left\{h^{\prime}, \ldots, h\right\}} O\left(1 /\left(h-h^{\prime}\right)\right) \cdot O(1 / \Delta) \\
& \leq \sum_{e \in E_{\text {top }}} O(1 / \Delta) \\
& =O\left(\left|E_{\text {top }}\right| / \Delta\right) \\
& =O\left(L / \Delta^{\prime}\right) \tag{6}
\end{align*}
$$

We next bound $\operatorname{Pr}\left[\mathcal{E}_{\text {bottom }}\right]$. Let $\{1, \ldots, \delta\}, j \in\left\{1, \ldots, \Delta^{\prime}\right\}, e \in E_{\text {bottom }}$, such that both $\mathcal{E}_{1}(e, i, j)$, and $\mathcal{E}_{2}(e, i)$ occur. As above, let $e=\{z, w\}$, and $h^{\prime}=\operatorname{depth}(\operatorname{nca}(z, w))$. Then, we must have $h^{\prime} \leq j \leq h$, which implies $j-h^{\prime} \leq h-h^{\prime} \leq L / 2$. Since $S_{i+1}$ is the ( $r, Y$ )-shift of $S_{i}$ for some $Y \subseteq E(P)$, with $r=\Delta / 3^{i}$, we obtain that $j-h^{\prime} \in[r, 6 r)$, which implies $3^{i} \geq 2 \Delta / L$. Let $R_{x}$ be the ray containing $x$, and let $\chi$ be the unique vertex in the intersection of $R_{x}$ with $\partial S_{i}$. Let also $\chi^{\prime}$ be the unique vertex in the intersection of $R_{x}$ with $\partial S_{\delta}$. For every $i^{\prime} \geq i$, the intersection of $\partial S_{i^{\prime}}$ with $R_{x}$ moves by at most $\Delta / 3^{i^{\prime}}$ along $R_{x}$, and therefore $d_{T}\left(\chi, \chi^{\prime}\right)<2 \Delta / 3^{i^{\prime}}=O(L)$. Since $\chi \in P_{j}$, and $j \in\left[h^{\prime}, h\right]$, it follows that depth $(\chi)$ can take at most $h^{\prime}-h+1$ different values. Therefore, $\chi^{\prime}$ can only lie inside a subpath $R_{x}^{\prime} \subseteq R_{x}$ of length $O\left(h^{\prime}-h\right)$. Applying lemma 6.1, we obtain

$$
\begin{align*}
\operatorname{Pr}\left[\mathcal{E}_{\text {bottom }}\right] & \leq \operatorname{Pr}\left[\chi^{\prime} \in R_{x}^{\prime}\right] \\
& \leq\left|V\left(R_{x}^{\prime}\right)\right| / \Delta^{\prime} \\
& =O\left(h^{\prime}-h\right) / \Delta^{\prime} \\
& =O\left(L / \Delta^{\prime}\right) . \tag{7}
\end{align*}
$$

Combining (6) \& (7) we conclude that $\|f(x)-f(y)\|_{1}=O\left(L / \Delta^{\prime}\right)=O\left(d(x, y) / \Delta^{\prime}\right)$, as required.
We now bound the contraction of $f$. For any $i \in\{1, \ldots, \delta\}$, let

$$
J_{i}=\left\{h-\frac{\Delta}{3^{i}}, \ldots, h-\frac{2}{3} \cdot \frac{\Delta}{3^{i}}\right\} .
$$



Lemma 6.9. Let $i \in\{1, \ldots, \delta\}, j \in J_{i}$, such that $S_{i}$ is $j$-straight. Let $Y \subseteq E\left(G^{\prime}\right)$, such that $S_{i+1}$ is the $\left(\Delta / 3^{i}, Y\right)$-shift of $S_{i}$, where $|Y \cap E(P)|$ is odd. Then, $\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(y)$.

