

Spatial mixing and approximation algorithms for graphs with bounded connective constant

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Abstract

The hard core model in statistical physics is a probability distribution on independent sets in a graph in which the weight of any independent set I is proportional to $\lambda^{|I|}$, where $\lambda > 0$ is the *vertex activity*. We show that there is an intimate connection between the *connective constant* of a graph and the phenomenon of strong spatial mixing (decay of correlations) for the hard core model; specifically, we prove that the hard core model with vertex activity $\lambda < \lambda_c(\Delta + 1)$ exhibits strong spatial mixing on any graph of connective constant Δ , irrespective of its maximum degree, and hence derive an FPTAS for the partition function of the hard core model on such graphs. Here $\lambda_c(d) := \frac{d^d}{(d-1)^{d+1}}$ is the critical activity for the uniqueness of the Gibbs measure of the hard core model on the infinite d -ary tree. As an application, we show that the partition function can be efficiently approximated with high probability on graphs drawn from the random graph model $\mathcal{G}(n, d/n)$ for all $\lambda < e/d$, even though the maximum degree of such graphs is unbounded with high probability.

We also improve upon Weitz's bounds for strong spatial mixing on bounded degree graphs [32] by providing a computationally simple method which uses known estimates of the connective constant of a lattice to obtain bounds on the vertex activities λ for which the hard core model on the lattice exhibits strong spatial mixing. Using this framework, we improve upon these bounds for several lattices including the Cartesian lattice in dimensions 3 and higher.

Our techniques also allow us to relate the threshold for the uniqueness of the Gibbs measure on a general tree to its *branching factor* [17].

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1 Introduction

1.1 Background

In spin systems on graphs, the property of *spatial mixing* (i.e., decay of correlations with distance) plays a central role. In statistical physics spatial mixing guarantees a unique Gibbs measure, and thus a single phase for the underlying physical model. In computer science spatial mixing implies the existence of efficient algorithms for approximating key combinatorial quantities. Much attention has therefore been focused on identifying ranges of parameters for which spatial mixing holds; interestingly, this is a case where the computer science perspective, motivated by algorithmic applications, has led to new insights into the behavior of physical models.

In this paper we contribute to this line of work, focusing on the *hard core* (or *weighted independent set*) model, which is one of the most widely studied classical examples (though our techniques actually apply more widely to other two-spin systems, such as the anti-ferromagnetic Ising model). The configurations of the hard core model are the independent sets of a graph $G = (V, E)$, each of which has a weight $w(I) = \lambda^{|I|}$, where λ is a positive parameter known as the *vertex activity*. The probability of occurrence of configuration I is determined by the *Gibbs distribution*:

$$\pi(I) = w(I)/Z.$$

Here the normalizing factor $Z = Z(G, \lambda) := \sum_I w(I)$ is called the *partition function*. As a natural generalization of counting, computing Z is a central problem in statistical physics and combinatorics, and is known to be #P-hard in most interesting cases. Considerable interest has been devoted to the problem of approximating Z , which is equivalent to sampling from the Gibbs distribution [12].

In the special case where G is the infinite d -ary tree, it is well known that the hard core model exhibits a phase transition: there exists a critical activity $\lambda_c(d) = \frac{d^d}{(d-1)^{d+1}}$ such that point-to-set correlations in the Gibbs distribution¹ decay to zero exponentially with distance when $\lambda < \lambda_c(d)$ (this decay is referred to as *weak spatial mixing*), and remain bounded away from zero when $\lambda > \lambda_c(d)$. This phase transition has been at the center of dramatic recent results relating phase transitions to the computational complexity of partition functions [29, 32].

A stronger notion of decay of correlations is that of *strong spatial mixing*. Here, the exponential decay of point-to-set correlations is required to hold even in the presence of arbitrary boundary conditions; i.e., correlations should still decay even when the configurations of some vertices (possibly close to those whose correlations are being measured) are fixed arbitrarily. Algorithmically, strong spatial mixing guarantees the existence of an efficient approximation algorithm (an FPTAS) for the partition function Z and other important quantities associated with the model. In a seminal paper [32], Weitz showed that for the hard core model, strong spatial mixing on the d -ary tree implies strong spatial mixing on *any* graph of degree at most $d+1$. Weitz further showed that weak spatial mixing is equivalent to strong spatial mixing for the hard-core model on the d -ary tree, thus establishing that a graph of maximum degree $d+1$ always exhibits strong spatial mixing, and hence admits an FPTAS for the partition function, for all $\lambda < \lambda_c(d)$.

At the time of publication of Weitz's paper, his bound for graphs of maximum degree $d+1$ in terms of the critical activity of the d -ary tree improved upon the best known bounds for spatial mixing even for such special classes of graphs as Cartesian lattices, which are the most widely studied in statistical physics. This remained the state-of-the-art until the recent work of Restrepo, Shin, Tetali, Vigoda and Yang [27], who improved upon Weitz's bounds in the special case of the 2-dimensional Cartesian lattice. Although in principle the methods of Restrepo *et al.* can be applied to any fixed lattice, such an application requires a

¹When G is an infinite graph the partition function is not well defined. However, the Gibbs distribution is still well-defined on any finite subset of G , and can be extended in a natural way to a measure on the whole of G . On the infinite d -ary tree, this extension is uniquely determined only when $\lambda \leq \lambda_c(d)$.

numerical search over a high-dimensional parameter space, and hence is computationally very intensive. Further, in contrast to Weitz’s bound, which depends solely upon the maximum degree of the graph, the bounds of Restrepo *et al.* (as well as those in the subsequent paper of Vera, Vigoda and Yang [30]) do not seem to depend upon any easily identifiable characteristic of the graph. Also, their methods are tailored to *fixed* lattices rather than to general classes of graphs (such as graphs drawn from the random graph family $\mathcal{G}(n, d/n)$).

1.2 Contributions

In this work we improve upon Weitz’s result for the hard core model in two ways. First, we relax the bounded degree restriction in Weitz’s result. Second, in the case of bounded degree graphs we improve Weitz’s bounds by taking into account more information about the structure of the graph. We achieve these goals by relating spatial mixing on a graph to its *connective constant*, a natural measure of the “effective degree” in a sense that we describe below. Since in many interesting families of graphs the connective constant is significantly smaller than the maximum degree, this will allow us to obtain tighter bounds.

For an infinite graph $G = (V, E)$, the connective constant $\Delta(G)$ is defined as

$$\sup_{v \in V} \limsup_{\ell \rightarrow \infty} N(v, \ell)^{1/\ell},$$

where $N(v, \ell)$ is the number of self-avoiding walks of length ℓ in G starting at v . This definition extends naturally to families of finite graphs (see Section 2.3). The set of self-avoiding walks originating at a vertex v can naturally be viewed as a tree rooted at v , known as the *self-avoiding walk (SAW) tree*. The connective constant can then be viewed as the *average arity* of this tree. Thus, for example, the connective constant of a graph of maximum degree $d + 1$ is at most d , though it can be much smaller. The connective constant is a well-studied quantity, especially for standard graph families such as lattices, and rigorous bounds on its value are known in many cases (see, for example, [1, 2, 10, 19, 26] and the recent breakthrough in [4]).

Our interest in the connective constant comes from Weitz’s construction in [32] establishing that the decay of correlations on a graph is always at least as rapid as on the corresponding SAW tree. Intuitively, the decay of correlations on this tree should in turn be related to the rate of growth with ℓ of the number of vertices at distance ℓ from the root (which is exactly $N(v, \ell)$). So far, in the case of general graphs, this intuition has been captured only by crudely bounding the growth of the number of vertices as d^ℓ , where $d + 1$ is the maximum degree of the graph. Our results show that by instead using the connective constant, one can obtain tighter relations between the rate of decay of correlations and the growth of the number of descendants. We say a few words about our proof techniques at the end of this section.

Our first result is an analog of Weitz’s bound, with the maximum degree replaced by the connective constant and a slightly stronger condition on the vertex activity λ .

Theorem 1.1. *Let $\lambda > 0$ and Δ be such that $\lambda < \lambda_c(\Delta + 1)$. Then:*

1. *The hard core model with vertex activity λ exhibits strong spatial mixing on any family of (finite or infinite) graphs with connective constant at most Δ .*
2. *There is an FPTAS for the partition function of the hard core model with vertex activity λ on any family of finite graphs with connective constant at most Δ .*

Remark 1.1. The condition on λ in the theorem is satisfied whenever $\lambda < \frac{e}{\Delta}$. This is asymptotically optimal for large Δ since $\lambda_c(\Delta) \sim e/\Delta$ as $\Delta \rightarrow \infty$. (We note that a trivial path coupling argument can be used to prove strong spatial mixing under the much stronger assumption $\lambda < 1/\Delta$.)

Remark 1.2. If we could replace $\lambda_c(\Delta + 1)$ in the above theorem by $\lambda_c(\Delta)$, this would exactly parallel Weitz’s bound with the maximum degree replaced by the connective constant and would be optimal for all Δ . Although we do not know if such a result holds in general, we do obtain such a tight bound for the special case of spherically symmetric trees; see Section 5.

The result in Theorem 1.1 has the advantage of being applicable without any bound on the maximum degree. This is in contrast to various recent results on the approximation of the hard core model partition function [14, 15, 27, 32], all of which require the maximum degree to be bounded by a constant. As an application of Theorem 1.1 we consider the random graph model $\mathcal{G}(n, d/n)$, which has constant average degree $d(1 - o(1))$ but unbounded maximum degree $\Theta(\log n / \log \log n)$ with high probability. We prove the following result (see Section 4 for a more precise formulation of the high probability statements).

Theorem 1.2. *Let $\epsilon > 0$, $d > 1$ and $\lambda < \frac{\epsilon}{d(1+\epsilon)}$ be fixed. Then:*

- *The hard core model with activity λ on a graph G drawn from $\mathcal{G}(n, d/n)$ exhibits strong spatial mixing with high probability.*
- *There exists a deterministic algorithm which on input $\mu > 0$, approximates the partition function of the hard core model with activity $\lambda > 0$ on a graph G drawn from $\mathcal{G}(n, d/n)$ within a multiplicative factor of $(1 \pm \mu)$, and which runs in time polynomial in n and $1/\mu$ with high probability (over the random choice of the graph).*

Similar results for $\mathcal{G}(n, d/n)$ have appeared in the literature in the context of rapid mixing of Glauber dynamics for the ferromagnetic Ising model [23], and also for the hard core model [5, 22]. Although the authors of [22] do not supply an explicit range of λ for which their rapid mixing results hold, an examination of their proofs suggests that necessarily $\lambda < O(1/d^2)$. Similarly, the results of [5] hold when $\lambda < 1/(2d)$. In contrast, our bound approaches the conjectured optimal value e/d . Further, unlike ours, the results of [5, 22, 23] are restricted to $\mathcal{G}(n, d/n)$ and certain other classes of sparse graphs.

While it is asymptotically optimal, Theorem 1.1 above is not strong enough to improve upon Weitz’s result in the important special case of small degree lattices. To do this, we adapt our techniques to take into account the maximum degree as well as the connective constant. Define the function $\nu_\lambda(d)$ by

$$\nu_\lambda(d) := \frac{d\tilde{x}_\lambda(d) - 1}{1 + \tilde{x}_\lambda(d)},$$

where $\tilde{x}_\lambda(d)$ is the unique positive solution of the fixed point equation $dx = 1 + \lambda/(1+x)^d$. We can now state our second general theorem.

