

# A Proof of the CSP Dichotomy Conjecture

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## Abstract

Many natural combinatorial problems can be expressed as constraint satisfaction problems. This class of problems is known to be NP-complete in general, but certain restrictions on the form of the constraints can ensure tractability. The standard way to parameterize interesting subclasses of the constraint satisfaction problem is via finite constraint languages. The main problem is to classify those subclasses that are solvable in polynomial time and those that are NP-complete. It was conjectured that if a constraint language has a weak near-unanimity polymorphism then the corresponding constraint satisfaction problem is tractable, otherwise it is NP-complete.

In the paper we present an algorithm that solves Constraint Satisfaction Problem in polynomial time for constraint languages having a weak near unanimity polymorphism, which proves the remaining part of the conjecture.

## 1 Introduction

The *Constraint Satisfaction Problem (CSP)* is the problem of deciding whether there is an assignment to a set of variables subject to some specified constraints. Formally, the *Constraint Satisfaction Problem* is defined as a triple  $\langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$ , where

- $\mathbf{X} = \{x_1, \dots, x_n\}$  is a set of variables,
- $\mathbf{D} = \{D_1, \dots, D_n\}$  is a set of the respective domains,
- $\mathbf{C} = \{C_1, \dots, C_m\}$  is a set of constraints,

where each variable  $x_i$  can take on values in the nonempty domain  $D_i$ , every *constraint*  $C_j \in \mathbf{C}$  is a pair  $(t_j, \rho_j)$  where  $t_j$  is a tuple of variables of length  $m_j$ , called the *constraint scope*, and  $\rho_j$  is an  $m_j$ -ary relation on the corresponding domains, called the *constraint relation*.

The question is whether there exists a *solution* to  $\langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$ , that is a mapping that assigns a value from  $D_i$  to every variable  $x_i$  such that for each constraint  $C_j$  the image of the constraint scope is a member of the constraint relation.

In this paper we consider only CSP over finite domains. The general CSP is known to be NP-complete [46, 50]; however, certain restrictions on the allowed form of constraints involved may ensure tractability (solubility in polynomial time) [28, 38, 39, 41, 17, 22]. Below we provide a formalization to this idea.

To simplify the formulation of the main result we assume that the domain of every variable is a finite set  $A$ . Later we will assume that the domain of every variable is a unary relation from the constraint language  $\Gamma$  (see below). By  $R_A$  we denote the set of all finitary relations on  $A$ , that is, subsets of  $A^m$  for some  $m$ . Thus, all the constraint relations are from  $R_A$ .

For a set of relations  $\Gamma \subseteq R_A$  by  $\text{CSP}(\Gamma)$  we denote the Constraint Satisfaction Problem where all the constraint relations are from  $\Gamma$ . The set  $\Gamma$  is called a *constraint language*. Another way to formalize the Constraint Satisfaction Problem is via conjunctive formulas. Every  $h$ -ary relation on  $A$  can be viewed as a predicate, that is, a mapping  $A^h \rightarrow \{0, 1\}$ . Suppose  $\Gamma \subseteq R_A$ , then  $\text{CSP}(\Gamma)$  is the following decision problem: given a formula

$$\rho_1(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \rho_s(v_{s,1}, \dots, v_{s,n_s}),$$

where  $\rho_1, \dots, \rho_s \in \Gamma$ , and  $v_{i,j} \in \{x_1, \dots, x_n\}$  for every  $i, j$ ; decide whether this formula is satisfiable.

It is well known that many combinatorial problems can be expressed as  $\text{CSP}(\Gamma)$  for some constraint language  $\Gamma$ . Moreover, for some sets  $\Gamma$  the corresponding decision problem can be solved in polynomial time; while for others it is NP-complete. It was conjectured that  $\text{CSP}(\Gamma)$  is either in P, or NP-complete [29].

**Conjecture 1.** *Suppose  $\Gamma \subseteq R_A$  is a finite set of relations. Then  $\text{CSP}(\Gamma)$  is either solvable in polynomial time, or NP-complete.*

We say that an operation  $f: A^n \rightarrow A$  *preserves* the relation  $\rho \in R_A$  of arity  $m$  if for any tuples  $(a_{1,1}, \dots, a_{1,m}), \dots, (a_{n,1}, \dots, a_{n,m}) \in \rho$  the tuple  $(f(a_{1,1}, \dots, a_{n,1}), \dots, f(a_{1,m}, \dots, a_{n,m}))$  is in  $\rho$ . We say that an operation *preserves a set of relations*  $\Gamma$  if it preserves every relation in  $\Gamma$ . A mapping  $f: A \rightarrow A$  is called an *endomorphism* of  $\Gamma$  if it preserves  $\Gamma$ .

**Theorem 1.1.** [37] *Suppose  $\Gamma \subseteq R_A$ . If  $f$  is an endomorphism of  $\Gamma$ , then  $\text{CSP}(\Gamma)$  is polynomially reducible to  $\text{CSP}(f(\Gamma))$  and vice versa, where  $f(\Gamma)$  is a constraint language with domain  $f(A)$  defined by  $f(\Gamma) = \{f(\rho) : \rho \in \Gamma\}$ .*

A constraint language is a *core* if every endomorphism of  $\Gamma$  is a bijection. It is not hard to show that if  $f$  is an endomorphism of  $\Gamma$  with minimal range, then  $f(\Gamma)$  is a core. Another important fact is that we can add all singleton unary relations to a core constraint language without increasing the complexity of its CSP. By  $\sigma_{=a}$  we denote the unary relation  $\{a\}$ .

**Theorem 1.2.** [17] *Let  $\Gamma \subseteq R_A$  be a core constraint language, and  $\Gamma' = \Gamma \cup \{\sigma_{=a} \mid a \in A\}$ . Then  $\text{CSP}(\Gamma')$  is polynomially reducible to  $\text{CSP}(\Gamma)$ .*

Therefore, to prove Conjecture 1 it is sufficient to consider only the case when  $\Gamma$  contains all unary singleton relations. In other words, all the predicates  $x = a$ , where  $a \in A$ , are in the constraint language  $\Gamma$ .

In [54] Schaefer classified all tractable constraint languages over two-element domain. In [19] Bulatov generalized the result for three-element domain. His dichotomy theorem was formulated in terms of a  $G$ -set. Later, the dichotomy conjecture was formulated in several different forms (see [17]).

The result of McKenzie and Maróti [47] allows us to formulate the dichotomy conjecture in the following nice way. An operation  $f$  on a set  $A$  is called a *weak near-unanimity operation* (WNU) if it satisfies  $f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, x, \dots, x, y)$  for all  $x, y \in A$ . An operation  $f$  is called *idempotent* if  $f(x, x, \dots, x) = x$  for all  $x \in A$ .

**Conjecture 2.** *Suppose  $\Gamma \subseteq R_A$  is a finite set of relations. Then  $\text{CSP}(\Gamma)$  can be solved in polynomial time if there exists a WNU preserving  $\Gamma$ ;  $\text{CSP}(\Gamma)$  is NP-complete otherwise.*

It is not hard to see that the existence of a WNU preserving  $\Gamma$  is equivalent to the existence of a WNU preserving a core of  $\Gamma$ , and also equivalent to the existence of an idempotent WNU preserving the core. Hence, Theorems 1.1 and 1.2 imply that it is sufficient to prove Conjecture 2 for a core and an idempotent WNU.

One direction of this conjecture follows from [47].

**Theorem 1.3.** [47] *Suppose  $\Gamma \subseteq R_A$  and  $\{\sigma_{=a} \mid a \in A\} \subseteq \Gamma$ . If there exists no WNU preserving  $\Gamma$ , then  $\text{CSP}(\Gamma)$  is NP-complete.*

The dichotomy conjecture was proved for many special cases: for CSPs over undirected graphs [34], for CSPs over digraphs with no sources or sinks [4], for constraint languages containing all unary relations [18], and many others. More information about the algebraic approach to CSP can be found in [6].

In this paper we present an algorithm that solves  $\text{CSP}(\Gamma)$  in polynomial time if  $\Gamma$  is preserved by an idempotent WNU, and therefore prove the dichotomy conjecture.

**Theorem 1.4.** *Suppose  $\Gamma \subseteq R_A$  is a finite set of relations. Then  $\text{CSP}(\Gamma)$  can be solved in polynomial time if there exists a WNU preserving  $\Gamma$ ;  $\text{CSP}(\Gamma)$  is NP-complete otherwise.*

Another proof of the dichotomy conjecture was announced by Andrei Bulatov [20, 21]. Even though both algorithms appeared at the same time, they are significantly different. Bulatov's algorithm uses full strength of the few subpowers algorithm [35], uses Maroti's trick for trees on top of Mal'tsev [48], while this one just checks some local consistency and solves linear equations over prime fields. Also Bulatov's algorithm works for infinite constraint languages, which is not the case for the algorithm presented in this paper. But its slight modification works even for infinite constraint languages [63].

The paper is organized as follows. In Section 2 we explain the algorithm informally and give an example showing how the algorithm works for a system of linear equations in  $\mathbb{Z}_4$ . In Section 3 we give main definitions, in Section 4 we give a formal description of the algorithm showing a pseudocode for most functions and explain the meaning of every function. In Section 5 we formulate all theorems that are necessary to prove the correctness of the algorithm. Then, we prove that on every algebra (domain) with a WNU operation there exists a subuniverse of one of four types, which is the main ingredient of the algorithm. Additionally, in this section we prove that some functions of the algorithm work properly and the algorithm actually works in polynomial time.

In Section 6 we give the remaining definitions. The proof of the main theorems is divided into three sections. In Section 7 we study properties of subuniverses of each of four types (absorbing, central, PC, and linear subuniverses). In Section 8 we prove all the auxiliary statements, and in the last section we prove the main theorems of this paper formulated in Section 5.

In Section 10, we discuss open questions and consequences of this result. In particular, we consider generalizations of the CSP such as Valued CSP, Infinite Domain CSP, Quantified CSP, Promise CSP and so on.

## 2 Outline of the algorithm

In this section we give an informal description of the algorithm and show how it works for a system of linear equations in  $\mathbb{Z}_4$ . The algorithm is based on the following three ingredients:

- Each domain has either one of three kinds of proper strong subsets (absorbing, central, polynomially complete) or an equivalence relation modulo which the domain is essentially a product of prime fields (Theorem 5.1).
- If a sufficient level of consistency (cycle consistency + irreducibility - see Section 3.6) is enforced, then we do not lose all the solutions when we reduce the domain to a proper strong subset (that is, if the original instance has a solution, then the reduced instance has a solution as well), which is guaranteed by Theorems 5.5 and 5.6.

- If we cannot reduce the domain in such a way, we are left with an instance whose each domain has an equivalence relation modulo which it is a product of prime fields, and all relations are affine subspaces. Now we have:

$A$  = the set of all solutions of the instance factorized by the equivalences;

$B$  = the set of all solutions of the factorized instance (where all domains and relations are factorized).

Both  $A$  and  $B$  are affine subspaces,  $A \subseteq B$ . We would like to know whether  $A$  is empty, what we can efficiently compute is  $B$  (using Gaussian elimination). The algorithm gradually makes  $B$  smaller (of smaller dimension), while maintaining the property  $A \subseteq B$ .

First, for some solution from  $B$  we check whether  $A$  has the same solution, which can be done by a recursive call of the algorithm for smaller domains. If  $A$  has it then we are done. If  $A$  has not then  $A \neq B$ . In this case we can make (see Theorem 5.7) the instance weaker maintaining the property  $A' \subsetneq B$  (here  $A'$  is  $A$  for the weaker instance) until the moment when

1.  $A'$  is a subspace of  $B$  of codimension one,
2. or  $A = A' = \emptyset$ ,
3. or the obtained instance is not linked (it splits into several instances on smaller domains, hence  $A'$  can be calculated using recursion).

In (1) and (2)  $A'$  can be computed by linearly many recursive calls of the algorithm for smaller domains. In fact,  $A'$  can be defined by a linear equation  $c_1x_1 + \dots + c_hx_h = c_0$  in a prime field  $\mathbb{Z}_p$ . Then the coefficients  $c_0, c_1, \dots, c_h$  can be learned (up to a multiplicative constant) by  $(p \cdot h + 1)$  queries of the form “ $(a_1, \dots, a_h) \in A'$ ?” (see Subsection 4.3 for more details). To check each query we just need to call the algorithm recursively for the smaller domains that are the equivalence classes corresponding to  $a_1, \dots, a_h$ .

We update  $B = A'$ , return back to the original instance and continue tightening  $B$ . We eventually stop when  $B = A$ , which gives us the answer to our question: if  $B \neq \emptyset$  then the original instance has a solution, if  $B = \emptyset$  then it has no solutions.

We demonstrate the work of the algorithm on a system of linear equations in  $\mathbb{Z}_4$ :

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 + x_4 = 2 \\ \quad \quad \quad x_1 + x_2 = 2 \\ \quad \quad \quad x_1 + x_2 + 2x_4 = 2 \end{cases} \quad (1)$$

All the relations (equations) are invariants of the WNU  $x_1 + \dots + x_5 \pmod{4}$ , therefore, this system of equations is an instance of  $\text{CSP}(\Gamma)$ , where  $\Gamma$  is the set of all relations of arity at most 4 preserved by the WNU. Hence, we can apply the algorithm.

First,  $\mathbb{Z}_4$  does not have a proper strong subset, which is why for every domain there should be an equivalence relation modulo which it is just a product of prime fields. In our example it is the modulo 2 equivalence relation.

We factorize our instance modulo 2, and obtain a system of linear equations in  $\mathbb{Z}_2$ , where  $x'_i = x_i \pmod{2}$  for every  $i$ .

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ \quad \quad \quad x'_1 + x'_2 = 0 \\ \quad \quad \quad x'_1 + x'_2 = 0 \end{cases} \quad (2)$$

Using Gaussian elimination we solve this system of equations in a field, choose independent variables  $x'_1$  and  $x'_3$ , and write the general solution (the set  $B$  in the informal description):  $x'_1 = x'_1, x'_2 = x'_1, x'_3 = x'_3, x'_4 = x'_1 + x'_3$ .

We choose any solution from  $B$ . Let it be  $(0, 0, 0, 0)$  for  $x'_1 = x'_3 = 0$ . Then we check whether (1) has a solution corresponding to  $(0, 0, 0, 0)$  by restricting every domain to the set  $\{0, 2\}$  ( $x_i \bmod 2 = 0$ ). We recursively call the algorithm for smaller domain and find out that (1) has no solutions inside  $\{0, 2\}$ . This means that  $(0, 0, 0, 0)$  does not belong to  $A$  from the informal description, therefore  $A \subsetneq B$ .

Then we try to make the instance weaker so that  $A' \subsetneq B$ , where  $A'$  is the intersection of  $B$  with the set of all solutions of the new instance factorized by the equivalences. Let us remove the last equation from (1) to obtain a new solution set  $A'$ .

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + x_2 = 2 \end{cases} \quad (3)$$

Again, by solving an instance on the 2-element domain  $\{0, 2\}$  we find out that (3) has no solutions corresponding to  $(0, 0, 0, 0)$ . Therefore, we have  $A' \subsetneq B$ .

We need to check that if we remove one more equation from (3), then we get  $A' = B$ . Thus, for every weaker instance we need to check that for any  $a_1, a_3 \in \mathbb{Z}_2$  there exists a solution corresponding to  $(x'_1, x'_3) = (a_1, a_3)$ . Since  $A'$  is an affine subspace, it is sufficient to check this for  $(x'_1, x'_3) = (0, 0)$ ,  $(x'_1, x'_3) = (1, 0)$ , and  $(x'_1, x'_3) = (0, 1)$ , i.e. for  $h + 1$  tuples, where  $h$  is the dimension of  $B$ . Again, to check a concrete solution from  $B$  we recursively call the algorithm for 2-element domains.

Since (3) is linked, Theorem 5.7 guarantees that the dimension of  $A'$  equals the dimension of  $B$  minus one or  $A'$  is empty. Hence, we need exactly one equation to describe all pairs  $(a_1, a_3)$  such that (3) has a solution corresponding to  $(x'_1, x'_3) = (a_1, a_3)$ . Let the equation be  $c_1x'_1 + c_3x'_3 = c_0$ . We need to find  $c_1, c_3$ , and  $c_0$ . Recursively calling the algorithm for smaller domains, we find out that (3) has a solution  $(3, 3, 0, 1)$  corresponding to  $(x'_1, x'_3) = (1, 0)$  (the solution  $(1, 1, 0, 1)$  from  $B$ ) but does not have a solution corresponding to  $(x'_1, x'_3) = (0, 1)$  (the solution  $(0, 0, 1, 1)$  from  $B$ ). We have

$$\begin{cases} c_1 \cdot 0 + c_3 \cdot 0 \neq c_0 \\ c_1 \cdot 1 + c_3 \cdot 0 = c_0, \\ c_1 \cdot 0 + c_3 \cdot 1 \neq c_0 \end{cases}$$

which implies that  $c_1 = 1, c_3 = 0, c_0 = 1$ , and the equation we are looking for is  $x'_1 = 1$ . Thus, we found  $A'$ .

We add this equation to (2) (update  $B = A'$ ) and solve the new system of linear equations in  $\mathbb{Z}_2$ .

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 = 1 \end{cases} \quad (4)$$

The general solution of this system (the new set  $B$ ) is  $x'_1 = 1, x'_2 = 1, x'_3 = x'_3, x'_4 = x'_3 + 1$ , where  $x'_3$  is an independent variable. Thus, we decreased the dimension of the solution set  $B$  by 1 and we still have the property that  $A \subseteq B$ . We go back to (1), and check whether it

has a solution corresponding to  $x'_3 = 0$  (the solution  $(1, 1, 0, 1)$  from  $B$ ). Again, by solving an instance on the 2-element domain we find out that  $(1, 1, 0, 1) \notin A$ . Therefore  $A \subsetneq B$ .

The remaining part of the procedure looks trivial but we want to follow the algorithm till the end to make it clear. Again, we try to make the instance weaker so that  $A' \subsetneq B$ . Let us remove the third equation from (1).

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases} \quad (5)$$

By solving this instance on smaller domains we find out that (5) has no solutions corresponding to  $x'_3 = 0$  (the solution  $(1, 1, 0, 1)$  of  $B$ ). Therefore, we obtained a new set  $A' \subsetneq B$ .

Then we try to remove one more equation from (5) maintaining the property  $A' \subsetneq B$ . We check for every weaker instance that for any  $a_3 \in \mathbb{Z}_2$  there exists a solution corresponding to  $x'_3 = a_3$ .

Again, the instance (5) is linked, and by Theorem 5.7 we need exactly one equation to describe all elements  $a_3$  such that (5) has a solution corresponding to  $x'_3 = a_3$ . Let the equation be  $c_3 x'_3 = c_0$ . We already checked that it does not hold for  $x'_3 = 0$ . By solving an instance on 2-element domains we find out that (5) has a solution  $(3, 3, 1, 0)$  corresponding to the solution  $(1, 1, 1, 0)$  from  $B$  and  $x'_3 = 1$ . Thus we have

$$\begin{cases} c_3 \cdot 0 \neq c_0 \\ c_3 \cdot 1 = c_0 \end{cases},$$

which implies  $c_3 = 1$ ,  $c_0 = 1$ , and the equation we are looking for is  $x'_3 = 1$  (we calculated  $A'$ ).

We add this equation to (4) (update  $B = A'$ ) and solve the new system of linear equations in  $\mathbb{Z}_2$ .

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 = 1 \\ x'_3 = 1 \end{cases} \quad (6)$$

The only solution of this system is  $(x'_1, x'_2, x'_3, x'_4) = (1, 1, 1, 0)$ . Thus, we decreased the dimension of the solution set  $B$  to 0 and we still have the property that  $A \subseteq B$ . It remains to check whether the original system (1) has a solution corresponding to the solution  $(1, 1, 1, 0)$  of  $B$ . Again, by solving an instance on the 2-element domain we find a solution  $(3, 3, 1, 0)$  of the original instance. Therefore,  $(1, 1, 1, 0) \in A$  and we finally reached the condition  $A = B$ .

### 3 Definitions

A set of operations is called a *clone* if it is closed under composition and contains all projections. For a set of operations  $M$  by  $\text{Clo}(M)$  we denote the clone generated by  $M$ .

An idempotent WNU  $w$  is called *special* if  $x \circ (x \circ y) = x \circ y$ , where  $x \circ y = w(x, \dots, x, y)$ . It is not hard to show that for any idempotent WNU  $w$  on a finite set there exists a special WNU  $w' \in \text{Clo}(w)$  (see Lemma 4.7 in [47]).

A relation  $\rho \subseteq A_1 \times \dots \times A_n$  is called *subdirect* if for every  $i$  the projection of  $\rho$  onto the  $i$ -th coordinate is  $A_i$ . For a relation  $\rho$  by  $\text{pr}_{i_1, \dots, i_s}(\rho)$  we denote the projection of  $\rho$  onto the coordinates  $i_1, \dots, i_s$ .

### 3.1 Algebras

An *algebra* is a pair  $\mathbf{A} := (A; F)$ , where  $A$  is a finite set, called *universe*, and  $F$  is a family of operations on  $A$ , called *basic operations of  $\mathbf{A}$* . In the paper we always assume that we have a special WNU  $w$  preserving all constraint relations. Therefore, every domain  $D$ , which is from the constraint language, can be viewed as an algebra  $(D; w)$ . By  $\text{Clo}(\mathbf{A})$  we denote the clone generated by all basic operations of  $\mathbf{A}$ .

An equivalence relation  $\sigma$  on the universe of an algebra  $\mathbf{A}$  is called a *congruence* if it is preserved by every operation of the algebra. A congruence (an equivalence relation) is called *proper*, if it is not equal to the full relation  $A \times A$ . A subuniverse is called *nontrivial* if it is proper and nonempty. We use standard universal algebraic notions of term operation, subalgebra, factor algebra, product of algebras, see [8]. We say that a subalgebra  $\mathbf{R} = (R; F_R)$  is a *subdirect subalgebra* of  $\mathbf{A} \times \mathbf{B}$  if  $R$  is a subdirect relation in  $A \times B$ .

### 3.2 Polynomially complete algebras

An algebra  $(A; F_A)$  is called *polynomially complete (PC)* if the clone generated by  $F_A$  and all constants on  $A$  is the clone of all operations on  $A$  (see [36, 45]).

### 3.3 Linear algebra

An idempotent finite algebra  $(A; w_A)$  is called *linear* (similar to affine in [31]) if it is isomorphic to  $(\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_s}; x_1 + \dots + x_m)$  for prime numbers  $p_1, \dots, p_s$ . Since  $\mathbf{A}/(\sigma \cap \tau)$  is always isomorphic to a subalgebra of  $\mathbf{A}/\sigma \times \mathbf{A}/\tau$ , and since linear algebras are closed under products and subalgebras by Corollary 7.20.1, for every idempotent finite algebra  $(B; w_B)$  there exists a least congruence  $\sigma$ , called *the minimal linear congruence*, such that  $(B; w_B)/\sigma$  is linear.

### 3.4 Absorption

Let  $B$  be a (probably empty) subuniverse of  $\mathbf{A} = (A; F_A)$ . We say that  $B$  *absorbs  $\mathbf{A}$*  if there exists  $t \in \text{Clo}(\mathbf{A})$  such that  $t(B, B, \dots, B, A, B, \dots, B) \subseteq B$  for any position of  $A$ . In this case we also say that  $B$  is an *absorbing subuniverse of  $\mathbf{A}$  with a term operation  $t$* . If the operation  $t$  can be chosen binary then we say that  $B$  is a binary absorbing subuniverse of  $\mathbf{A}$ . For more information about absorption and its connection with CSP see [3].

### 3.5 Center

Suppose  $\mathbf{A} = (A; w_A)$  is a finite algebra with a special WNU operation.  $C \subseteq A$  is called a *center* if there exists an algebra  $\mathbf{B} = (B; w_B)$  with a special WNU operation of the same arity and a subdirect subalgebra  $(R; w_R)$  of  $\mathbf{A} \times \mathbf{B}$  such that there is no nontrivial binary absorbing subuniverse in  $\mathbf{B}$  and  $C = \{a \in A \mid \forall b \in B: (a, b) \in R\}$ . This notion was motivated by central relations defining maximal clones on finite sets (see section 5.2.5 in [44]) and it is very similar to ternary absorption (see Corollary 7.10.2).

### 3.6 CSP instance

An instance of the constraint satisfaction problem is called a *CSP instance*. Sometimes we use the same letter for a CSP instance and for the set of all constraints of this instance. For a variable  $z$  by  $D_z$  we denote the domain of the variable  $z$ .

We say that  $z_1 - C_1 - z_2 - \dots - C_{l-1} - z_l$  is a *path* in a CSP instance  $\Theta$  if  $z_i, z_{i+1}$  are in the scope of  $C_i$  for every  $i$ . We say that a *path*  $z_1 - C_1 - z_2 - \dots - C_{l-1} - z_l$  *connects  $b$  and*

$c$  if there exists  $a_i \in D_{z_i}$  for every  $i$  such that  $a_1 = b$ ,  $a_l = c$ , and the projection of  $C_i$  onto  $z_i, z_{i+1}$  contains the tuple  $(a_i, a_{i+1})$ .

A CSP instance is called *1-consistent* if every constraint of the instance is subdirect. A CSP instance is called *cycle-consistent* if it is 1-consistent and for every variable  $z$  and  $a \in D_z$  any path starting and ending with  $z$  in  $\Theta$  connects  $a$  and  $a$ . Other types of local consistency and its connection with the complexity of CSP are considered in [43]. A CSP instance  $\Theta$  is called *linked* if for every variable  $z$  occurring in the scope of a constraint of  $\Theta$  and every  $a, b \in D_z$  there exists a path starting and ending with  $z$  in  $\Theta$  that connects  $a$  and  $b$ .

Suppose  $\mathbf{X}' \subseteq \mathbf{X}$ . Then we can define a projection of  $\Theta$  onto  $\mathbf{X}'$ , that is a CSP instance where variables are elements of  $\mathbf{X}'$  and constraints are projections of the constraints of  $\Theta$  onto the intersection of their scopes with  $\mathbf{X}'$ , ignoring any constraint whose scope does not intersect  $\mathbf{X}'$ . We say that an instance  $\Theta$  is *fragmented* if the set of variables  $\mathbf{X}$  can be divided into 2 disjoint sets  $\mathbf{X}_1$  and  $\mathbf{X}_2$  such that each of them contains a variable from the scope of a constraint of  $\Theta$ , and the constraint scope of any constraint of  $\Theta$  either has variables only from  $\mathbf{X}_1$ , or only from  $\mathbf{X}_2$ . Thus, if an instance is fragmented, then it can be divided into several nontrivial instances.

A CSP instance  $\Theta$  is called *irreducible* if any instance  $\Theta' = (\mathbf{X}', \mathbf{D}', \mathbf{C}')$  such that  $\mathbf{X}' \subseteq \mathbf{X}$ ,  $D'_x = D_x$  for every  $x \in \mathbf{X}'$ , and every constraint of  $\Theta'$  is a projection of a constraint from  $\Theta$  on some set of variables is fragmented, or linked, or its solution set is subdirect.

We say that a constraint  $C_1 = ((y_1, \dots, y_t); \rho_1)$  is *weaker or equivalent to* a constraint  $C_2 = ((z_1, \dots, z_s); \rho_2)$  if  $\{y_1, \dots, y_t\} \subseteq \{z_1, \dots, z_s\}$  and  $C_2$  implies  $C_1$ . In other words, the second condition says that the solution set to  $\Theta_1 := (\{z_1, \dots, z_s\}, (D_{z_1}, \dots, D_{z_s}), C_1)$  contains the solution set to  $\Theta_2 := (\{z_1, \dots, z_s\}, (D_{z_1}, \dots, D_{z_s}), C_2)$ . We say that  $C_1$  is *weaker than*  $C_2$  if  $C_1$  is weaker or equivalent to  $C_2$  but  $C_1$  does not imply  $C_2$ .