Proof. Since $j \in J_{i}$, it follows that $j \leq h-\Delta / 3^{i}$. Let $x^{\prime}$ be the ancestor of $x$ with $d_{G}\left(x, x^{\prime}\right)=\Delta / 3^{i}$, and let $y^{\prime}$ be the ancestor of $y$ with $d_{G}\left(y, y^{\prime}\right)=\Delta / 3^{i}$. Since $S_{i}$ is $j$-straight, it follows that $x^{\prime}, y^{\prime} \in S_{i}$. Suppose that $S_{i+1}$ is an odd shift of $S_{i}$; the case where $S_{i+1}$ is an even shift of $S_{i}$ is completely symmetric by exchanging $x$, and $y$. Since $S_{i+1}$ is the odd $\left(\Delta / 3^{i}, Y\right)$-shift of $S_{i}$, and $|Y \cap E(P)|$ is odd, it follows that $x \in S_{i+1}$, and $y \notin S_{i+1}$. Since $S_{\delta} \supset \ldots \supset S_{i+1}$, we obtain $x \in S_{\delta}$. Let $W$ be the subpath of the ray containing $y$, and $y^{\prime}$. We have that for any $k>1$, the cut $S_{i+k}$ contains a prefix of $W$ of length at most $\Delta / 3^{i+1}+\ldots+\Delta / 3^{\delta}<2 \Delta / 3^{i+1} \leq h-j=d_{G}\left(y, y^{\prime}\right)$, and therefore $y$ is not in $S_{\delta}$, concluding the proof.

For any $t \in\left\{h^{\prime}, \ldots, h\right\}$, let

$$
E_{t}=\{\{z, w\} \in E(P): \operatorname{depth}(\operatorname{nca}(z, w)) \leq t\}
$$

Lemma 6.10. There exists $t^{*} \in\{h-L / 2, \ldots, h-L / 4\}$, such that $L / 2 \leq\left|E_{t^{*}}\right| \leq L$.
Proof. By lemma 6.3 we have $\left|E_{h-L / 2}\right| \leq L$. Let $x^{\prime}$ be the ancestor of $x$ with $d_{G}\left(x, x^{\prime}\right)=L / 4$, and let $y^{\prime}$ be the ancestor of $y$ with $d_{G}\left(y, y^{\prime}\right)=L / 4$. We have $\left|E_{h-L / 4}\right| \geq d_{G}\left(x^{\prime}, y^{\prime}\right) \geq d_{G}(x, y)-$ $d_{G}\left(x, x^{\prime}\right)-d_{G}\left(y, y^{\prime}\right) \geq L / 2$, and the assertion follows.

For any $i \in\{1, \ldots, \delta\}$, and for any $j \in\left\{1, \ldots, \Delta^{\prime}\right\}$, let

$$
B_{i, j}=\bigcup_{t=j-6 \Delta / 3^{i}}^{j-\Delta / 3^{i}} E_{t}
$$

Intuitively, the set $B_{i, j}$ contains all edges in $H$ that could possibly bend $S_{i}$, when $S_{i}$ is $j$-straight.
Lemma 6.11. There exists $i^{*} \in\{1, \ldots, \delta\}$, with $\Delta / 3^{i^{*}}=\Omega(L)$, and such that

$$
\begin{equation*}
\bigcup_{i=1}^{i^{*}-1} \bigcup_{j \in J_{i}} B_{i, j}=O(L) \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\bigcup_{i=1}^{i^{*}} \bigcup_{j \in J_{i}} B_{i, j}=\Omega(L) \tag{9}
\end{equation*}
$$

Proof. It is straightforward to verify that

$$
\bigcup_{i=1}^{\delta} \bigcup_{j \in J_{i}} B_{i, j}=E(P)
$$

Let $t^{*}$ be as in lemma 6.10. It follows that by setting

$$
i^{*}=\min \left\{i \in\{1, \ldots, \delta\}: h-\Delta / 3^{i} \leq t^{*}\right\}
$$

we have $E_{t^{*}} \subseteq \bigcup_{i=1}^{i^{*}} \bigcup_{j \in J_{i}} B_{i, j}$, and therefore conditions (8) \& (9) are satisfied. Moreover, we have $\Delta / 3^{i^{*}} \geq h-t^{*} \geq L / 4=\Omega(L)$, as required.