Theorem 1.3. *Let d be a positive integer, and let λ and Δ be such that $\nu_\lambda(d)\Delta < 1$. Then:*

1. *The hard-core model with vertex activity $\lambda' \leq \lambda$ exhibits strong spatial mixing on any family of (finite or infinite) graphs with maximum degree at most $d + 1$ and connective constant at most Δ .*
2. *For any $\lambda' \leq \lambda$, there is an FPTAS for the partition function of the hard-core model with vertex activity λ' on any family of finite graphs with maximum degree at most $d + 1$ and connective constant at most Δ .*

Remark 1.3. The connective constant Δ of any infinite graph of maximum degree $d + 1$ can be at most d . This, combined with the fact (proved later) that $\nu_\lambda(d)d < 1$ whenever $\lambda < \lambda_c(d)$, ensures that the bound for spatial mixing in Theorem 1.3 is always better than that obtained from Weitz’s result. Thus, the above theorem extends the applicability of Weitz’s results to a larger range of vertex activities.

Using Theorem 1.3, we are able to improve upon the best known spatial mixing bounds for various lattices, including the Cartesian lattice in three and higher dimensions, as shown in Table 1. The table shows, for each lattice, the best known upper bound for the connective constant and the strong spatial mixing (SSM) bounds we obtain using these values in Theorem 1.3. In the table, a value α in the “ Δ ” column means that SSM is shown to hold for the appropriate lattice whenever $\lambda \leq \lambda_c(\alpha)$; the corresponding value $\lambda_c(\alpha)$ (rounded to three decimal places) appears in the adjacent “ λ ” column. The last pair of columns give the previously best known SSM bounds.

As is evident from the table, our general result gives improvements on known SSM bounds for all lattices except the 2-dimensional Cartesian lattice \mathbb{Z}^2 . For that lattice, our bound still improves on Weitz’s bound but is not as strong as the bound obtained in [27] by more numerically intensive methods tailored to this special case. We note, however, that any improvement in the bound on the connective constant of the corresponding SAW tree immediately yields an improvement in our SSM bound; we illustrate this with a simple example in Appendix B, where we also give details of the derivation of the results in the table.

Lattice	Conn. constant		Max. degree	Our SSM bound		Previous best SSM bound	
				Δ	λ	Δ	λ
\mathbb{T}	4.251	419 [2]	6	4.325	0.937	5	0.762 [32]
\mathbb{H}	1.847	760 [4]	3	1.884	4.706	2	4.0 [32]
\mathbb{Z}^2	2.679	193 [26]	4	2.731	2.007	2.502	2.48 [27,30]
\mathbb{Z}^3	4.7387	[26]	6	4.765	0.816	5	0.762 [32]
\mathbb{Z}^4	6.8040	[26]	8	6.818	0.506	7	0.490 [32]
\mathbb{Z}^5	8.8602	[26]	10	8.868	0.367	9	0.360 [32]
\mathbb{Z}^6	10.8886	[31]	12	10.894	0.288	11	0.285 [32]

Table 1: Strong spatial mixing bounds for various lattices. (\mathbb{Z}^D is the D -dimensional Cartesian lattice; \mathbb{T} and \mathbb{H} denote the triangular and honeycomb lattices respectively.)

Finally, we apply our techniques to get tighter results for correlation decay in some other important cases. The first of these is strong spatial mixing on *spherically symmetric trees* (rooted trees in which the degree of each vertex is dependent only upon its distance from the root). Such trees have been studied before, for example by Lyons [18]. For these trees, we improve upon Theorem 1.1 to show that strong spatial mixing holds as long as $\lambda < \lambda_c(\Delta)$, which is optimal as a function of Δ . As an application, we present an FPTAS for the partition function of the hard core model on bounded degree bipartite graphs that improves upon the parameters promised by Weitz’s algorithm [32]; see Section 5.1 for details. We also consider the question of uniqueness of Gibbs measure on general trees. Uniqueness is a weaker notion than spatial mixing, requiring correlations to decay to zero with distance but not necessarily at an exponential rate (see Section 5.2 for a formal definition). We show that the threshold for uniqueness of the Gibbs measure of the hard core model on a general tree can be related to the *branching factor* of the tree, another natural notion of average arity that has appeared in the study of uniqueness of Gibbs measure for models such as the ferromagnetic Ising model [17] and is slightly stronger than the connective constant. The details of these results can be found in Section 5.2.

We close this section with a few remarks on our proof techniques, which begin with the message approach first used by Restrepo *et al.* [27]. The key idea here is to consider a function, called a *message*, of the occupation probabilities, and to study how “errors” in this message propagate through the standard tree recurrence. All previous applications of this method [14, 15, 27, 28] establish correlation decay on the tree by showing that, for an appropriate range of parameters and for an appropriately defined message,

the error at every vertex in the tree is at most some constant fraction of the *maximum* error at its children. This is an inherently worst case analysis that is oblivious to the more detailed structure of the tree. At a very high level, our main technical innovation gets around this issue by considering the decay of a smoother statistic (an ℓ_q norm for $q < \infty$) of the errors at the children rather than the maximum error. We present a general, message-independent framework for achieving this in Section 3, and then in Section 4 we instantiate the framework for a specific message (previously used in [15]) to obtain the proofs of our main results. We note that our message framework extends in a straightforward manner to other two-spin systems such as the anti-ferromagnetic Ising model; this extension is deferred to a future paper.

1.3 Related work

Luby and Vigoda [16] were the first to give a general approximation algorithm (an FPRAS) for the hard core model partition function valid for $\lambda < \frac{1}{d-3}$ on graphs of maximum degree d . In a breakthrough result, Weitz [32] proved that for the hard core model, uniqueness of the Gibbs measure on the infinite d -ary tree implies that all graphs of degree at most $d + 1$ exhibit strong spatial mixing, and further that there is an FPTAS for the partition function of the hard core model on such graphs. Weitz’s results have since been extended to the anti-ferromagnetic Ising model with an arbitrary external field [28], and later to general anti-ferromagnetic two-spin systems [15]. On the other hand, in the special case of the hard core model on the two dimensional Cartesian lattice \mathbb{Z}^2 , Restrepo, Shin, Tetali, Vigoda and Yang [27] improved upon the threshold for strong spatial mixing obtained using a direct application of Weitz’s result. Their numerical bound for \mathbb{Z}^2 has recently been improved by Vera, Vigoda and Yang [30], using methods that are similar in spirit to those in [27], but which require even more intensive and sophisticated numerical computations. As observed earlier, although the methods of Restrepo *et al.* [27] (as well those of Vera *et al.* [30]) can in principle be applied to any given lattice, in practice such an application would require a significant amount of computer-assisted numeric and symbolic computations, especially for higher dimensional lattices. Further, unlike ours, their methods do not seem to generalize to arbitrary graph families (such as the random graph family $\mathcal{G}(n, d/n)$).

All of the results above pertain to the bounded degree case. Li, Lu and Yin [14,15] extended some of the above results for two-spin systems to the setting of general graphs without any bound on the maximum degree, but only in the regime of parameters for which uniqueness of the Gibbs measure holds on the d -ary tree for *all* d . In the case of the hard core model, however, the latter condition fails to hold for any non-trivial vertex activity. Our results, in contrast, impose a condition on the activity only in terms of a natural notion of effective degree. Further, this condition is asymptotically optimal as observed earlier.

Mossel and Sly [22] consider the question of sampling from the hard core distribution on unbounded degree graphs such as $\mathcal{G}(n, d/n)$ using Glauber dynamics. Under some conditions, their results also extend to other families of sparse graphs with bounded “tree excess.” Although the authors do not give an explicit bound on the vertex activity λ under which their results hold, an examination of the proofs suggests that necessarily $\lambda < O(1/d^2)$. Efthymiou [5] has recently improved upon [22] by exhibiting rapid mixing for a Gibbs sampler under the condition $\lambda < 1/(2d)$. However, both these bounds still remain far from our bound of e/d . Further, as noted above, the above results do not seem to hold for all graphs of bounded connective constant. Hayes and Vigoda [9] also considered the question of sampling from the hard core model on special classes of unbounded degree graphs. They showed that for regular graphs on n vertices of degree $d(n) = \Omega(\log n)$ and of girth greater than 6, the Glauber dynamics for the hard core model mixes rapidly for $\lambda < (1 - \epsilon)e/d(n)$ (where ϵ is an arbitrary positive constant). Their results are incomparable to ours; while our Theorem 1.1 requires neither the condition that the graph should be regular nor any lower bounds on its degree or girth, it does require additional information about the graph in the form of its connective constant. However, when the connective constant is available, then irrespective of the maximum degree of the graph or its girth, the theorem affords an FPTAS for the partition function.

Much more progress has been made on relating spatial mixing to notions of average degree in the case of the zero field *ferromagnetic* Ising model. Lyons [17] demonstrated that on an arbitrary tree, the *branching factor* exactly determines the threshold for uniqueness of the Gibbs measure for this model. For the ferromagnetic Ising model on general graphs, Mossel and Sly [21, 23] proved results analogous to our Theorem 1.1. However, the arguments of both [17] and [21, 23] seem to rely heavily on special properties of the ferromagnetic Ising model, and do not appear to be easily extensible to the case of “repulsive” spin systems such as the hard core model. Further, an FPRAS for the partition function of the ferromagnetic Ising model, without any restrictions on the degree, is already known [6, 11].

In work related to [17] above, Pemantle and Steif [25] define the notion of a *robust phase transition* (RPT) and relate the threshold for RPT for various “symmetric” models such as the zero field Potts model and the Heisenberg model on general trees to the branching factor of the tree. In the results of both [17] and [25], an important ingredient seems to be the existence of a symmetry group on the set of spins under whose action the underlying measure remains invariant. In contrast, in the hard core model, the two possible spin states of a vertex (“occupied” and “unoccupied”) do not admit any such symmetry.

The first reference to the connective constant occurs in classical papers by Hammersley and Morton [8], Hammersley and Broadbent [3] and Hammersley [7]. Since then, several natural combinatorial questions concerning the number and other properties of self-avoiding walks in various lattices have been studied in depth; see the monograph of Madras and Slade [19] for a survey. Much work has been devoted especially to finding rigorous upper and lower bounds for the connective constant of various lattices [1, 2, 10, 13, 26]. Heuristic techniques from physics have also been brought to bear upon this question. For example, Nienhuis [24] conjectured on the basis of heuristic arguments that the connective constant of the honeycomb lattice \mathbb{H} must be $\sqrt{2 + \sqrt{2}}$. In a celebrated recent breakthrough, Duminil-Copin and Smirnov [4] rigorously proved Nienhuis’ conjecture.

2 Preliminaries

2.1 The hard core model on trees and graphs

In this section, we introduce some standard notions associated with the hard core model on trees and graphs. Our notation is similar to that used in other recent works on the subject [14, 15, 27, 28, 32].

Given a graph $G = (V, E)$, a *boundary condition* will refer to a partially specified independent set in G . Formally, a boundary condition $\sigma = (S, I)$ is a subset $S \subseteq V$ along with an independent set I on S .