The following remark justifies weakening constraints of the instance in the algorithm (this remark follows from Lemma 6.1).

**Remark 1.** Suppose  $\Theta = \langle \mathbf{X}; \mathbf{D}; \mathbf{C} \rangle$  and  $\Theta' = \langle \mathbf{X}'; \mathbf{D}'; \mathbf{C}' \rangle$  are CSP instances such that  $\mathbf{X}' \subseteq \mathbf{X}$ ,  $D'_x = D_x$  for every  $x \in \mathbf{X}'$ , and every constraint of  $\Theta'$  is weaker or equivalent to a constraint of  $\Theta$ . If  $\Theta$  is cycle-consistent and irreducible, then so is  $\Theta'$ .

We say that a variable  $y_i$  of the constraint  $((y_1, \dots, y_t); \rho)$  is *dummy* if  $\rho$  does not depend on its  $i$ -th variable.

**Remark 2.** Adding a dummy variable to a constraint and removing of a dummy variable do not affect the property of being cycle-consistent and irreducible.

Let  $D'_i \subseteq D_i$  for every  $i$ . A constraint  $C$  of  $\Theta$  is called *crucial in*  $(D'_1, \dots, D'_n)$  if it has no dummy variables,  $\Theta$  has no solutions in  $(D'_1, \dots, D'_n)$  but the replacement of  $C \in \Theta$  by all weaker constraints gives an instance with a solution in  $(D'_1, \dots, D'_n)$ . A CSP instance  $\Theta$  is called *crucial in*  $(D'_1, \dots, D'_n)$  if it has at least one constraint and every constraint of  $\Theta$  is crucial in  $(D'_1, \dots, D'_n)$ .

**Remark 3.** Suppose  $\Theta$  has no solutions in  $(D'_1, \dots, D'_n)$ . We can replace each constraint by its projection onto its non-dummy variables. Then we iteratively replace every constraint by all weaker constraints having no dummy variables until it is crucial. Finally, we get a CSP instance that is crucial in  $(D'_1, \dots, D'_n)$ .



## 4 Algorithm

### 4.1 Main part

Suppose we have a constraint language  $\Gamma_0$  that is preserved by an idempotent WNU operation. As it was mentioned before,  $\Gamma_0$  is also preserved by a special WNU operation  $w$ . Let  $k_0$  be the maximal arity of the relations in  $\Gamma_0$ . By  $\Gamma$  we denote the set of all relations of arity at most  $k_0$  that are preserved by  $w$ . Obviously,  $\Gamma_0 \subseteq \Gamma$ , therefore every instance of  $\text{CSP}(\Gamma_0)$  is an instance of  $\text{CSP}(\Gamma)$ .

In this section we provide an algorithm that solves  $\text{CSP}(\Gamma)$  in polynomial time. Suppose we have a CSP instance  $\Theta = \langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$ , where  $\mathbf{X} = \{x_1, \dots, x_n\}$  is a set of variables,  $\mathbf{D} = \{D_1, \dots, D_n\}$  is a set of the respective domains,  $\mathbf{C} = \{C_1, \dots, C_q\}$  is a set of constraints. Let the arity of the WNU  $w$  be equal to  $m$ .

The main part of the algorithm (function **SOLVE**) is an iterative loop; in each pass through the loop, the algorithm calls a subroutine **ANSWERORREDUCE** whose job is to find a reduction of a domain or to terminate with the final answer. The reduction returned by the function should satisfy the following property: if  $\Theta$  has a solution, then it has a solution after the reduction. If the reduction was found then we apply the function **REDUCE**, which takes an instance  $\Theta = (\mathbf{X}, \mathbf{D}, \mathbf{C})$  and a domain set  $\mathbf{D}' = (D'_1, \dots, D'_n)$ , and returns a new instance  $(\mathbf{X}, \mathbf{D}', \mathbf{C}')$ , where  $\mathbf{C}' = \{((x_{i_1}, \dots, x_{i_s}), \rho \cap (D'_{i_1} \times \dots \times D'_{i_s})) \mid ((x_{i_1}, \dots, x_{i_s}), \rho) \in \mathbf{C}\}$ .

---

```

1: function SOLVE( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta = (\mathbf{X}, \mathbf{D}, \mathbf{C})$ ,  $\mathbf{X} = (x_1, \dots, x_n)$ ,  $\mathbf{D} = (D_1, \dots, D_n)$ 
3:   repeat
4:     Output := ANSWERORREDUCE( $\Theta$ )
5:     if Output = "Solution" then return "Solution"
6:     if Output = "No solution" then return "No solution"
7:     if Output =  $(x_i, U)$  then  $\triangleright \emptyset \neq U \subset D_i$ 
8:        $\Theta := \text{REDUCE}(\Theta, (D_1, \dots, D_{i-1}, U, D_{i+1}, \dots, D_n))$   $\triangleright \text{Set } D_i = U$ 
9:   until Done

```

---

The function **ANSWERORREDUCE** (see the pseudocode) checks different types of consistency such as cycle-consistency and irreducibility, and reduce a domain if the instance is not consistent. If it is consistent, then either it reduces a domain to a proper strong subset, or it uses **SOLVILINEARCASE** to solve the remaining case.

First, the function **ANSWERORREDUCE** checks whether the instance  $\Theta$  is cycle-consistent (function **CHECKCYCLECONSISTENCY**). If it is not cycle-consistent then either some domain can be reduced, or the instance has no solutions. In both cases we terminate the function and return the result. If it is cycle-consistent then we go on.

If the size of every domain is one it returns that a solution was found.

Then we check whether the instance is irreducible (function **CHECKIRREDUCIBILITY**). If it is not irreducible then we return how to reduce some domain or return that there is no solutions, otherwise we go on.

After that we check a different type of consistency (function **CHECKWEAKERINSTANCE**). We make a copy of  $\Theta$ , and simultaneously replace every constraint by all weaker constraints without dummy variables. Recursively calling the algorithm, we check that the obtained instance has a solution with  $x_i = b$  for every  $i \in \{1, 2, \dots, n\}$  and  $b \in D_i$ . If not, reduce  $D_i$  to the projection onto  $x_i$  of the solution set of the obtained instance. Otherwise, go on.

By Theorem 5.5 we cannot pass from an instance having solutions to an instance having no solutions when reduce a domain to a nontrivial binary absorbing subuniverse or to a nontrivial

---

```

1: function ANSWERORREDUCE( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta = (\mathbf{X}, \mathbf{D}, \mathbf{C})$ ,  $\mathbf{X} = (x_1, \dots, x_n)$ ,  $\mathbf{D} = (D_1, \dots, D_n)$ 
3:   Output := CHECKCYCLECONSISTENCY( $\Theta$ )
4:   if Output  $\neq$  "Ok" then return Output
5:   if  $|D_i| = 1$  for every  $i$  then return "Solution"
6:   Output := CHECKIRREDUCIBILITY( $\Theta$ )
7:   if Output  $\neq$  "Ok" then return Output
8:   Output := CHECKWEAKERINSTANCE( $\Theta$ )
9:   if Output  $\neq$  "Ok" then return Output
10:  if  $B_i$  is a nontrivial binary absorbing subuniverse of  $D_i$  then return  $(x_i, B_i)$ 
11:  if  $C_i$  is a nontrivial center of  $D_i$  then return  $(x_i, C_i)$ 
12:  if  $\sigma$  is a proper congruence on  $D_i$  and  $(D_i; w)/\sigma$  is polynomially complete then
    Choose an equivalence class  $E$  of  $\sigma$ 
    return  $(x_i, E)$ 
  return SOLVELINEARCASE( $\Theta$ )

```

---

center. Thus, if  $D_i$  has a nontrivial binary absorbing subuniverse  $B_i \subsetneq D_i$  for some  $i$ , then we reduce  $D_i$  to  $B_i$ . Similarly, if  $D_i$  has a nontrivial center  $C_i \subsetneq D_i$  for some  $i$ , then we reduce  $D_i$  to  $C_i$ .

By Theorem 5.6 we cannot pass from an instance having solutions to an instance having no solutions when reduce a domain to an equivalence class of a proper congruence  $\sigma$  such that  $(D_i; w)/\sigma$  is polynomially complete. Thus, if such a congruence on  $D_i$  exists, we reduce  $D_i$  to its equivalence class.

By Theorem 5.1, it remains to consider the case when on every domain  $D_i$  of size greater than 1 there exists a proper congruence  $\sigma$  such that  $(D_i; w)/\sigma$  is isomorphic to  $(\mathbb{Z}_p; x_1 + \dots + x_m)$  for some  $p$ . In this case the problem is solved by the function SOLVELINEARCASE, which will be described in the next subsection. A detailed description and a pseudocode for the functions CHECKCYCLECONSISTENCY, CHECKIRREDUCIBILITY, and CHECKWEAKERINSTANCE will be given in Subsection 4.4

## 4.2 Linear case

In this section we define the function SOLVELINEARCASE (see the pseudocode). For every  $i$  let  $\sigma_i$  be the minimal linear congruence on  $D_i$ , which is the smallest congruence  $\sigma$  such that  $(D_i; w)/\sigma$  is linear. Then  $(D_i; w)/\sigma_i$  is isomorphic to  $(\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_l}; x_1 + \dots + x_m)$  for prime numbers  $p_1, \dots, p_l$ . Recall that we apply the function SOLVELINEARCASE only if  $\sigma_i$  is proper for every  $i$  such that  $|D_i| > 1$ . We will show that modulo these congruences the instance can be viewed as a system of linear equations in fields.

We denote  $D_i/\sigma_i$  by  $L_i$  and define a new CSP instance  $\Theta_L$  with domains  $L_1, \dots, L_n$  as follows. To every constraint  $((x_{i_1}, \dots, x_{i_s}); \rho) \in \Theta$  we assign a constraint  $((x'_{i_1}, \dots, x'_{i_s}); \rho')$ , where  $\rho' \subseteq L_{i_1} \times \dots \times L_{i_s}$  and  $(E_1, \dots, E_s) \in \rho' \Leftrightarrow (E_1 \times \dots \times E_s) \cap \rho \neq \emptyset$ . The constraints of  $\Theta_L$  are all constraints that are assigned to the constraints of  $\Theta$ . The function generating the instance  $\Theta_L$  from  $\Theta$  is called FACTORIZEINSTANCE in the pseudocode. Note that  $\Theta_L$  is a CSP instance but not necessarily an instance in the constraint language  $\Gamma$ .

Since each  $L_i$  is isomorphic to some  $\mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_s}$ , we may define a natural bijective mapping  $\psi : \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r} \rightarrow L_1 \times \dots \times L_n$ , and assign a variable  $z_i$  to every  $\mathbb{Z}_{p_i}$ . Since every relation on  $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r}$  preserved by  $x_1 + \dots + x_m$  is known (see Lemma 7.20) to be a conjunction of linear equations, the instance  $\Theta_L$  can be viewed as a system of linear equations

over  $z_1, \dots, z_r$ . Note that every equation is an equation in  $\mathbb{Z}_p$  but  $p$  can be different for different equations, and only variables with the same domain  $\mathbb{Z}_p$  may appear in one equation.

---

```

1: function SOLVELINEARCASE( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta = (\mathbf{X}, \mathbf{D}, \mathbf{C})$ ,  $\mathbf{X} = (x_1, \dots, x_n)$ ,  $\mathbf{D} = (D_1, \dots, D_n)$ 
3:    $\Theta_L := \text{FACTORIZEINSTANCE}(\Theta)$ 
4:    $Eq := \emptyset$  ▷ The equations we add to  $\Theta_L$ 
5:   repeat
6:      $\phi := \text{SOLVELINEARSYSTEM}(\Theta_L \cup Eq)$ 
▷  $\phi(\mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k})$  is the solution set of  $\Theta_L \cup Eq$ 
7:     if  $\phi = \emptyset$  then return “No solution”
8:     if  $\text{SOLVE}(\text{REDUCE}(\Theta, \phi(0, 0, \dots, 0))) = \text{“Solution”}$  then return “Solution”
9:     else if  $k=0$  then return “No solution” ▷  $\Theta_L$  has just one solution
10:     $\Theta' := \text{REMOVETRIVIALITIES}(\Theta)$ 
11:    repeat ▷ Try to weaken  $\Theta'$ 
12:       $\text{Changed} := \text{false}$ 
13:      for  $C \in \Theta'$  do
14:         $\Omega := \text{REMOVETRIVIALITIES}(\text{WEAKENCONSTRAINT}(\Theta', C))$ 
15:        if  $\neg \text{CHECKALLTUPLES}(\Omega, \phi)$  then
16:           $\Theta' := \Omega$ 
17:           $\text{Changed} := \text{true}$ 
18:          break
19:      until  $\neg \text{Changed}$  ▷  $\Theta'$  cannot be weakened anymore
20:      if  $\Theta'$  is not linked then
21:         $Eq := Eq \cup \text{FINDEQUATIONSNONLINKED}(\Theta')$ 
22:      else
23:         $Eq := Eq \cup \{\text{FINDONEEQUATIONLINKED}(\Theta', \phi)\}$ 
24:    until Done

```

---

As it was described in Section 2, we consider the set  $A$ , which is the solution set of  $\Theta$  factorized by the congruences  $\sigma_1, \dots, \sigma_n$ , and the set  $B$ , which is the solution set of  $\Theta_L$ . We know that  $A \subseteq B$  and we want to check whether  $A$  is empty. We iteratively add new equations to the set  $\Theta_L$  maintaining the property that  $A \subseteq B$ , and therefore reduce the dimension of  $B$ . We start with the empty set of equations  $Eq$  (line 4 of the pseudocode).

Then we apply the function `SOLVELINEARSYSTEM` that solves the system of linear equations  $\Theta_L \cup Eq$  using Gaussian elimination. If the system has no solutions then  $\Theta$  has no solutions and we are done. Otherwise, we choose independent variables  $y_1, \dots, y_k$ , then the general solution (the set  $B$ ) can be written as an affine mapping  $\phi: \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k} \rightarrow L_1 \times \dots \times L_n$ . Denote  $Z = \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$ , then any solution of  $\Theta_L \cup Eq$  can be obtained as  $\phi(a_1, \dots, a_k)$  for some  $(a_1, \dots, a_k) \in Z$ .

Note that for any tuple  $(a_1, \dots, a_k) \in Z$  we can check recursively whether  $\Theta$  has a solution in  $\phi(a_1, \dots, a_k)$  (i.e. whether  $\phi(a_1, \dots, a_k) \in A$ ). To do this, we just need to reduce the domains to the solution (function `REDUCE`) and solve an easier CSP instance (on smaller domains). Similarly, we can check whether  $\Theta$  has a solution in  $\phi(a_1, \dots, a_k)$  for every  $(a_1, \dots, a_k) \in \mathbb{Z}$  (i.e. whether  $A = B$ ). Since  $A$  and  $B$  are subuniverses of  $L_1 \times \dots \times L_n$  (almost subspaces), we just need to check the existence of a solution in  $\phi(0, \dots, 0)$  and  $\phi(0, \dots, 0, 1, 0, \dots, 0)$  for any position of 1. See the pseudocode of the function `CHECKALLTUPLES` for the last procedure.

Let us go back to the function `SOLVELINEARCASE`. After solving the linear system we

---

```

1: function CHECKALLTUPLES( $\Theta, \phi$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ , a solution of a linear system of equations  $\phi$ 
3:   if SOLVE(REDUCE( $\Theta, \phi(0, \dots, 0)$ )) = “No solution” then return false
4:   for  $i = 1, 2, \dots, k$  do
5:      $t := (\underbrace{0, \dots, 0}_i, 1, 0, \dots, 0)$ 
6:     if SOLVE(REDUCE( $\Theta, \phi(t)$ )) = “No solution” then return false
   return true;

```

---

check whether there exists a solution of  $\Theta$  corresponding to the solution  $\phi(0, 0, \dots, 0)$  of  $\Theta_L \cup Eq$ . If  $k = 0$ , i.e.  $\Theta_L \cup Eq$  has only one solution, then we denote this solution by  $\phi(0, 0, \dots, 0)$ . If  $\Theta$  has a solution in  $\phi(0, \dots, 0)$ , then it remains to return the result “Solution”. If it has no solutions and  $k = 0$  then return the result “No solution”.

At this point (line 10 of the pseudocode of SOLVELINEARCASE), we have the property that the set  $B$  is of dimension at least 1, and  $A \neq B$  since we found a solution  $\phi(0, \dots, 0)$  of the system of linear equations without the corresponding solution of  $\Theta$ .

Then we iteratively remove from  $\Theta$  all constraints that are weaker than some other constraints of  $\Theta$ , remove all constraints without non-dummy variables, and replace every constraint by its projection onto non-dummy variables. This procedure we denote by the function REMOVETRIVIALITIES. In the pseudocode of SOLVELINEARCASE we denote the obtained instance by  $\Theta'$ .

Then we try to make the constraints of  $\Theta'$  weaker maintaining the property that  $A' \neq B$ , where  $A'$  is the solution set of  $\Theta'$  factorized by the congruences  $\sigma_1, \dots, \sigma_n$ . Precisely, we choose a constraint  $C$ , replace it by all weaker constraints without dummy variables (function WEAKENCONSTRAINT), apply REMOVETRIVIALITIES, and check using the function CHECKALLTUPLES whether  $A' = B$ . If not, then we replace  $\Theta'$  by the new weaker instance.

Suppose we cannot make any constraint weaker maintaining the property  $A' \neq B$ . Then  $\Theta'$  has no solutions in  $\phi(b_1, \dots, b_k)$  for some  $(b_1, \dots, b_k) \in Z$ , but if we replace any constraint  $C \in \Theta'$  by all weaker constraints, then we get an instance that has a solution in  $\phi(a_1, \dots, a_k)$  for every  $(a_1, \dots, a_k) \in Z$ . Therefore,  $\Theta'$  is crucial in  $\phi(b_1, \dots, b_k)$ . Note that by Lemma 6.1 the instance  $\Theta'$  is still cycle-consistent and irreducible. Also,  $\Theta'$  is not fragmented because it is crucial.

Then, in line 20 of the function SOLVELINEARCASE we have two options.

If  $\Theta'$  is not linked then using the function FINDEQUATIONSNONLINKED we calculate its solution set factorized by the congruences (the set  $A'$ ). This solution set can be defined by a set of linear equations, which we add to  $Eq$  and therefore replace  $B$  by  $A' \cap B$ . Thus, we made  $B$  smaller and we still have the property  $A \subseteq B$ , since  $A'$  is the factorized solution set of the instance  $\Theta'$ , which is weaker than  $\Theta$ .

If  $\Theta'$  is linked then by Theorem 5.7 either  $A' = \emptyset$ , or the dimension of  $A'$  is equal to the dimension of  $B$  minus 1, which allows us to find a new linear equation by polynomially many queries “Does there exist a solution of  $\Theta'$  in  $\phi(a_1, \dots, a_k)$ ?”. We calculate this new equation by the function FINDONEEQUATIONLINKED, which will be defined in the next section as well as the function FINDEQUATIONSNONLINKED. Note that the new equation can be “ $0 = 1$ ” if  $A' = \emptyset$ .

After new equations found, we go back to line 6 of the function SOLVELINEARCASE and solve a system of linear equations again. Since every time we reduce the dimension of  $B$  by at least one, the procedure will stop in at most  $r$  steps.

### 4.3 Finding linear equations

In this section we define the functions `FINDONEEQUATIONLINKED`, `FINDONEEQUATIONNONLINKED`, and `FINDEQUATIONSNONLINKED`, which allow us to find new equations defining the set  $A'$ .

---

```

1: function FINDONEEQUATIONLINKED( $\Theta, \phi$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ , a solution of a system of linear equations  $\phi$ 
3:    $t := \emptyset$  ▷ We search for a tuple  $t$  outside of the solution set
4:   if SOLVE(REDUCE( $\Theta, \phi(0, \dots, 0)$ )) = “No solution” then
5:      $t := (0, \dots, 0)$ 
6:   else
7:     for  $i = 1, 2, \dots, k$  do
8:        $t' := (0, \dots, 0, \underbrace{1}_i, 0, \dots, 0)$ 
9:       if SOLVE(REDUCE( $\Theta, \phi(t')$ )) = “No solution” then
10:         $t := t'$ 
11:      break
12:   if  $t = \emptyset$  then return “ $0 = 0$ ”
13:   for  $i = 1, 2, \dots, k$  do
14:      $b_i := 0$ 
15:     for  $a \in \mathbb{Z}_{q_i} \setminus \{t(i)\}$  do
16:        $t' := t$ 
17:        $t'(i) := a$ 
18:       if SOLVE(REDUCE( $\Theta, \phi(t')$ )) = “Solution” then
19:         $b_i := 1/(a - t(i))$ 
return “ $b_1(y_1 - t(1)) + \dots + b_k(y_k - t(k)) = 1$ ”

```

---

First, we explain how the function `FINDONEEQUATIONLINKED` works. Suppose  $V$  is an affine subspace of  $\mathbb{Z}_p^k$  of dimension  $k - 1$ , thus  $V$  is the solution set of a linear equation  $c_1y_1 + \dots + c_ky_k = c_0$ . Then the coefficients  $c_0, c_1, \dots, c_k$  can be learned (up to a multiplicative constant) by  $(p \cdot k + 1)$  queries of the form “ $(a_1, \dots, a_k) \in V$ ?” as follows. First, we need at most  $(k + 1)$  queries to find a tuple  $(t_1, \dots, t_k) \notin V$ . To do this we just check all tuples with 0s and at most one 1 (lines 4-11 of the pseudocode). Then, to find this equation it is sufficient to check for every  $a$  and every  $i$  whether the tuple  $(t_1, \dots, t_{i-1}, a, t_{i+1}, \dots, t_k)$  satisfies this equation (lines 13-19 of the pseudocode). Here the query is performed by the reduction of all domains to the corresponding solution (the function `REDUCE`) and a recursive call of the main function `SOLVE`.

As we said before, we may define a natural bijective mapping  $\psi : \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r} \rightarrow L_1 \times \dots \times L_n$ , and assume that all relations from  $\Theta_L$  and  $E_q$  are systems of linear equations over  $z_1, \dots, z_r$ . Below we explain how the function `FINDEQUATIONSNONLINKED` calculates the solution set of  $\Theta'$  factorized by the congruences (the set  $A'$ ) if  $\Theta'$  is not linked. It describes the solution set by linear equations over  $z_1, \dots, z_r$ .

We start with an empty set of equations  $E$  and claim that the first variable is independent, by  $I$  we denote the set of independent variables (see the pseudocode of `FINDEQUATIONSNONLINKED`). Assume that we already found all the equations over  $z_1, \dots, z_{j-1}$ , i.e. we described the projection of  $A'$  onto  $z_1, \dots, z_{j-1}$ .

Then the projection of  $A'$  onto the independent variables and the  $j$ -th variable is either full or of codimension 1. Thus, we can learn this equation by queries of the form “Does there exist  $v \in A'$  such that  $\text{pr}_{I \cup \{j\}}(v) = (a_1, \dots, a_h)$ ?” in the same way as we did in `FINDONEEQUATIONLINKED`, but now we use `FINDONEEQUATIONNONLINKED`. The only

---

```

1: function FINDEQUATIONSNONLINKED( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:    $I := \{1\}$  ▷  $I$  is the set of independent variables
4:    $E := \emptyset$  ▷ We start with an empty set of equations
5:   for  $j = 1, 2, \dots, r$  do
6:      $e := \text{FINDONEEQUATIONNONLINKED}(\Theta, I \cup \{j\})$ 
7:     if  $e = "0 = 0"$  then ▷  $j$ -th variable is independent
8:        $I := I \cup \{j\}$ 
9:     else if  $e = "0 = 1"$  then return "No solution"
10:    else
11:       $E := E \cup e$  ▷ Add the equation we found
  return  $E$ 

```

---

difference in these functions is how we check a query: in `FINDEQUATIONSNONLINKED` we use the function `CHECKTUPLE` instead of `REDUCE` and `SOLVE` (see the pseudocode).

If the new equation was found and this equation is not trivial then we add this equation to  $E$  and claim that  $z_j$  is not independent. If the equation we found is " $0 = 0$ " then add  $x_j$  to the set of independent variables and go to the next variable.

---

```

1: function FINDONEEQUATIONNONLINKED( $\Theta, I$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ ,  $I = \{i_1, \dots, i_h\}$  a set of variables
3:    $t := \emptyset$  ▷ We search for a tuple  $t$  outside of the solution set
4:   if  $\neg \text{CHECKTUPLE}(\Theta, I, (0, \dots, 0))$  then
5:      $t := (0, \dots, 0)$ 
6:   else
7:     for  $j = 1, 2, \dots, h$  do
8:        $t' := (\underbrace{0, \dots, 0}_j, 1, 0, \dots, 0)$ 
9:       if  $\neg \text{CHECKTUPLE}(\Theta, I, t')$  then
10:         $t := t'$ 
11:        break
12:   if  $t = \emptyset$  then return " $0 = 0$ "
13:   for  $j = 1, 2, \dots, h$  do
14:      $b_j := 0$ 
15:     for  $a \in \mathbb{Z}_{p_{i_j}} \setminus \{t(j)\}$  do
16:        $t' := t$ 
17:        $t'(j) := a$ 
18:       if  $\text{CHECKTUPLE}(\Theta, I, t')$  then
19:         $b_j := 1/(a - t(j))$ 
  return " $b_1(z_{i_1} - t(1)) + \dots + b_h(z_{i_h} - t(h)) = 1$ "

```

---

It remains to explain how the function `CHECKTUPLE` works. As an input it takes an instance  $\Theta$ , a set of variables  $I$ , and a tuple  $t$  of length  $|I|$ . The restriction of the variables from  $I$  to the tuple  $t$  implies the restrictions  $L'_1, \dots, L'_n$  of the domains  $L_1, \dots, L_n$ . Put  $D'_i = \bigcup_{E \in L'_i} E$  for every  $i$ . Then we add unary constraints  $x_i \in D'_i$  to  $\Theta$  and solve the obtained instance by the function `SOLVENONLINKED`, which works only for non-linked instances and will be defined in the next section.

---

```

1: function CHECKTUPLE( $\Theta, I, t$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ ,  $I$  a subset of variables,  $t$  a tuple of length  $|I|$ 
3:    $R := \{\alpha \in \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r} \mid \text{pr}_I(\alpha) = t\}$  ▷ We don't really calculate  $R$ 
4:   for  $i = 1, 2, \dots, n$  do
5:      $D'_i := \bigcup_{E \in \text{pr}_i(\psi(R))} E$  ▷ We calculate  $D'_i$ 
6:   if SOLVENONLINKED( $\Theta \wedge (x_1 \in D'_1) \wedge \dots \wedge (x_n \in D'_n)$ ) = "Solution" then return
   true
7:   else return false

```

---

## 4.4 Remaining functions

In this subsection we define the functions CHECKCYCLECONSISTENCY, CHECKIRREDUCIBILITY, and CHECKWEAKERINSTANCE which were used in Subsection 4.1, and function SOLVENONLINKED from Subsection 4.3.

First, we define the function CHECKCYCLECONSISTENCY. To check cycle-consistency it is sufficient to use constraint propagation providing a variant of (2,3)-consistency (see the pseudocode). First, for every pair of variables  $(x_i, x_j)$  we consider the intersections of projections of all constraints onto these variables. The corresponding relation we denote by  $\rho_{i,j}$ . Then, for every  $i, j, k \in \{1, 2, \dots, n\}$  we replace  $\rho_{i,j}$  by  $\rho'_{i,j}$  where  $\rho'_{i,j}(x, y) = \exists z \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$ .

We repeat this procedure while we can change some  $\rho_{i,j}$ . If in the end we get a relation  $\rho_{i,j}$  that is not subdirect in  $D_i \times D_j$ , then we can either reduce  $D_i$  or  $D_j$ , or, if  $\rho_{i,j}$  is empty, state that there are no solutions. If every relation  $\rho_{i,j}$  is subdirect in  $D_i \times D_j$ , then we claim (see Lemma 5.3) that the original CSP instance is cycle-consistent.