Lemma 6.12. Let $i \in\left\{1, \ldots, i^{*}\right\}$, and $j \in J_{i}$. Then, $\operatorname{Pr}\left[S_{i}\right.$ is $j$-straight $]=\Omega\left(1 / \Delta^{\prime}\right)$.
Proof. We use a coupling argument. Let $E^{*}=\bigcup_{i=1}^{i^{*}-1} \bigcup_{j \in J_{i}} B_{i, j}$. Our embedding uses a random process $\sigma$ of sampling $S_{1}, \ldots, S_{i}$. We define a modified random process $\sigma^{\prime}$ for sampling monotone cuts $S_{1}^{\prime}, \ldots, S_{i}^{\prime}$ as follows. The process $\sigma^{\prime}$ uses the same algorithm as $\sigma$, with the only difference that when taking $S_{\iota+1}^{\prime}$ to be the $(r, Y)$-shift of $S_{\iota}^{\prime}$, where $Y$ is chosen from a set $E^{\prime}=E^{\prime}\left(S_{\iota}^{\prime}\right)$ independently with probability $1 / r$, we instead chose $Y^{\prime}$ from the set $E^{\prime} \backslash E^{*}$. In other words, we execute the same algorithm, but we always ignore the edges in $E^{*}$ when computing shifts.

Since we ignore the edges in $E^{*}$, the final cut $S_{i}^{\prime}$ is always either $j^{\prime}$-straight, for some $j^{\prime} \leq j$, or it contains $P_{j}$. Arguing as in the proof of lemma 6.1, it is straightforward to verify that every vertex $u \in V(G)$ appears in $\partial_{V} S_{i}$ with probability $1 / \Delta^{\prime}$. Therefore, $\operatorname{Pr}\left[S_{i}^{\prime}\right.$ is $j$-straight $]=1 / \Delta^{\prime}$.

We can now re-define the original random process $\sigma$ as follows. At every step, when computing a $(r, Y)$-shift of some cut $S$, we first pick $Y^{\prime}$ according to the choices made in $\sigma^{\prime}$, and then we augment $Y^{\prime}$ to a new set $Y$ by adding independently, and with probability $1 / r$ every edge from $E^{*}$ that was ignored in $\sigma^{\prime}$ at this step. It is straightforward to check that this definition results in the same process $\sigma$.

Let us say that the coupling of these two processes fails if at some step the processes $\sigma$, and $\sigma^{\prime}$ deviate. Recall that we obtain a set $S_{\iota+1}$ by taking a $(r, Y)$-shift of a set $S_{\iota}$, for some $r=\Delta / 3^{\iota}$. Every edge in $E^{*}$ is eligible for appearing in such a set $Y$ at most $O(1)$ times during the process. Moreover, since $\iota \leq i^{*}$, we have that every eligible edge is chosen in a set $Y$ with probability $1 / r=O\left(3^{i^{*}} / \Delta\right)=O(1 / L)$. It follows that for any execution of $\sigma^{\prime}$, the coupling does not fail with at least some constant probability $q>0$. Thus, $\operatorname{Pr}\left[S_{i}\right.$ is $j$-straight $] \geq q \cdot \operatorname{Pr}\left[S_{i}^{\prime}\right.$ is $j$-straight $]=\Omega\left(1 / \Delta^{\prime}\right)$, as required.

We will use the following simple fact about the parity of the sum of independent Bernoulli random variables.

Proposition 6.13. There exists $c>0$, such that the following holds. Let $p, k>0$, and let $X_{1}, \ldots, X_{k}$ be a collection of independent Bernoulli random variables, such that for any $i \in$ $\{1, \ldots, k\}$, we have $\operatorname{Pr}\left[X_{i}=1\right]=p$. Then, $\operatorname{Pr}\left[\sum_{i=1}^{k} X_{i}\right.$ is odd $]>\min \{1 / 4, c p k\}$.

Lemma 6.14 (Contraction of horizontal pairs). Let $x, y \in V(G)$, such that $\operatorname{depth}(x)=\operatorname{depth}(y)$. Then, $\|f(x)-f(y)\|_{1}=\Omega\left(d(x, y) / \Delta^{\prime}\right)$.