Definition 2.1. (Occupation probability and occupation ratio). Consider the hard core model with vertex activity $\lambda > 0$ on a graph G , and let v be a vertex in G . Given a boundary condition $\sigma = (S, I_S)$ on G , the *occupation probability* $p_v(\sigma, G)$ at the vertex v is the probability that v is included in an independent set I sampled according to the hard core distribution conditioned on the event that I restricted to S coincides with I_S . The *occupation ratio* $R_v(\sigma, G)$ is then defined as

$$R_v(\sigma, G) = \frac{p_v(\sigma, G)}{1 - p_v(\sigma, G)}.$$

In the special case where the graph G is a tree, both the occupation probability and the occupation ratio admit a simple recurrence in terms of similar quantities on subtrees. Formally, let T be an arbitrary tree rooted at a vertex ρ . Denote the children of ρ as $\rho_1, \rho_2, \dots, \rho_d$, and let T_{ρ_i} denote the subtree of T rooted at ρ_i . Let σ be an arbitrary boundary condition on T , and let σ_i be its restriction to T_{ρ_i} . We denote by R_i the occupation ratio $R_{\rho_i}(\sigma_i, T_{\rho_i})$ of the root ρ_i of the tree T_{ρ_i} . It is well known—and easy to

show—that the R_i obey the following recurrence (see, for example, [32]):

$$R_\rho(\sigma, T) = f_{d,\lambda}(R_1, R_2, \dots, R_d) := \lambda \prod_{i=1}^d \frac{1}{1 + R_i}. \quad (1)$$

With a slight abuse of notation, we also use the same notation to denote a symmetric one-argument version of $f_{d,\lambda}$ defined as $f_{d,\lambda}(x) := f_{d,\lambda}(x, x, \dots, x)$.

In order to analyze the convergence properties of the above recurrence, it is often convenient to consider the evolution of a suitable function of the occupation ratio, called a *message* (also known as a *statistic* or *potential*) [14, 15, 27, 28], as opposed to the occupation ratio itself.

Definition 2.2. (Message). A *message* is a strictly increasing, continuously differentiable function $\phi : (0, \infty) \rightarrow \mathbb{R}$, such that the derivative of ϕ on any interval of the form $(0, M]$ is bounded away from 0.

Note that the conditions on ϕ imply that it has a continuously differentiable inverse.

In what follows, given a recurrence f for the quantity R_v , we will denote by f^ϕ the recurrence for the quantity $\phi(R_v)$. Formally,

$$f^\phi(x_1, x_2, \dots, x_d) := \phi(f(\psi(x_1), \psi(x_2), \dots, \psi(x_d))),$$

where ψ is the inverse of ϕ . Similarly, for a one-argument recurrence f , $f^\phi(x) := \phi(f(\psi(x)))$.

Definition 2.3. (Strong Spatial Mixing [32]). The hard core model with a fixed vertex activity $\lambda > 0$ is said to exhibit *strong spatial mixing* on a family \mathcal{F} of graphs if for any graph G in \mathcal{F} , any vertex v in G , and any two boundary conditions σ and τ on G which differ only at a distance of at least ℓ from v , we have

$$|R_v(\sigma, G) - R_v(\tau, G)| = \exp(-\Omega(\ell)).$$

In the definition, the family \mathcal{F} might consist of a single infinite graph.

2.2 Locally finite trees

We introduce some notation and terminology for locally finite trees. Let T be a locally finite, but possibly infinite, tree rooted at some vertex ρ . For two vertices u and v , the statement “ u is an ancestor of v ” is denoted by $u < v$, while $u \leq v$ will denote the statement that “either $u = v$ or u is an ancestor of v ”. For example, $\rho \leq v$ for all vertices v in T . For any two vertices u, v in T we denote by $d(u, v)$ the distance between u and v . Further, $|v| = d(\rho, v)$ denotes the distance of a vertex v from the root ρ .

We will need the notion of a *cutset* (see, for example, Lyons [17]) in an infinite tree.

Definition 2.4. (Cutset). Let T be any locally finite infinite tree rooted at a vertex ρ . A *cutset* is a finite set of vertices C such that (i) any infinite path starting at ρ must intersect C ; and (ii) no vertex in C is an ancestor of another vertex in C .

Remark 2.1. Notice that if C is a cutset, then so is the set of children of vertices in C .

For a cutset C we define its distance from the root ρ as the minimum distance between the root ρ and any vertex v in C , and denote this distance by $d(\rho, C)$. Further, we denote by $T_{<C}$ the restriction of T to vertices which are not descendants of vertices in C , and by $T_{\leq C}$ the further restriction of $T_{<C}$ to vertices not in C .

2.3 Self-avoiding walks and the connective constant

As discussed in the introduction, the connective constant is a natural notion of “effective degree” that has been especially well studied in the case of lattices. For a vertex v in a locally finite graph G , we will denote by $N(v, \ell)$ the number of self-avoiding walks of length ℓ starting at v . The connective constant of a graph captures the rate of growth of $N(v, \ell)$ as a function of ℓ .

Definition 2.5. (Connective constant: infinite graphs). Let $G = (V, E)$ be a locally finite infinite graph. The *connective constant* $\Delta(G)$ is defined as $\sup_{v \in V} \limsup_{\ell \rightarrow \infty} N(v, \ell)^{1/\ell}$.

Remark 2.2. For vertex-transitive graphs (such as Cartesian lattices), the supremum over v can be removed without changing the definition. Moreover, for such graphs, the lim sup can be replaced by a limit. We use lim sup in order to avoid issues about the existence of the limit for more general classes of graphs.

The definition can be easily extended to finite graphs. For algorithmic applications, it is natural to define the connective constant for a family of graphs parametrized by size.

Definition 2.6. (Connective constant: finite graphs). Let \mathcal{F} be a family of finite graphs. The connective constant of \mathcal{F} is at most Δ if $\sum_{i=1}^{\ell} N(v, i) = O(\Delta^{\ell})$. More formally, we say that the connective constant of \mathcal{F} is at most Δ if there exist constants a and c such that for any graph $G = (V, E)$ in \mathcal{F} and any vertex v in G , we have $\sum_{i=1}^{\ell} N(v, i) \leq c\Delta^{\ell}$ for all $\ell \geq a \log |V|$.

Note that the connective constant of a graph of maximum degree $d + 1$ is at most d . However, the connective constant can be much smaller than the maximum degree. For example, the maximum degree of a graph drawn from $\mathcal{G}(n, d/n)$ is $\Theta(\log n / \log \log n)$ with high probability; however, it is easy to show (as we do in the proof of Theorem 1.2 below) that for any fixed $\epsilon > 0$, the connective constant is at most $d(1 + \epsilon)$ with high probability. Similarly, for the 2-dimensional Cartesian lattice (which has maximum degree 4) the connective constant is less than 2.68 [2, 26].

The set of self-avoiding walks starting at a vertex v in a graph G can be naturally represented as a tree rooted at v , where each vertex in the tree at distance ℓ from v corresponds to a distinct self-avoiding walk of length ℓ . Any vertex u in this tree can also be naturally identified (many-to-one) with the vertex in G at which the corresponding self-avoiding walk ends.

Weitz [32] showed that by fixing certain vertices of this tree to be occupied or unoccupied, one obtains a tree $T_{SAW}(v, G)$ (which we will often refer to as the “Weitz SAW tree”) such that, for any boundary condition σ in G , one has

$$R_v(\sigma, G) = R_v(\sigma, T_{SAW}(v, G)),$$

where on the right hand side, by a slight abuse of notation, we denote again by σ the natural translation of the boundary condition σ on G to $T_{SAW}(v, G)$. This result forms the cornerstone of all recent results that use correlation decay for two-spin system on trees to derive results for other graphs [14, 15, 27, 28, 32]. The definition of $T_{SAW}(v, G)$ implies that its connective constant is always upper-bounded by the connective constant of G . However, it can also be lower than that of G because of the additional boundary conditions introduced in Weitz’s construction.

Notation. For any bivariate function $g(x, y)$, we will denote the partial derivative $\frac{\partial^{i+j} g}{\partial x^i \partial y^j}$ evaluated at $x = a, y = b$ as $g^{(i,j)}(a, b)$.

3 Messages on a tree

In this section, we study the behavior of a general recurrence f on a tree. As before, T is a tree rooted at a vertex ρ , and the occupation ratios R_ρ and R_i are defined in Section 2.1. We begin with a version of the

mean value theorem adapted to our setting. Given a message ϕ (as in Definition 2.2), let $\Phi = \phi'$ denote the derivative of ϕ ; notice that $\Phi(x) > 0$ for all non-negative x since ϕ is strictly increasing.

Lemma 3.1. (Mean value theorem). *Consider two vectors \mathbf{x} and \mathbf{y} in $\phi([0, \infty))^d$. Then there exists a vector $\mathbf{z} \in [0, \infty)^d$ such that*

$$\left| f_{d,\lambda}^\phi(\mathbf{x}) - f_{d,\lambda}^\phi(\mathbf{y}) \right| \leq \Phi(f_{d,\lambda}(\mathbf{z})) \sum_{i=1}^d \frac{|y_i - x_i|}{\Phi(z_i)} \left| \frac{\partial f_{d,\lambda}}{\partial z_i} \right|,$$

where by a slight abuse of notation we denote by $\frac{\partial f_{d,\lambda}}{\partial z_i}$ the partial derivative of $f_{d,\lambda}(R_1, R_2, \dots, R_d)$ with respect to R_i evaluated at $\mathbf{R} = \mathbf{z}$.

We defer the proof of this lemma to Appendix C. For the special case of the tree recurrence of the hard core model (eq. 1), Lemma 3.1 implies that

$$\left| f_{d,\lambda}^\phi(\mathbf{x}) - f_{d,\lambda}^\phi(\mathbf{y}) \right| \leq f_{d,\lambda}(\mathbf{z}) \Phi(f_{d,\lambda}(\mathbf{z})) \sum_{i=1}^d \frac{|y_i - x_i|}{(1 + z_i) \Phi(z_i)}. \quad (2)$$

The first step of our approach is similar to that taken in the papers [14, 15, 27, 28] in that we will use an appropriate message—along with the estimate in Lemma 3.1—to argue that the “distance” between two input message vectors \mathbf{x} and \mathbf{y} at the children of a vertex shrinks by a constant factor at each step of the recurrence. Previous works on the subject [14, 15, 27, 28] show such a decay on some version of the ℓ_∞ norm of the “error” vector $\mathbf{x} - \mathbf{y}$: this is achieved by bounding the appropriate dual ℓ_1 norm of the gradient of the recurrence. Our intuition is that in order to achieve a bound in terms of a global quantity such as the connective constant, it should be advantageous to use a more global measure of the error such as an ℓ_q norm, for some $q < \infty$.

In line with the above plan, we first prove the following lemma, specialized here to the case of the hard core model. For ease of notation, we assume throughout that the vertex activity $\lambda > 0$ is fixed and suppress dependence on λ .