---

```

1: function CHECKCYCLECONSISTENCY( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:   for  $i, j \in \{1, 2, \dots, n\}$  do ▷ Calculate binary projections  $\rho_{i,j}$ 
4:      $\rho_{i,j} := D_i \times D_j$ 
5:     for  $C \in \Theta$  do
6:        $\rho_{i,j} := \rho_{i,j} \cap \text{pr}_{x_i, x_j} C$  ▷  $\text{pr}_{x_i, x_j} C$  is the projection of  $C$  onto  $x_i, x_j$ 
7:   repeat ▷ Propagate constraints to reduce  $\rho_{i,j}$ 
8:     Changed := false
9:     for  $i, j, k \in \{1, 2, \dots, n\}$  do
10:       $\rho'_{i,j}(x, y) := \exists z \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$ 
11:      if  $\rho_{i,j} \neq \rho'_{i,j}$  then
12:         $\rho_{i,j} := \rho'_{i,j}$ 
13:        Changed := true
14:   until  $\neg \text{Changed}$  ▷ We cannot reduce  $\rho_{i,j}$  anymore
15:   for  $i, j \in \{1, 2, \dots, n\}$  do
16:     if  $\rho_{i,j} = \emptyset$  then return "No solution"
17:     if  $\text{pr}_1(\rho_{i,j}) \neq D_i$  then return  $(x_i, \text{pr}_1(\rho_{i,j}))$ 
18:     if  $\text{pr}_2(\rho_{i,j}) \neq D_j$  then return  $(x_j, \text{pr}_2(\rho_{i,j}))$ 
   return "Ok"

```

---

Let us explain how CHECKIRREDUCIBILITY works. For every  $k \in \{1, 2, \dots, n\}$  and every maximal congruence  $\sigma_k$  on  $D_k$  we do the following. We start with the partition  $\sigma_k$  of the  $k$ -th variable, so we put  $I = \{k\}$  (line 4 of the pseudocode), which is the set of variables with a partition. Then we try to extend the partition of  $D_k$  to other domains. We choose

a constraint having  $x_k$  in the scope, choose another variable  $x_j$ , and consider the projection of  $C$  onto  $x_k, x_j$ , which we denote by  $\delta$ . Since  $\sigma_k$  is maximal, we may have two possibilities: either all equivalence classes of  $\sigma_k$  are connected in  $\delta$ , or none of the equivalence classes are connected in  $\delta$ . In the second case the partition of  $D_k$  generates a partition of  $D_j$  with the same number of classes, and we add  $j$  to  $I$  (lines 10-15 of the pseudocode).

We continue this procedure while we can add new variables to  $I$ . As a result we get a set  $I$  and a partition of  $D_i$  for every  $i \in I$ . Put  $\mathbf{X}' = \{x_i \mid i \in I\}$ . Then, the projection of  $\Theta$  onto  $\mathbf{X}'$  can be split into several instances on smaller domains, and each of them can be solved using recursion. Thus, we can check whether the solution set of the projection of the instance onto  $\mathbf{X}'$  is subdirect or empty. If it is empty then we state that there are no solutions. If it is not subdirect, then we can reduce the corresponding domain. If it is subdirect, then we go to the next  $k \in \{1, 2, \dots, n\}$  and the next maximal congruence  $\sigma_k$  on  $D_k$ , and repeat the procedure. If for all  $k$  and all maximal congruences the solution set of the obtained instance is subdirect, then the instance is irreducible (see Lemma 5.4).

---

```

1: function CHECKIRREDUCIBILITY( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:   for  $k = 1, \dots, n$  do
4:     for  $\sigma_k = \{E_k^1, \dots, E_k^t\}$  is a maximal congruence on  $D_k$  do
5:        $I := \{k\}$ 
6:       repeat
7:          $\text{Changed} := \text{false}$ 
8:         for  $C \in \Theta$ ,  $i \in I$ ,  $j \notin I$  such that  $x_i$  and  $x_j$  are in the scope of  $C$  do
9:            $\delta := \text{pr}_{x_i, x_j} C$   $\triangleright \text{pr}_{x_i, x_j} C$  is the projection of  $C$  onto  $x_i, x_j$ 
10:          for  $u = 1, 2, \dots, t$  do  $\triangleright$  Calculate the partition on  $D_j$ 
11:             $E_j^u := \{b \in D_j \mid \exists a \in E_i^u : (a, b) \in \delta\}$ 
12:            if  $E_j^1, \dots, E_j^t$  are disjoint then
13:               $I := I \cup \{j\}$ 
14:               $\text{Changed} := \text{true}$ 
15:              break
16:          until  $\neg \text{Changed}$ 
17:          for  $i \in I$  do
18:             $D'_i := \emptyset$ 
19:            for  $a \in D_i$  do
20:              Choose  $u$  such that  $a \in E_i^u$ 
21:              for  $j = 1, 2, \dots, n$  do
22:                if  $j = i$  then
23:                   $E_j := \{a\}$ 
24:                else if  $j \in I$  then
25:                   $E_j := E_j^u$ 
26:                else
27:                   $E_j := D_j$ 
28:               $\mathbf{X}' := \{x_i \mid i \in I\}$ 
29:              if  $\text{SOLVE}(\text{pr}_{\mathbf{X}'}(\text{REDUCE}(\Theta, (E_1, \dots, E_n)))) = \text{"Solution"}$  then
30:                 $D'_i := D'_i \cup \{a\}$ 
31:              if  $D'_i = \emptyset$  then return "No solution"
32:              else if  $D'_i \neq D_i$  then return  $(x_i, D'_i)$ 
return "Ok"

```

---



Define the function `CHECKWEAKERINSTANCE`, which checks that if we simultaneously weaken every constraint then the solution set of the obtained instance is subdirect. Thus, we weaken every constraint of  $\Theta$  (function `WEAKENEVERYCONSTRAINT` in the pseudocode), that is, we make a copy of  $\Theta$ , and replace each constraint by all weaker constraints without dummy variables. Recursively calling the algorithm, check that the obtained instance has a solution with  $x_i = b$  for every  $i \in \{1, 2, \dots, n\}$  and  $b \in D_i$ . If not, reduce  $D_i$  to the projection onto  $x_i$  of the solution set of the obtained instance. Otherwise, go on.

---

```

1: function CHECKWEAKERINSTANCE( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:    $\Theta' = \text{WEAKENEVERYCONSTRAINT}(\Theta)$ 
4:   for  $i = 1, \dots, n$  do
5:      $D'_i := \emptyset$ 
6:     for  $a \in D_i$  do
7:       Output := SOLVE(REDUCE( $\Theta'$ , ( $D_1, \dots, D_{i-1}, \{a\}, D_{i+1}, \dots, D_n$ )))
8:       if Output = "Solution" then
9:          $D'_i := D'_i \cup \{a\}$ 
10:    if  $D'_i = \emptyset$  then return "No solution"
11:    else if  $D'_i \neq D_i$  then return ( $x_i, D'_i$ )
    return "Ok"

```

---

It remains to define the function `SOLVENONLINKED`, which solves an instance that is not linked and not fragmented (see the pseudocode). Such an instance can be split into several instances on smaller domains. First, we consider the set  $\mathbf{X}'$  of all variables appearing in the constraints of the instance and take the projection of the instance onto  $\mathbf{X}'$ . Then we consider each linked component, that is, elements that can be connected by a path in the instance. Since the instance is cycle-consistent, the division into linked components defines a congruence on every domain (see Lemma 6.2), and each block of this congruence is a subuniverse of the domain. Thus, each linked component can be viewed as a CSP instance in a constraint language  $\Gamma$  on smaller domains, which can be solved using the recursion. If at least one of them has a solution, then the original instance has a solution.

---

```

1: function SOLVENONLINKED( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:    $\mathbf{X}' := \text{Var}(\Theta)$  ▷ Choose variables that appear in  $\Theta$ 
4:    $\Theta' := \text{pr}_{\mathbf{X}'}(\Theta)$  ▷ Remove variables that never occur
5:   for a linked component ( $D'_1, \dots, D'_{n'}$ ) of  $\Theta'$  do
6:     if SOLVE(REDUCE( $\Theta'$ , ( $D'_1, \dots, D'_{n'}$ ))) = "Solution" then return "Solution"
    return "No solution"

```

---

## 5 Correctness of the Algorithm

### 5.1 Rosenberg completeness theorem

The main idea of the algorithm is based on a beautiful result obtained by Ivo Rosenberg in 1970, who found all maximal clones on a finite set. Applying this result to the clone generated by a WNU together with all constant operations, we can show that every algebra with a WNU operation has a nontrivial binary absorbing subuniverse, or a nontrivial center, or it is polynomially complete or linear modulo some proper congruence.

**Theorem 5.1.** *Suppose  $\mathbf{A} = (A; w)$  is a finite algebra, where  $w$  is a special WNU of arity  $m$ . Then one of the following conditions holds:*

1. *there exists a nontrivial binary absorbing subuniverse  $B \subsetneq A$ ,*
2. *there exists a nontrivial center  $C \subsetneq A$ ,*
3. *there exists a proper congruence  $\sigma$  on  $A$  such that  $(A; w)/\sigma$  is polynomially complete,*
4. *there exists a proper congruence  $\sigma$  on  $A$  such that  $(A; w)/\sigma$  is isomorphic to  $(\mathbb{Z}_p; x_1 + \dots + x_m)$ .*

*Proof.* Let us prove this statement by induction on the size of  $A$ . If we have a nontrivial binary absorbing subuniverse in  $A$  then there is nothing to prove. Assume that  $A$  has no nontrivial binary absorbing subuniverse. Let  $M$  be the clone generated by  $w$  and all constant operations on  $A$ . If  $M$  is the clone of all operations, then  $(A; w)$  is polynomially complete.

Otherwise, by Rosenberg's Theorem [53],  $M$  belongs to one of the following maximal clones.

1. Maximal clone of monotone operations, that is, the clone of operations preserving a partial order relation with the greatest and the least element;
2. Maximal clone of autodual operations, that is, the clone of operations preserving the graph of a permutation of a prime order without a fixed element;
3. Maximal clone defined by an equivalence relation;
4. Maximal clone of quasi-linear operations;
5. Maximal clone defined by a central relation;
6. Maximal clone defined by an  $h$ -regularly generated (or  $h$ -universal) relation.

Let us consider all the cases.

1. As we assumed, there is no nontrivial binary absorbing subuniverse on  $A$ . Hence, the least element of the partial order can be viewed as a center by letting  $\mathbf{B} = \mathbf{A}$  and using the partial order relation as a subdirect subuniverse of  $\mathbf{A} \times \mathbf{B}$  (the least element is connected with all other elements in the partial order relation). Thus, we have a nontrivial center in  $A$ .
2. Constants are not autodual operations. This case cannot happen.
3. Let  $\delta$  be a maximal congruence on  $\mathbf{A}$ . We consider a factor algebra  $(A; w)/\delta$  and apply the inductive assumption.
  - (a) If  $\mathbf{A}/\delta$  has a binary absorbing subuniverse  $B' \subseteq A/\delta$ , then  $\bigcup_{E \in B'} E$  is a binary absorbing subuniverse of  $A$  with the same term operation.
  - (b) If  $\mathbf{A}/\delta$  has a nontrivial center  $C' \subseteq A/\delta$  witnessed by a subdirect relation  $R' \subseteq A/\delta \times B$ , then  $\bigcup_{E \in C'} E$  is a nontrivial center of  $A$  witnessed by  $R = \bigcup_{(E, b) \in R'} E \times \{b\}$ .
  - (c) Suppose  $(\mathbf{A}/\delta)/\sigma$  is polynomially complete. Since  $\delta$  is a maximal congruence,  $\sigma$  is the equality relation and  $\mathbf{A}/\delta$  is polynomially complete.

- (d) Suppose  $(\mathbf{A}/\delta)/\sigma$  is isomorphic to  $(\mathbb{Z}_p; x_1 + \dots + x_m)$ . Since  $\delta$  is a maximal congruence,  $\sigma$  is the equality relation and  $\mathbf{A}/\delta$  is isomorphic to  $(\mathbb{Z}_p; x_1 + \dots + x_m)$ .
4. By Lemma 6.4 from [62], we know that  $w(x_1, \dots, x_m) = x_1 + \dots + x_m$ , where  $+$  is the operation in an abelian group. We assume that  $\mathbf{A}$  has no nontrivial congruences, otherwise we refer to case (3). Then the algebra  $\mathbf{A}$  is simple and isomorphic to  $(\mathbb{Z}_p; x_1 + \dots + x_m)$  for a prime number  $p$ .
5. Let  $\rho$  be a central relation of arity  $k$  preserved by  $w$ . It is not hard to see that the existence of a nontrivial binary absorbing subuniverse on  $\underbrace{\mathbf{A} \times \dots \times \mathbf{A}}_{k-1}$  implies the existence of a nontrivial binary absorbing subuniverse on  $\mathbf{A}$  (see Lemma 7.3). Since there is no nontrivial binary absorbing subuniverse on  $\mathbf{A}$  and the relation  $\rho$  contains all tuples  $(b_1, \dots, b_k)$  such that  $b_1$  is from the center of  $\rho$ , the center of  $\rho$  is a center of  $A$  by letting  $\mathbf{B} = \underbrace{\mathbf{A} \times \dots \times \mathbf{A}}_{k-1}$ .
6. By Corollary 5.10 from [62] this case cannot happen.

□

## 5.2 The algorithm is polynomial

**Lemma 5.2.** *The depth of the recursion in the algorithm is less than  $|A| + |\Gamma|$ .*

*Proof.* We use the recursion in the functions SOLVELINEARCASE, FINDONEEQUATIONLINKED, CHECKALLTUPLES, CHECKIRREDUCIBILITY, CHECKWEAKERINSTANCE, SOLVENONLINKED. In each of them but CHECKWEAKERINSTANCE we reduce all domains of size greater than 1 before using the recursion and we never increase the domain. Therefore, every path in the recursion tree contains at most  $|A|$  calls of the function SOLVE in the above functions.

Let us consider the function CHECKWEAKERINSTANCE. First, we introduce a partial order on the set of relations in  $\Gamma$ . We say that  $\rho_1 \leq \rho_2$  if one of the following conditions hold

1. the arity of  $\rho_1$  is less than the arity of  $\rho_2$ .
2. the arities of  $\rho_1$  and  $\rho_2$  are equal,  $\text{pr}_i(\rho_1) \subseteq \text{pr}_i(\rho_2)$  for every  $i$ ,  $\text{pr}_j(\rho_1) \neq \text{pr}_j(\rho_2)$  for some  $j$ .
3. the arities of  $\rho_1$  and  $\rho_2$  are equal,  $\text{pr}_i(\rho_1) = \text{pr}_i(\rho_2)$  for every  $i$ , and  $\rho_1 \supseteq \rho_2$ .

We can check that in the algorithm we never make any relation bigger, and every time we use recursion in CHECKWEAKERINSTANCE we make every constraint relation strictly smaller. Since our constraint language  $\Gamma$  is finite, every path in the recursion tree contains at most  $|\Gamma|$  calls of the function SOLVE in CHECKWEAKERINSTANCE. Therefore the depth of the recursion tree is bounded by  $|A| + |\Gamma|$ . □

**Corollary 5.2.1.** *The algorithm is polynomial.*

*Proof.* Since the depth of the recursive tree is bounded by  $|A| + |\Gamma|$ , it remains to show that each loop in each function is polynomial.

In the function SOLVE we go through the loop at most  $n \cdot |A|$  times, which is polynomially many.

In the function SOLVELINEARCASE we go through the external **repeat** loop at most  $r$  times, where  $r$  is the dimension of  $L_1 \times \dots \times L_n$ . Therefore,  $r$  is bounded by  $|A| \cdot n$ . We go

through the inner **repeat** loop at most  $|\Gamma| \cdot N$  times, where  $N$  is the number of constraints of the instance.

In the function **CHECKCYCLECONSISTENCY** we go through the **repeat** loop at most  $|\Gamma| \cdot n^2$  times, because every time we change at least one relation  $\rho_{i,j}$ , which is from  $\Gamma$ , and we have  $n^2$  of them.

In the function **CHECKIRREDUCIBILITY** we go through the **repeat** loop at most  $n$  times, since we always add an element to  $I$ .

All other loops are **for** loops, and polynomial bounds for them follow from the description of the algorithm. Therefore, the algorithm is polynomial.  $\square$

### 5.3 Correctness of the auxiliary functions

**Lemma 5.3.** *If the function **CHECKCYCLECONSISTENCY** returns “Ok” then the instance is cycle-consistent, if it returns “No solution” then the instance has no solutions, if it returns  $(x_i, D)$  then any solution of the instance has  $x_i \in D$ .*

*Proof.* Assume that the function returned “Ok”. Since every relation  $\rho_{i,j}$  in the end of the algorithm is subdirect, the instance is 1-consistent. Consider a path  $x_{i_1} - C_1 - x_{i_2} - \dots - x_{i_{l-1}} - C_{l-1} - x_{i_l}$  starting and ending with  $x_{i_1} = x_{i_l}$ . Since the projection of  $C_j$  onto  $x_{i_j}, x_{i_{j+1}}$  contains  $\rho_{i_j, i_{j+1}}$  for every  $j$ , to show that the instance is cycle-consistent, it is sufficient to prove that the formula

$$\delta(x_{i_1}) = \exists x_{i_2} \dots \exists x_{i_{l-1}} \rho_{i_1, i_2}(x_{i_1}, x_{i_2}) \wedge \dots \wedge \rho_{i_{l-1}, i_l}(x_{i_{l-1}}, x_{i_l})$$

defines  $D_{i_1}$ . This follows from the fact that we terminated the function when for all  $i, j, k$

$$\rho_{i,j}(x, y) = \exists z \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y).$$

The remaining part follows from the fact that all the constraints  $\rho_{i,j}(x_i, x_j)$  were derived from the original constraints, and therefore they should hold for any solution.  $\square$

**Lemma 5.4.** *If the function **CHECKIRREDUCIBILITY** returns “Ok” then the instance is irreducible, if it returns “No solution” then the instance has no solutions, if it returns  $(x_i, D)$  then any solution of the instance has  $x_i \in D$ .*

*Proof.* Assume that **CHECKIRREDUCIBILITY** returned “Ok” but the instance is not irreducible. Then, there exists an instance  $\Theta'$  such that every constraint of  $\Theta'$  is a projection of a constraint from the original instance  $\Theta$  on some set of variables, and  $\Theta'$  is not fragmented, not linked, and its solution set is not subdirect. Let  $\mathbf{X}'$  be the set of all variables occurring in  $\Theta'$ . Choose a variable  $x_k \in \mathbf{X}'$ . If we consider the set of all pairs  $(a, b) \in D_k^2$  such that  $a$  and  $b$  can be connected by a path in  $\Theta'$  then we get a congruence (see Lemma 6.2). Since  $\Theta'$  is not linked, there should be a maximal congruence  $\sigma_k$  containing the congruence. This congruence was chosen in the line 4 of the pseudocode.

Since  $\Theta'$  is not fragmented, there exists a path in  $\Theta'$  from  $x_k$  to any other variable from  $\mathbf{X}'$ . Following this path we can always define a partition on the next variable using the partition on the previous one. Since every constraint of  $\Theta'$  is a projection of a constraint from  $\Theta$ , we could define the same partitions on  $\Theta$  (see the pseudocode of the function). We just need to show that on every domain  $D_i$  we can generate a unique partition using  $\sigma_k$  (the order in which we add elements to  $I$  and the way how we choose constraints is not important). Consider two paths from  $x_k$  to  $x_i$  defining two partitions. We glue together the beginnings of these paths and get a path from  $x_i$  to  $x_i$  connecting these partitions. Since the instance is cycle-consistent, these partitions should be equal. Thus, we showed that starting from the congruence  $\sigma_k$  (in

the pseudocode) we get a unique partition on every variable  $x_i \in \mathbf{X}'$ . Therefore, we actually checked in the algorithm that the solution set of  $\Theta'$  is subdirect, which gives us a contradiction. Hence,  $\Theta$  is irreducible.

The remaining part follows from the fact that  $D'_i$  is the set of all possible evaluations of  $x_i$  in solutions of a weaker instance.  $\square$

## 5.4 Main theorems without a proof

To explain the correctness of the algorithm in Section 4 we used the following main facts, which will be proved in Section 9.

**Theorem 5.5.** *Suppose  $\Theta$  is a cycle-consistent irreducible CSP instance, and  $B$  is a nontrivial binary absorbing subuniverse or a nontrivial center of  $D_i$ . Then  $\Theta$  has a solution if and only if  $\Theta$  has a solution with  $x_i \in B$ .*

**Theorem 5.6.** *Suppose  $\Theta$  is a cycle-consistent irreducible CSP instance, there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on  $D_j$  for every  $j$ ,  $(D_i; w)/\sigma$  is a polynomially complete algebra, and  $E$  is an equivalence class of  $\sigma$ . Then  $\Theta$  has a solution if and only if  $\Theta$  has a solution with  $x_i \in E$ .*

**Theorem 5.7.** *Suppose the following conditions hold:*

1.  $\Theta$  is a linked cycle-consistent irreducible CSP instance with domain set  $(D_1, \dots, D_n)$ ;
2. there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on  $D_j$  for every  $j$ ;
3. if we replace every constraint of  $\Theta$  by all weaker constraints then the obtained instance has a solution with  $x_i = b$  for every  $i$  and  $b \in D_i$  (the obtained instance has a subdirect solution set);
4.  $L_i = D_i/\sigma_i$  for every  $i$ , where  $\sigma_i$  is the minimal linear congruence on  $D_i$ ;
5.  $\phi : \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k} \rightarrow L_1 \times \dots \times L_n$  is a homomorphism, where  $q_1, \dots, q_k$  are prime numbers;
6. if we replace any constraint of  $\Theta$  by all weaker constraints then for every  $(a_1, \dots, a_k) \in \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$  there exists a solution of the obtained instance in  $\phi(a_1, \dots, a_k)$ .

Then  $\{(a_1, \dots, a_k) \mid \Theta \text{ has a solution in } \phi(a_1, \dots, a_k)\}$  is either empty, or is full, or is an affine subspace of  $\mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$  of codimension 1 (the solution set of a single linear equation).

## 6 The Remaining Definitions

### 6.1 Variety of algebras

We consider the variety of all algebras  $\mathbf{A} = (A; w)$  such that  $w$  is a special WNU operation of arity  $m$ . As it was mentioned in Section 3 every domain  $D$  will be viewed as a finite algebra  $(D; w)$  from this variety. Note that in the remainder of this paper any claim or assumption “ $\rho$  is a relation” should be understood as “ $\rho$  is a subalgebra of  $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ ” for the corresponding finite algebras  $\mathbf{A}_1, \dots, \mathbf{A}_n$  from this variety.

## 6.2 Additional notations

For a relation  $\rho \subseteq A_1 \times \dots \times A_n$  and a congruence  $\sigma$  on  $A_i$ , we say that the  $i$ -th variable of the relation  $\rho$  is *stable under  $\sigma$*  if  $(a_1, \dots, a_n) \in \rho$  and  $(a_i, b_i) \in \sigma$  imply  $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$ . We say that a relation is *stable under  $\sigma$*  if every variable of this relation is stable under  $\sigma$ .

We say that a congruence  $\sigma$  is *irreducible* if it is proper and it cannot be represented as an intersection of other binary relations  $\delta_1, \dots, \delta_s$  stable under  $\sigma$ . For an irreducible congruence  $\sigma$  on a set  $A$  by  $\sigma^*$  we denote the minimal binary relation  $\delta \supsetneq \sigma$  stable under  $\sigma$ .

For a relation  $\rho$  by  $\text{Con}(\rho, i)$  we denote the binary relation  $\sigma(y, y')$  defined by

$$\exists x_1 \dots \exists x_{i-1} \exists x_{i+1} \dots \exists x_n \rho(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \wedge \rho(x_1, \dots, x_{i-1}, y', x_{i+1}, \dots, x_n).$$

For a constraint  $C = \rho(x_1, \dots, x_n)$  by  $\text{Con}(C, x_i)$  we denote  $\text{Con}(\rho, i)$ . For a set of constraints  $\Omega$  by  $\text{Con}(\Omega, x)$  we denote the set  $\{\text{Con}(C, x) \mid C \in \Omega\}$ .

A congruence  $\sigma$  on  $\mathbf{A}$  is called a *PC congruence* if  $\mathbf{A}/\sigma$  is a PC algebra without a nontrivial binary absorbing subuniverse or center. For an algebra  $\mathbf{A}$  by  $\text{ConPC}(\mathbf{A})$  we denote the intersection of all PC congruences. A subuniverse  $A' \subseteq A$  is called a *PC subuniverse* if  $A' = E_1 \cap \dots \cap E_s$ , where each  $E_i$  is an equivalence class of a PC congruence. Note that a PC subuniverse can be empty or full.

A congruence  $\sigma$  on  $\mathbf{A}$  is called *linear* if  $\mathbf{A}/\sigma$  is a linear algebra. For an algebra  $\mathbf{A}$  by  $\text{ConLin}(\mathbf{A})$  we denote the minimal linear congruence. A subuniverse of  $\mathbf{A}$  is called a *linear subuniverse* if it is stable under  $\text{ConLin}(\mathbf{A})$ . Note that we could not define a PC subuniverse in the same way because not every subuniverse stable under  $\text{ConPC}(\mathbf{A})$  is a PC subuniverse of  $\mathbf{A}$  (see Subsection 7.3).

A subuniverse  $B \subseteq A$  is called a *one-of-four subuniverse* if it is a binary absorbing subuniverse, a center, a PC subuniverse, or a linear subuniverse. We say that  $B$  is a one-of-four subuniverse of *absorbing type*, *central type*, *PC type*, or *linear type*, respectively. A subuniverse of type  $\mathcal{T}$  is called *minimal* if it is a minimal nontrivial subuniverse of this type. Note that a minimal PC/linear subuniverse is a block of  $\text{ConPC}(\mathbf{A})/\text{ConLin}(\mathbf{A})$ .

## 6.3 pp-formula, subconstraint, coverings

Every variable  $x$  appearing in the paper has its domain, which we denote by  $D_x$ . In the paper we usually identify a CSP instance and a set of constraints. For an instance  $\Omega$  by  $\text{Var}(\Omega)$  we denote the set of all variables occurring in constraints of  $\Omega$  (the set of all variables  $\mathbf{X}$  is not important, all the properties of the instance depend only on the variables that actually occur in the instance). For an instance  $\Omega$  and two sets of variables  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  by  $\Omega_{x_1, \dots, x_n}^{y_1, \dots, y_n}$  we denote the instance obtained from  $\Omega$  by replacement of every variable  $x_i$  by  $y_i$ .

Sometimes we write an instance  $\{C_1, \dots, C_n\}$  as a conjunctive formula  $C_1 \wedge \dots \wedge C_n$ . We say that an instance is a *tree-formula* if there is no a path  $z_1 - C_1 - z_2 - \dots - z_{l-1} - C_{l-1} - z_l$  such that  $l \geq 3$ ,  $z_1 = z_l$ , and all the constraints  $C_1, \dots, C_{l-1}$  are different.