Proof. For any $i \in\left\{1, \ldots, i^{*}\right\}$, and for any $j \in J_{i}$, let $\mathcal{W}_{1}(i, j)$ denote the event that $S_{i}$ is $j$-straight. By lemma 6.12 we have

$$
\operatorname{Pr}\left[\mathcal{W}_{1}(i, j)\right]=\Omega\left(1 / \Delta^{\prime}\right) .
$$

Let $\mathcal{W}_{2}(i, j)$ denote the event that there exists $Y \subseteq E\left(G^{\prime}\right)$, such that $|Y \cap E(P)|$ is odd, and $S_{i+1}$ is the $\left(\Delta / 3^{i}, Y\right)$-shift of $S_{i}$. Conditioned on the event that $S_{i}$ is $j$-straight, we have that $S_{i+1}$ is the $\left(\Delta / 3^{i}, Y\right)$-shift of $S_{i}$, for some random $Y \subset E\left(G^{\prime}\right)$, with $Y \cap E(P) \subseteq B_{i, j}$, where every element of $B_{i, j}$ is chosen independently with probability $p=3^{i} / \Delta$. Applying Proposition 6.13 we deduce that

$$
\operatorname{Pr}\left[\mathcal{W}_{2}(i, j) \mid \mathcal{W}_{1}(i, j)\right]=\Omega\left(\min \left\{1 / 4,\left|B_{i, j}\right| 3^{i} / \Delta\right\}\right)=\Omega\left(\min \left\{1 / 4,\left|E_{j}\right| /(h-j)\right\}\right) .
$$

Consider some $e=\{z, w\} \in B_{i, j}$, with depth $(\operatorname{nca}(z, w))=h^{\prime \prime}$. The edge $e$ appears in $B_{i^{\prime}, j^{\prime}}$, for at least $\Omega\left(h-h^{\prime \prime}\right)$ different values of $j^{\prime} \in \bigcup_{i^{\prime}=1}^{i^{*}} J_{i^{\prime}}$. Arguing as in the proof of lemma 6.8, we can show that for every such value $j^{\prime}$, we have $h-j^{\prime}=\Theta\left(h-h^{\prime \prime}\right)$. Therefore,

$$
\sum_{i \in\left\{1, \ldots, i^{*}\right\}} \sum_{j \in J_{i}}\left|B_{i, j}\right| /(h-j)=\Omega\left(\left|\bigcup_{i \in\left\{1, \ldots, i^{*}\right\}} \bigcup_{j \in J_{i}} B_{i, j}\right|\right)=\Omega(L)
$$

Combining the above with lemma 6.9, we obtain

$$
\begin{aligned}
\|f(x)-f(y)\|_{1} & =\operatorname{Pr}\left[\mathbf{1}_{S_{\delta}}(x) \neq \mathbf{1}_{S_{\delta}}(x)\right] \\
& \geq \sum_{i \in\left\{1, \ldots, i^{*}\right\}} \sum_{j \in J_{i}} \operatorname{Pr}\left[\mathcal{W}_{2}(i, j) \mid \mathcal{W}_{1}(i, j)\right] \cdot \operatorname{Pr}\left[\mathcal{W}_{1}(i, j)\right] \\
& =\Omega\left(1 / \Delta^{\prime}\right) \min \left\{1 / 4, \sum_{i \in\{1, \ldots, \delta\}} \sum_{j \in J_{i}}\left|B_{i, j}\right| /(h-j)\right\} \\
& =\Omega\left(L / \Delta^{\prime}\right) \\
& =\Omega\left(d_{G}(x, y) / \Delta^{\prime}\right),
\end{aligned}
$$

as required.

### 6.3 Distortion of general pairs of points

Lemma 6.15 (Embedding pyramids into $L_{1}$ ). There exists a universal constant $c>1$, such that every pyramid graph admits an embedding into $L_{1}$ with distortion at most $c$.

Proof. We will show that the embedding $g=f \oplus f_{0}$ has constant distortion on $G$. Let $x, y \in V(G)$. Assume w.l.o.g. that depth $(x) \geq \operatorname{depth}(y)$. Let $R_{x}$ be the ray containing $x$, and let $x^{\prime}$ be the unique vertex in $R_{x}$, with depth $\left(x^{\prime}\right)=\operatorname{depth}(y)$. By lemmas $6.8 \& 6.14$ we have that there exist universal constants $\alpha>\beta>0$, such that for any

$$
\begin{equation*}
\beta d_{G}\left(x^{\prime}, y\right) / \Delta \leq\left\|f\left(x^{\prime}\right)-f(y)\right\|_{1} \leq \alpha d_{G}\left(x^{\prime}, y\right) / \Delta \tag{10}
\end{equation*}
$$