Lemma 3.2. *Let ϕ be a message and let $\Phi = \phi'$ be its derivative. Let p and q be positive reals such that $\frac{1}{p} + \frac{1}{q} = 1$. Define the functions $S_{\phi,p}$ and $\Xi_{\phi,q}(d, x)$ as follows:*

$$S_{\phi,p}(x) := \left(\frac{e^{-x}}{\Phi(e^x - 1)} \right)^p; \quad \Xi_{\phi,q}(d, x) := d^{q-1} \left(\frac{\Phi(f_d(x)) f_d(x)}{(1+x)\Phi(x)} \right)^q.$$

We further define $\xi_{\phi,q}(d) := \sup_{x \geq 0} \Xi_{\phi,q}(d, x)$. If $S_{\phi,p}$ is a concave function on the non-negative reals, then for any two vectors \mathbf{x}, \mathbf{y} in $\phi(\mathbb{R}^+)^d$, we have

$$\left| f_d^\phi(\mathbf{x}) - f_d^\phi(\mathbf{y}) \right|^q \leq \xi_{\phi,q}(d) \|\mathbf{x} - \mathbf{y}\|_q^q.$$

Proof. The concavity of $S_{\phi,p}(x)$ for non-negative x , combined with Jensen’s inequality, implies that for any vector $\mathbf{z} \in [0, \infty)^d$, and $Z = \prod_{i=1}^d (1 + z_i)^{1/d} - 1$,

$$\frac{1}{d} \sum_{i=1}^d \left(\frac{1}{(1 + z_i) \Phi(z_i)} \right)^p = \frac{1}{d} \sum_{i=1}^d S_{\phi,p}(\ln(1 + z_i)) \leq S_{\phi,p} \left(\frac{1}{d} \sum_{i=1}^d \ln(1 + z_i) \right) = \left(\frac{1}{(1 + Z) \Phi(Z)} \right)^p. \quad (3)$$

Besides, it is easy to verify that $f_d(Z) = f_d(\mathbf{z})$. Now, we apply Lemma 3.1 (specifically, its consequence in eq. (2)). Assume that \mathbf{z} is as defined in that lemma, and again let Z denote $\prod_{i=1}^d (1 + z_i)^{1/d} - 1$. We then have

$$\begin{aligned} \left| f_d^\phi(\mathbf{x}) - f_d^\phi(\mathbf{y}) \right| &= \Phi(f_d(\mathbf{z})) f_d(\mathbf{z}) \sum_{i=1}^d \frac{|y_i - x_i|}{(1 + z_i)\Phi(z_i)} \quad \text{using eq. (2)} \\ &\leq d^{1/p} \Phi(f_d(Z)) f_d(Z) \left(\frac{1}{d} \sum_{i=1}^d \left(\frac{1}{(1 + z_i)\Phi(z_i)} \right)^p \right)^{1/p} \|\mathbf{x} - \mathbf{y}\|_q, \\ &\quad \text{using } f_d(Z) = f_d(\mathbf{z}) \text{ and Hölder's inequality} \\ &\leq \frac{d^{1/p} \Phi(f_d(Z)) f_d(Z)}{(1 + Z)\Phi(Z)} \|\mathbf{x} - \mathbf{y}\|_q, \text{ using eq. (3).} \end{aligned}$$

Raising both sides to the q th power, using $\frac{1}{p} + \frac{1}{q} = 1$, and the definitions of the function Ξ and ξ , we get the claimed inequality. \square

Let ϕ be a message satisfying the conditions of Lemma 3.2. Lemma 3.2 then implies the following general lemma on propagation of “errors” in locally finite infinite trees. As before, we denote by $R_\rho(\sigma)$ the occupation probability of the root ρ in the hard core model with boundary condition σ . We consider the dependence of $R_\rho(\sigma)$ on the boundary conditions σ which are fixed everywhere except at some cutset C . For technical reasons, we will assume that the conditions are actually allowed to differ not on the cutset C itself, but on the cutset C' which is the set of children of the vertices in C .

Lemma 3.3. *Let T be a locally finite tree rooted at ρ . Let C be a cutset in T at distance at least 1 from the root, and C' the cutset comprising of all children of vertices in C . Consider two arbitrary boundary conditions σ and τ on $T_{\leq C'}$ which differ only on C' . If ϕ and q satisfy the conditions of Lemma 3.2, we have*

$$|R_\rho(\sigma) - R_\rho(\tau)|^q \leq \frac{c_0 \xi_{\phi,q}(d_\rho)}{\alpha'} \sum_{v \in C} \alpha^{|v|},$$

where d_ρ is the degree of the root ρ , c_0 is a constant depending only upon q , λ and the message ϕ , while $\alpha = \sup \xi_{\phi,q}(d)$, where the supremum is taken over the arities d of all vertices in T except the root ρ , and $\alpha' = \inf_{d \geq 1} \xi_{\phi,q}(d)$.

For a proof of this lemma, see Appendix C.

4 A special message

We now instantiate the approach outlined in Section 3 to prove Theorems 1.1 and 1.3. Our message is the same as that used in [15]; we choose

$$\phi(x) := \sinh^{-1}(\sqrt{x}), \text{ so that } \Phi(x) := \phi'(x) = \frac{1}{2\sqrt{x(1+x)}}. \quad (4)$$

Notice that ϕ is a strictly increasing, continuously differentiable function on $(0, \infty)$, and also satisfies the technical condition on the derivative Φ as required in the definition of a message. Moreover, we choose $p = q = 2$. Our choices are designed to satisfy the conditions of Lemma 3.2. We now proceed to show that this is indeed the case.

Observation 4.1. *The function $S_{\phi,2}(x)$, when ϕ is as defined in eq. (4), is concave for $x \geq 0$.*

Proof. For $x \geq 0$, and Φ defined in eq. (4) we have $S_{\phi,2}(x) = 4(1 - e^{-x})$, which is concave. \square

In order to see what Lemma 3.2 implies with our message, we first analyze the function $\xi(d) := \xi_{\phi,2}(d)$ in some detail. The proof of the following simple lemma follows from arguments in [15]; however, we include a proof here for completeness. In what follows, we drop the subscripts ϕ and q since these will always be clear from the context.

Lemma 4.2. *Consider the hard core model with any fixed vertex activity $\lambda > 0$. With ϕ as defined in eq. (4), we have $\xi(d) = \Xi(d, \tilde{x}_\lambda(d))$, where $\tilde{x}_\lambda(d)$ is the unique solution to*

$$d\tilde{x}_\lambda(d) = 1 + f_{d,\lambda}(\tilde{x}_\lambda(d)). \quad (5)$$

Proof. Plugging in Φ from eq. (4) in the definition of Ξ and using $q = 2$, we get

$$\Xi(d, x) = \frac{dx}{1+x} \frac{f_{d,\lambda}(x)}{1+f_{d,\lambda}(x)}.$$

Taking the partial derivative with respect to the second argument, we get

$$\Xi^{(0,1)}(d, x) = \frac{\Xi(d, x)}{x(1+x)(1+f_{d,\lambda}(x))} [1 + f_{d,\lambda}(x) - dx].$$

For fixed d , the quantity outside the square brackets is always positive, while the expression inside the square brackets is strictly decreasing in x . Thus, any zero of the expression in the brackets will be a unique maximum of Ξ . The fact that such a zero exists follows by noting that the partial derivative is positive at $x = 0$ and negative as $x \rightarrow \infty$. Thus, $\Xi(d, x)$ is maximized at $\tilde{x}_\lambda(d)$ as defined above, and hence $\xi(d) = \Xi(d, \tilde{x}_\lambda(d))$, as claimed. \square

We now define $\nu_\lambda(d) := \xi(d)$. We first show that this definition agrees with the one used in the statement of Theorem 1.3, and then derive some monotonicity properties of the function ν .

Lemma 4.3. *For a given $\lambda > 0$ and a positive integer d let $\tilde{x}_\lambda(d)$ be the unique solution to eq. (5). We then have*

1. $\nu_\lambda(d) = \frac{d\tilde{x}_\lambda(d)-1}{1+\tilde{x}_\lambda(d)}$ and $\nu_\lambda(d) = \frac{1}{d}$ when $\lambda = \lambda_c(d)$.
2. $\nu_\lambda(d)$ is increasing in d for fixed $\lambda > 0$.
3. $\nu_\lambda(d)$ is increasing in λ for fixed $d > 0$.

The proof of the above lemma is somewhat technical, and is deferred to Appendix A. We now proceed with the proof of Theorem 1.3.

Proof of Theorem 1.3. Let \mathcal{F} be any family of finite or infinite graphs with connective constant Δ and maximum degree $d+1$. We prove the result for any fixed λ such that $\nu_\lambda(d)\Delta = 1 - \epsilon$ for some fixed $\epsilon > 0$. The result will then follow for all $\lambda' \leq \lambda$, since by item 3 in Lemma 4.3 we then have $\nu_{\lambda'}(d)\Delta \leq 1 - \epsilon$ for every such λ' .

We first prove that the hard core model with these parameters exhibits strong spatial mixing on this family of graphs. Let G be any graph from \mathcal{F} , v any vertex in G , and consider any boundary conditions σ and τ on G which differ only at a distance of at least ℓ from v . We consider the Weitz self-avoiding walk tree $T_{SAW}(v, G)$ rooted at v (as defined in Section 2.3). As before, we denote again by σ (respectively, τ) the translation of the boundary condition σ (respectively, τ) on G to $T_{SAW}(v, G)$. From Weitz's theorem, we then have that $R_v(\sigma, G) = R_v(\sigma, T_{SAW}(v, G))$ (respectively, $R_v(\tau, G) = R_v(\tau, T_{SAW}(v, G))$).

Consider first the case where G is infinite. Let C_ℓ denote the cutset in $T_{SAW}(v, G)$ consisting of all vertices at distance ℓ from v . Since G has connective constant at most Δ , it follows that for ℓ large enough, we have $|C_\ell| \leq \Delta^\ell (1 - \epsilon/2)^{-\ell}$. Notice that the maximum degree of $T_{SAW}(v, G)$ is $d + 1$, and hence every vertex except for the root has arity at most d . In the notation of Lemma 3.3, $\nu_\lambda(d)\Delta = 1 - \epsilon$ then implies that $\alpha \leq (1 - \epsilon)/\Delta$. Further, since ν is increasing in d , we also have $\alpha' := \inf_{d \geq 1} \nu(d) = \nu(1) > 0$. Applying Lemma 3.3 we then get

$$\begin{aligned} |R_v(\sigma, G) - R_v(\tau, G)|^2 &= |R_v(\sigma, T_{SAW}(v, G)) - R_v(\tau, T_{SAW}(v, G))|^2 \\ &\leq \frac{c_0}{\nu_\lambda(1)} \nu_\lambda(d_v) \sum_{u \in C_\ell} \left(\frac{1 - \epsilon}{\Delta} \right)^\ell, \text{ where } d_v \text{ is the degree of } v \text{ in } G \\ &\leq \frac{c_0}{\nu_\lambda(1)} \nu_\lambda(d + 1) \left(\frac{1 - \epsilon}{1 - \epsilon/2} \right)^\ell, \\ &\quad \text{using } |C_\ell| \leq \Delta^\ell (1 - \epsilon/2)^{-\ell} \text{ and } \nu_\lambda(d_v) \leq \nu_\lambda(d + 1), \end{aligned}$$

which establishes strong spatial mixing in G , since $1 - \epsilon < 1 - \epsilon/2$.