An expression  $\exists y_1 \dots \exists y_s (C_1 \wedge \dots \wedge C_n)$  is called a *positive primitive formula (pp-formula)*. To simplify, we use a notation  $\Omega(x_1, \dots, x_n)$  to write the pp-formula  $\exists y_1 \dots \exists y_s \Omega$ , where  $\Omega$  is an instance (or a conjunction of constraints) and  $y_1, \dots, y_s$  are all variables occurring in  $\Omega$  except for  $x_1, \dots, x_n$ . Then, we say that a pp-formula  $\Omega(x_1, \dots, x_n)$  defines a relation  $\rho$  if  $\rho(x_1, \dots, x_n) = \exists y_1 \dots \exists y_s \Omega$ . Sometimes, if it is convenient, we write  $\Omega(x_1, \dots, x_n)$  meaning the relation defined by the pp-formula. A pp-formula  $\Omega(x_1, \dots, x_n)$  is called a *subconstraint of  $\Theta$*  if  $\Omega \subseteq \Theta$ , and  $\Omega$  and  $\Theta \setminus \Omega$  do not have common variables except for  $x_1, \dots, x_n$ . Note that all relations that can be defined by a pp-formula are preserved by the WNU (see [32, 13, 14]).

For a formula  $\Omega$  by  $\text{Coverings}(\Omega)$  we denote the set of all formulas  $\Omega'$  such that there exists a mapping  $S : \text{Var}(\Omega') \rightarrow \text{Var}(\Omega)$  satisfying the following conditions:

1. the domain of any variable  $x$  from  $\Omega'$  is equal to the domain of  $S(x)$  in  $\Omega$ ;
2. for every constraint  $((x_1, \dots, x_n); \rho)$  of  $\Omega'$ ,  $((S(x_1), \dots, S(x_n)); \rho)$  is a constraint of  $\Omega$ ;
3. if a variable  $x$  appears in both  $\Omega$  and  $\Omega'$  then  $S(x) = x$ .

Similarly, by  $\text{ExpCov}(\Omega)$  (*Expanded Coverings*) we denote the set of all formulas  $\Omega'$  such that there exists a mapping  $S : \text{Var}(\Omega') \rightarrow \text{Var}(\Omega)$  satisfying the following conditions:

1. the domain of any variable  $x$  from  $\Omega'$  is equal to the domain of  $S(x)$  in  $\Omega$ ;
2. for every constraint  $((x_1, \dots, x_n); \rho)$  of  $\Omega'$  either the variables  $S(x_1), \dots, S(x_n)$  are different and the constraint  $((S(x_1), \dots, S(x_n)); \rho)$  is weaker or equivalent to some constraint of  $\Omega$ , or  $S(x_1) = \dots = S(x_n)$  and  $\{(a, a, \dots, a) \mid a \in D_{x_1}\} \subseteq \rho$ ;
3. if a variable  $x$  appears in both  $\Omega$  and  $\Omega'$  then  $S(x) = x$ .

For a variable  $x$  we say that  $S(x)$  is *the parent of  $x$* .

The following easy facts about coverings can be derived from the definition.

1. every time we replace some constraints by weaker constraints we get an expanded covering of the original instance;
2. any solution of the original instance can be naturally expanded to a solution of a covering (expanded covering);
3. suppose  $\Omega$  is a covering (expanded covering) of a 1-consistent instance and  $\Omega$  is a tree-formula, then the solution set of  $\Omega$  is subdirect;
4. the union (union of all constraints) of two coverings (expanded coverings) is also a covering (expanded covering);
5. a covering (expanded covering) of a covering (expanded covering) is a covering (expanded covering).

Another important property is formulated in the following lemma.

**Lemma 6.1.** *Suppose  $\Theta$  is a cycle-consistent irreducible CSP instance and  $\Theta' \in \text{ExpCov}(\Theta)$ . Then  $\Theta'$  is cycle-consistent and irreducible.*

*Proof.* Let us prove that  $\Theta'$  is cycle-consistent. Consider a path in  $\Theta'$  starting and ending with  $z$ . Since  $\Theta'$  is an expanded covering, for every constraint of  $\Theta'$  either there exists a corresponding constraint in  $\Theta$ , or this constraint is reflexive (contains all tuples  $(a, a, \dots, a)$ ). Thus, to transform the path in  $\Theta'$  to a path in  $\Theta$  it is sufficient to replace every variable  $x$  in the path by  $S(x)$  (from the definition of expanded coverings), remove all reflexive constraints, and replace the remaining constraints by the corresponding constraints from  $\Theta$ . Since  $\Theta$  is cycle-consistent, the obtained path connects  $a$  with  $a$  for any  $a \in D_z$ . Since constraints in the path in  $\Theta'$  are weaker or equivalent to constraints in the path in  $\Theta$  and relations we removed are reflexive, the path in  $\Theta'$  also connects  $a$  with  $a$  for every  $a \in D_z$ .

Let us show that  $\Theta'$  is irreducible. Assume the converse, then there exists an instance  $\Omega'$  consisting of projections of constraints from  $\Theta'$  that is not linked, not fragmented, and its solution set is not subdirect. By  $\Omega$  we denote the set of corresponding projections of constraints from  $\Theta$  corresponding to the constraints of  $\Omega'$  (we ignore reflexive constraints from  $\Omega'$ ). To be more accurate, suppose a constraint  $C'' \in \Omega'$  is equal to  $\text{pr}_{\mathbf{X}}(C')$  for a constraint  $C' \in \Theta'$

and a set of variable  $\mathbf{X}$ , and  $C'$  is weaker or equivalent to a constraint  $C \in \Theta$ . Then we add the constraint  $\text{pr}_{S(\mathbf{X})}(C)$  to  $\Omega$ .

Let us show that  $\Omega$  is not linked. Assume the contrary. For any path in  $\Omega$  connecting elements  $a$  and  $b$  of  $D_x$  we can build a path connecting  $a$  and  $b$  in  $\Omega'$  in the following way. We replace every constraint of  $\Omega$  by the corresponding constraint of  $\Omega'$ , and glue them with any path in  $\Omega'$  starting and ending with the corresponding variables having the same parent. Since  $\Omega'$  is not fragmented, we can always do this. Since  $\Omega$  is cycle-consistent, the obtained path connects  $a$  and  $b$  in  $\Omega'$ . Thus,  $\Omega$  is not linked. Any solution of  $\Omega$  can be naturally extended to a solution of  $\Omega'$ , hence the solution set of  $\Omega$  cannot be subdirect. Since  $\Omega'$  is not fragmented,  $\Omega$  is also not fragmented. Thus,  $\Omega$  is not linked, not fragmented, and its solution set is not subdirect, which contradicts the fact that  $\Theta$  is irreducible.  $\square$

For an instance  $\Theta$  and its variable  $x$  by  $\text{LinkedCon}(\Theta, x)$  we denote the binary relation on the set  $D_x$  defined as follows:  $(a, b) \in \text{LinkedCon}(\Theta, x)$  if there exists a path in  $\Theta$  that connects  $a$  and  $b$ .

**Lemma 6.2.** *Suppose  $\Theta$  is a cycle-consistent CSP instance,  $x \in \text{Var}(\Theta)$ . Then there exists a path in  $\Theta$  connecting all pairs  $(a, b) \in \text{LinkedCon}(\Theta, x)$  and  $\text{LinkedCon}(\Theta, x)$  is a congruence.*

*Proof.* Since the instance is cycle-consistent, gluing all the paths starting and ending at  $x$  we can build a path connecting all pairs  $(a, b) \in \text{LinkedCon}(\Theta, x)$ . The set of all pairs  $(a, b)$  connected by this path can be defined by a pp-formula, therefore it is an invariant relation, which is also reflexive (by cycle-consistency) and transitive (we can glue paths).  $\square$

## 6.4 Critical, key relations, and parallelogram property

We say that a relation  $\rho$  has the *parallelogram property* if any permutation of its variables gives a relation  $\rho'$  satisfying

$$\forall \alpha_1, \beta_1, \alpha_2, \beta_2: (\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2 \in \rho' \Rightarrow \alpha_1\alpha_2 \in \rho').$$

Note that the parallelogram property plays an important role in universal algebra (see [40] for more details).

We say that the  *$i$ -th variable of a relation  $\rho$  is rectangular*, if for every  $(a_i, b_i) \in \text{Con}(\rho, i)$  and  $(a_1, \dots, a_n) \in \rho$  we have  $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$ . We say that a relation is *rectangular* if all of its variables are rectangular. The following facts can be easily seen: if the  $i$ -th variable of a subdirect relation  $\rho$  is rectangular then  $\text{Con}(\rho, i)$  is a congruence; if a relation has the parallelogram property then it is rectangular.

A relation  $\rho \subseteq A_1 \times \dots \times A_n$  is called *essential* if it cannot be represented as a conjunction of relations with smaller arities. It is easy to see that any relation  $\rho$  can be represented as a conjunction of essential relations that are projections of  $\rho$  on some sets of variables (See Lemma 4.2 in [59]).

A relation  $\rho \subseteq A_1 \times \dots \times A_n$  is called *critical* if it cannot be represented as an intersection of other subalgebras of  $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$  and it has no dummy variables. This notion was introduced in [40] but appeared in [61, 57] by the name maximal. For a critical relation  $\rho$  the minimal relation  $\rho'$  (a subalgebra of  $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ ) such that  $\rho' \supsetneq \rho$  is called *the cover of  $\rho$* .

Suppose  $\rho \subseteq A_1 \times \dots \times A_h$ . A tuple  $\Psi = (\psi_1, \psi_2, \dots, \psi_h)$ , where  $\psi_i : A_i \rightarrow A_i$ , is called a *unary vector-function*. We say that  $\Psi$  *preserves  $\rho$*  if  $\Psi \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} := \begin{pmatrix} \psi_1(a_1) \\ \psi_2(a_2) \\ \vdots \\ \psi_h(a_h) \end{pmatrix} \in \rho$  for every

$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} \in \rho$ . We say that  $\rho$  is a *key relation* if there exists a tuple  $\beta \in (A_1 \times \dots \times A_h) \setminus \rho$  such



that for every  $\alpha \in (A_1 \times \cdots \times A_n) \setminus \rho$  there exists a vector-function  $\Psi$  which preserves  $\rho$  and gives  $\Psi(\alpha) = \beta$ . A tuple  $\beta$  is called a *key tuple* for  $\rho$ . The notion key relation was introduced in [62], where such relations were characterized for all algebras having a WNU term operation.

A constraint is called *critical/essential/key* if the constraint relation is critical/essential/key. The notions critical, crucial, essential, and key relation are related to each other, namely, we can observe:

1. if  $C$  is a constraint in a CSP instance and  $C$  is crucial in some  $(D_1, \dots, D_n)$  then the constraint relation of  $C$  is critical;
2. every critical relation of arity greater than 1 is essential;
3. every critical relation of arity greater than 1 is a key relation (see Lemma 2.4 in [62]).

The notions essential, critical, and key relations (see [62] for their comparison) proved their efficiency in clone theory and universal algebra (see [61, 57, 59, 60, 58, 40]). Instead of considering all relations we consider only relations with one of these properties, and this is still the general case because any relation can be represented as a conjunction of essential/key/critical relations. For instance, we can always assume that all constraint relations are critical.

## 6.5 Reductions

Suppose the domain set of an instance  $\Theta$  is  $D = (D_1, \dots, D_n)$ . A domain set  $D' = (D'_1, \dots, D'_n)$  is called a *reduction of  $\Theta$*  if  $D'_i$  is a subuniverse of  $D_i$  for every  $i$ . Note that to avoid unnecessary bold font starting at this subsection we do not use it for domain sets. Thus, every time we write  $D$  without a subscript we mean a domain set or a reduction. Note that any reduction of  $\Theta$  can be naturally extended to a covering (expanded covering) of  $\Theta$ , thus we assume that any reduction is automatically defined on any covering (expanded covering).

A reduction  $D' = (D'_1, \dots, D'_n)$  is called *1-consistent* if the instance obtained after reduction of every domain is 1-consistent.

We say that  $D'$  is an *absorbing reduction*, if there exists a term operation  $t$  such that  $D'_i$  is a binary absorbing subuniverse of  $D_i$  with the term operation  $t$  for every  $i$ . We say that  $D'$  is a *central reduction*, if  $D'_i$  is a center of  $D_i$  for every  $i$ . We say that  $D'$  is a *PC/linear reduction*, if  $D'_i$  is a PC/linear subuniverse of  $D_i$  and  $D_i$  does not have a nontrivial binary absorbing subuniverse or a nontrivial center for every  $i$ . Additionally, we say that  $D'$  is a *minimal central/PC/linear reduction* if  $D'$  is a minimal center/PC/linear subuniverse of  $D_i$  for every  $i$ . We say that  $D'$  is a *minimal absorbing reduction* for a term operation  $t$  if  $D'$  is a minimal absorbing subuniverse of  $D_i$  with  $t$  for every  $i$ .

A reduction is called *nonlinear* if it is an absorbing, central, or PC reduction. A reduction  $D'$  is called *one-of-four reduction* if it is an absorbing, central, PC, or linear reduction such that  $D' \neq D$ .

We usually denote reductions by  $D^{(j)}$  for some  $j$  (or by  $D^{(\top)}$ ). In this case by  $C^{(j)}$  we denote the constraint obtained after the reduction of the constraint  $C$ . Similarly, by  $\Theta^{(j)}$  we denote the instance obtained after the reduction of every constraint of  $\Theta$ . For a relation  $\rho$  by  $\rho^{(j)}$  we denote the relation  $\rho$  restricted to the corresponding domains of  $D^{(j)}$ . Sometimes we write  $(a_1, \dots, a_n) \in D^{(j)}$  meaning that every  $a_i$  belongs to the corresponding  $D_x^{(j)}$ .

A *strategy* for a CSP instance  $\Theta$  with a domain set  $D$  is a sequence of reductions  $D^{(0)}, \dots, D^{(s)}$ , where  $D^{(j)} = (D_1^{(j)}, \dots, D_n^{(j)})$ , such that  $D^{(0)} = D$  and  $D^{(j)}$  is a one-of-four 1-consistent reduction of  $\Theta^{(j-1)}$  for every  $j \geq 1$ . A strategy is called *minimal* if every reduction in the sequence is minimal.

## 6.6 Bridges

Suppose  $\sigma_1$  and  $\sigma_2$  are congruences on  $D_1$  and  $D_2$ , respectively. A relation  $\rho \subseteq D_1^2 \times D_2^2$  is called a *bridge* from  $\sigma_1$  to  $\sigma_2$  if the first two variables of  $\rho$  are stable under  $\sigma_1$ , the last two variables of  $\rho$  are stable under  $\sigma_2$ ,  $\text{pr}_{1,2}(\rho) \supsetneq \sigma_1$ ,  $\text{pr}_{3,4}(\rho) \supsetneq \sigma_2$ , and  $(a_1, a_2, a_3, a_4) \in \rho$  implies

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$

An example of a bridge is the relation  $\rho = \{(a_1, a_2, a_3, a_4) \mid a_1, a_2, a_3, a_4 \in \mathbb{Z}_4 : a_1 - a_2 = 2a_3 - 2a_4\}$ . We can check that  $\rho$  is a bridge from the equality relation (0-congruence) and (*mod* 2) equivalence relation. For example, we have  $\text{pr}_{1,2}\rho$  is (*mod* 2)-equivalence relation,  $\text{pr}_{3,4}\rho$  is full relation.

The notion of a bridge is strongly related to other notions in Universal Algebra and Tame Congruence Theory such as similarity and centralizers (see [56] for the detailed comparison).

For a bridge  $\rho$  by  $\tilde{\rho}$  we denote the binary relation defined by  $\tilde{\rho}(x, y) = \rho(x, x, y, y)$ .

The following lemma shows how we can compose bridges.

**Lemma 6.3.** *Suppose  $\sigma_1, \sigma_2, \sigma_3$  are irreducible congruences,  $\rho_1$  is a bridge from  $\sigma_1$  to  $\sigma_2$ ,  $\rho_2$  is a bridge from  $\sigma_2$  to  $\sigma_3$ . Then the formula*

$$\rho(x_1, x_2, z_1, z_2) = \exists y_1 \exists y_2 \rho_1(x_1, x_2, y_1, y_2) \wedge \rho_2(y_1, y_2, z_1, z_2)$$

*defines a bridge from  $\sigma_1$  to  $\sigma_3$ . Moreover,  $\tilde{\rho} = \tilde{\rho}_1 \circ \tilde{\rho}_2$ .*

*Proof.* Stability of the first two variables under  $\sigma_1$  and of the last two variables under  $\sigma_3$  follows from the definition.

Let us prove that  $\text{pr}_{1,2}(\rho) \supsetneq \sigma_1$  (the inclusion  $\text{pr}_{3,4}(\rho) \supsetneq \sigma_3$  can be proved in the same way). By the definition, for every  $a$  there exists  $b$  such that  $(a, a, b, b) \in \rho_1$ , and for every  $b$  there exists  $c$  such that  $(b, b, c, c) \in \rho_2$ . Then  $(a, a, c, c) \in \rho$ , and since the first two variables of  $\rho_1$  are stable under  $\sigma_1$  we obtain  $\text{pr}_{1,2}(\rho) \supseteq \sigma_1$ . Since  $\sigma_2$  is irreducible,  $\text{pr}_{3,4}(\rho_1) \supseteq \sigma_2^*$  and  $\text{pr}_{1,2}(\rho_2) \supseteq \sigma_2^*$ . Choose  $(b_1, b_2) \in \sigma_2^*$ , then there exist  $a_1, a_2, c_1, c_2$  such that  $(a_1, a_2, b_1, b_2) \in \rho_1$  and  $(b_1, b_2, c_1, c_2) \in \rho_2$ . Then  $(a_1, a_2, c_1, c_2) \in \rho$ , which means that  $\text{pr}_{1,2}(\rho) \supsetneq \sigma_1$ .

Suppose  $(a_1, a_2, c_1, c_2) \in \rho$ . If  $(a_1, a_2) \in \sigma_1$  then, since  $\rho_1$  is a bridge, the corresponding values of  $y_1$  and  $y_2$  are equivalent modulo  $\sigma_2$ . Since  $\rho_2$  is a bridge we obtain that  $c_1$  and  $c_2$  are equivalent modulo  $\sigma_3$ .

The equation  $\tilde{\rho} = \tilde{\rho}_1 \circ \tilde{\rho}_2$  follows directly from the definition of  $\rho$ .  $\square$

A bridge  $\rho \subseteq D^4$  is called *reflexive* if  $(a, a, a, a) \in \rho$  for every  $a \in D$ .

We say that two congruences  $\sigma_1$  and  $\sigma_2$  on a set  $D$  are *adjacent* if there exists a reflexive bridge from  $\sigma_1$  to  $\sigma_2$ .

**Remark 4.** *Since we can always put  $\rho(x_1, x_2, x_3, x_4) = \sigma(x_1, x_3) \wedge \sigma(x_2, x_4)$ , any proper congruence  $\sigma$  is adjacent with itself.*

A reflexive bridge  $\rho$  from an irreducible congruence  $\sigma_1$  to an irreducible congruence  $\sigma_2$  is called *optimal* if there does not exist a reflexive bridge  $\rho'$  from  $\sigma_1$  to  $\sigma_2$  such that  $\tilde{\rho}' \supsetneq \tilde{\rho}$ . Suppose  $\rho$  is a reflexive bridge from  $\sigma_1$  to  $\sigma_2$ . then we can build a new bridge

$$\rho'(x_1, x_2, y_1, y_2) = \exists x'_1 \exists x'_2 \exists y'_1 \exists y'_2 [\rho(x_1, x_2, y'_1, y'_2) \wedge \rho(x'_1, x'_2, y'_1, y'_2) \wedge \rho(x'_1, x'_2, y_1, y_2)]$$

from  $\sigma_1$  to  $\sigma_2$  such that  $\tilde{\rho}' = \tilde{\rho} \circ \tilde{\rho}^{-1} \circ \tilde{\rho}$ . Note that because of the reflexivity,  $\tilde{\rho}$  contains the equality relation. Thus, if  $\rho$  is optimal, then  $\tilde{\rho}$  is a congruence. For an irreducible congruence  $\sigma$  by  $\text{Opt}(\sigma)$  we denote the congruence  $\tilde{\rho}$  for an optimal bridge  $\rho$  from  $\sigma$  to  $\sigma$ . Since we can compose two reflexive bridges,  $\text{Opt}(\sigma)$  is unique and therefore well-defined. For a set of irreducible congruences  $\mathfrak{C}$  put  $\text{Opt}(\mathfrak{C}) = \{\text{Opt}(\sigma) \mid \sigma \in \mathfrak{C}\}$ .

**Lemma 6.4.** *Suppose  $\sigma_1$  and  $\sigma_2$  are irreducible adjacent congruences. Then  $\text{Opt}(\sigma_1) = \text{Opt}(\sigma_2)$ .*

*Proof.* Let  $\rho_1$  be an optimal bridge from  $\sigma_1$  to  $\sigma_1$ ,  $\rho_2$  be an optimal bridge from  $\sigma_2$  to  $\sigma_2$ , and  $\rho$  be a reflexive bridge from  $\sigma_1$  to  $\sigma_2$ .

Assume that  $\text{Opt}(\sigma_2) \not\subseteq \text{Opt}(\sigma_1)$ , that is  $\tilde{\rho}_2 \not\subseteq \tilde{\rho}_1$ . Using Lemma 6.3, we compose bridges  $\rho_1$ ,  $\rho$ ,  $\rho_2$ , and  $\rho$  (in this order) to obtain a reflexive bridge  $\rho'_1$  from  $\sigma_1$  to  $\sigma_1$ . Since  $\tilde{\rho}'_1 \supseteq \tilde{\rho}_1 \cup \tilde{\rho}_2$ , we get a contradiction with the fact that  $\rho_1$  is optimal.  $\square$

We say that two rectangular constraints  $C_1$  and  $C_2$  are *adjacent* in a common variable  $x$  if  $\text{Con}(C_1, x)$  and  $\text{Con}(C_2, x)$  are adjacent. A formula is called *connected* if every constraint in the formula is critical and rectangular, and the graph, whose vertexes are constraints and edges are adjacent constraints, is connected. Note that this connectedness is not related to the paths from one variable to another connecting two elements. Recall that if for every  $a, b$  there exists a path that connects  $a$  and  $b$ , then the instance is called linked (see Section 3.6).

It can be shown (see Corollary 8.22.1) that every two constraints with a common variable in a connected instance are adjacent.

## 7 Absorption, Center, PC Congruence, and Linear Congruence

### 7.1 Binary Absorption

**Lemma 7.1.** [1] *Suppose  $\rho$  is defined by a pp-formula  $\Omega(x_1, \dots, x_n)$  and  $\Omega'$  is obtained from  $\Omega$  by replacement of some constraint relations  $\sigma_1, \dots, \sigma_s$  by constraint relations  $\sigma'_1, \dots, \sigma'_s$  such that  $\sigma'_i$  absorbs  $\sigma_i$  with a term operation  $t$  for every  $i$ . Then the relation defined by  $\Omega'(x_1, \dots, x_n)$  absorbs  $\rho$  with the term operation  $t$ .*

**Corollary 7.1.1.** *Suppose  $\theta$  is a congruence of  $A$ .*

1. *If  $B$  is an absorbing subuniverse of  $A$ , then  $\{b/\theta \mid b \in B\}$  is an absorbing subuniverse of  $A/\theta$  with the same term.*
2. *If  $A$  has no nontrivial (binary) absorbing subuniverse, then neither does  $A/\theta$ .*

**Corollary 7.1.2.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a relation such that  $\text{pr}_1(\rho) = A_1$  and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is an absorbing subuniverse in  $A_i$  with a term  $t$  for every  $i$ . Then  $C$  is an absorbing subuniverse in  $A_1$  with the term  $t$ .*

*Proof.* It is not hard to see that the sets  $C$  and  $A_1$  can be defined by the following pp-formulas

$$(x_1 \in C) = \exists x_2 \dots \exists x_n [(x_1 \in C_1) \wedge \dots \wedge (x_n \in C_n) \wedge \rho(x_1, \dots, x_n)],$$

$$(x_1 \in A_1) = \exists x_2 \dots \exists x_n [(x_1 \in A_1) \wedge \dots \wedge (x_n \in A_n) \wedge \rho(x_1, \dots, x_n)].$$

It remains to apply Lemma 7.1.  $\square$

**Lemma 7.2.** *Suppose  $\kappa_A \subseteq A \times A$  is the equality relation,  $\sigma \supseteq \kappa_A$ , and  $\omega$  is a nontrivial binary absorbing subuniverse in  $\sigma$ . Then  $\omega \cap \kappa_A \neq \emptyset$ .*

*Proof.* We prove the lemma by induction on the size of  $A$ . Suppose  $\omega$  absorbs  $\sigma$  with a binary absorbing term operation  $f$ .

Assume that there exists a nontrivial binary absorbing subuniverse  $B \subsetneq A$  with the absorbing operation  $f$ . For any  $(b_1, b_2) \in \omega$  and  $b \in B$  we have  $(f(b_1, b), f(b_2, b)) \in \omega \cap (B \times B)$ . Then by Lemma 7.1,  $\omega \cap (B \times B)$  is a nontrivial absorbing subuniverse in  $\sigma \cap (B \times B)$ , and we can restrict  $\sigma$  and  $\omega$  to  $B$  and apply the inductive assumption.

Thus, we assume that there does not exist a nontrivial binary absorbing subuniverse  $B \subsetneq A$  with the absorbing operation  $f$ . By Lemma 7.1,  $\text{pr}_1(\omega)$  and  $\text{pr}_2(\omega)$  binary absorb  $A$ , then  $\text{pr}_1(\omega) = \text{pr}_2(\omega) = A$ . Now, the statement of the lemma could be derived from [5, Theorem 6] but we will finish the argument because it is simple.

For every  $b \in A$  we consider  $A_b = \{a \mid (a, b) \in \sigma\}$  and  $C_b = \{a \mid (a, b) \in \omega\}$ . Since  $\text{pr}_2(\omega) = A$ ,  $C_b \neq \emptyset$  for every  $b$ . By Lemma 7.1  $C_b$  is a binary absorbing subuniverse in  $A_b$  with  $f$ . Therefore  $A_b \neq A$  or  $A_b = C_b = A$ . In the latter case we have  $(b, b) \in \omega$ , which completes this case.

Assume that  $A_b \neq A$  for some  $b$ . Since  $\sigma \supseteq \kappa_A$ , we have  $b \in A_b$  and  $(A_b \times A_b) \cap \omega \supseteq (C_b \times \{b\}) \cap \omega \neq \emptyset$ . Then we restrict  $\sigma$  and  $\omega$  to  $A_b$  and apply the inductive assumption.  $\square$

**Lemma 7.3.** *Suppose  $\rho$  is a nontrivial absorbing subuniverse of  $A_1 \times \cdots \times A_n$ . Then for some  $i$  there exists a nontrivial absorbing subuniverse  $B_i$  in  $A_i$  with the same term.*

*Proof.* We prove this lemma by induction on the arity of  $\rho$ . If the projection of  $\rho$  onto the first coordinate is not  $A_1$  then by Lemma 7.1 this projection is an absorbing subuniverse with the same term. Otherwise, we choose any element  $a \in A_1$  such that  $\rho$  does not contain all tuples starting with  $a$ , and consider  $\rho' = \{(a_2, \dots, a_n) \mid (a, a_2, \dots, a_n) \in \rho\}$ , which, by Lemma 7.1, is a nontrivial absorbing subuniverse in  $A_2 \times \cdots \times A_n$  with the same term. It remains to apply the inductive assumption.  $\square$

A relation  $\rho \subseteq A^n$  is called *C-essential* if  $\rho \cap (C^{i-1} \times A \times C^{n-i}) \neq \emptyset$  for every  $i$  but  $\rho \cap C^n = \emptyset$ . A relation  $\rho \subseteq A_1 \times \cdots \times A_n$  is called  $(C_1, \dots, C_n)$ -essential if  $\rho \cap (C_1 \times \cdots \times C_{i-1} \times A_i \times C_{i+1} \times \cdots \times C_n) \neq \emptyset$  for every  $i$  but  $\rho \cap (C_1 \times \cdots \times C_n) = \emptyset$ .