Note that

$$
\begin{equation*}
d_{G}\left(x, x^{\prime}\right)=\operatorname{depth}(x)-\operatorname{depth}\left(x^{\prime}\right)=\operatorname{depth}(x)-\operatorname{depth}(y) \geq d_{G}(x, y) . \tag{11}
\end{equation*}
$$

Thus, we have

$$
\begin{align*}
\|f(x)-f(y)\|_{1} & \leq\left\|f(x)-f\left(x^{\prime}\right)\right\|_{1}+\left\|f\left(x^{\prime}\right)-f(y)\right\|_{1}  \tag{12}\\
& \leq d_{G}(x, y) / \Delta+\alpha d_{G}\left(x^{\prime}, y\right) / \Delta  \tag{13}\\
& \leq d_{G}(x, y) / \Delta+\alpha d_{G}\left(x^{\prime}, x\right) / \Delta+\alpha d_{G}(x, y) / \Delta \\
& =(\alpha+1) d_{G}\left(x, x^{\prime}\right) / \Delta+\alpha d_{G}(x, y) / \Delta \\
& \leq(\alpha+1) d_{G}(x, y) / \Delta+\alpha d_{G}(x, y) / \Delta  \tag{14}\\
& =(2 \alpha+1) d_{G}(x, y) / \Delta, \tag{15}
\end{align*}
$$

where (12) follows by the triangle inequality, (13) by lemma 6.2 \& (10), and (14) by (11). By (15) we have

$$
\begin{align*}
\|g(x)-g(y)\|_{1} & =\|f(x)-f(y)\|_{1}+\left\|f_{0}(x)-f_{0}(y)\right\|_{1} \\
& \leq(2 \alpha+1) d_{G}(x, y) / \Delta+d_{G}(x, y) / \Delta \\
& =(2 \alpha+2) d_{G}(x, y) / \Delta . \tag{16}
\end{align*}
$$

This bounds the expansion of $g$. It remains to bound the contraction of $g$.
Let $\gamma=\frac{\beta}{4(2 \alpha+1)}$. Assume first that $d_{G}(x, y) \geq \gamma d_{G}(x, y)$. We have

$$
\begin{align*}
\|g(x)-g(y)\|_{1} & \geq\left\|f_{0}(x)-f_{0}(y)\right\|_{1} \\
& =d_{G}\left(x^{\prime}, y\right) / \Delta \\
& \geq \gamma d_{G}(x, y) / \Delta \tag{17}
\end{align*}
$$

Next, assume that $d_{G}(x, y)<\gamma d_{G}(x, y)$. We have

$$
\begin{align*}
\|g(x)-g(y)\|_{1} & \geq\|f(x)-f(y)\|_{1} \\
& \geq\left\|f\left(x^{\prime}\right)-f(y)\right\|_{1}-\left\|f(x)-f\left(x^{\prime}\right)\right\|_{1} \\
& \geq \beta d_{G}\left(x^{\prime}, y\right) / \Delta-(2 \alpha+1) d_{G}\left(x, x^{\prime}\right) / \Delta  \tag{18}\\
& >(1-\gamma) \beta d_{G}(x, y) / \Delta-\gamma(2 \alpha+1) d_{G}(x, y) / \Delta \\
& >\frac{1}{2} \beta d_{G}(x, y) / \Delta \tag{19}
\end{align*}
$$

where (18) follows by (10) \& (15). Combining (17) \& (19), we obtain that for all $x, y \in V(G)$

$$
\begin{equation*}
\|g(x)-g(y)\|_{1} \geq \frac{\beta}{4(2 \alpha+1)} d_{G}(x, y) / \Delta . \tag{20}
\end{equation*}
$$

From (16) \& 20) we conclude that the distortion of $g$ is at most $4(2 \alpha+1)(2 \alpha+2) / \beta=O(1)$, concluding the proof.