We now consider the case when \mathcal{F} is a family of finite graphs, and G is a graph from \mathcal{F} of n vertices. Since the connective constant of the family is Δ , there exist constants a and c such that for $\ell \geq a \log n$, $\sum_{i=1}^\ell N(v, i) \leq c\Delta^\ell$. We now proceed with the same argument as in the infinite case, but choosing $\ell \geq a \log n$. The cutset C_ℓ is again chosen to be the set of all vertices at distance ℓ from v in $T_{SAW}(v, G)$, so that $|C_\ell| \leq c\Delta^\ell$. As before, we then have for $\ell > a \log n$,

$$\begin{aligned} |R_v(\sigma, G) - R_v(\tau, G)|^2 &= |R_v(\sigma, T_{SAW}(v, G)) - R_v(\tau, T_{SAW}(v, G))|^2 \\ &\leq \frac{c_0}{\nu_\lambda(1)} \nu_\lambda(d_v) \sum_{u \in C_\ell} \left(\frac{1 - \epsilon}{\Delta} \right)^\ell, \text{ where } d_v \text{ is the degree of } v \text{ in } G \\ &\leq c \frac{c_0}{\nu_\lambda(1)} \nu_\lambda(d + 1) (1 - \epsilon)^\ell, \text{ using } |C_\ell| \leq c\Delta^\ell \text{ and } \nu_\lambda(d_v) \leq \nu_\lambda(d + 1), \quad (6) \end{aligned}$$

which establishes the requisite strong spatial mixing bound.

In order to prove the algorithmic part, we first recall an observation of Weitz [32] that an FPTAS for the “non-occupation” probabilities $1 - p_v$ under arbitrary boundary conditions is sufficient to derive an FPTAS for the partition function. We further note that if the vertex v is not already fixed by a boundary condition, then $1 - p_v = \frac{1}{1 + R_v} \geq \frac{1}{1 + \lambda}$, since R_v lies in the interval $[0, \lambda]$ for any such vertex. Hence, an additive approximation to R_v with error μ implies a multiplicative approximation to $1 - p_v$ within a factor of $1 \pm \mu(1 + \lambda)$. Thus, an algorithm that produces in time polynomial in n and $1/\mu$ an *additive* approximation to R_v with error at most μ immediately gives an FPTAS for $1 - p_v$, and hence, by Weitz’s observation, also for the partition function. To derive such an algorithm, we again use the tree $T_{SAW}(v, G)$ considered above. Suppose we require an additive approximation with error at most μ to $R_v(\sigma, G) = R_v(\sigma, T_{SAW}(v, G))$. We notice first that $R_v = 0$ if and only if there is a neighbor of v that is fixed to be occupied in the boundary condition σ . In this case, we simply return 0. Otherwise, we expand $T_{SAW}(v, G)$ up to depth ℓ for some $\ell \geq a \log n$ to be specified later. Notice that this subtree can be explored in time $O\left(\sum_{i=1}^\ell N(v, i)\right)$ which is $O(\Delta^\ell)$ since the connective constant is at most Δ .

We now consider two extreme boundary conditions σ_+ and σ_- on C_ℓ : in σ_+ (respectively, σ_-) all vertices in C_ℓ that are not already fixed by σ are fixed to “occupied” (respectively, unoccupied). The form of the recurrence ensures that the true value $R_v(\sigma)$ lies between the values $R_v(\sigma_+)$ and $R_v(\sigma_-)$. We compute the recurrence for both these boundary conditions on the tree. The analysis leading to eq. (6) ensures that, since $\ell \geq a \log n$, we have

$$|R_v(\sigma_+, G) - R_v(\sigma_-, G)| \leq M_1 \exp(-M_2 \ell)$$

for some fixed positive constants M_1 and M_2 . Now, assume without loss of generality that $R_v(\sigma_+) \geq R_v(\sigma_-)$. By the preceding observations, we then have

$$R_v(\sigma) \leq R_v(\sigma_+) \leq R_v(\sigma) + M_1 \exp(-M_2 \ell).$$

By choosing $\ell = a \log n + O(1) + O(\log(1/\mu))$, we get the required $\pm\mu$ approximation. Further, by the observation above, the algorithm runs in time $O(\Delta^\ell)$, which is polynomial in n and $1/\mu$ as required. \square

We now prove Theorem 1.1.

Proof of Theorem 1.1. The spatial mixing part of the Theorem follows directly from Theorem 1.3 once we prove that $\lambda < \lambda_c(\Delta + 1)$ implies that $\sup_{d \geq 1} \nu_\lambda(d) \Delta < 1$. We now proceed with this verification. For ease of notation, we will denote $\tilde{x}_\lambda(d)$ (as defined by eq. (5)) by \tilde{x} in the proof.

Let D be such that $\lambda = \lambda_c(D)$. Since $\lambda < \lambda_c(\Delta + 1)$, we have $D > \Delta + 1 > 1$, and hence there exists an $\epsilon > 0$ such that $\nu_\lambda(d) \Delta \leq \frac{1}{1+\epsilon} \nu_\lambda(d)(D-1)$ for all $d \geq 1$. Thus, in order to establish that $\sup_{d \geq 1} \nu_\lambda(d) \Delta < 1$, we only need to show that $\nu_\lambda(d)(D-1) < 1$ for all $d \geq 1$. Using the definition $\nu_\lambda(d) = \frac{d\tilde{x}-1}{1+\tilde{x}}$, this translates into the following requirement for \tilde{x} :

$$(d(D-1) - 1)\tilde{x} < D.$$

Now, the definition of \tilde{x} implies that $\tilde{x} \geq 1/d > 0$, so this condition is trivially satisfied if $d(D-1) \leq 1$. Hence, we assume from now on that $d(D-1) - 1 > 0$. In this case, the above condition translates to

$$\tilde{x} < x^* := \frac{D}{d(D-1) - 1}.$$

We now notice that eq. (5) for \tilde{x} can be written as $P_{d,D}(\tilde{x}) = 0$, where $P_{d,D}(x) := dx - 1 - f_{d,\lambda_c(D)}(x)$ is a strictly increasing function of x . Thus the above requirement translates to $P_{d,D}(x^*) > 0$, which simplifies to the requirement

$$\left[1 + \frac{D}{d(D-1) - 1}\right]^{d+1} > \left(\frac{D}{D-1}\right)^D \quad \text{for all } d \geq 1. \quad (7)$$

Since the left hand side of eq. (7) is a decreasing function of d , we only need to verify the condition in the limit $d \rightarrow \infty$. This is equivalent to the condition

$$e^{\frac{D}{D-1}} > \left(\frac{D}{D-1}\right)^D,$$

which in turn is equivalent to the inequality $e^{1/(D-1)} > 1 + 1/(D-1)$, and the latter is true for all $D > 1$. Thus we see that whenever $\Delta + 1 < D$, or equivalently, when $\lambda_c(D) = \lambda < \lambda_c(\Delta + 1)$, we have $\nu_\lambda(d) \Delta < 1$.

In order to prove the claim that $\lambda_c(\Delta + 1) \geq \frac{e}{\Delta}$ in the remark following Theorem 1.1, we notice that

$$\lambda_c(\Delta + 1) = \frac{1}{\Delta} \left(1 + \frac{1}{\Delta}\right)^{\Delta+1} \geq \frac{e}{\Delta},$$

since $(1 + 1/x)^{x+1} > e$ for $x \geq 0$.

The proof of the algorithmic part of the theorem is identical to the proof of the algorithmic part of Theorem 1.3. As before, the depth ℓ up to which the self-avoiding walk tree needs to be explored in order to additively approximate R_v up to an error of at most μ is $a \log n + O(\log(1/\mu)) = O(\log n + \log(1/\mu))$, and hence the running time $O(\Delta^\ell)$ is still polynomial in n and $1/\mu$. \square

Finally, we apply Theorem 1.1 to the random graph model $\mathcal{G}(n, d/n)$, and prove Theorem 1.2. We first restate Theorem 1.2 to make the statements about the various high probability events more precise.

Theorem 1.2. (Restated). *Let $\epsilon, \beta > 0, d > 1$ and $\lambda < \frac{\epsilon}{d(1+\epsilon)}$ be fixed. Then:*

- *The hard core model with activity λ on a graph G drawn from $\mathcal{G}(n, d/n)$ exhibits strong spatial mixing with probability at least $1 - n^{-\beta}$.*
- *There exists a deterministic algorithm which, on input $\mu > 0$, approximates the partition function of the hard core model with activity $\lambda > 0$ on a graph G drawn from $\mathcal{G}(n, d/n)$ within a multiplicative factor of $(1 \pm \mu)$, and which runs in time polynomial in n and $1/\mu$ with probability at least $1 - n^{-\beta}$ (over the random choice of the graph).*

Proof. Both parts of the Theorem follow easily from Theorem 1.1 once we prove that graphs drawn from $\mathcal{G}(n, d/n)$ have connective constant at most $d(1 + \epsilon/2)$ with probability at least $1 - n^{-\beta}$.

Recall that $N(v, \ell)$ is the number of self-avoiding walks of length ℓ starting at v . Suppose $\ell \geq a \log n$, where a is a constant depending upon the parameters ϵ, β and d which will be specified later. We first observe that

$$\mathbb{E} \left[\sum_{i=1}^{\ell} N(v, i) \right] \leq \sum_{i=1}^{\ell} \left(\frac{d}{n} \right)^i n^i \leq d^{\ell} \frac{d}{d-1},$$

and hence by Markov's inequality, we have $\sum_{i=1}^{\ell} N(v, i) \leq d^{\ell} \frac{d}{d-1} (1 + \epsilon/2)^{\ell}$ with probability at least $1 - (1 + \epsilon/2)^{-\ell}$. By choosing a such that $a \log(1 + \epsilon/2) \geq \beta + 2$, we see that this probability is at least $1 - n^{-(\beta+2)}$. By taking a union bound over all ℓ with $a \log n \leq \ell \leq n$ and over all vertices v , we see that the connective constant Δ is at most $d(1 + \epsilon/2)$ with probability at least $1 - n^{-\beta}$. Since $\lambda \leq \frac{\epsilon}{d(1+\epsilon)} < \frac{\epsilon}{d(1+\epsilon/2)}$, we therefore see that with probability at least $1 - n^{-\beta}$, the conditions of the first part of Theorem 1.1 are satisfied, and the graph sampled from $\mathcal{G}(n, d/n)$ exhibits strong spatial mixing. This proves the part about strong spatial mixing.

The algorithmic part is proved using the same algorithm as in the proofs of Theorems 1.1 and 1.3. Provided the above bound on the connective constant holds, that algorithm will terminate in time $n^{\gamma} \text{poly}(1/\mu)$ and produce an estimate of the partition function accurate up to a factor of $1 \pm \mu$, where γ is a constant dependent only upon β, λ and ϵ . Since the required connective constant bound holds with probability at least $1 - n^{-\beta}$, we therefore see that the algorithm terminates in time $n^{\gamma} \text{poly}(1/\mu)$ with at least this probability. \square

5 More results on trees

In this section, we improve upon the bounds in Theorem 1.1 in the special case of spherically symmetric trees and then present an application of this improvement to the problem of counting independent sets in bounded degree bipartite graphs. We also show that in the case of general trees, uniqueness of the Gibbs measure for the hard core model can be related to the *branching factor*: a slightly stronger notion than the connective constant that has been used before in the context of the zero field Ising, Potts and Heisenberg models on trees [17, 25].

5.1 Spherically symmetric trees

We first consider improvements on the bounds of Theorem 1.1 in the special case of spherically symmetric trees. Recall that a rooted tree is called *spherically symmetric* if the degree of any vertex in the tree depends only upon its distance from the root.

We now consider an infinite spherically symmetric subtree rooted at ρ . Let d_i denote the arity of vertices at distance i from the root. We then have $N(\rho, \ell) = \prod_{i=0}^{\ell-1} d_i$. The *connective constant with respect to ρ* , denoted by Δ_ρ , is defined as $\Delta_\rho = \limsup_{\ell \rightarrow \infty} N(\rho, \ell)^{1/\ell}$. Notice that this number is always at most the true connective constant, which is the supremum of this quantity over all vertices in the tree.