**Lemma 7.4.** [1] *Suppose  $C$  is a subuniverse of  $A$ . Then  $C$  absorbs  $A$  with an operation of arity  $n$  if and only if there does not exist a  $C$ -essential relation  $\rho \subseteq A^n$ .*

**Lemma 7.5.** *Suppose  $D^{(1)}$  is an absorbing reduction of a CSP instance  $\Theta$  and a relation  $\rho \subseteq D_{i_1} \times \cdots \times D_{i_n}$  is subdirect, where  $D_{i_1}, \dots, D_{i_n}$  are domains of variables from  $\Theta$ . Then  $\rho^{(1)}$  is not empty.*

*Proof.* It is sufficient to apply the binary absorbing term operation  $t$  to all the tuples of  $\rho$  using term  $t(x_1, t(x_2, t(x_3, \dots, t(x_{s-1}, x_s))))$ , where  $s = |\rho|$ . The resulting tuple will be from  $\rho^{(1)}$ , which means that  $\rho^{(1)}$  is not empty.  $\square$

## 7.2 Center

**Lemma 7.6.** *Suppose  $\rho$  is defined by a pp-formula  $\Omega(x_1, \dots, x_n)$  and  $\Omega'$  is obtained from  $\Omega$  by replacement of some constraint relations  $\sigma_1, \dots, \sigma_s$  by constraint relations  $\sigma'_1, \dots, \sigma'_s$  such that  $\sigma'_i$  is a center of  $\sigma_i$  for every  $i$ . Then the relation defined by  $\Omega'(x_1, \dots, x_n)$  is a center of  $\rho$ .*

*Proof.* Suppose  $\Omega'(x_1, \dots, x_n)$  defines a relation  $\rho'$ . Suppose  $B_i$  and  $R_i$  are the corresponding algebra and binary relation such that  $\sigma'_i = \{c \mid \forall b \in B_i: (c, b) \in R_i\}$ . Let  $|B_i| = n_i$  for every  $i$ . Let  $\Upsilon$  be obtained from  $\Omega$  by replacement of every constraint  $\sigma_i(y_1, \dots, y_t)$  by

$$R_i((y_1, \dots, y_t), z_{i,1}) \wedge \cdots \wedge R_i((y_1, \dots, y_t), z_{i,n_i}).$$

Suppose  $\Upsilon((x_1, \dots, x_n), (z_{1,1}, \dots, z_{s,n_s}))$  defines a relation  $R$ . It is not hard to see that  $\rho' = \{c \mid \forall b \in (B_1^{n_1} \times \dots \times B_s^{n_s}) : (c, b) \in R\}$ . By Lemma 7.3, there is no nontrivial binary absorbing subuniverse on  $B_1^{n_1} \times \dots \times B_s^{n_s}$ . This proves that  $\rho'$  is a center of  $\rho$ .  $\square$

**Corollary 7.6.1.** *Suppose  $\theta$  is a congruence of  $A$*

1. *If  $B$  is a center of  $A$ , then  $\{b/\theta \mid b \in B\}$  is a center of  $A/\theta$ .*
2. *If  $A$  has no nontrivial center, then neither does  $A/\theta$ .*

**Corollary 7.6.2.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a relation such that  $\text{pr}_1(\rho) = A_1$  and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is a center in  $A_i$  for every  $i$ . Then  $C$  is a center in  $A_1$ .*

**Corollary 7.6.3.** *Suppose  $C_i$  is a center of  $D_i$  for every  $i$ . Then  $C_1 \times \dots \times C_n$  is a center of  $D_1 \times \dots \times D_n$ .*

**Corollary 7.6.4.** *Suppose  $C_1$  and  $C_2$  are centers of  $D$ . Then  $C_1 \cap C_2$  is a center of  $D$ .*

**Lemma 7.7.** *Suppose  $\rho$  is a nontrivial center of  $A_1 \times \dots \times A_n$ . Then for some  $i$  there exists a nontrivial center  $C_i$  of  $A_i$ .*

*Proof.* We prove by induction on the arity of  $\rho$ . If the projection of  $\rho$  onto the first coordinate is not  $A_1$  then by Lemma 7.6 this projection is a center.

Otherwise, we choose any element  $a \in A_1$  such that  $\rho$  does not contain all tuples starting with  $a$ . Then we consider  $\rho' = \{(a_2, \dots, a_n) \mid (a, a_2, \dots, a_n) \in \rho\}$ , which, by Lemma 7.6, is a nontrivial center of  $A_2 \times \dots \times A_n$ . It remains to apply the inductive assumption.  $\square$

In the proof of the following two lemmas we assume that a center  $C$  is defined by  $C = \{a \in A \mid \forall b \in B : (a, b) \in R\}$  for a subalgebra  $R$  of  $\mathbf{A} \times \mathbf{B}$ . For an element  $a \in A$  we put  $a^+ = \{b \mid (a, b) \in R\}$ . Also, we introduce a quasi-order on elements of  $A$ . We say that  $y_1 \leq y_2$  if  $y_1^+ \subseteq y_2^+$ , and  $y_1 \sim y_2$  if  $y_1^+ = y_2^+$ . Note that if  $b_1, b_2, \dots, b_m \geq c$ , then  $w(b_1^+, \dots, b_m^+) \supseteq w(c^+, \dots, c^+) \supseteq c^+$ , and therefore  $w(b_1, \dots, b_m) \geq c$ .

**Lemma 7.8.** *Suppose  $(c_1, \dots, c_m) \in A^m$ ,  $c_i \in C$  for every  $i \neq j$ , and  $c_j \notin C$ . Then  $w(c_1, \dots, c_m) > c_j$ .*

*Proof.* Assume the contrary, then  $w(c_1, \dots, c_m) \sim c_j$  and  $w(\underbrace{B, \dots, B}_{i-1}, c_j^+, \underbrace{B, \dots, B}_{m-i}) \subseteq c_j^+$ .

This is enough to imply that  $c_j^+$  is a binary absorbing subuniverse with the term  $x \circ y = w(x, x, \dots, x, y)$ . In fact, if  $b_1 \in B$  and  $b_2 \in c_j^+$ , then we can write  $b_1 \circ b_2 = w(b_1, \dots, b_1, b_2, b_1, \dots, b_1)$  with  $b_2$  in the  $j$ -th spot; if  $b_1 \in c_j^+$  and  $b_2 \in B$ , then we can write  $b_1 \circ b_2 = w(b_1, \dots, b_1, b_2, b_1, \dots, b_1)$  with one of the  $b_1$ 's in the  $j$ -th spot. In both cases we obtain  $b_1 \circ b_2 \in c_j^+$ . Contradiction.  $\square$

**Lemma 7.9.** *Suppose  $w$  is a special WNU of arity  $m$ ,  $C$  is a nontrivial center in  $A$ ,  $\delta \subseteq A^s$  is  $C$ -essential. Then  $s < m^{|A|}$ .*

*Proof.* Choose  $\alpha_1, \dots, \alpha_s \in \delta$  such that  $\alpha_i \in C^{i-1} \times A \times C^{s-i}$  for every  $i$ . We start with the matrix  $M_1$  whose columns are tuples  $\alpha_1, \dots, \alpha_s$ . Then we build a matrix  $M_2$  whose columns are tuples  $w(\alpha_1, \dots, \alpha_m), w(\alpha_{m+1}, \dots, \alpha_{2m}), w(\alpha_{2m+1}, \dots, \alpha_{3m}), \dots$ . Then we apply the WNU  $w$  to the corresponding columns of the previous matrix to define a new matrix  $M_3$ . We continue this way until we get a matrix with less than  $m$  columns. Note that the next matrix has  $m$  times less columns than the previous one. It is not hard to see that every row of every matrix has at most one element that is not from the center. Moreover, by Lemma 7.8, the noncentral element in the  $i$ -th row of the  $(j+1)$ -th matrix is greater than the noncentral element in the  $i$ -th row of the  $j$ -th matrix. This means that the  $|A|$ -th matrix, if it exists, has only central elements, which contradicts our assumptions. Hence, it does not exist and  $s < m^{|A|}$ .  $\square$

Combining this result with Lemma 7.4, we obtain the following corollary.

**Corollary 7.9.1.** *Suppose  $C$  is a center of  $A$ . Then  $C$  is an absorbing subuniverse of  $A$ .*

The following lemma is a stronger version of an original lemma suggested by Marcin Kozik.

**Lemma 7.10.** *Suppose  $C_1 \subseteq A_1$  and  $C_2 \subseteq A_2$  are centers,  $B$  is a subuniverse of  $D$ , and a relation  $\rho \subseteq A_1 \times D^l \times A_2$  is  $(C_1, B, \dots, B, C_2)$ -essential. Then there exists a relation  $\rho' \subseteq A_1 \times D^{2l} \times A_1$  that is  $(C_1, B, \dots, B, C_1)$ -essential.*

*Proof.* Assume that  $\rho$  is a minimal relation (with respect to inclusion) that is  $(C_1, B, \dots, B, C_2)$ -essential. Put  $E = \text{pr}_{l+2}(\rho \cap (C_1 \times B^l \times A_2))$ . Since  $\rho$  is minimal, for any  $b \in E$  the algebra generated by  $\{b\} \cup C_2$  contains  $\text{pr}_{l+2}(\rho)$  (otherwise we would restrict the  $(l+2)$ -th variable of  $\rho$  to this algebra). Fix  $b \in E$ .

Let  $\sigma$  be the subalgebra of  $A_2 \times A_2$  generated by  $\{b\} \times C_2 \cup C_2 \times C_2 \cup C_2 \times \{b\}$ . Since our algebras are idempotent, for any  $c \in \text{pr}_{l+2}(\rho)$  we have  $\{c\} \times C_2 \subseteq \sigma$ . Put

$$\rho'(x, y_1, \dots, y_l, y'_1, \dots, y'_l, x') = \exists z \exists z' \rho(x, y_1, \dots, y_l, z) \wedge \rho(x', y'_1, \dots, y'_l, z') \wedge \sigma(z, z').$$

Let us show that  $\rho'$  is  $(C_1, B, \dots, B, C_1)$ -essential. Since  $\rho$  is  $(C_1, B, \dots, B, C_2)$ -essential, for any  $i \in \{1, \dots, l+1\}$  there exists a tuple  $(a_1, \dots, a_{l+2})$  such that only its  $i$ -th element is not from the corresponding set of  $(C_1, B, \dots, B, C_2)$ . Since  $b \in E$ , there exists  $c_1, \dots, c_{l+1}$  such that  $(c_1, \dots, c_{l+1}, b) \in \rho \cap (C_1 \times B^l \times A_2)$ . Then  $(a_1, \dots, a_{l+1}, c_2, \dots, c_{l+1}, c_1) \in \rho'$  (it is sufficient to put  $z = a_{l+2}$  and  $z' = b$ ). Thus, for any  $i \in \{1, \dots, l+1\}$  we build a tuple from  $\rho'$  such that only its  $i$ -th element is not from the corresponding set of  $(C_1, B, \dots, B, C_1)$ . In the same way we can build such a tuple for each  $i \in \{l+2, \dots, 2l+2\}$ .

To prove that  $\rho'$  is  $(C_1, B, \dots, B, C_1)$ -essential it remains to show that  $(C_1 \times B^{2l} \times C_1) \cap \rho' = \emptyset$ . Assume the converse, let a tuple from the intersection be obtained by sending  $z$  to  $d$  and  $z'$  to  $d'$ . Clearly,  $d, d' \in E$  and  $\{e \in A_2 \mid (e, d') \in \sigma\} \supseteq \{d\} \cup C_2$ , therefore  $\{e \in A_2 \mid (e, d') \in \sigma\} \supseteq \text{pr}_{l+2}(\rho)$ . Hence,  $\{e \in A_2 \mid (b, e) \in \sigma\} \supseteq \{d'\} \cup C_2$  and  $\{e \in A_2 \mid (b, e) \in \sigma\} \supseteq \text{pr}_{l+2}(\rho)$ .

Thus,  $(b, b) \in \sigma$  and there exists an  $n$ -ary term  $t$  such that

$$t(b, b, \dots, b, c_1, \dots, c_i) = b, \quad t(c'_1, \dots, c'_j, b, b, \dots, b) = b,$$

where  $i+j \geq n$  and  $c_1, \dots, c_i, c'_1, \dots, c'_j \in C_2$ . Suppose  $R \subseteq A_2 \times G$  is a binary relation from the definition of the center  $C_2$ ,  $b^+ = \{a \mid (b, a) \in R\}$ . Since  $t$  preserves  $R$ , we have

$$t(b^+, b^+, \dots, b^+, \underbrace{G, \dots, G}_i) \subseteq b^+, \quad t(\underbrace{G, \dots, G}_j, b^+, b^+, \dots, b^+) \subseteq b^+,$$

and therefore  $b^+$  absorbs  $G$  with the binary term  $t(\underbrace{x, \dots, x}_j, y, \dots, y)$ . This contradiction completes the proof.  $\square$

**Corollary 7.10.1.** *Suppose  $C_1 \subseteq A_1$  and  $C_2 \subseteq A_2$  are centers and  $B \subseteq D$  is an absorbing subuniverse. Then there does not exist  $(C_1, B, C_2)$ -essential relation  $\rho \subseteq A_1 \times D \times A_2$ .*

*Proof.* Assume that such a relation  $\rho$  exists. Iteratively applying Lemma 7.10 to  $\rho$  we can obtain a  $(C_1, B, \dots, B, C_1)$ -essential relation  $\rho_l \subseteq A_1 \times D^l \times A_1$  for  $l = 2, 4, 8, \dots$ . If we restrict the first and the last variables of  $\rho_l$  to  $C_1$  and consider the projection onto the remaining variables we get a  $B$ -essential relation of arity  $l$ . Since we can make  $l$  as large as we need, we get a contradiction with Lemma 7.4 and the fact that  $B$  is an absorbing subuniverse.  $\square$

**Corollary 7.10.2.** *Suppose  $C$  is a center of  $A$ . Then  $C$  is a ternary absorbing subuniverse of  $A$ .*

*Proof.* Assume that  $C$  is not a ternary absorbing subuniverse then by Lemma 7.4, there exists a  $C$ -essential relation of arity 3. By Corollary 7.9.1,  $C$  is an absorbing subuniverse of  $A$ , then by Corollary 7.10.1 such a relation cannot exist.  $\square$

**Corollary 7.10.3.** *Suppose  $C_i$  is a center of  $A_i$  for  $i \in \{1, 2, \dots, k\}$  and  $k \geq 3$ . Then there does not exist a  $(C_1, \dots, C_k)$ -essential relation  $\rho \subseteq A_1 \times \dots \times A_k$ .*

*Proof.* If such a relation  $\rho$  exists then restricting all but the first three variables of  $\rho$  to the corresponding centers and projecting the result onto the first three variables we obtain  $(C_1, C_2, C_3)$ -essential relation, which cannot exist by Corollary 7.10.1.  $\square$

### 7.3 PC Subuniverse

**Lemma 7.11.** *Suppose  $A$  is a PC algebra and  $\rho \subseteq A^n$  is a relation containing all the constant tuples  $(a, \dots, a)$ . Then  $\rho$  can be represented as a conjunction of binary relations of the form  $x_i = x_j$ .*

*Proof.* All constant operations preserve  $\rho$ , and together with the constant operations the algebra  $A$  generates all operations on the set  $A$ . Then  $\rho$  is preserved by all operations on  $A$ , and therefore,  $\rho$  is diagonal (see Theorem 2.9.3 from [44]) and it can be represented as a conjunction of binary relations of the form  $x_i = x_j$ .  $\square$

**Lemma 7.12.** *Suppose  $\rho \subseteq A \times B$  is a subdirect relation and  $A$  is a PC algebra. Then either for every  $b \in B$  there exists a unique  $a \in A$  such that  $(a, b) \in \rho$ , or there exists  $b \in B$  such that  $(a, b) \in \rho$  for every  $a \in A$ .*

*Proof.* Put  $\sigma_l(x_1, x_2, \dots, x_l) = \exists y \rho(x_1, y) \wedge \dots \wedge \rho(x_l, y)$ . It is not hard to see that  $\sigma_l$  contains all constant tuples. Therefore, Lemma 7.11 implies that  $\sigma_2$  is either full, or the equality relation.

If  $\sigma_2$  is the equality relation, then for every  $b \in B$  there exists a unique  $a \in A$  such that  $(a, b) \in \rho$ .

Suppose  $\sigma_2$  is full. Then we consider the minimal  $l$ , if it exists, such that  $\sigma_l$  is not full. Since  $\sigma_{l-1}$  is full, the relation  $\sigma_l$  contains all tuples whose elements are not different. Then Lemma 7.11 implies that  $\sigma_l$  is a full relation, which means that  $\sigma_l$  is a full relation for every  $l$ . Substituting  $l = |A|$  and  $\{x_1, \dots, x_l\} = A$  in the definition of  $\sigma_l$  we obtain that there exists  $b$  such that  $(a, b) \in \rho$  for every  $a \in A$ .  $\square$

**Lemma 7.13.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a subdirect relation,  $A_i$  is a PC algebra for every  $i \in \{2, \dots, n\}$ , and there is no nontrivial binary absorbing subuniverse or nontrivial center on  $A_i$  for every  $i \in \{1, \dots, n\}$ . Then  $\rho$  can be represented as a conjunction of binary relations  $\delta_1, \dots, \delta_k$  such that  $\text{Con}(\delta_l, j)$  is the equality relation whenever the domain of the  $j$ -th variable of  $\delta_l$  is a PC algebra.*

This lemma says that the relation  $\rho$  can be represented by constraints from the first coordinate to an  $i$ -th coordinate such that the  $i$ -th coordinate is uniquely determined by the first (also we can define the corresponding PC congruence on the first coordinate using this relation) and by bijective binary constraints between pairs of coordinates other than first. Also, it says that in a subdirect product of PC algebras without a nontrivial binary absorbing subuniverse or center (even  $A_1$  is a PC algebra) we can choose some essential coordinates which can have any value, each other coordinate is uniquely determined by exactly one of them (in a bijective way).

*Proof.* We prove by induction on the arity of  $\rho$ . If  $\rho$  is binary, Lemma 7.12 implies that there exists a nontrivial binary absorbing subuniverse on  $A_2$ , or there exists a nontrivial center on  $A_1$  witnessed by  $\rho$ , or the second coordinate of  $\rho$  is uniquely determined by the first, or  $\rho$  is full. First two conditions contradict our assumptions, the last two conditions are what we need.

Assume that  $\rho$  is not essential, then it can be represented as a conjunction of essential relations satisfying the same properties. By the inductive assumption, each of them can be represented as a conjunction of binary relations. It remains to join these binary relations to complete the proof for this case.

Assume that  $\rho$  is essential. The projection of  $\rho$  onto any proper set of variables gives a relation of a smaller arity satisfying the same properties. By the inductive assumption, the relation of a smaller arity can be represented as a conjunction of binary relations  $\delta_1, \dots, \delta_k$  such that  $\text{Con}(\delta_l, j)$  is the equality relation whenever the domain of the  $j$ -th variable of  $\delta_l$  is a PC algebra. In each relation  $\delta_i$  one variable (let it be the  $u$ -th variable of  $\rho$ ) is uniquely determined by another, and therefore the relation  $\rho$  can be represented as a conjunction of  $\delta_i$  and the projection of  $\rho$  onto all variables but  $u$ -th, which cannot happen with an essential relation. Therefore, each projection of  $\rho$  onto any proper set of variables is a full relation.

Let us consider the relation  $\rho \subseteq (A_1 \times \dots \times A_{n-1}) \times A_n$  as a binary relation. By Lemma 7.12 we have one of the following two situations.

Case 1: there exist  $b_1, \dots, b_{n-1}$  such that  $(b_1, \dots, b_{n-1}, a) \in \rho$  for every  $a \in A_n$ . We consider the maximal  $s$  such that  $\rho(b_1, \dots, b_s, x_{s+1}, \dots, x_n)$  is not a full relation. It is easy to see that  $s \leq n-2$  and  $s$  exists. Let  $R(x_{s+1}, \dots, x_n) = \rho(b_1, \dots, b_s, x_{s+1}, \dots, x_n)$ . Since the projection of  $\rho$  onto any proper subset of variables is full,  $R$  is a subdirect relation. By Lemma 7.3, there is no nontrivial binary absorbing subuniverse on  $A_{s+2} \times \dots \times A_n$ , then we get a nontrivial center  $C$  on  $A_{s+1}$  defined by  $C = \{a_{s+1} \in A_{s+1} \mid \forall a_{s+2} \dots \forall a_n : (a_{s+1}, a_{s+2}, \dots, a_n) \in R\}$  and witnessed by  $R$ .

Case 2: for every  $a_1, \dots, a_{n-1}$  there exists a unique  $b$  such that  $(a_1, \dots, a_{n-1}, b) \in \rho$ . We can show in the same way that for any  $(a_1, a_3, \dots, a_n)$  there exists a unique  $b$  such that  $(a_1, b, a_3, \dots, a_n) \in \rho$ . Let us consider the relation  $\zeta$  defined by

$$\zeta(z_1, z_2, z_3, z_4) = \exists x_1 \exists x_2 \dots \exists x_{n-1} \exists x'_1 \exists x'_2 \rho(x_1, x_2, x_3, \dots, x_{n-1}, z_1) \wedge \\ \rho(x_1, x'_2, x_3, \dots, x_{n-1}, z_2) \wedge \rho(x'_1, x_2, x_3, \dots, x_{n-1}, z_3) \wedge \rho(x'_1, x'_2, x_3, \dots, x_{n-1}, z_4).$$

Since any projection of  $\rho$  onto any proper subset of variables is a full relation, any projection of  $\zeta$  onto 3 variables is a full relation. Since  $\rho$  is subdirect,  $\zeta$  contains all constant tuples. Then Lemma 7.11 implies that  $\zeta$  is a full relation. Suppose  $a \neq b$  and  $(a, a, a, b) \in \zeta$  witnessed by  $x_1, \dots, x_{n-1}, x'_1, x'_2$ . Since  $z_1 = z_2 = a$ , we have  $x_2 = x'_2$  and therefore  $z_3 = z_4$ , that is  $a = b$ . Contradiction.  $\square$

**Corollary 7.13.1.** *Suppose  $\sigma_1, \dots, \sigma_k$  are all PC congruences on  $A$ . Put  $A_i = A/\sigma_i$ , and define  $\psi : A \rightarrow A_1 \times \dots \times A_k$  by  $\psi(a) = (a/\sigma_1, \dots, a/\sigma_k)$ . Then*

1.  $\psi$  is surjective, hence  $A/\text{ConPC}(A) \cong A_1 \times \dots \times A_k$ ;
2. the PC subuniverses are the sets of the form  $\psi^{-1}(S)$ , where  $S \subseteq A_1 \times \dots \times A_k$  is a relation definable by unary constraints of the form  $x_j = a_j$ ;
3. for each nonempty PC subuniverse  $B$  of  $A$  there is a congruence  $\theta$  of  $A$  such that  $B$  is an equivalence class of  $\theta$  and  $A/\theta$  is isomorphic to a product of PC algebras having no nontrivial binary absorbing subuniverse or center.



*Proof.* Consider the image  $\psi(A)$ , which is a subdirect subuniverse of  $A_1 \times \cdots \times A_k$ . By Lemma 7.13, this relation can be represented as a conjunction of binary relations whose one coordinate uniquely determines another (in a bijective way). This means that congruences  $\sigma_i$  corresponding to these coordinates should be equal, which contradicts the definition. Then  $\psi(A)$  is a full relation and  $\psi$  is surjective.

Claim (2) follows directly from the definition of a PC subuniverse.

To prove (3) consider the intersection of all congruences whose equivalence classes we intersected to define the PC subuniverse. Then, in the same way as in (1) we can prove the isomorphism.  $\square$

**Corollary 7.13.2.** *Suppose  $\rho \subseteq A_1 \times \cdots \times A_n$  is a subdirect relation, there is no nontrivial binary absorbing subuniverse or nontrivial center on  $A_1$ , and  $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$ , where  $C_i$  is a PC subuniverse in  $A_i$  for every  $i$ . Then  $C$  is a PC subuniverse in  $A_1$ .*

*Proof.* By the previous corollary for every  $i$  we choose PC algebras  $A_{i,1}, \dots, A_{i,k_i}$  and a mapping  $\psi_i : A_i \rightarrow A_{i,1} \times \cdots \times A_{i,k_i}$  such that  $A_i / \text{ConPC}(A_i) \cong A_{i,1} \times \cdots \times A_{i,k_i}$ . Define  $\phi : A_1 \times \cdots \times A_n \rightarrow A_1 \times \prod_{i,j} A_{i,j}$  by  $\phi(a_1, \dots, a_n) = (a_1, \psi_1(a_1), \dots, \psi_n(a_n))$ . Let  $\gamma = C_1 \times \cdots \times C_n$ ,  $\rho' = \phi(\rho)$ ,  $\gamma' = \phi(\gamma)$ . We can check that  $\text{pr}_1(\rho \cap \gamma) = \text{pr}_1(\rho' \cap \gamma')$ , then it is sufficient to show that  $\text{pr}_1(\rho' \cap \gamma')$  is a PC subuniverse of  $A_1$ .

Since  $\rho'$  is subdirect, by Lemma 7.13 it can be represented by binary constraints from the first coordinate to an  $i$ -th coordinate such that the  $i$ -th coordinate is uniquely determined by the first, and by bijective binary constraints between pairs of coordinates other than first. The relation  $\gamma'$  can be represented by constraints of the form  $x_{i,j} = a_{i,j}$  and canonical constraints saying that the  $j$ -th element of  $\psi_1(x_1)$  is equal to  $x_{1,j}$ . To calculate  $\text{pr}_1(\rho' \cap \gamma')$  we join constraints of these two representations. Let us explain how any constraint from  $x_1$  to  $x_{i,j}$  in this representation looks like. There exists a congruence  $\sigma$  on  $A_1$  such that  $A_1/\sigma$  is a PC algebra isomorphic to  $A_{i,j}$ , then the constraint assigns to all elements of each equivalence class of  $\sigma$  the corresponding element of  $A_{i,j}$ . All other constraints of this representations are of the form  $x_{i,j} = a_{i,j}$  or bijective constraints between two coordinates. This implies that  $\text{pr}_1(\rho' \cap \gamma')$  is an intersection of equivalence classes of PC congruences, that is,  $\text{pr}_1(\rho' \cap \gamma')$  is a PC subuniverse.  $\square$

**Corollary 7.13.3.** *Suppose  $C_i$  is a PC subuniverse of  $A_i$  for  $i \in \{1, 2, \dots, n\}$  and  $n \geq 3$ . Then there does not exist a subdirect  $(C_1, \dots, C_n)$ -essential relation  $\rho \subseteq A_1 \times \cdots \times A_n$ .*

*Proof.* Assume that such a relation  $\rho$  exists. By Corollary 7.13.1 for every  $i$  we choose PC algebras  $A_{i,1}, \dots, A_{i,k_i}$  and a mapping  $\psi_i : A_i \rightarrow A_{i,1} \times \cdots \times A_{i,k_i}$  such that  $A_i / \text{ConPC}(A_i) \cong A_{i,1} \times \cdots \times A_{i,k_i}$ . Define  $\phi : A_1 \times \cdots \times A_n \rightarrow \prod_{i,j} A_{i,j}$  by  $\phi(a_1, \dots, a_n) = (\psi_1(a_1), \dots, \psi_n(a_n))$ . Let  $\gamma_i = C_1 \times \cdots \times C_{i-1} \times A_i \times C_{i+1} \times \cdots \times C_n$  for every  $i$  and  $\gamma = C_1 \times \cdots \times C_n$ . Put  $\rho' = \phi(\rho)$ ,  $\gamma' = \phi(\gamma)$ , and  $\gamma'_i = \phi(\gamma_i)$  for every  $i$ .