### 6.4 Proof of the main result

Combining the above results, we can now prove our main theorem.
Proof. Proof of theorem 1.2 Let $(X, d)$ be a planar metric of non-positive curvature. Using lemma 2.4 the metric $(X, d)$ admits an embedding into some funnel $G$ with distortion $c_{1}=O(1)$. Using lemma 3.4 we can find a stochastic embedding of $G$ into a distribution $\mathcal{F}$ over pyramids with distortion $c_{2}=O(1)$. By lemma 6.15 every pyramid in the support of $\mathcal{F}$ admits an embedding into $L_{1}$ with distortion $c_{3}=O(1)$. Combining with lemma 1.4 we obtain that $G$ admits an embedding into $L_{1}$ with distortion $c_{2} c_{3}$. Therefore ( $X, d$ ) admits an embedding into $L_{1}$ with distortion $\gamma=c_{1} c_{2} c_{3}=O(1)$, concluding the proof.

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## References

[AR98] Yonatan Aumann and Yuval Rabani. An $O(\log k)$ approximate min-cut max-flow theorem and approximation algorithm. SIAM J. Comput., 27(1):291-301 (electronic), 1998.
[BS] Mario Bonk and Oded Schramm. Embeddings of gromov hyperbolic spaces. Geom. Funct. Anal, 10:266-306.
[BS05a] Sergei Buyalo and Viktor Schroeder. A product of trees as universal space for hyperbolic groups. arXiv.org, September 2005.
[BS05b] Sergei Buyalo and Viktor Schroeder. Embedding of hyperbolic spaces in the product of trees. Geometriae Dedicata, 113(1):75-93, 2005.
$\left[\mathrm{CGN}^{+} 06\right]$ Chandra Chekuri, Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Embedding $k$-outerplanar graphs into $l_{1}$. SIAM J. Discrete Math., 20(1):119-136 (electronic), 2006.
[CJLV08] Amit Chakrabarti, Alexander Jaffe, James R. Lee, and Justin Vincent. Embeddings of topological graphs: Lossy invariants, linearization, and 2-sums. In FOCS, pages 761-770, 2008.
[CSW13] Chandra Chekuri, F. Bruce Shepherd, and Christophe Weibel. Flow-cut gaps for integer and fractional multiflows. J. Comb. Theory, Ser. B, 103(2):248-273, 2013.
[DL97] Michel Marie Deza and Monique Laurent. Geometry of cuts and metrics, volume 15 of Algorithms and Combinatorics. Springer-Verlag, Berlin, 1997.
[GNRS04] Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Cuts, trees and $l_{1}$-embeddings of graphs. Combinatorica, 24(2):233-269, 2004.
[KL06] Robert Krauthgamer and James R. Lee. Algorithms on negatively curved spaces. In FOCS, pages 119-132, 2006.
[KPR93] Philip N. Klein, Serge A. Plotkin, and Satish Rao. Excluded minors, network decomposition, and multicommodity flow. In STOC, pages 682-690, 1993.
[LLR95] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. Combinatorica, 15(2):215-245, 1995.
[LP13] James R. Lee and Daniel Poore. On the 2-sum embedding conjecture. In SoCG, 2013.
[LR99] Tom Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. J. ACM, 46(6):787-832, 1999.
[LR10] James R. Lee and Prasad Raghavendra. Coarse differentiation and multi-flows in planar graphs. Discrete $\mathcal{E}$ Computational Geometry, 43(2):346-362, 2010.
[LS09] James R. Lee and Anastasios Sidiropoulos. On the geometry of graphs with a forbidden minor. In STOC, pages 245-254, 2009.
[OS81] Haruko Okamura and P.D. Seymour. Multicommodity flows in planar graphs. Journal of Combinatorial Theory, Series B, 31(1):75-81, 1981.
[Pap05] A. Papadopoulos. Metric spaces, convexity and nonpositive curvature. Irma Lectures in Mathematics and Theoretical Physics Series. European Mathematical Society Publishing House, 2005.
[Rao99] Satish Rao. Small distortion and volume preserving embeddings for planar and euclidean metrics. In Symposium on Computational Geometry, pages 300-306, 1999.
[TL97] W.P. Thurston and S. Levy. Three-dimensional geometry and topology. 1 (1997). Princeton mathematical series. Princeton University Press, 1997.


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[^1]:    ${ }^{1}$ This connection was pointed out by James R. Lee.