Theorem 5.1. *Let T be a locally finite spherically symmetric tree rooted at ρ , whose connective constant with respect to ρ is Δ_ρ . If $\lambda < \lambda_c(\Delta_\rho)$, then for any two boundary conditions σ and τ on T which differ only at depth at least ℓ in T , we have*

$$|R_\rho(\sigma, T) - R_\rho(\tau, T)| = \exp(-\Omega(\ell)).$$

Remark 5.1. Note that Theorem 5.1 implies that on a spherically symmetric tree with connective constant Δ “as observed from the root”, the correlation between the state of the root and a set of vertices at a distance ℓ from the root decays exponentially whenever $\lambda < \lambda_c(\Delta)$, and this rate of decay holds irrespective of any fixed boundary conditions. This bound is tight as a function of Δ , and improves upon the bound $\lambda < \lambda_c(\Delta + 1)$ obtained from a direct application of Theorem 1.1. Note, however, that this does not show strong spatial mixing as we have defined it, since the decay of correlations is shown only for the root. Nevertheless, as we show in Corollary 5.4 below, this version of strong spatial mixing can still be applied to derive an FPTAS for the estimation of the partition function of the hard core model on bounded degree bipartite graphs for a range of activities larger than that promised by Weitz’s result [32], or our Theorems 1.1 and 1.3.

We begin by proving a convexity condition on the function ν defined in Lemma 4.3 with respect to the message ϕ defined in eq. (4); this condition will be a crucial ingredient in the proofs in this section. Throughout, we consider the hard core model with some fixed vertex activity $\lambda > 0$. Since the potential function ϕ and the vertex activity λ are always going to be clear from the context, we will drop the corresponding suffixes for ease of notation.

Lemma 5.2. *The function $H(x) := \log(e^x \nu(e^x))$ is concave in the interval $[0, \infty)$.*

The proof of this lemma is somewhat technical and can be found in Appendix D.

To take advantage of Lemma 5.2, we define the function $\chi(d) := d \cdot \nu(d)$. Lemma 5.2 along with Jensen’s inequality then implies that, for any $d_i \geq 1$ and non-negative constants β_i summing up to 1, we have

$$\prod_{i=0}^{\ell-1} \chi(d_i)^{\beta_i} \leq \chi\left(\prod_{i=0}^{\ell-1} d_i^{\beta_i}\right). \quad (8)$$

We now prove the following analog of Lemma 3.3 for the case of spherically symmetric trees. In addition to the notation used in Lemma 3.3, we will also need the following: for $j \leq \ell$ we define $\Delta_{j,\ell} := (\prod_{i=j}^{\ell-1} d_i)^{1/(\ell-j)}$ (with the convention that $\Delta_{\ell,\ell} = 1$). Intuitively, $\Delta_{j,\ell}$ gives an estimate of the average arity of the subtree of depth $\ell - j$ rooted at a vertex at depth j . Thus, for example, $N(\rho, \ell) = \Delta_{0,\ell}^\ell$. Further, using eq. (8), we see that for $j \leq \ell - 1$,

$$\chi(d_j) \chi(\Delta_{j+1,\ell})^{\ell-j-1} \leq \chi(\Delta_{j,\ell})^{\ell-j}. \quad (9)$$

Lemma 5.3. *Let T be a locally finite spherically symmetric tree rooted at ρ . For $\ell \geq 1$, let C_ℓ be the cutset consisting of the vertices at distance exactly ℓ from ρ . Let $C' = C_{\ell+1}$ be the cutset comprising the children of vertices in C_ℓ . Consider two arbitrary boundary conditions σ and τ on $T_{\leq C'}$ which differ only on C' . With χ and $\Delta_{j,\ell}$ as defined above, we have*

$$|R_\rho(\sigma) - R_\rho(\tau)|^2 \leq c_2 \cdot \chi(\Delta_{0,\ell})^\ell,$$

where c_2 is a constant depending only upon λ (and the message ϕ).

Proof. The proof is similar in structure to the proof of Lemma 3.3. The main difference is the application of the more delicate concavity condition provided by Lemma 5.2 and eq. (9) to aggregate the decay obtained at the children of a vertex.

As before, for a vertex v in $T_{\leq C'}$, we will denote by T_v the subtree rooted at v and containing all the descendants of v , and by $R_v(\sigma)$ (respectively, $R_v(\tau)$) the occupation probability $R_v(\sigma, T_v)$ (respectively, $R_v(\tau, T_v)$) of the vertex v in the subtree T_v under the boundary condition σ (respectively, τ) restricted to T_v . Further, we will denote by C_v (respectively C'_v) the restriction of the cutset C (respectively, C') to T_v . We consider again the quantities

$$L := \min_{x \in [0, \lambda]} \phi'(x) = \frac{1}{\sqrt{\lambda(1+\lambda)}}, \text{ and } M := \phi(\lambda) - \phi(0) = \sinh^{-1}(\sqrt{\lambda}),$$

where the latter bounds are based on our specific message ϕ defined in eq. (4).

As in the proof of Lemma 3.3, we proceed by induction on the structure of T_ρ . We will show that for any vertex v in T_ρ which is at a distance $j \leq \ell$ from ρ , and thus has arity d_j , we have

$$|\phi(R_v(\sigma)) - \phi(R_v(\tau))|^2 \leq c_1 \chi(\Delta_{j, \ell})^{\ell-j}. \quad (10)$$

where $c_1 = M^2$. To get the claim of the lemma, we notice that since $\rho \notin C$, the form of the recurrence for the hard core model implies that both $R_\rho(\sigma)$ and $R_\rho(\tau)$ are in the interval $[0, \lambda]$. It then follows that $|R_v(\sigma) - R_v(\tau)| \leq \frac{1}{L} |\phi(R_v(\sigma)) - \phi(R_v(\tau))|$. Hence, taking $v = \rho$ in eq. (10) and then setting $c_2 = c_1/L^2$, the claim of the lemma follows.

We now proceed to prove eq. (10). The base case of the induction consists of vertices v which are either fixed by a boundary condition or which are in C_ℓ . In the first case, since the vertex is not in C_ℓ , we have $R_v(\sigma) = R_v(\tau)$ (since σ and τ differ only on $C_{\ell+1}$) and hence the claim is trivially true. In case $v \in C_\ell$, all the children of v must lie in $C_{\ell+1}$. The form of the recurrence of the hard core model then implies that both $R_v(\sigma)$ and $R_v(\tau)$ lie in the interval $[0, \lambda]$, so that we have

$$|\phi(R_v(\sigma)) - \phi(R_v(\tau))|^2 \leq (\phi(\lambda) - \phi(0))^2 = M^2, \text{ as required.}$$

We now proceed to the inductive case. Consider a vertex v at a distance $j \leq \ell - 1$ from ρ . Let v_1, v_2, \dots, v_{d_j} be the children of v , which satisfy eq. (10) by induction. Applying Lemma 3.2 (and noticing that $\nu(d) = \xi(d)$ and $d \cdot \nu(d) = \chi(d)$) followed by the induction hypothesis, we then have,

$$\begin{aligned} |\phi(R_v(\sigma)) - \phi(R_v(\tau))|^2 &\leq \nu(d_j) \sum_{i=1}^{d_j} |\phi(R_{v_i}(\sigma)) - \phi(R_{v_i}(\tau))|^2, \text{ using Lemma 3.2} \\ &\leq c_1 \chi(d_j) \chi(\Delta_{j+1, \ell})^{\ell-j-1}, \text{ using the induction hypothesis} \\ &\leq c_1 \chi(\Delta_{j, \ell})^{\ell-j}, \text{ using eq. (9).} \end{aligned}$$

This completes the induction. □

Proof of Theorem 5.1. Recall that $\chi(d) = d \cdot \nu(d)$. We then see from Lemma 4.3 that $\chi(d)$ is increasing in d and $\chi(\Delta_\rho) < 1$ for $\lambda < \lambda_c(\Delta_\rho)$. Note that $\Delta_{0, \ell} \leq \Delta_\rho$ for ℓ large enough, and hence the claim in the theorem follows from Lemma 5.3. □

Corollary 5.4. *Consider the family of bipartite graphs in which the “left” side has degree at most $(d_1 + 1)$ and the “right” side has degree at most $(d_2 + 1)$. Let $0 < \lambda < \lambda_c(\sqrt{d_1 d_2})$. Then,*

- *The hard core model with vertex activity λ exhibits strong spatial mixing on this family of graphs.*

- There is an FPTAS for the partition function of the hard core model with vertex activity λ for graphs in this family.

Notice that for this class of graphs, the above corollary improves upon Weitz's result [32] (which is valid only for $\lambda < \lambda_c(\max(d_1, d_2))$), as well as Theorem 1.3: the latter would require $\nu(\max(d_1, d_2))\sqrt{d_1 d_2} \leq 1$, while the requirement in the above corollary is equivalent to $\nu(\sqrt{d_1 d_2})\sqrt{d_1 d_2} \leq 1$.

Proof. The proof is similar in structure to the proofs of Theorem 1.1 and 1.3, except that the tighter Lemma 5.3 is used to analyze the decay of correlations in place of Lemma 3.3 used in those proofs.

We first prove that the hard core model with these parameters exhibits strong spatial mixing on this family of graphs. Let G be any graph from \mathcal{F} , and let v be any vertex in G , and consider any boundary conditions σ and τ in G which differ only at a distance of at least $\ell + 1$ from v . Consider the Weitz self-avoiding walk tree [32] $T_{SAW}(v, G)$ rooted at v . Without loss of generality, assume that v lies on the “left” side of the graph.

We observe that $T_{SAW}(v, G)$ is a subtree of a spherically symmetric tree \mathcal{T} in which the root has degree $d_1 + 1$, other vertices at even distance from the root have arity d_1 , and vertices at odd distance from the root have arity d_2 . Notice that we may, in fact, view $T_{SAW}(v, G)$ as just the tree \mathcal{T} by adding in the extra boundary condition that the vertices of \mathcal{T} that are not present in $T_{SAW}(v, G)$ are fixed to be “unoccupied”. This latter modification does not modify any of the occupation probabilities. With a slight abuse of notation, we refer to \mathcal{T} with these boundary conditions as $T_{SAW}(v, G)$. Further, as before, we denote the extra boundary conditions on $T_{SAW}(v, G)$ corresponding to the boundary condition σ (respectively, τ) on G by the same letter σ (respectively, τ). From Weitz's theorem, we then know that $R_v(\sigma, G) = R_v(\sigma, T_{SAW}(v, G))$ (respectively, $R_v(\tau, G) = R_v(\tau, T_{SAW}(v, G))$).