Since  $\rho'$  is subdirect, by Lemma 7.13  $\rho'$  can be represented by bijective binary constraints between pairs. By Corollary 7.13.1  $\gamma'$  can be represented by constraints of the form  $x_{i,j} = a_{i,j}$ . If  $\rho \cap \gamma = \emptyset$ , then  $\rho' \cap \gamma' = \emptyset$ , which can only happen if two unary constraints defining  $\gamma'$  assign contradictory values to variables with respect to the binary constraints defining  $\rho'$ . Since  $k \geq 3$ , we can choose  $l$  such that  $\gamma'_l$  includes the two contradictory unary constraints. Then  $\rho' \cap \gamma'_l = \emptyset$  and  $\rho \cap \gamma_l = \emptyset$ , which gives a contradiction.  $\square$

**Lemma 7.14.** *Suppose  $\sigma \supseteq \sigma_1 \cap \cdots \cap \sigma_n$ , where  $\sigma_1, \dots, \sigma_n$  are PC congruences on  $D$  and  $\sigma$  is a proper congruence on  $D$ . Then there exists  $I \subseteq \{1, 2, \dots, n\}$  such that  $\sigma = \bigcap_{i \in I} \sigma_i$ .*

*Proof.* Consider a  $2n$ -ary relation  $R \subseteq D/\sigma_1 \times \cdots \times D/\sigma_n \times D/\sigma_1 \times \cdots \times D/\sigma_n$  consisting of all tuples  $(a/\sigma_1, \dots, a/\sigma_n, b/\sigma_1, \dots, b/\sigma_n)$ , where  $(a, b) \in \sigma$ . By Lemma 7.13,  $R$  can be represented as a conjunction of binary bijective relations. Since  $(a/\sigma_1, \dots, a/\sigma_n, a/\sigma_1, \dots, a/\sigma_n) \in R$

$R$  for every  $a \in D$ , we conclude that all these binary relations are equalities. This implies that  $\sigma = \bigcap_{i \in I} \sigma_i$  for some  $I \subseteq \{1, 2, \dots, n\}$ .  $\square$

**Lemma 7.15.** *For every  $D$  the algebra  $D/\text{ConPC}(D)$  has no nontrivial binary absorbing subuniverse or center.*

*Proof.* By Corollary 7.13.1,  $D/\text{ConPC}(D) \cong A_1 \times \dots \times A_k$ , where  $A_i$  is a PC algebra without a nontrivial binary absorbing subuniverse or center. Then Lemmas 7.3 and 7.7 imply that there cannot be a nontrivial binary absorbing subuniverse or center on  $D$ .  $\square$

**Lemma 7.16.** *Suppose  $\sigma$  is a PC congruence on  $A_1 \times A_2$ , there is no nontrivial binary absorbing subuniverse or center on  $A_1$  and  $A_2$ . Then there exist  $i \in \{1, 2\}$  and a PC congruence  $\sigma_i$  on  $A_i$  such that  $\sigma = \{(\alpha, \beta) \mid (\text{pr}_i(\alpha), \text{pr}_i(\beta)) \in \sigma_i\}$ .*

*Proof.* First, consider  $S \subseteq A_1 \times (A_1 \times A_2)/\sigma$  consisting of all pairs  $(a_1, (a_1, a_2)/\sigma)$  such that  $a_1 \in A_1, a_2 \in A_2$ . Since there is no nontrivial binary absorbing subuniverse or center on  $A_1$ , Lemma 7.13 implies that either  $S$  is a full relation, or  $\text{Con}(S, 2)$  is the equality relation. In the latter case the congruence  $\sigma$  depends only on the first coordinate, that is, there exists a PC congruence  $\sigma_1$  on  $A_1$  such that  $\sigma = \{(\alpha, \beta) \mid (\text{pr}_1(\alpha), \text{pr}_1(\beta)) \in \sigma_1\}$ , which completes this case.

Thus, we assume that  $S$  is a full relation and for every  $a_1 \in A_1$  and every equivalence class  $E$  of  $\sigma$  there exists  $a_2 \in A_2$  such that  $(a_1, a_2) \in E$ . In the same way we assume that for every  $a_2 \in A_2$  and every equivalence class  $E$  of  $\sigma$  there exists  $a_1 \in A_1$  such that  $(a_1, a_2) \in E$ .

Choose an element  $c_1 \in A_1$ . By  $\sigma_2$  we denote the congruence  $\{(a_2, a'_2) \mid ((c_1, a_2), (c_1, a'_2)) \in \sigma\}$ . As it follows from the above assumptions,  $A_2/\sigma_2 \cong (A_1 \times A_2)/\sigma$ . Consider the ternary relation  $\rho \subseteq A_1 \times (A_1 \times A_2)/\sigma \times A_2/\sigma_2$  consisting of all the tuples  $(a_1, (a_1, a_2)/\sigma, a_2/\sigma_2)$ , where  $a_1 \in A_1$  and  $a_2 \in A_2$ . As we already know, the projection of  $\rho$  onto any two coordinates is a full relation. Then Lemma 7.13 implies that  $\rho$  is a full relation, which contradicts the fact that  $(c_1, (c_1, a_2)/\sigma, b_2/\sigma_2) \notin \rho$  for any  $(a_2, b_2) \notin \sigma_2$ .  $\square$

**Corollary 7.16.1.** *Suppose  $\sigma$  is a PC congruence on  $A_1 \times A_2 \times \dots \times A_n$ , there is no nontrivial binary absorbing subuniverse or center on  $A_i$  for every  $i$ . Then there exist  $i \in \{1, 2, \dots, n\}$  and a PC congruence  $\sigma_i$  on  $A_i$  such that  $\sigma = \{(\alpha, \beta) \mid (\text{pr}_i(\alpha), \text{pr}_i(\beta)) \in \sigma_i\}$ .*

*Proof.* We prove this corollary by induction on  $n$ . For  $n = 2$  it follows from Lemma 7.16. By Lemmas 7.3, 7.7, there is no nontrivial binary absorbing subuniverse or center on  $A_2 \times \dots \times A_n$ . We apply Lemma 7.16 to  $A_1 \times (A_2 \times \dots \times A_n)$  to get a PC congruence on  $A_1$  or on  $A_2 \times \dots \times A_n$ . In the latter case we apply the inductive assumption to complete the proof.  $\square$

**Lemma 7.17.** *Suppose  $B$  is a PC subuniverse on  $A_1 \times \dots \times A_n$ , and there is no nontrivial binary absorbing subuniverse or center on  $A_i$  for every  $i$ . Then there exists a PC subuniverse  $B_i$  on  $A_i$  for every  $i$  such that  $B = B_1 \times \dots \times B_n$ .*

*Proof.* Assume that  $B = E_1 \cap \dots \cap E_t$ , where  $E_i$  is an equivalence class of a PC congruence  $\sigma_i$  on  $A_1 \times \dots \times A_n$  for every  $i$ . By Corollary 7.16.1, for every  $i$  there exists  $s_i$  and a congruence  $\sigma'_i$  on  $A_{s_i}$  such that  $\sigma_i = \{(\alpha, \beta) \mid (\text{pr}_{s_i}(\alpha), \text{pr}_{s_i}(\beta)) \in \sigma'_i\}$ . Then there exists an equivalence class  $E'_i$  of  $\sigma'_i$  such that  $E_i = A_1 \times \dots \times A_{s_i-1} \times E'_i \times A_{s_i+1} \times \dots \times A_n$ . Hence, the intersection  $E_1 \cap \dots \cap E_t$  is equal to  $B_1 \times \dots \times B_n$  for PC subuniverses  $B_1, \dots, B_n$ .  $\square$

**Lemma 7.18.** *Suppose  $\rho \subseteq A \times B$  is a subdirect relation,  $A$  is a PC algebra without nontrivial binary absorbing subuniverse or center, and  $C = \{b \in B \mid \forall a \in A: (a, b) \in \rho\}$ . Then  $C$  binary absorbs  $B$ .*

*Proof.* Suppose  $A = \{a_1, \dots, a_k\}$ . Let us consider the matrix  $M$  whose rows are the tuples  $(\underbrace{a, a, \dots, a}_{k+1}, b, a_1, \dots, a_k)$  and  $(b, a_1, \dots, a_k, \underbrace{a, a, \dots, a}_{k+1})$  for all  $a, b \in A$ . The  $2k+2$  columns of this matrix we denote by  $\alpha_1, \dots, \alpha_{2k+2}$ . By  $\beta$  we denote the tuple of length  $2k^2$  such that the  $i$ -th element of  $\beta$  equals  $b$  from the corresponding row. By Lemma 7.13, the relation generated by  $\alpha_1, \dots, \alpha_{2k+2}$  is a full relation. Hence, there exists a term operation  $f$  such that  $f(\alpha_1, \dots, \alpha_{2k+2}) = \beta$ . Let us show that  $C$  absorbs  $B$  with the term operation defined by  $h(x, y) = f(\underbrace{x, \dots, x}_{k+1}, y, \dots, y)$ . Suppose  $d \in B$ ,  $c \in C$ . Assume that

$h(d, c) = e \notin C$ . Choose elements  $a, a' \in A$  such that  $(a, e) \notin \rho$  and  $(a', d) \in \rho$ . Consider the row  $(a', \dots, a', a, a_1, \dots, a_k)$  from the matrix. We know that  $f$  returns  $a$  on this tuple and  $f(\underbrace{d, \dots, d}_{k+1}, c, \dots, c) = e$ , which contradicts the fact that  $f$  preserves  $\rho$ . Thus,  $h(d, c) \in C$ .

In the same way we can prove that  $h(c, d) \in C$  for every  $d \in B$ ,  $c \in C$ .  $\square$

**Lemma 7.19.** *Suppose  $\rho \subseteq A \times B \times B$  is a subdirect relation,  $A$  is a PC algebra without a nontrivial binary absorbing subuniverse or center, and for every  $b \in B$  there exists  $a \in A$  such that  $(a, b, b) \in \rho$ . Then for every  $a \in A$  there exists  $b \in B$  such that  $(a, b, b) \in \rho$ .*

*Proof.* We prove the lemma by induction on the size of  $B$ .

By Lemma 7.12, only two situations are possible: either there exist  $c_1, c_2 \in B$  such that  $(a, c_1, c_2) \in \rho$  for every  $a \in A$ , or for each  $(b_1, b_2) \in \text{pr}_{2,3}(\rho)$  there exists a unique  $a \in A$  such that  $(a, b_1, b_2) \in \rho$ .

Case 1. There exist  $c_1, c_2 \in B$  such that  $(a, c_1, c_2) \in \rho$  for every  $a \in A$ . Put  $D = \{(b, c) \mid \forall a \in A: (a, b, c) \in \rho\}$ . By Lemma 7.18,  $D$  is a binary absorbing subuniverse in the projection of  $\rho$  onto the last two variables. By Lemma 7.2, there exists  $(b, b) \in D$ . This completes this case.

Case 2. For each  $(b_1, b_2) \in \text{pr}_{2,3}(\rho)$  there exists a unique  $a \in A$  such that  $(a, b_1, b_2) \in \rho$ . Let  $\delta_1$  be the projection of  $\rho$  onto the first two variables. By Lemma 7.12 we have one of two situations.

Case 2A. For every  $b \in B$  there exists a unique  $a$  such that  $(a, b) \in \delta_1$ . Since  $\rho$  is subdirect, for every  $a$  there exists  $(a, b, b') \in \rho$ , which implies that  $(a, b, b) \in \rho$  and completes this case.

Case 2B. There exists an element  $b$  such that  $(a, b) \in \delta_1$  for every  $a \in A$ . Consider the relation  $\delta_2(x, y_2) = \rho(x, b, y_2)$ . If  $\text{pr}_2(\delta_2) \neq B$ , then we restrict the last two variables of  $\rho$  to  $\text{pr}_2(\delta_2)$  and apply the inductive assumption. Assume that  $\text{pr}_2(\delta_2) = B$ . By the definition of the second case we know that for every  $c \in B$  there exists a unique  $a$  such that  $(a, c) \in \delta_2$ . Then  $\sigma = \text{Con}(\delta_2, 2)$  is a proper congruence such that  $B/\sigma \cong A$ . If  $\sigma$  is the equality relation, then  $B \cong A$ , and, by Lemma 7.13,  $\rho$  can be represented by binary bijective constraints. If the first coordinate of  $\rho$  is uniquely defined by the second or the third, then it is equivalent to the case 2A, which we already considered. If the first coordinate of  $\rho$  does not depend on the others, then the claim is trivial.

If  $\sigma$  is not the equality relation, then we consider the relation  $\rho'$  obtained from  $\rho$  by factorization of the last two variables by  $\sigma$ , that is,  $\rho' \subseteq A \times B/\sigma \times B/\sigma$  contains all tuples  $(a, b/\sigma, b'/\sigma)$  such that  $(a, b, b') \in \rho$ . By the inductive assumption for any  $a \in A$  there exists  $E \in B/\sigma$  such that  $(a, E, E) \in \rho'$ . By Lemma 7.12, we have one of the following situations. Case 1. There exists  $E \in B/\sigma$  such that for every  $a \in A$  we have  $(a, E, E) \in \rho'$ . Then we restrict the last two variables of  $\rho$  to  $E$  and apply the inductive assumption. Case 2. For every  $E \in B/\sigma$  there exists a unique  $a \in A$  such that  $(a, E, E) \in \rho'$ . In this case for any  $a \in A$  we choose  $E$  such that  $(a, E, E) \in \rho'$ . By the uniqueness of  $a$  we have  $(a, b, b) \in \rho$  for any  $b \in E$ , which completes the proof.  $\square$

## 7.4 Linear Subuniverse

We have the following well-known fact from linear algebra [33].

**Lemma 7.20.** *Suppose  $\rho \subseteq (\mathbb{Z}_{p_1})^{n_1} \times \cdots \times (\mathbb{Z}_{p_k})^{n_k}$ , where  $p_1, \dots, p_k$  are distinct prime numbers dividing  $m - 1$  and  $\mathbb{Z}_{p_i} = (\mathbb{Z}_{p_i}; x_1 + \cdots + x_m)$  for every  $i$ . Then  $\rho = L_1 \times \cdots \times L_k$  where each  $L_i$  is an affine subspace of  $(\mathbb{Z}_{p_i})^{n_i}$ .*

**Corollary 7.20.1.** *The set of linear algebras is closed under taking subalgebras, quotients, and finite products.*

**Lemma 7.21.** *A linear algebra has no nontrivial absorbing subuniverse, nontrivial center, or nontrivial PC subuniverse.*

*Proof.* Let us prove that a linear algebra  $A$  has no nontrivial absorbing subuniverse, which by Corollary 7.9.1 implies that  $A$  has no nontrivial center. By Lemma 7.3, it is sufficient to show that  $\mathbb{Z}_p$  has no nontrivial absorbing subuniverse. Every term operation in  $\mathbb{Z}_p$  can be represented as  $a_1x_1 + \cdots + a_lx_l$ , and for each  $a_l \neq 0$  fixing all variables but  $x_l$  to some values gives a bijective mapping, which means that this term cannot witness an absorption.

Since linear algebras (by Corollary 7.20.1) are closed under quotients, to prove that it does not have a nontrivial PC subuniverse it is sufficient to prove that a linear algebra  $A$  cannot be a PC algebra. Assume that  $A$  is isomorphic to  $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$  for prime numbers  $p_1, \dots, p_k$ , and  $\psi : A \rightarrow \mathbb{Z}_{p_1}$  is the canonical mapping. Let  $\rho$  be the set of all tuples  $(a, b, c, d)$  such that  $\psi(a) + \psi(b) = \psi(c) + \psi(d)$ . We can check that  $\rho$  is preserved by  $w$  and all constants but not all operations, therefore  $A$  cannot be polynomially complete.  $\square$

**Lemma 7.22.** *Suppose  $\rho \subseteq A_1 \times A_2$  is a subdirect relation,  $A_2$  is a linear algebra, and there is no nontrivial binary absorbing subuniverse on  $A_1$ . Then for all  $a, b \in A_1$  we have*

$$|\{c \mid (a, c) \in \rho\}| = |\{c \mid (b, c) \in \rho\}|.$$

*Proof.* Assume the contrary, then we choose all elements  $a$  with the maximal  $|\{c \mid (a, c) \in \rho\}|$ . Denote the set of such elements by  $C$ .

Since  $w(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_m)$  is a bijection on  $A_2$  for every  $a_1, \dots, a_m \in A_2$ , we have  $w(A_1, \dots, A_1, C, A_1, \dots, A_1) \subseteq C$ . Hence  $w(x, \dots, x, y)$  is a binary absorbing operation and  $C$  is a binary absorbing subuniverse.  $\square$

**Lemma 7.23.** *Suppose  $A$  is a linear algebra. Then  $w(a, b, \dots, b) = a$  for every  $a, b \in A$ .*

*Proof.* Suppose  $A \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$ . Since the WNU  $w$  is special and idempotent, each  $p_i$  divides  $m - 1$ . Therefore,  $w(a, b, \dots, b) = a$  for every  $a, b \in A$ .  $\square$

**Lemma 7.24.** *Suppose  $\rho \subseteq A_1 \times A_2$  is a subdirect relation,  $A_2$  is a linear algebra, and there is no nontrivial binary absorbing subuniverse on  $A_1$ . Then  $\rho$  has the parallelogram property.*

*Proof.* First, we define a relation  $\sigma_k$  for every  $k \geq 2$  by

$$\sigma_k(y_1, \dots, y_k) = \exists x \rho(x, y_1) \wedge \cdots \wedge \rho(x, y_k).$$

By Lemma 7.23,  $w(a, b, \dots, b, b) = a$  and  $w(b, b, \dots, b, c) = c$  for any  $a, b, c \in A_2$ , therefore  $(a, b), (b, c), (b, b) \in \sigma_2$  implies  $(a, c) \in \sigma_2$ . Since  $\sigma_2$  is reflexive and symmetric, it is a congruence.

Let us show by induction on  $k$  that  $\sigma_k(y_1, \dots, y_k) = \bigwedge_{i=2}^k \sigma_2(y_1, y_i)$ . For  $k = 2$  it is obvious. Consider a tuple  $(a_1, \dots, a_k)$  such that  $(a_i, a_j) \in \sigma_2$  for any  $i, j$ . By the inductive assumption

for  $k - 1$  we have  $(a_1, a_1, a_3, \dots, a_k), (a_1, a_2, a_1, a_4, \dots, a_k), (a_1, a_1, a_1, a_4, \dots, a_k) \in \sigma_k$ . If we apply the term operation  $g(x, y, z) = w(x, y, z, \dots, z)$  to these three tuples (in the same order) we obtain  $(a_1, \dots, a_k)$ , which means that  $(a_1, \dots, a_k) \in \sigma_k$ . Thus  $\sigma_k(y_1, \dots, y_k) = \bigwedge_{i=2}^k \sigma_2(y_1, y_i)$  for every  $k$ .

Substituting  $\{y_1, \dots, y_k\} = E$  in the definition of  $\sigma_k$  for an equivalence class  $E$  of  $\sigma_2$  we derive that there exists  $c \in A_1$  such that  $(c, d) \in \rho$  for any  $d \in E$ . Then it follows from Lemma 7.22, that  $\rho$  has the parallelogram property.  $\square$

**Corollary 7.24.1.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a relation such that  $\text{pr}_1(\rho) = A_1$ , there is no nontrivial binary absorbing subuniverse on  $A_1$ , and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is a linear subuniverse of  $A_i$  for every  $i$ . Then  $C$  is a linear subuniverse of  $A_1$ .*

*Proof.* Let  $\psi : A_1 \times \dots \times A_n \rightarrow A_1 \times A_2 / \text{ConLin}(A_2) \times \dots \times A_n / \text{ConLin}(A_n)$  be a natural homomorphism. Put  $\gamma = C_1 \times \dots \times C_n$ ,  $\rho' = \psi(\rho)$ ,  $\gamma' = \psi(\gamma)$ . We can check that  $\text{pr}_1(\rho \cap \gamma) = \text{pr}_1(\rho' \cap \gamma')$ . The relation  $\rho'$  can be viewed as a subdirect subalgebra of  $A_1 \times B$ , where  $B = \text{pr}_{2, \dots, n}(\rho')$  is a linear algebra by Corollary 7.20.1. Then  $D = \text{pr}_{2, \dots, n}(\gamma') \cap B$  can be viewed as a subalgebra of  $B$ . We need to show that  $\text{pr}_1(\rho' \cap (C_1 \times D))$  is a linear subuniverse of  $A_1$ . By Lemma 7.24, the binary relation  $\rho'$  has the parallelogram property, then  $\rho'$  induces an isomorphism  $A_1 / \sigma_1 \cong B / \sigma_2$ , where  $\sigma_1 = \text{Con}(\rho', 1)$ ,  $\sigma_2 = \text{Con}(\rho', 2)$ . Note that  $\sigma_1$  is a linear congruence since  $B$  is a linear algebra. Hence,  $D_1 = \{a \in A_1 \mid \exists d \in D : (a, d) \in \rho'\}$  is stable under  $\sigma_1$  (and under  $\text{ConLin}(A_1)$ ). Then  $\text{pr}_1(\rho \cap \gamma) = \text{pr}_1(\rho' \cap \gamma') = C_1 \cap D_1$  is stable under  $\text{ConLin}(A_1)$ , which completes the proof.  $\square$

## 7.5 Common properties

In this subsection we list some properties that are common for all types of one-of-four subuniverses.

**Lemma 7.25.** *Suppose  $R \subseteq D_1 \times \dots \times D_n$  is a subdirect relation,  $B_i$  is a one-of-four subuniverse of  $D_i$  of type  $\mathcal{T}$  for every  $i \in \{1, \dots, n\}$ ; if  $\mathcal{T}$  is the absorbing type then the absorbing subuniverses are witnessed by the same term operation. Then  $R \cap (B_1 \times \dots \times B_n)$  is a one-of-four subuniverse of  $R$  of type  $\mathcal{T}$ .*

*Proof.* If  $\mathcal{T}$  is the absorbing type, then the statement follows from Lemma 7.1, if  $\mathcal{T}$  is the central type, then the statement follows from Lemma 7.6.

Suppose  $\mathcal{T}$  is the linear type, then put  $\sigma_i = \text{ConLin}(D_i)$  for each  $i \in \{1, \dots, n\}$ . First, extend every  $\sigma_i$  naturally on  $D = D_1 \times \dots \times D_n$  and denote the obtained congruence  $\sigma'_i$  so that  $D / \sigma'_i \cong D_i / \sigma_i$ . Since linear algebras are closed under taking subalgebras and quotients (Corollary 7.20.1),  $\sigma = \sigma'_1 \cap \dots \cap \sigma'_n$  is a linear congruence and  $B_1 \times \dots \times B_n$  is stable under this congruence. Therefore,  $\sigma \cap (R \times R)$  is a linear congruence and  $R \cap (B_1 \times \dots \times B_n)$  is stable under it. This completes this case.

It remains to consider the case when  $\mathcal{T}$  is the PC type. Let  $\delta_1, \dots, \delta_t$  be the set of all PC congruences on  $D_1, \dots, D_n$  we need to define  $B_1, \dots, B_n$ . For every  $i \in \{1, 2, \dots, t\}$  by  $\delta'_i$  we denote  $\delta_i$  naturally extended on  $D = D_1 \times \dots \times D_n$ , by  $E_i$  we denote the equivalence class of  $\delta'_i$  containing  $B_1 \times \dots \times B_n$ . Since  $R$  is subdirect,  $R / \delta'_i \cong D / \delta'_i$  and  $R / \delta'_i$  is a PC algebra without a nontrivial binary absorbing subuniverse or center. Since  $R \cap (B_1 \times \dots \times B_n) = R \cap (E_1 \cap \dots \cap E_t)$ , the set  $R \cap (B_1 \times \dots \times B_n)$  is a PC subuniverse of  $R$ .  $\square$

**Lemma 7.26.** *Suppose  $\sigma$  is a congruence on  $D$ ,  $B$  is a one-of-four subuniverse of  $D$  stable under  $\sigma$ . Then  $\{b / \sigma \mid b \in B\}$  is a one-of-four subuniverse of  $D / \sigma$  of the same type as  $B$ .*

*Proof.* For a binary subuniverse and a center it follows from Corollaries 7.1.1 and 7.6.1, respectively. Suppose  $B$  is a linear subuniverse. Let  $\delta$  be the minimal congruence containing both  $\sigma$  and  $\text{ConLin}(D)$ . By Corollary 7.20.1,  $D/\delta$  is a linear algebra. Since  $B$  is stable under  $\delta$ ,  $\{b/\sigma \mid b \in B\}$  is a linear subuniverse of  $D/\sigma$ .

It remains to consider the case when  $B$  is a PC subuniverse of  $D$ , that is,  $B = E_1 \cap \dots \cap E_s$ , where  $E_i$  is an equivalence class of a PC congruence  $\sigma_i$  for every  $i$ . Let  $\delta$  be the minimal congruence containing  $\sigma$  and  $\sigma_1 \cap \dots \cap \sigma_s$ . By Lemma 7.14,  $\delta$  is an intersection of PC congruences  $\delta_1, \dots, \delta_t$ . Since  $B$  is stable under  $\delta$  and  $B$  is an equivalence class of  $\sigma_1 \cap \dots \cap \sigma_s$ ,  $B$  is an equivalence class of  $\delta$ . Hence,  $\{b/\sigma \mid b \in B\}$  is an intersection of the equivalence classes of congruences on  $D/\sigma$  corresponding to  $\delta_1, \dots, \delta_t$ .  $\square$

The following corollaries (proved earlier) state that if we restrict all coordinates of a relation to one-of-four subuniverses of type  $\mathcal{T}$  then we restrict its projection onto the first coordinate to a subuniverse of type  $\mathcal{T}$ . The only difference is that for PC subuniverse we require the relation to be subdirect and without nontrivial binary absorbing subuniverse or center on every coordinate, and for linear subuniverse the first coordinate should be without a nontrivial binary absorbing subuniverse.

**Corollary 7.1.2.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a relation such that  $\text{pr}_1(\rho) = A_1$  and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is an absorbing subuniverse in  $A_i$  with a term  $t$  for every  $i$ . Then  $C$  is an absorbing subuniverse in  $A_1$  with the term  $t$ .*

**Corollary 7.6.2.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a relation such that  $\text{pr}_1(\rho) = A_1$  and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is a center in  $A_i$  for every  $i$ . Then  $C$  is a center in  $A_1$ .*

**Corollary 7.13.2.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a subdirect relation, there is no nontrivial binary absorbing subuniverse or nontrivial center on  $A_1$ , and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is a PC subuniverse in  $A_i$  for every  $i$ . Then  $C$  is a PC subuniverse in  $A_1$ .*

**Corollary 7.24.1.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a relation such that  $\text{pr}_1(\rho) = A_1$ , there is no nontrivial binary absorbing subuniverse on  $A_1$ , and  $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$ , where  $C_i$  is a linear subuniverse of  $A_i$  for every  $i$ . Then  $C$  is a linear subuniverse of  $A_1$ .*

Another common property is that we cannot have  $(C_1, \dots, C_k)$ -essential relation of arity greater than 2 if  $C_1, \dots, C_k$  are subuniverses of a fixed type (any but linear). Note that for PC subuniverses we additionally require the relation to be subdirect. From these claims it can be derived that for the nonlinear case (see Corollary 9.2.1) it is sufficient to check cycle consistency (all calculations are on binary relations) to guarantee a solution.

**Lemma 7.27.** *Suppose  $C_i$  is a nontrivial binary absorbing subuniverse of  $A_i$  with a term  $t$  for  $i \in \{1, 2, \dots, k\}$ ,  $k \geq 2$ . Then there does not exist a  $(C_1, \dots, C_k)$ -essential relation  $\rho \subseteq A_1 \times \dots \times A_k$ .*

*Proof.* Assume that such a relation exists. To get a contradiction it is sufficient to apply term  $t$  to a tuple from  $A_1 \times C_2 \times \dots \times C_k$  and a tuple from  $C_1 \times \dots \times C_{k-1} \times A_k$ .  $\square$

The following two corollaries were proved earlier.

**Corollary 7.10.3.** *Suppose  $C_i$  is a center of  $A_i$  for  $i \in \{1, 2, \dots, k\}$ ,  $k \geq 3$ . Then there does not exist a  $(C_1, \dots, C_k)$ -essential relation  $\rho \subseteq A_1 \times \dots \times A_k$ .*

**Corollary 7.13.3.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a subdirect relation,  $n \geq 3$ ,  $C_i$  is a PC subuniverse in  $A_i$ . There does not exist a  $(C_1, \dots, C_n)$ -essential relation.*

## 7.6 Interaction

Here we explain how one-of-four subuniverses of different types interact with each other.

**Lemma 7.28.** *Suppose  $B_1$  is a binary absorbing, central, or linear subuniverse of  $D$ ,  $B_2$  is a subuniverse of  $D$ . Then  $B_1 \cap B_2$  is a binary absorbing, central, or linear subuniverse of  $B_2$ , respectively.*

*Proof.* If  $B_1$  is a binary absorbing subuniverse or a center, then the claim follows from Lemmas 7.1 and 7.6, respectively. If  $B_1$  is a linear subuniverse, then by Corollary 7.20.1  $B_2/\text{ConLin}(D)$  is a linear algebra, hence  $B_1 \cap B_2$  is a linear subuniverse of  $B_2$ .  $\square$

**Lemma 7.29.** *Suppose  $B_1$  and  $B_2$  are nonempty one-of-four subuniverses of  $D$ ,  $B_1 \cap B_2 = \emptyset$ . Then  $B_1$  and  $B_2$  are subuniverses of the same type.*

*Proof.* Assume the converse. Consider all possible cases.

Case 1.  $B_1$  is a linear subuniverse,  $B_2$  is a binary absorbing subuniverse. By Corollary 7.1.1  $\{b/\text{ConLin}(D) \mid b \in B_2\}$  is a binary absorbing subuniverse on  $D/\text{ConLin}(D)$ . By Lemma 7.21 this subuniverse should be trivial, which contradicts the fact that  $B_1 \cap B_2 = \emptyset$  and  $B_1$  is stable under  $\text{ConLin}(D)$ .

Case 2.  $B_1$  is a linear subuniverse,  $B_2$  is a center. By Corollary 7.6.1  $\{b/\text{ConLin}(D) \mid b \in B_2\}$  is a center of  $D/\text{ConLin}(D)$ . By Lemma 7.21 this subuniverse should be trivial, which contradicts the fact that  $B_1 \cap B_2 = \emptyset$  and  $B_1$  is stable under  $\text{ConLin}(D)$ .

Case 3.  $B_1$  is a linear subuniverse,  $B_2$  is a PC subuniverse. Let  $S \subseteq (D/\text{ConLin}(D)) \times D$  consist of all the tuples  $(c/\text{ConLin}(D), c)$ , where  $c \in D$ . By Lemma 7.21, there is no nontrivial binary absorbing subuniverse or center on  $D/\text{ConLin}(D)$ . Hence, by Corollary 7.13.2, the restriction of the second variable to  $B_2$  implies the restriction of the first variable to a PC subuniverse. Since  $B_1 \cap B_2 = \emptyset$ , this restriction is nontrivial. Thus, there exists a nontrivial PC subuniverse on  $D/\text{ConLin}(D)$ , which contradicts Lemma 7.21.

Case 4.  $B_1$  is a PC subuniverse,  $B_2$  is a binary absorbing subuniverse. By Corollary 7.1.1 the set  $\{b/\text{ConPC}(D) \mid b \in B_2\}$  is a binary absorbing subuniverse of  $D/\text{ConPC}(D)$ . By Lemma 7.15 this subuniverse should be trivial, which contradicts the fact that  $B_1 \cap B_2 = \emptyset$  and  $B_1$  is a PC subuniverse.

Case 5.  $B_1$  is a PC subuniverse,  $B_2$  is a center. By Corollary 7.6.1  $\{b/\text{ConPC}(D) \mid b \in B_2\}$  is a center of  $D/\text{ConPC}(D)$ . By Lemma 7.15 this subuniverse should be trivial, which contradicts the fact that  $B_1 \cap B_2 = \emptyset$  and  $B_1$  is a PC subuniverse.

Case 6.  $B_1$  is a binary absorbing subuniverse,  $B_2$  is a center. Suppose  $R \subseteq D \times G$  is the binary relation from the definition of the center  $B_2$ , and denote  $b^+ = \{a \mid (b, a) \in R\}$  for every  $b \in D$ . We prove this case by induction on the size of  $D$ . Assume that  $b_1^+ \neq b_2^+$  for some  $b_1, b_2 \in B_1$ . Choose an element  $c \in b_1^+ \setminus b_2^+$  (or in  $b_2^+ \setminus b_1^+$ ). Put  $D' = \{a \mid (a, c) \in R\}$ . Note that  $D' \subsetneq D$ ,  $D' \cap B_1 \neq \emptyset$ ,  $D' \cap B_2 = B_2$ . Thus, we obtain subuniverses  $B_1 \cap D'$  and  $B_2$  of a smaller set  $D'$  that are a binary absorbing subuniverse and a center (by Lemma 7.28), respectively. It remains to apply the inductive assumption to  $B_1 \cap D'$  and  $B_2$ . Let us consider the case when  $b_1^+ = b_2^+$  for any  $b_1, b_2 \in B_1$ . Since  $B_1 \cap B_2 = \emptyset$ ,  $b^+ \neq G$  for every  $b \in B_1$ . Let  $f$  be the binary absorbing operation. Choose  $b \in B_1$  and  $e \in B_2$ . Then  $f(b, e) = b_1 \in B_1$  and  $f(e, b) = b_2 \in B_1$ , which means that  $f(b^+, G) \subseteq b_1^+ = b^+$ ,  $f(G, b^+) \subseteq b_2^+ = b^+$ . This contradicts the definition of a center, saying that there is no nontrivial binary absorbing subuniverse on  $G$ .  $\square$

**Theorem 7.30.** *Suppose  $B_1$  and  $B_2$  are one-of-four subuniverses of  $D$  of types  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , respectively. Then  $B_1 \cap B_2$  is a one-of-four subuniverse of  $B_2$  of type  $\mathcal{T}_1$ .*

*Proof.* If  $B_1$  is not a PC subuniverse, then the claim follows from Lemma 7.28. Assume that  $B_1$  is a PC subuniverse of  $D$ .

Let  $\sigma_1, \dots, \sigma_t$  be the set of all PC congruences on  $D$ . Assume that  $B_2$  is not a PC subuniverse. Every equivalence class  $E$  of  $\sigma_i$  is a PC subuniverse. Then Lemma 7.29 implies that  $E$  has a nonempty intersection with  $B_2$ . Therefore  $B_2/\sigma_i \cong D/\sigma_i$  and  $\sigma_i \cap (B_2 \times B_2)$  is a PC congruence on  $B_2$  for every  $i$ . Hence,  $B_1 \cap B_2$  is a PC subuniverse of  $B_2$ , which completes this case.

If  $B_2$  is also a PC subuniverse, then by Corollary 7.13.1,  $B_1 \cap B_2$  is a PC subuniverse of  $B_2$ .  $\square$

**Lemma 7.31.** *Suppose  $D = A_0 = B_0$ ,  $s \geq 1$ ,  $t \geq 0$ ,  $A_i$  is a one-of-four subuniverse of  $A_{i-1}$  for every  $i \in \{1, \dots, s\}$ , and  $B_i$  is a one-of-four subuniverse of  $B_{i-1}$  for every  $i \in \{1, \dots, t\}$ . Then  $A_s \cap B_t$  is a one-of-four subuniverse of  $A_{s-1} \cap B_t$  of the same type as  $A_s$ .*

*Proof.* We prove this lemma by induction on  $s + t$ . Let  $A_s$  be a one-of-four subuniverse of  $A_{s-1}$  of type  $\mathcal{T}$ . For  $t = 0$  the claim follows from the statement. Assume that  $t \geq 1$ . By the inductive assumption,  $A_{s-1} \cap B_t$  and  $A_s \cap B_{t-1}$  are one-of-four subuniverses of  $A_{s-1} \cap B_{t-1}$ , and the second of them is of type  $\mathcal{T}$ . Then by Theorem 7.30, their intersection  $A_s \cap B_t$  is a one-of-four subuniverse of  $A_{s-1} \cap B_t$  of type  $\mathcal{T}$ .  $\square$

**Lemma 7.32.** *Suppose  $R \subseteq A_0 \times B_0$  is a subdirect relation,  $B_i$  is a one-of-four subuniverse of  $B_{i-1}$  for every  $i \in \{1, \dots, t\}$ ,  $A_1$  is a one-of-four subuniverse of  $A_0$ . Then  $\text{pr}_2(R \cap (A_1 \times B_t))$  is a one-of-four subuniverse of  $\text{pr}_2(R \cap (A_1 \times B_{t-1}))$  of the same type as  $B_t$ .*

*Proof.* By Lemma 7.25,  $R \cap (A_0 \times B_i)$  is a one-of-four subuniverse of  $R \cap (A_0 \times B_{i-1})$  of the same type as  $B_i$ , and  $R \cap (A_1 \times B_0)$  is a one-of-four subuniverse of  $R$ . By Lemma 7.31,  $R \cap (A_1 \times B_t)$  is a one-of-four subuniverse of  $R \cap (A_1 \times B_{t-1})$  of the same type as  $B_t$ . Let  $\sigma$  be the congruence on  $R \cap (A_1 \times B_0)$  such that two elements are equivalent whenever their projections onto the second coordinate are equal. Then  $R \cap (A_1 \times B_t)$  is stable under  $\sigma$  for every  $i$ . By Lemma 7.26,  $\text{pr}_2(R \cap (A_1 \times B_t))$  is a one-of-four subuniverse of  $\text{pr}_2(R \cap (A_1 \times B_{t-1}))$  of the same type as  $B_t$ .  $\square$

**Theorem 7.33.** *Suppose  $B_1, \dots, B_n$  are one-of-four subuniverses of  $D$ , and  $B_1 \cap \dots \cap B_n = \emptyset$ . Then there exists  $I \subseteq \{1, \dots, n\}$  with  $\bigcap_{i \in I} B_i = \emptyset$  satisfying one of the following conditions:*

1.  $|I| \leq 2$  and all subuniverses  $B_i$ , where  $i \in I$ , are of the same type;
2.  $B_i$  is a linear subuniverse for every  $i \in I$ ;
3.  $B_i$  is a binary absorbing subuniverse for every  $i \in I$ .

*Proof.* Let us prove by induction on  $n$ . For  $n = 1$  it is trivial. For  $n = 2$  it follows from Lemma 7.29. If  $\bigcap_{i \in I} B_i = \emptyset$  for some  $I \subsetneq \{1, 2, \dots, n\}$ , then applying the inductive assumption to  $\bigcap_{i \in I} B_i$  we obtain the required property. Thus, we assume that if we remove one one-of-four subuniverse from the intersection  $B_1 \cap \dots \cap B_n$  we get a nonempty set.

Let us show that all subuniverses should be of the same type. Put  $C_i = B_i \cap B_n$  for every  $i \in \{1, 2, \dots, n-1\}$ . By Lemma 7.30,  $C_i$  is a one-of-four subuniverse of  $B_n$  of the same type as  $B_i$ . Applying the inductive assumption to  $C_1 \cap \dots \cap C_{n-1} = \emptyset$ , we derive that  $C_1, \dots, C_{n-1}$  are of the same type, hence  $B_1, \dots, B_{n-1}$  are of the same type. Similarly we can show that  $B_2, \dots, B_n$  are of the same type, and therefore, since  $n \geq 3$ , all of them are of the same types.

Assume that all subuniverses  $B_1, \dots, B_n$  are centers or PC subuniverses. Let  $R$  be the  $n$ -ary relation consisting of all tuples  $(a, a, \dots, a)$ . Then  $R$  is a  $(B_1, \dots, B_n)$ -essential relation, which contradicts Corollary 7.10.3 for centers and Corollary 7.13.3 for PC subuniverses.  $\square$



## 8 Proof of the Auxiliary Statements

### 8.1 One-of-four reductions

**Lemma 8.1.** *Suppose  $D^{(1)}$  is a one-of-four reduction for an instance  $\Theta$  of type  $\mathcal{T}$ , which is not the PC type. Then  $\Theta^{(1)}(z)$  is a one-of-four subuniverse of  $\Theta(z)$  of type  $\mathcal{T}$  for every variable  $z$ .*

*Proof.* Let  $\text{Var}(\Theta) = \{x_1, \dots, x_t\}$  and  $\Theta(x_1, \dots, x_t)$  define the relation  $R$ . By Lemma 7.28,  $D_{x_i}^{(1)} \cap \text{pr}_i(R)$  is a one-of-four subuniverse of  $\text{pr}_i(R)$  of type  $\mathcal{T}$  for every  $i$ . Considering  $R$  as a subdirect relation on smaller domains and applying Corollaries 7.1.2, 7.6.2, and 7.24.1 we conclude that  $\Theta^{(1)}(z)$  is a one-of-four subuniverse of  $\Theta(z)$  of type  $\mathcal{T}$ .  $\square$

**Lemma 8.2.** *Suppose  $D^{(1)}$  is a PC reduction for a 1-consistent instance  $\Theta$ , for every variable  $y$  appearing at least twice in  $\Theta$  the pp-formula  $\Theta(y)$  defines  $D_y$ , and  $\Theta(z)$  defines  $D_z$  for a variable  $z$ . Then  $\Theta^{(1)}(z)$  is a PC subuniverse of  $D_z$ .*

*Proof.* First, we rename the variables in  $\Theta$  so that every variable occurs just once and denote the obtained instance by  $\Theta_0$ . Then we identify variables back to obtain the original instance step by step. Thus, we get a sequence  $\Theta_0, \Theta_1, \Theta_2, \dots, \Theta_s$  such that  $\Theta_{i+1}$  is obtained from  $\Theta_i$  by identifying of two variables and  $\Theta_s = \Theta$ . Let us show by induction on  $i$  that for every variable  $z$  the set  $\Theta_i(z) \cap D_z^{(1)}$  is a PC subuniverse of  $\Theta_i(z)$ . For  $i = 0$  it follows from the fact that  $\Theta$  is 1-consistent, and therefore,  $\Theta_0(z)$  defines the full  $D_z$ .

Assume that  $\Theta_{i+1}$  is obtained from  $\Theta_i$  by identifying of  $y$  and  $y'$ , and the variable in  $\Theta$  corresponding to  $y$  and  $y'$  is  $y$ . We know that for every variable  $z$  appearing at least twice in  $\Theta$ ,  $\Theta(z)$  defines  $D_z$ . Hence  $\Theta_{i+1}(y)$  also defines  $D_y$ . Thus, we just need to show that for any variable  $z$  different from  $y$  and  $y'$  the set  $\Theta_{i+1}(z) \cap D_z^{(1)}$  is a PC subuniverse of  $\Theta_{i+1}(z)$ . By the inductive assumption  $\Theta_i(z) \cap D_z^{(1)}$  is a PC subuniverse of  $\Theta_i(z)$ . Then  $\Theta_i(z) \cap D_z^{(1)} = E_1 \cap \dots \cap E_t$ , where  $E_j$  is an equivalence class of a PC congruence  $\sigma_j$  on  $\Theta_i(z)$  for every  $j$ . Let  $S \subseteq \Theta_i(z)/\sigma_j \times D_y \times D_y$  be the relation consisting of all tuples  $(a/\sigma_j, b, b')$  such that  $\Theta_i$  has a solution with  $z = a$ ,  $y = b$ ,  $y' = b'$ . Since the variable  $y$  appears at least twice in  $\Theta$ ,  $\Theta(y)$  defines a full relation. Hence, the relation  $S$  is subdirect and for every  $b \in D_y$  there exists  $E$  such that  $(E, b, b) \in S$ . Lemma 7.19 implies that for every equivalence class  $E$  of  $\sigma_j$  there exists  $b$  such that  $\Theta_i$  has a solution with  $z \in E$  and  $y = y' = b$ , which means that there exists a solution of  $\Theta_{i+1}$  with  $z \in E$ . Therefore,  $\Theta_i(z)/\sigma_j \cong \Theta_{i+1}(z)/\sigma_j$ , which implies that  $\Theta_{i+1}(z) \cap D_z^{(1)}$  is a PC subuniverse of  $\Theta_{i+1}(z)$ . This completes the inductive step.

Since  $\Theta = \Theta_s$ , we proved that  $\Theta(z) \cap D_z^{(1)}$  is a PC subuniverse of  $\Theta(z)$  for every variable  $z$  of  $\Theta$ .

Suppose  $\text{Var}(\Theta) = \{x_1, \dots, x_t\}$ ,  $\Theta(x_1, \dots, x_t)$  defines a relation  $R$ . Then  $R$  can be viewed as a subdirect relation if we reduce the domain of every variable  $x_i$  to  $\Theta(x_i)$ . By Corollary 7.13.2, for any variable  $z$  with  $\Theta(z) = D_z$  we obtain that  $\Theta^{(1)}(z)$  is a PC subuniverse of  $D_z$ .  $\square$

**Lemma 8.3.** *Suppose  $D^{(1)}$  is a minimal absorbing, central, or linear reduction for an instance  $\Theta$ , and  $\Theta(x_1, \dots, x_n)$  defines a full relation. Then  $\Theta^{(1)}(x_1, \dots, x_n)$  defines a full relation or an empty relation.*

*Proof.* If  $\Theta^{(1)}(x_1, \dots, x_n)$  defines an empty relation, then there is nothing to prove. Assume that  $\Theta^{(1)}(x_1, \dots, x_n)$  is not empty.

We prove by induction on  $n$ . For  $n = 1$  by Lemma 8.1  $\Theta^{(1)}(x_1)$  is a subuniverse of  $\Theta(x_1)$  of the corresponding type. By the minimality of the reduction  $D^{(1)}$  the pp-formula  $\Theta^{(1)}(x_1)$  defines  $D_{x_1}^{(1)}$ .

Let us prove the induction step. For each  $i \in \{1, \dots, n-1\}$  choose  $a_i \in D_{x_i}^{(1)}$ . By the inductive assumption,  $\Theta^{(1)}(x_1, \dots, x_{n-1})$  defines a full relation, hence there exists a solution of  $\Theta^{(1)}$  having  $x_i = a_i$  for every  $i \in \{1, \dots, n-1\}$ .

Add the constraint  $x_i = a_i$  to  $\Theta$  for every  $i \in \{1, \dots, n-1\}$  and denote the obtained instance by  $\Omega$ . By the condition of this lemma  $\Omega(x_n)$  defines  $D_{x_n}$ . By Lemma 8.1,  $\Omega^{(1)}(x_n)$  defines a one-of-four subuniverse of  $D_{x_n}$  of the corresponding type, which by the minimality of the reduction  $D^{(1)}$  implies that  $\Omega^{(1)}(x_n)$  defines  $D_{x_n}^{(1)}$ . Since we chose  $a_1, \dots, a_{n-1}$  arbitrary, this means that  $\Theta^{(1)}(x_1, \dots, x_n)$  defines a full relation.  $\square$

**Lemma 8.4.** *Suppose  $D^{(1)}$  is a minimal PC reduction for a 1-consistent instance  $\Theta$ , for every variable  $y$  appearing at least twice in  $\Theta$  the pp-formula  $\Theta(y)$  defines  $D_y$ , and  $\Theta(x_1, \dots, x_n)$  defines a full relation. Then  $\Theta^{(1)}(x_1, \dots, x_n)$  defines a full relation or an empty relation.*

*Proof.* If  $\Theta^{(1)}(x_1, \dots, x_n)$  defines an empty relation, then there is nothing to prove. Assume that  $\Theta^{(1)}(x_1, \dots, x_n)$  is not empty.

First, we join variables  $x_1, \dots, x_n$  into one variable  $X$  with domain  $D_{x_1} \times \dots \times D_{x_n}$ . We replace  $x_1, \dots, x_n$  by  $X$  and change all constraints containing one of the variables  $x_1, \dots, x_n$  correspondingly. The obtained instance we denote by  $\Omega$ . Since  $\Theta(x_1, \dots, x_n)$  defines a full relation, the instance  $\Omega$  is 1-consistent.

Second, we define a reduction  $D^{(1)}$  on the domain of the new variable  $X$  by  $D_X^{(1)} = D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}$ . Let us show that this is a PC reduction. By Lemma 7.25,  $D_X^{(1)}$  is a PC subuniverse of  $D_X$ . By Lemmas 7.3 and 7.7, there is no nontrivial binary absorbing subuniverse or center on  $D_X$ . Thus,  $D^{(1)}$  is a PC reduction for  $\Omega$ . By Lemma 8.2,  $\Omega^{(1)}(X)$  is a PC subuniverse of  $D_X$ . By Lemma 7.17,  $\Omega^{(1)}(X) = B_1 \times \dots \times B_n$ , where  $B_i$  is a PC subuniverse of  $D_{x_i}$  for every  $i$ . By the minimality of  $D^{(1)}$  on  $\Theta$  we obtain that  $B_i = D_{x_i}^{(1)}$ . Hence,  $\Theta^{(1)}(x_1, \dots, x_n)$  defines a full relation.  $\square$

**Lemma 8.5.** *Suppose  $D^{(1)}$  is a one-of-four minimal reduction of an instance  $\Theta$ ,  $\rho(x_1, \dots, x_n)$  is a subdirect constraint of  $\Theta$ , and  $\rho^{(1)}$  is not empty. Then  $\rho^{(1)}$  is subdirect.*

*Proof.* We need to show that  $\text{pr}_i(\rho \cap (D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)})) = D_{x_i}^{(1)}$ . By Corollaries 7.1.2, 7.6.2, 7.13.2, 7.24.1,  $B_i = \text{pr}_i(\rho \cap (D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}))$  is a one-of-four subuniverse of  $D_{x_i}$  of the same type. Since  $\rho^{(1)}$  is not empty,  $B_i$  is not empty. Since  $D_{x_i}^{(1)}$  is a minimal subuniverse of this type, we have  $B_i = D_{x_i}^{(1)}$ .  $\square$

**Lemma 8.6.** *Suppose  $D^{(1)}$  is a one-of-four minimal reduction for a cycle-consistent irreducible CSP instance  $\Theta$ , and  $\Theta^{(1)}$  has a solution. Then  $\Theta^{(1)}$  is cycle-consistent and irreducible.*

*Proof.* Consider a path  $P$  in  $\Theta$  starting and ending with one variable  $x$ . By  $\Omega$  we denote its covering  $z_1 - Q_1 - z_2 - \dots - Q_{l-1} - z_l$  (which is also a covering of  $\Theta$ ) that is obtained from  $P$  by renaming the variables so that every variable except for  $z_2, \dots, z_{l-1}$  occurs just once,  $z_2, \dots, z_{l-1}$  occur twice. Thus,  $z_1$  and  $z_l$  are different but  $S(z_1) = S(z_l) = x$  in the definition of the covering. By  $\Omega'$  we denote the formula obtained from  $\Omega$  by substituting  $z_1$  for  $z_l$ .

First, we prove that  $P$  connects  $a$  with  $a$  in  $\Theta^{(1)}$  for every  $a \in D_x^{(1)}$ . Since  $\Theta$  is cycle-consistent,  $\Omega'(z_1)$  defines  $D_x$ . Since  $\Theta^{(1)}$  has a solution,  $\Omega^{(1)}(z_1)$  defines a nonempty relation. By Lemmas 8.3 and 8.4,  $\Omega^{(1)}(z_1)$  defines  $D_x^{(1)}$ , which means that  $P$  connects  $a$  with  $a$  in  $\Theta^{(1)}$  for every  $a \in D_x^{(1)}$ . Hence,  $\Theta^{(1)}$  is cycle-consistent.

Assume that  $P$  connects any two elements of  $D_x$ , which means that  $\Omega(z_1, z_l)$  defines a full relation. Since  $\Theta^{(1)}$  has a solution,  $\Omega^{(1)}(z_1, z_l)$  defines a nonempty relation. By Lemmas 8.3, 8.4,  $\Omega^{(1)}(z_1, z_l)$  also defines a full relation, which means that  $P$  connects any two elements of  $D_x^{(1)}$  in  $\Theta^{(1)}$ .

Let us prove that  $\Theta^{(1)}$  is irreducible. Consider an instance  $\Upsilon_1 = \{C'_1, \dots, C'_s\}$  consisting of projections of constraints from  $\Theta^{(1)}$  such that it is not fragmented and not linked. Let  $\text{Var}(\Upsilon_1) = \{x_1, \dots, x_n\}$ . By the definition for each constraint  $C'_i$  we can find a constraint  $C_i \in \Theta$  such that  $C'_i$  is a projection of  $C_i^{(1)}$  onto some variables. Let  $\Upsilon_2$  consist of the projections of  $C_1, \dots, C_s$  onto the same variables as in  $\Upsilon_1$ , and  $\Upsilon \in \text{Coverings}(\Theta)$  is obtained from  $\{C_1, \dots, C_s\}$  by renaming variables so that each variable except for  $x_1, \dots, x_n$  appears just once. Then the pp-formulas  $\Upsilon_1(x_1, \dots, x_n)$  and  $\Upsilon^{(1)}(x_1, \dots, x_n)$  define the same relation,  $\Upsilon_2(x_1, \dots, x_n)$  and  $\Upsilon(x_1, \dots, x_n)$  define the same relation. Since  $\Upsilon_1$  is not fragmented, both  $\Upsilon$  and  $\Upsilon_2$  are not fragmented. Also, by Lemma 6.1, both  $\Upsilon$  and  $\Upsilon_2$  are cycle-consistent and irreducible.

Assume that  $\Upsilon_2$  is linked. By Lemma 6.2 there exists a path that connects any two elements of  $D_{x_1}$  in  $\Upsilon_2$ . Then there exists a corresponding path within the variables  $x_1, \dots, x_n$  of  $\Upsilon$  connecting any two elements of  $D_{x_1}$ . As we showed earlier this path, reduced to  $D^{(1)}$ , also connects any two elements of  $D_{x_1}^{(1)}$  in  $\Upsilon^{(1)}$ . The same path can be used to connect any two elements of  $D_{x_1}^{(1)}$  in  $\Upsilon_1$ , which contradicts our assumption that  $\Upsilon_1$  is not linked.