Now, let C_ℓ denote the cutset in $T_{SAW}(v, G)$ consisting of all vertices at distance ℓ from v , for some $\ell \geq 1$. Applying Lemma 5.3 and adjusting for effects due to the parity of ℓ as well as the possibly higher arity (at most $\max(d_1 + 1, d_2 + 1)$) at the root, we then have

$$|R_v(\sigma, G) - R_v(\tau, G)|^2 = |R_v(\sigma, T_{SAW}(v, G)) - R_v(\tau, T_{SAW}(v, G))|^2 \leq c_2 M_0^2 \chi(\Delta)^\ell \quad (11)$$

where $M_0 = \max(\chi(d_1 + 1), \chi(d_2 + 1)) / \chi(\Delta)$ and $\Delta = \sqrt{d_1 d_2}$ are constants, and c_2 is the constant in the statement of Lemma 5.3. Recalling that $\chi(\Delta) = \Delta \nu(\Delta)$, we see from Lemma 4.3 that $\chi(\Delta) < 1$ for $\lambda < \lambda_c(\Delta)$. Combining with eq. (11), this establishes SSM in the regime $\lambda < \lambda_c(\sqrt{d_1 d_2})$ and hence proves the first part of the corollary.

The proof of the algorithmic part of the corollary is virtually identical to the proof of the algorithmic part of Theorem 1.3. As before, the depth ℓ up to which the self-avoiding walk tree needs to be explored in order to additively approximate R_v up to an error of at most μ is $O(1) + O(\log(1/\mu)) = O(\log n + \log(1/\mu))$, and hence the running time $\Delta^{O(\ell)}$ is still polynomial in n and $1/\mu$. \square

5.2 Branching factor and uniqueness of the Gibbs measure on general trees

We close with an application of our results to finding thresholds for the uniqueness of the Gibbs measure of the hard core model on locally finite infinite trees. Our bounds will be stated in terms of the *branching factor*, which, as indicated above, has been shown to be the appropriate parameter for establishing phase transition thresholds for symmetric models such as the ferromagnetic Ising, Potts and Heisenberg models [17, 25]. We begin with a general definition of the notion of uniqueness of Gibbs measure (see, for example, the survey article of Mossel [20]). Let T be a locally finite infinite tree rooted at ρ , and let C be a cutset in T . Consider the hard core model with vertex activity $\lambda > 0$ on T . We define the *discrepancy* $\delta(C)$ of C as follows. Let σ and τ be boundary conditions in T which fix the state of the vertices on C , but not of any vertex in $T_{<C}$. Then, $\delta(C)$ is the maximum over all such σ and τ of the quantity $R_\rho(\sigma, T) - R_\rho(\tau, T)$.

Definition 5.1. (Uniqueness of Gibbs measure). The hard core model with vertex activity $\lambda > 0$ is said to exhibit *uniqueness of Gibbs measure* on T if there exists a sequence of cutsets $(B_i)_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} d(\rho, B_i) \rightarrow \infty$ and such that $\lim_{i \rightarrow \infty} \delta(B_i) = 0$.

Remark 5.2. Our definition of *uniqueness* here is similar in form to those used by Lyons [17] and Pemantle and Steif [25]. Notice, however, that the recurrence for the hard core model implies that the discrepancy is “monotonic” in the sense that if cutsets C and D are such that $C < D$ (i.e., every vertex in D is the descendant of some vertex in C) then $\delta(C) > \delta(D)$. This ensures that the choice of the sequence $(B_i)_{i=1}^\infty$ in the definition above is immaterial. For example, uniqueness is defined by Mossel [20] in terms of the cutsets C_ℓ consisting of vertices at distance exactly ℓ from the root. However, the above observation shows that for the hard core model, Mossel’s definition is equivalent to the one presented here.

We now define the notion of the branching factor of an infinite tree.

Definition 5.2. (Branching factor [17, 18, 25]). Let T be an infinite tree. The branching factor $\text{br}(T)$ is defined as follows:

$$\text{br}(T) := \inf \left\{ b > 0 \mid \inf_C \sum_{v \in C} b^{-|v|} = 0 \right\},$$

where the second infimum is taken over all cutsets C .

To clarify this definition, we consider some examples. If T is a d -ary tree, then $\text{br}(T) = d$. Further, by taking the second infimum over the cutsets C_ℓ of vertices at distance ℓ from the root, it is easy to see that the branching factor is never more than the connective constant. Further, Lyons [18] observes that in the case of spherically symmetric trees, one can define the branching factor as $\liminf_{\ell \rightarrow \infty} N(\rho, \ell)^\ell$.

We are now ready to state and prove our results on the uniqueness of the hard-core model on general trees.

Theorem 5.5. *Let T be an infinite tree rooted at ρ with branching factor b . The hard core model with vertex activity $\lambda > 0$ exhibits uniqueness of Gibbs measure on T if at least one of the following conditions is satisfied:*

- $\lambda < \lambda_c(b + 1)$.
- T is a spherically symmetric tree and $\lambda < \lambda_c(b)$.

Notice that the result for spherically symmetric trees is tight. We conjecture, however, that the general case can also be improved to $\lambda_c(b)$.

Proof of Theorem 5.5. We first consider the case of general trees. We will apply Lemma 3.3 specialized to the message in eq. (4). From the proof of item 2 of Theorem 1.1, we recall that when $\lambda < \lambda_c(b + 1)$, $\alpha := \sup_{d \geq 1} \nu(d) < \frac{1}{b}$, while $\alpha' := \inf_{d \geq 1} \nu(d) = \nu(1)$ is a positive constant. Suppose $\alpha = \frac{1}{b(1+\epsilon)}$ for some $\epsilon > 0$. Applying Lemma 3.3 to an arbitrary cutset C , we then get

$$\delta(C)^2 \leq M_0 \sum_{v \in C} [(1 + \epsilon)b]^{-|v|}, \tag{12}$$

where M_0 is a constant. Since $b(1 + \epsilon) > \text{br}(T)$, the definition of $\text{br}(T)$ implies that we can find a sequence $(B_i)_{i=1}^\infty$ of cutsets such that

$$\lim_{i \rightarrow \infty} \sum_{v \in B_i} [(1 + \epsilon)b]^{-|v|} = 0.$$

Further, such a sequence must satisfy $\lim_{i \rightarrow \infty} d(\rho, B_i) = \infty$, since otherwise the limit above would be positive. Combining with eq. (12), this shows that $\lim_{i \rightarrow \infty} \delta(B_i) = 0$, which completes the proof of this case.

For the case of the spherically symmetric tree, we will use Lyons's observation [18] that for such trees

$$b = \text{br}(T) = \liminf_{n \rightarrow \infty} N(v, \ell)^{1/\ell}. \quad (13)$$

Again, from the fact that $\lambda < \lambda_c(b)$, and the properties of the function ν proved in Lemma 4.3, we see that there exists an $\epsilon > 0$ such that for $b_1 < b(1 + \epsilon)$,

$$\chi(b_1) = b_1 \cdot \nu(b_1) \leq 1 - \epsilon. \quad (14)$$

Further, eq. (13) implies that there exists a strictly increasing sequence $(\ell_i)_{i=1}^{\infty}$, such that the cutsets C_{ℓ_i} (of vertices at distance exactly ℓ_i from the root) satisfy

$$|C_{\ell_i}|^{1/\ell_i} = N(\rho, \ell_i)^{1/\ell_i} = \Delta_{0, \ell_i} < b(1 + \epsilon), \quad (15)$$

where $\Delta_{0, \ell} = N(\rho, \ell)^{1/\ell}$ is as defined above. Applying Lemma 5.3 to a cutset C_{ℓ_i} , we then have

$$\delta(C_{\ell_i})^2 \leq M_1 \chi(\Delta_{0, \ell_i})^{\ell_i},$$

where M_1 is a constant. Combining with eqs. (14) and (15), we then see that $\lim_{i \rightarrow \infty} \delta(C_{\ell_i}) = 0$, which completes the proof. \square

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A Monotonicity properties of ν_λ

In this section, we prove Lemma 4.3.

Proof of Lemma 4.3. The expression for $\nu_\lambda(d)$ follows by using the definitions $\nu_\lambda(d) = \xi(d) = \Xi(d, \tilde{x}_\lambda(d))$ and then using eq. (5) from Lemma 4.2. To prove the rest of item 1, we observe that when $\lambda = \lambda_c(d) = \frac{d^d}{(d-1)^{d+1}}$, $\tilde{x}_\lambda(d) = \frac{1}{d-1}$ is the unique solution to eq. (5) (indeed, the potential function ϕ was chosen in [15] to achieve this condition). Plugging this into the definition of ν_λ , we see that $\nu_\lambda(d) = \frac{1}{d}$, as claimed.

We now proceed to prove the other two items. For item 2, we compute the derivative $\nu'_\lambda(d)$ using the chain rule, and show that it is positive (for ease of notation, we denote $\tilde{x}_\lambda(d)$ as \tilde{x}):

$$\begin{aligned}
\nu'_\lambda(d) &= \Xi^{(1,0)}(d, \tilde{x}) + \Xi^{(0,1)}(d, \tilde{x}) \frac{d\tilde{x}}{dd} \\
&= \Xi^{(1,0)}(d, \tilde{x}), \text{ since } \Xi^{(0,1)}(d, \tilde{x}) = 0 \text{ by definition of } \tilde{x} \\
&= \Xi(d, \tilde{x}) \left[\frac{1}{d} - \frac{\log(1 + \tilde{x})}{1 + f_{d,\lambda}(\tilde{x})} \right] \\
&= \frac{\Xi(d, \tilde{x})}{d\tilde{x}} [\tilde{x} - \log(1 + \tilde{x})] > 0, \text{ since } \tilde{x} > 0 \text{ for } d > 0.
\end{aligned} \tag{16}$$

Here, we use $1 + f_{d,\lambda}(\tilde{x}) = d\tilde{x}$ to get the last equality.

To prove item 3, we notice first that for fixed d , $\nu_\lambda(d) = \frac{d\tilde{x}_\lambda(d)-1}{1+\tilde{x}_\lambda(d)}$ increases with $\tilde{x}_\lambda(d)$. Thus, we only need to establish that $\tilde{x}_\lambda(d)$ is increasing in λ when d is fixed. We first observe that eq. (5) implies that

$$\lambda = (d\tilde{x}_\lambda(d) - 1)(1 + \tilde{x}_\lambda(d)). \tag{17}$$

Since $\lambda > 0$, we must have $d\tilde{x}_\lambda(d) - 1 \geq 0$, in which case the right hand side of eq. (17) increases with $\tilde{x}_\lambda(d)$. This shows that as λ increases, so must $\tilde{x}_\lambda(d)$. This completes the proof. \square

B Description of numerical results

In this section, we describe how the numerical bounds presented in Table 1 were obtained. All of the bounds are direct applications of Theorem 1.3 using published upper bounds on the connective constant for the appropriate graph. For the Cartesian lattices $\mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^4, \mathbb{Z}^5$ and \mathbb{Z}^6 , and the triangular lattice \mathbb{T} , the exact connective constant is not known, but rigorous upper and lower bounds are available in the literature [19, 31]. In a recent breakthrough, Duminil-Copin and Smirnov [4] rigorously established that connective constant of the honeycomb lattice \mathbb{H} is $\sqrt{2 + \sqrt{2}}$. In order to apply Theorem 1.3 for a given lattice of maximum degree $d + 1$ and connective constant Δ , we choose a given λ , solve eq. (5) for $\tilde{x}_\lambda(d)$ (this is a polynomial equation of degree $d + 1$, and hence can be solved efficiently for all these lattices which have small degree), and then compute $\nu_\lambda(d)\Delta$ to check if it is less than 1. The monotonicity of $\nu_\lambda(d)$ in λ (Lemma 4.3) then allows us to search easily for the best possible λ . Each of these computations for a particular lattice takes a few seconds on *Mathematica* [33] running on a laptop with a 2.8 GHz Intel[®] Core[™] 2 Duo CPU, and 4GB of RAM.

As we pointed out in the introduction, any improvement in the connective constant of a lattice (or that of its corresponding Weitz SAW tree) will immediately lead to an improvement in our bounds. We demonstrate this here in the special case of \mathbb{Z}^2 , by using a tighter combinatorial analysis of the connective constant for the Weitz SAW tree of this lattice to obtain a somewhat better bound than the one presented in Table 1. The Weitz SAW tree adds additional boundary conditions to the SAW tree of the lattice, and hence allows a smaller number of self-avoiding walks, and therefore can have a smaller connective constant than that of the lattice itself. Further, the proof of Theorem 1.3 is only in terms of the SAW tree, and hence the bounds there clearly hold if the connective constant of the SAW tree is used in place of the connective constant of the lattice.

To upper bound the connective constant of the Weitz SAW tree, we use the method of *finite memory self-avoiding walks* [19]—these are walks which are constrained only to not have cycles of length up to some finite length L . Clearly, the number of such walks of any given length ℓ upper bounds $N(v, \ell)$. In order to bring the boundary conditions on the Weitz SAW tree into play, we further enforce the constraint that the walk is not allowed to make any moves which will land it in a vertex fixed to be “unoccupied” by Weitz’s boundary conditions. Such a walk can be in one of a finite number k (depending upon L) of states, such that the number of possible moves it can make to state j while respecting the above constraints is some finite number M_{ij} . The $k \times k$ matrix $M = (M_{ij})_{i,j \in [k]}$ is called the *branching matrix* [27]. We therefore get $N(v, \ell) \leq e_1^T M^\ell \mathbf{1}$, where $\mathbf{1}$ denotes the all 1’s vector, and e_1 denotes the co-ordinate vector for the state of the zero-length walk.

Since the entries of M are non-negative, the Perron-Frobenius theorem implies that one of the maximum magnitude eigenvalues of the matrix M is a positive real number γ . Using the Jordan canonical form, one then sees that

$$\limsup_{\ell \rightarrow \infty} N(v, \ell)^{1/\ell} \leq \limsup_{\ell \rightarrow \infty} (e_1^T M^\ell \mathbf{1})^{1/\ell} \leq \max(\gamma, 1).$$

Hence, the largest real eigenvalue γ of M gives a bound on the connective constant of the Weitz SAW tree. Using the matrix M corresponding to walks in which cycles of length at most $L = 14$ are avoided, we get that the connective constant of the Weitz SAW tree is at most 2.5384. Using this bound, and applying Theorem 1.3 as described above, we get the bounds 2.614 and 2.185 for Δ and λ respectively, in the notation of the table.

C Proofs omitted from Section 3

Proof of Lemma 3.1. Define $F(t) = f_{d,\lambda}^\phi(t\mathbf{x} + (1-t)\mathbf{y})$ for $t \in [0, 1]$. By the scalar mean value theorem applied to F , we have

$$f_{d,\lambda}^\phi(\mathbf{x}) - f_{d,\lambda}^\phi(\mathbf{y}) = F(1) - F(0) = F'(s), \text{ for some } s \in [0, 1].$$

Let ψ denote the inverse of the message ϕ : the derivative of ψ is given by $\psi'(y) = \frac{1}{\Phi(\psi(y))}$, where Φ is the derivative of ϕ . We now define the vector \mathbf{z} by setting $z_i = \psi(sx_i + (1-s)y_i)$ for $1 \leq i \leq d$. We then have

$$\begin{aligned} \left| f_{d,\lambda}^\phi(\mathbf{x}) - f_{d,\lambda}^\phi(\mathbf{y}) \right| &= |F'(s)| = \left| \left\langle \nabla f_{\lambda,d}^\phi(s\mathbf{x} + (1-s)\mathbf{y}), \mathbf{x} - \mathbf{y} \right\rangle \right| \\ &= \Phi(f_{d,\lambda}(\mathbf{z})) \left| \sum_{i=1}^d \frac{x_i - y_i}{\Phi(z_i)} \frac{\partial f_{d,\lambda}}{\partial z_i} \right|, \quad \text{using the chain rule} \\ &\leq \Phi(f_{d,\lambda}(\mathbf{z})) \sum_{i=1}^d \frac{|y_i - x_i|}{\Phi(z_i)} \left| \frac{\partial f_{d,\lambda}}{\partial z_i} \right|, \quad \text{as claimed.} \end{aligned}$$

We recall that for simplicity, we are using here the somewhat non-standard notation $\frac{\partial f}{\partial z_i}$ for the value of the partial derivative $\frac{\partial f}{\partial R_i}$ at the point $\mathbf{R} = \mathbf{z}$. \square

Proof of Lemma 3.3. We first set up some notation. For a vertex v in $T_{\leq C'}$, we will denote by T_v the subtree rooted at v and containing all the descendants of v . By a slight abuse of notation, we denote by $R_v(\sigma)$ (respectively, $R_v(\tau)$) the occupation probability $R_v(\sigma, T_v)$ (respectively, $R_v(\tau, T_v)$) of the vertex v in the subtree T_v under the boundary condition σ (respectively, τ) restricted to T_v . Further, we will denote by C_v (respectively C'_v) the restriction of the cutset C (respectively, C') to T_v . We also define the following two quantities related to the message ϕ :

$$L := \min_{x \in [0, \lambda]} \phi'(x), \text{ and } M := \phi(\lambda) - \phi(0)$$

Notice that both these quantities are finite and positive because of the constraints in the definition of a message.

By induction on the structure of T_ρ , we will now show that for any vertex v in T_ρ which is at a distance δ_v from ρ , and has arity d_v , one has

$$|\phi(R_v(\sigma)) - \phi(R_v(\tau))|^q \leq \frac{c_1 \xi(d_v)}{\alpha'} \sum_{u \in C_v} \alpha^{|u| - \delta_v}, \quad (18)$$

where $c_1 = M^q$. To get the claim of the lemma, we notice that since $\rho \notin C$, the form of the recurrence for the hard core model implies that both $R_\rho(\sigma)$ and $R_\rho(\tau)$ are in the interval $[0, \lambda]$. It then follows that $|R_v(\sigma) - R_v(\tau)| \leq \frac{1}{L} |\phi(R_v(\sigma)) - \phi(R_v(\tau))|$. Hence, taking $v = \rho$ in eq. (18) and then setting $c_0 = c_1/L^q$, the claim of the lemma follows.

We now proceed to prove eq. (18). The base case of the induction consists of vertices v which are either of arity 0 or which are in C . In the first case (which includes the case where v is fixed by the boundary condition), we clearly have $R_v(\sigma) = R_v(\tau)$, and hence the claim is trivially true. In the second case, we have $C_v = \{v\}$, and all the children of v must lie in C' . The form of the recurrence of the hard core model then implies that both $R_v(\sigma)$ and $R_v(\tau)$ lie in the interval $[0, \lambda]$, so that we have

$$|\phi(R_v(\sigma)) - \phi(R_v(\tau))|^q \leq (\phi(\lambda) - \phi(0))^q = M^q \leq \frac{M^q \xi(d_v)}{\alpha'}, \text{ since } \alpha' \leq \xi(d_v).$$

We now proceed to the inductive case. Let v_1, v_2, \dots, v_{d_v} be the children of v , which satisfy eq. (18) by induction. Applying Lemma 3.2 followed by the induction hypothesis, we then have,

$$\begin{aligned} |\phi(R_v(\sigma)) - \phi(R_v(\tau))|^q &\leq \xi(d_v) \sum_{i=1}^{d_v} |\phi(R_{v_i}(\sigma)) - \phi(R_{v_i}(\tau))|^q, \text{ using Lemma 3.2} \\ &\leq \frac{c_1 \xi(d_v)}{\alpha'} \sum_{i=1}^{d_v} \xi(d_{v_i}) \sum_{u \in C_{v_i}} \alpha^{|u| - \delta_{v_i}}, \text{ using the induction hypothesis} \\ &\leq \frac{c_1 \xi(d_v)}{\alpha'} \sum_{u \in C_v} \alpha^{|u| - \delta_v}, \text{ using } \xi(d_{v_i}) \leq \alpha \text{ and } \delta_{v_i} = \delta_v + 1. \end{aligned}$$

This completes the induction. □

D Proof of Lemma 5.2

Proof of Lemma 5.2. Define the function L as

$$L(x) := \frac{\nu'(x)}{\nu(x)}.$$

Notice that by item 2 in Lemma 4.3, $L(x) \geq 0$ for $x \geq 1$. In particular, $L(e^z) \geq 0$ for $z \geq 0$. We can now write the second derivative of H as follows:

$$H''(x) = e^z L(e^x) \left(1 + e^x \frac{L'(e^x)}{L(e^x)} \right).$$

Since $L(e^z) \geq 0$ for non-negative z as observed above, in order to show that H is concave for $x \geq 0$, we only need to show that

$$d \frac{L'(d)}{L(d)} \leq -1 \text{ for all } d \geq 1 \tag{19}$$

We now proceed to analyze the function L . We recall that by the definition of ν following Lemma 4.2, we have $\nu(d) = \Xi(d, \tilde{x}(d))$, where Ξ is as defined in that lemma and $\tilde{x}(d)$ is the unique non-negative solution of $\Xi^{(0,1)}(d, x) = 0$. For ease of notation, we denote $\tilde{x}(d)$ by \tilde{x} in what follows. We now have

$$\begin{aligned} L(d) &= \frac{\nu'(d)}{\nu(d)} = \frac{\nu'(d)}{\Xi(d, \tilde{x})} \\ &= \frac{1}{d\tilde{x}} [\tilde{x} - \log(1 + \tilde{x})], \text{ using the derivation of eq. (16) above.} \end{aligned}$$

Before proceeding further, we evaluate $\tilde{x}' := \frac{d\tilde{x}}{dd}$. Since $d\tilde{x} = 1 + f_d(\tilde{x})$, differentiation gives

$$\tilde{x} + d\tilde{x}' = -f_d(\tilde{x}) \left[\frac{d\tilde{x}'}{1 + \tilde{x}} + \log(1 + \tilde{x}) \right]$$

which in turn yields (using $d\tilde{x} = 1 + f_d(\tilde{x})$)

$$\tilde{x}' = -\frac{(1 + \tilde{x}) [f_d(\tilde{x}) \log(1 + \tilde{x}) + \tilde{x}]}{d(1 + d)\tilde{x}}.$$

Since $\tilde{x} \geq 0$, this shows that $\tilde{x}' \leq 0$. We now differentiate $L(d)$ to get

$$\frac{L'(d)}{L(d)} = -\frac{1}{d} + \tilde{x}' \left[\frac{(1 + \tilde{x}) \log(1 + \tilde{x}) - \tilde{x}}{\tilde{x}(1 + \tilde{x})(\tilde{x} - \log(1 + \tilde{x}))} \right].$$

Since $\tilde{x} \geq 0$, we have both $(\tilde{x} - \log(1 + \tilde{x})) \geq 0$ and $(1 + \tilde{x}) \log(1 + \tilde{x}) - \tilde{x} \geq 0$. Combining this with the observation above that $\tilde{x}' \leq 0$, this shows that $d \frac{L'(d)}{L(d)} \leq -1$ for $d \geq 1$. As observed in the discussion following eq. (19), this implies that H is concave for $x \geq 0$. \square