Suppose  $\Upsilon_2$  is not linked. Since  $\Upsilon$  is irreducible, the solution set of  $\Upsilon_2$  is subdirect. Thus, for each variable  $x_i$  (these are only variables appearing more than once in  $\Upsilon$ ) we have  $\Upsilon(x_i) = \Upsilon_2(x_i) = D_{x_i}$ . Then by Lemmas 8.3, 8.4,  $\Upsilon^{(1)}(x_i)$  defines  $D_{x_i}^{(1)}$  or an empty set. It cannot be empty because  $\Theta^{(1)}$  has a solution, therefore we have  $\Upsilon_1(x_i) = \Upsilon^{(1)}(x_i) = D_{x_i}^{(1)}$  for every  $i$ , and the solution set of  $\Upsilon_1$  is subdirect, which completes the proof.  $\square$

**Lemma 8.7.** *Suppose  $D^{(1)}$  is a minimal absorbing or central reduction for  $\Theta$ , the solution set of  $\Theta$  is subdirect,  $D_{x_1} = D_{x_2}$ ,  $D_{x_1}^{(1)} = D_{x_2}^{(1)}$ , both  $\Theta(x_1, x_2)$  and  $\Theta^{(1)}(x_1, x_2)$  define reflexive symmetric relations, and  $\Theta(x_1, x_2)$  contains  $(a, b) \in D_{x_1}^{(1)} \times D_{x_2}^{(1)}$ . Then  $a$  and  $b$  are linked in the relation defined by  $\Theta^{(1)}(x_1, x_2)$ .*

*Proof.* Let  $\text{Var}(\Theta) = \{x_1, x_2, y_1, \dots, y_t\}$ ,  $\Theta(x_1, x_2, y_1, \dots, y_t)$  define a relation  $R$ . The relation  $R$  can be viewed as a ternary relation  $R \subseteq D_{x_1} \times D_{x_2} \times (D_{y_1} \times \dots \times D_{y_t})$ . By Lemmas 7.1 and 7.6,  $G := D_{y_1}^{(1)} \times \dots \times D_{y_t}^{(1)}$  is a one-of-four subuniverse of  $D_{y_1} \times \dots \times D_{y_t}$  of the same type as the reduction  $D^{(1)}$ . Let

$$R'(Y, Y', Y'') = \exists x_1 \exists x_2 R(a, x_1, Y) \wedge R(x_1, x_2, Y') \wedge R(x_2, b, Y'') \wedge x_1 \in D_{x_1}^{(1)} \wedge x_2 \in D_{x_1}^{(1)}.$$

Since  $\Theta(x_1, x_2)$  contains  $(a, b)$  and  $\Theta^{(1)}(x_1, x_2)$  defines a reflexive relation, there exist  $B_1, B'_1 \in G$  such that  $(B_1, B'_1, B''_1) \in R'$  (put  $x_1 = x_2 = a$ ). Similarly, there exist  $B'_2, B''_2 \in G$  such that  $(B_2, B'_2, B''_2) \in R'$  (put  $x_1 = x_2 = b$ ), and  $B_3, B'_3 \in G$  such that  $(B_3, B'_3, B''_3) \in R'$  (put  $x_1 = a, x_2 = b$ ). By Lemma 7.27 and Corollary 7.10.3  $R'$  cannot be  $G$ -essential, which means that  $R \cap (G \times G \times G) \neq \emptyset$ . Hence  $a$  and  $b$  are linked (by a path of length 3) in  $\Theta^{(1)}(x_1, x_2)$ .  $\square$

**Lemma 8.8.** *Suppose  $D^{(1)}$  is a minimal PC reduction for  $\Theta$ , the solution set of  $\Theta$  is subdirect,  $D_{x_1} = D_{x_2}$ ,  $D_{x_1}^{(1)} = D_{x_2}^{(1)}$ , both  $\Theta(x_1, x_2)$  and  $\Theta^{(1)}(x_1, x_2)$  define reflexive symmetric relations, and  $\Theta(x_1, x_2)$  contains  $(a, b) \in D_{x_1}^{(1)} \times D_{x_2}^{(1)}$ . Then  $a$  and  $b$  are linked in the relation defined by  $\Theta^{(1)}(x_1, x_2)$ .*

*Proof.* Suppose  $y$  is a variable of  $\Theta$  and  $\sigma$  is a PC congruence on  $D_y$ . Consider a relation  $\rho \subseteq D_{x_1} \times D_y/\sigma$  consisting of all the tuples  $(c, C)$  such that there exists a solution of  $\Theta$  with  $x_1 = c$  and  $y \in C$ . Since there is no nontrivial binary absorbing subuniverse or center on  $D_{x_1}$ , by Lemma 7.13, either  $\rho$  is a full relation, or  $\text{Con}(\rho, 2)$  is the equality relation. In the first case it does not matter what value we substitute for the variable  $x_1$  the variable  $y$  can be at any equivalence class of  $\sigma$ . In the second case the equivalence class is uniquely determined by

the variable  $x_1$ . Moreover, since  $D_{x_1}^{(1)}$  is a minimal PC subuniverse, the equivalence class is the same for all elements of  $D_{x_1}^{(1)}$ . Since  $\Theta^{(1)}$  has a solution, this equivalence class is the class containing  $D_y^{(1)}$ . Later, we will specify whether a PC congruence is of the *first type* (from the first case) or of the *second type* (from the second case).

Let  $\text{Var}(\Theta) = \{x_1, x_2, y_1, \dots, y_t\}$ . Let  $R$  be the relation defined by  $\Theta(x_1, x_2, y_1, \dots, y_t)$ . By  $\Upsilon$  denote the following formula

$$R(a, x_2, y_1, \dots, y_t) \wedge R(x_1, x_2, y'_1, \dots, y'_t) \wedge R(x_1, x'_2, z_1, \dots, z_t) \wedge R(b, x'_2, z'_1, \dots, z'_t) \wedge x_1 \in D_{x_1}^{(1)}.$$

Consider a congruence  $\sigma$  of the first type on the domain of any variable  $y$  of  $\Upsilon$ .

Assume that  $y \in \{x_2, y_1, \dots, y_t, y'_1, \dots, y'_t\}$ . It follows from the definition of the first type that for any equivalence class  $E$  of  $\sigma$  there exists a solution of  $\Upsilon$  such that  $x_1 = a$ ,  $x'_2 = b$ ,  $y_i = y'_i$  for every  $i$ , and  $y \in E$ .

Similarly, assume that  $y \in \{x'_2, z_1, \dots, z_t, z'_1, \dots, z'_t\}$ . For any equivalence class  $E$  of  $\sigma$  there exists a solution of  $\Upsilon$  such that  $x_1 = x_2 = b$ ,  $z_i = z'_i$  for every  $i$ , and  $y \in E$ .

Thus, we showed that in both cases  $\Upsilon(y)/\sigma \cong D_y/\sigma$ . Let  $E$  be the equivalence class of  $\sigma$  containing  $D_y^{(1)}$ . By  $\delta$  we denote the extension of  $\sigma$  onto the solution set of  $\Upsilon$ , and by  $E_\sigma$  we denote the equivalence class of  $\delta$  corresponding to  $E$ . Since  $\Upsilon(y)/\sigma \cong D_y/\sigma$  and  $D_y/\sigma$  is a PC algebra without a nontrivial binary absorbing subuniverse or center,  $E_\sigma$  is a PC subuniverse of the solution set of  $\Upsilon$ .

Consider the intersection of  $E_\sigma$  for all PC congruences  $\sigma$  of the first type. If this intersection is not empty, then there exists a solution of  $\Upsilon$  such that any element of this solution is in the equivalence class containing  $D_y^{(1)}$  for any PC congruence of the first type. Since  $a, b \in D_{x_1}^{(1)}$  and  $x_1 \in D_{x_1}^{(1)}$  in the definition of  $\Upsilon$ , the same is true for any PC congruence of the second type. Since  $D_y^{(1)}$  is the intersection of all equivalence classes containing  $D_y^{(1)}$  of all PC congruences for any variable  $y$ , the solution is in  $D^{(1)}$ , which means that  $a$  and  $b$  are linked in  $\Theta^{(1)}(x_1, x_2)$ .

Assume that the intersection of  $E_\sigma$  for all PC congruences  $\sigma$  of the first type is empty. By Theorem 7.33 there should be two congruences  $\sigma$  and  $\sigma'$  such that  $E_\sigma \cap E_{\sigma'} = \emptyset$ . Let  $y$  and  $y'$  be the variables of  $\Upsilon$  corresponding to  $\sigma$  and  $\sigma'$ . Consider several cases.

Case 1.  $y, y' \in \{x_2, x'_2, y'_1, \dots, y'_t, z_1, \dots, z_t, z'_1, \dots, z'_t\}$ . Since  $\Theta^{(1)}(x_1, x_2)$  defines a reflexive relation,  $\Upsilon$  has a solution with  $x_2 = x_1 = x'_2 = b$  and all the variables  $y'_1, \dots, y'_t, z_1, \dots, z_t, z'_1, \dots, z'_t$  are from  $D^{(1)}$ . This contradicts the fact that  $E_\sigma \cap E_{\sigma'} = \emptyset$ .

Case 2.  $y, y' \in \{x_2, x'_2, y_1, \dots, y_t, y'_1, \dots, y'_t, z_1, \dots, z_t\}$ . Similarly,  $\Upsilon$  has a solution with  $x_1 = x_2 = x'_2 = a$  and all the variables  $y_1, \dots, y_t, y'_1, \dots, y'_t, z_1, \dots, z_t$  are from  $D^{(1)}$ . This contradicts the fact that  $E_\sigma \cap E_{\sigma'} = \emptyset$ .

Case 3.  $y \in \{y_1, \dots, y_t\}$ ,  $y' \in \{z'_1, \dots, z'_t\}$ . Similarly,  $\Upsilon$  has a solution with  $x_1 = x_2 = a$ ,  $x'_2 = b$  and all the variables  $y_1, \dots, y_t, y'_1, \dots, y'_t, z'_1, \dots, z'_t$  are from  $D^{(1)}$ . Again, this contradicts the fact that  $E_\sigma \cap E_{\sigma'} = \emptyset$ .  $\square$

**Lemma 8.9.** *Suppose  $D^{(1)}$  is a minimal nonlinear reduction for  $\Theta$ , the solution set of  $\Theta$  is subdirect,  $\Theta^{(1)}$  is not empty,  $\Theta(x_1, x_2)$  defines a relation containing  $(a, b) \in D_{x_1}^{(1)} \times D_{x_2}^{(1)}$ . Then  $a$  and  $b$  are linked in the relation defined by  $\Theta^{(1)}(x_1, x_2)$ .*

*Proof.* By Lemma 8.5, the solution set of  $\Theta^{(1)}$  is subdirect. Let  $\text{Var}(\Theta) = \{x_1, x_2, y_1, \dots, y_t\}$ ,  $R$  be the relation defined by  $\Theta(x_1, x_2, y_1, \dots, y_t)$ . Let  $\Omega$  be the following instance

$$R(x_1, x_2, y_1, \dots, y_t) \wedge R(x'_1, x_2, y'_1, \dots, y'_t).$$

Since the solution sets of  $\Theta$  and  $\Theta^{(1)}$  are subdirect, the solution sets of  $\Omega$  and  $\Omega^{(1)}$  are also subdirect. Also, there should be a solution of  $\Theta^{(1)}$  with  $x_2 = b$ . Let  $x_1 = b'$  in this solution. Then  $\Omega(x_1, x'_1)$  contains  $(a, b')$ . Since both  $\Omega(x_1, x'_1)$  and  $\Omega^{(1)}(x_1, x'_1)$  define symmetric reflexive relations, Lemmas 8.7 and 8.8 imply that  $a$  and  $b'$  are linked in  $\Omega^{(1)}(x_1, x'_1)$ . Since  $(b', b)$  is in  $\Theta^{(1)}(x_1, x_2)$ , we derive that  $a$  and  $b$  are linked in  $\Theta^{(1)}(x_1, x_2)$ , which completes the proof.  $\square$

## 8.2 Properties of $\text{Con}(\rho, x)$

**Lemma 8.10.** *Suppose  $\rho$  is a critical rectangular relation of arity  $n \geq 2$ ,  $\rho'$  is the cover of  $\rho$ . Then  $\text{Con}(\rho', 1) \supsetneq \text{Con}(\rho, 1)$ , for  $n > 2$  we also have  $\text{Con}(\text{pr}_{1,2}(\rho), 1) \supsetneq \text{Con}(\rho, 1)$ ,*

*Proof.* For every  $i \in \{1, 2, \dots, n\}$  we define  $\rho_i(x_1, \dots, x_n) = \exists x'_i \rho(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$ . Since  $\rho$  is critical, it has no dummy variables, therefore  $\rho \subsetneq \rho_i$  for every  $i$ . Also  $\rho \subsetneq \bigcap_i \rho_i$ . Choose a tuple  $(a_1, \dots, a_n) \in \rho' \setminus \rho$ . Since  $\rho'$  is the cover of  $\rho$  we have  $\rho' \subseteq \bigcap_i \rho_i$ . Since this tuple is in  $\rho_i$ , for every  $i$  there is  $b_i$  such that  $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$ . Then  $(a_1, b_1) \in \text{Con}(\rho', 1)$ , which means by the rectangularity of  $\rho$  that  $\text{Con}(\rho', 1) \supsetneq \text{Con}(\rho, 1)$ . For  $n > 2$  we have  $(b_1, a_2, \dots, a_n), (a_1, \dots, a_{n-1}, b_n) \in \rho$ , hence  $(a_1, b_1) \in \text{Con}(\text{pr}_{1,2}(\rho), 1)$  and therefore  $\text{Con}(\text{pr}_{1,2}(\rho), 1) \supsetneq \text{Con}(\rho, 1)$ .  $\square$

**Lemma 8.11.** *Suppose  $\rho$  is a critical subdirect relation and the  $i$ -th variable of  $\rho$  is rectangular. Then  $\text{Con}(\rho, i)$  is an irreducible congruence.*

*Proof.* To simplify notations assume that  $i = 1$ . Put  $\sigma = \text{Con}(\rho, 1)$ . As we mentioned in Section 6.4,  $\sigma$  should be a congruence. Assume that it is not an irreducible congruence. Consider binary relations  $\delta_1, \dots, \delta_s \supsetneq \sigma$  stable under  $\sigma$  such that  $\delta_1 \cap \dots \cap \delta_s = \sigma$ . Put

$$\rho_j(x_1, \dots, x_n) = \exists x'_1 \rho(x'_1, x_2, \dots, x_n) \wedge \delta_j(x_1, x'_1).$$

Consider a tuple  $(x_1, \dots, x_n)$  in the intersection of  $\rho_1, \dots, \rho_s$ . Since  $\delta_j$  is stable under  $\sigma = \text{Con}(\rho, 1)$ , we may assume that  $x'_1$  takes the same value in the definition of every  $\rho_j$ . Then  $(x_1, x'_1)$  should be in  $\delta_j$  for every  $j$ , which implies that  $(x_1, x'_1) \in \sigma$  and  $(x_1, \dots, x_n) \in \rho$ . Hence, the intersection of  $\rho_1, \dots, \rho_s$  gives  $\rho$ . Since  $\rho \subsetneq \rho_j$  for every  $j$ , this contradicts the fact that  $\rho$  is critical.  $\square$

For a relation  $\rho$  of arity  $n$  by  $\text{UnPol}(\rho)$  we denote the set of all unary vector-functions preserving the relation  $\rho$ .

**Lemma 8.12.** *Suppose a pp-formula  $\Omega(x_1, \dots, x_n)$  defines a relation  $\rho$ ,  $\alpha \in D_{x_1} \times \dots \times D_{x_n}$ , and  $\rho' = \{f(\alpha) \mid f \in \text{UnPol}(\rho)\}$ . Then there exists  $\Omega' \in \text{Coverings}(\Omega)$  such that  $\Omega'(x_1, \dots, x_n)$  defines  $\rho'$ .*

*Proof.* Suppose  $\alpha = (a_1, \dots, a_n)$ . We introduce new variables  $x_i^a$  for every  $i \in \{1, 2, \dots, n\}$  and  $a \in D_{x_i}$ . By  $\Upsilon$  we denote the following formula  $\bigwedge_{(b_1, \dots, b_n) \in \rho} \rho(x_1^{b_1}, \dots, x_n^{b_n})$ . This formula can be understood in the following way. If we encode a unary vector function by variables so that  $f(b_1, \dots, b_n) = (x_1^{b_1}, \dots, x_n^{b_n})$  for every  $b_1, \dots, b_n$ , then the formula says that the vector function preserves  $\rho$ . Then  $\rho'$  can be defined by a pp-formula  $\Upsilon(x_1^{a_1}, \dots, x_n^{a_n})$ . To obtain the formula  $\Omega'$  it is sufficient to replace each  $\rho(x_1^{b_1}, \dots, x_n^{b_n})$  by a copy of  $\Omega$  (replacing  $x_1, \dots, x_n$  with  $x_1^{b_1}, \dots, x_n^{b_n}$ ) and then replace  $x_1^{a_1}, \dots, x_n^{a_n}$  with  $x_1, \dots, x_n$ .  $\square$

**Corollary 8.12.1.** *Suppose a pp-formula  $\Omega(x_1, \dots, x_n)$  defines a relation without a tuple  $\alpha \in D_{x_1} \times \dots \times D_{x_n}$ ,  $\Sigma$  is the set of all relations defined by  $\Upsilon(x_1, \dots, x_n)$  where  $\Upsilon \in \text{Coverings}(\Omega)$ , and  $\rho$  is an inclusion-maximal relation in  $\Sigma$  without the tuple  $\alpha$ . Then  $\alpha$  is a key tuple for  $\rho$ .*

*Proof.* For every tuple  $\beta \notin \rho$  we consider  $\rho_\beta := \{f(\beta) \mid f \in \text{UnPol}(\rho)\}$ . Since  $f$  can be a constant mapping to a tuple from  $\rho$  and an identity, we have  $\rho_\beta \supsetneq \rho$  for every  $\beta$ . By Lemma 8.12,  $\rho_\beta$  should be in  $\Sigma$ . Since  $\rho$  is inclusion-maximal,  $\alpha \in \rho_\beta$ . Therefore, any  $\beta$  can be mapped to  $\alpha$  by a unary vector-function preserving  $\rho$ , which means that  $\alpha$  is a key tuple for  $\rho$ .  $\square$

The next lemma shows that we can apply the operation Con and a nonlinear reduction  $D^{(1)}$  to a pp-formula  $\Upsilon(x_1, \dots, x_n)$  in any order, the result will be the same. For the linear reduction a slight modification of the statement is required (see Lemma 8.14).

**Lemma 8.13.** *Suppose  $D^{(1)}$  is a minimal nonlinear reduction for an instance  $\Upsilon$ , the solution set of  $\Upsilon$  is subdirect, and  $\Upsilon^{(1)}(x_1, \dots, x_n)$  defines a subdirect rectangular relation. Then for every  $i$*

$$(\text{Con}(\Upsilon(x_1, \dots, x_n), i))^{(1)} = \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), i).$$

*Proof.* Put  $\sigma_0 = \text{Con}(\Upsilon(x_1, \dots, x_n), i)$ ,  $\sigma_1 = \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), i)$ . Let  $\{x_1, \dots, x_n, y_1, \dots, y_s\}$  be the set of all variables of  $\Upsilon$ . Let  $\Xi = \Upsilon \wedge \Upsilon_{x_i, y_1, \dots, y_s}^{x'_i, y'_1, \dots, y'_s}$ . We can check that  $\sigma_0$  is defined by  $\Xi(x_i, x'_i)$ , and  $\sigma_1$  is defined by  $\Xi^{(1)}(x_i, x'_i)$ . Since  $\Upsilon^{(1)}(x_1, \dots, x_n)$  defines a rectangular relation,  $\sigma_1$  is a congruence. It follows from the definition that  $\sigma_0^{(1)} \supseteq \sigma_1$ . Let us prove the backward inclusion. Choose a pair  $(a, b) \in \sigma_0^{(1)}$ . Since  $\sigma_0$  is defined by  $\Xi(x_i, x'_i)$ , by Lemma 8.9,  $a$  and  $b$  should be linked in  $\Xi^{(1)}(x_i, x'_i)$ . Since  $\sigma_1$  is a congruence,  $a$  and  $b$  can be linked only if  $(a, b) \in \sigma_1$ , which means that  $\sigma_0^{(1)} = \sigma_1$ .  $\square$

**Lemma 8.14.** *Suppose  $D^{(1)}$  is a minimal linear reduction for  $\Upsilon$ ,  $\Upsilon^{(1)}(x_1, \dots, x_n)$  defines a subdirect rectangular relation,  $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, v_1, \dots, v_r\}$ , and  $\Omega = \Upsilon \wedge \bigwedge_{i=1}^r \sigma_i(v_i, u_i)$ , where  $\sigma_i = \text{ConLin}(D_{v_i})$ . Then  $(\text{Con}(\Omega(x_1, \dots, x_n, u_1, \dots, u_r), j))^{(1)} = \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), j)$  for every  $j$ .*

*Proof.* Without loss of generality assume that  $j = 1$ . Since the reduction  $D^{(1)}$  is minimal, we have the following inclusion

$$(\text{Con}(\Omega(x_1, \dots, x_n, u_1, \dots, u_r), 1))^{(1)} \supseteq \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), 1).$$

Let us prove the backward inclusion. Suppose  $\Omega(x_1, \dots, x_n, u_1, \dots, u_r)$  and  $\Upsilon^{(1)}(x_1, \dots, x_n)$  define the relations  $\rho'$  and  $\rho$  respectively. Choose  $a, b \in D_{x_1}^{(1)}$  such that  $(a, b) \in \text{Con}(\rho', 1)$ . For some  $\beta$  we have  $a\beta, b\beta \in \rho'$ . Since  $\rho$  is subdirect, there exist  $\alpha_a$  and  $\alpha_b$  in  $D^{(1)}$  such that  $a\alpha_a, b\alpha_b \in \rho'$ . Since  $w$  preserves  $\rho'$ ,

$$\begin{aligned} w(a, a, \dots, a)w(\alpha_a, \beta, \dots, \beta) &\in \rho', \\ w(a, b, \dots, b)w(\alpha_a, \beta, \dots, \beta) &\in \rho', \\ w(b, \dots, b, a)w(\alpha_b, \beta, \dots, \beta) &\in \rho', \\ w(b, b, \dots, b)w(\alpha_b, \beta, \dots, \beta) &\in \rho'. \end{aligned}$$

By Lemma 7.23,  $w(\alpha_a, \beta, \dots, \beta)$  and  $w(\alpha_b, \beta, \dots, \beta)$  belong to  $D^{(1)}$ . Then, for  $c = w(a, b, \dots, b) = w(b, \dots, b, a)$  we have  $(a, c), (c, b) \in \text{Con}(\rho, 1)$ . Since  $\rho$  is rectangular, we have  $(a, b) \in \text{Con}(\rho, 1)$ .  $\square$

### 8.3 Adding linear variable

Below we formulate few statements from [62] that will help us to prove the main property of a bridge. This property will be the main ingredient of the proof of the fact that  $A'$  from the informal description of the algorithm should be of codimension 1.

A relation  $\rho \subseteq A^n$  is called *strongly rich* if for every tuple  $(a_1, \dots, a_n)$  and every  $j \in \{1, \dots, n\}$  there exists a unique  $b \in A$  such that  $(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \in \rho$ . We will need two statements from [62].

Recall that for any bridge  $\rho$  by  $\tilde{\rho}$  we denote the binary relation defined by  $\tilde{\rho}(x, y) = \rho(x, x, y, y)$ .

**Theorem 8.15.** [62] Suppose  $\rho \subseteq A^n$  is a strongly rich relation preserved by an idempotent WNU. Then there exists an abelian group  $(A; +)$  and bijective mappings  $\phi_1, \phi_2, \dots, \phi_n : A \rightarrow A$  such that

$$\rho = \{(x_1, \dots, x_n) \mid \phi_1(x_1) + \phi_2(x_2) + \dots + \phi_n(x_n) = 0\}.$$

**Lemma 8.16.** [62] Suppose  $(G; +)$  is a finite abelian group, the relation  $\sigma \subseteq G^4$  is defined by  $\sigma = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 = a_3 + a_4\}$ , and  $\sigma$  is preserved by an idempotent WNU  $f$ . Then  $f(x_1, \dots, x_n) = t \cdot x_1 + t \cdot x_2 + \dots + t \cdot x_n$  for some  $t \in \{1, 2, 3, \dots\}$ .

**Theorem 8.17.** Suppose  $\sigma \subseteq A^2$  is a congruence,  $\rho$  is a bridge from  $\sigma$  to  $\sigma$  such that  $\tilde{\rho}$  is a full relation,  $\text{pr}_{1,2}(\rho) = \omega$ ,  $\omega$  is a minimal relation stable under  $\sigma$  such that  $\omega \supsetneq \sigma$ . Then there exists a prime number  $p$  and a relation  $\zeta \subseteq A \times A \times \mathbb{Z}_p$  such that  $\text{pr}_{1,2}\zeta = \omega$  and  $(a_1, a_2, b) \in \zeta$  implies that  $(a_1, a_2) \in \sigma \Leftrightarrow (b = 0)$ .

*Proof.* Since the relations  $\rho$  and  $\omega$  are stable under  $\sigma$ , we consider  $A/\sigma$  instead of  $A$  and assume that  $\sigma$  is the equality relation.

Without loss of generality we assume that  $\rho(x_1, x_2, y_1, y_2) = \rho(y_1, y_2, x_1, x_2)$  and  $(a, b, a, b) \in \rho$  for any  $(a, b) \in \omega$ . Otherwise, we consider the relation  $\rho'$  instead of  $\rho$ , where

$$\rho'(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2 \rho(x_1, x_2, z_1, z_2) \wedge \rho(y_1, y_2, z_1, z_2).$$

We prove by induction on the size of  $A$ . Assume that for some subuniverse  $A' \subsetneq A$  we have  $(A' \times A') \cap (\omega \setminus \sigma) \neq \emptyset$ . By  $\sigma'$  we denote the equality relation on  $A'$ . By  $\omega'$  we denote a minimal relation such that  $\sigma' \subsetneq \omega' \subseteq (A' \times A') \cap \omega$ . Since  $\text{pr}_{1,2}(\rho \cap (\omega' \times \omega')) = \omega' \supsetneq \sigma'$ , the relation  $\rho \cap (\omega' \times \omega')$  is a bridge from  $\sigma'$  to  $\sigma'$ . The inductive assumption for  $\rho \cap (\omega' \times \omega')$  implies that there exists a relation  $\zeta' \subseteq A' \times A' \times \mathbb{Z}_p$  such that  $(x_1, x_2, 0) \in \zeta' \Leftrightarrow (x_1, x_2) \in \sigma'$  and  $\text{pr}_{1,2}(\zeta') = \omega'$ . Put

$$\zeta(x_1, x_2, z) = \exists y_1 \exists y_2 \rho(x_1, x_2, y_1, y_2) \wedge \zeta'(y_1, y_2, z).$$

By the minimality of  $\omega$ , we have  $\text{pr}_{1,2}(\zeta) = \omega$ . The remaining property of  $\zeta$  follows from the fact that  $\rho$  is a bridge and the properties of  $\zeta'$ .

Thus, we may assume that for any subuniverse  $A' \subsetneq A$  we have  $(A' \times A') \cap (\omega \setminus \sigma) = \emptyset$ .

Consider a pair  $(a_1, a_2) \in \omega \setminus \sigma$ . Let  $A' = \{a \mid (a_1, a) \in \omega\}$ . Since  $\omega \supsetneq \sigma$ , we have  $a_1 \in A'$ , and therefore  $(a_1, a_2) \in (A' \times A') \cap (\omega \setminus \sigma) \neq \emptyset$  and  $A' = A$ . Thus,  $\{a \mid (a_1, a) \in \omega\} = \{a \mid (a, a_2) \in \omega\} = A$ . Hence, any element connected in  $\omega$  to some other element is connected to all elements. Therefore,  $(a_1, a), (a, a_2) \in \omega$  for every  $a \in A \setminus \{a_1, a_2\}$ , which for  $|A| > 2$  implies that  $\omega = A \times A$ .

If  $|A| = 2$  and  $\omega \neq A \times A$  then  $\omega = \{(a, a), (a, b), (b, b)\}$  and  $\rho$  is uniquely defined. We know [52] that any clone on a 2-element domain containing an idempotent WNU operation contains majority operation, conjunction, disjunction, or minority operation. None of them preserve  $\rho$ , which contradicts our assumptions.

Thus, we proved that  $\omega = A \times A$  and  $A$  has no proper subuniverses of size at least 2.

Note the remaining part of the proof could also be derived from known facts of commutator theory. In fact, it follows from the properties of  $\rho$  that  $\sigma$  (the equality) is an equivalence block of a congruence on  $A^2$ , which means that  $A$  is Abelian. Using Abelianess for Taylor varieties (since we have a WNU), we could also define the required ternary relation  $\zeta$  (see [8] for more details). Nevertheless, we do not want to introduce new algebraic notions, and give a proof based on two claims from [62].

Let us show that for any  $a_1, a_2, a_3 \in A$  there exists a unique  $a_4$  such that  $(a_1, a_2, a_3, a_4) \in \rho$ . For every  $a \in A$  put  $\lambda_a(x_1, x_2) = \exists y_2 \rho(x_1, x_2, a, y_2)$ . It is easy to see that  $\sigma \subsetneq \lambda_a \subseteq \omega$ . Therefore  $\lambda_a = \omega = A \times A$  for every  $a$ . We consider the unary relation defined by  $\delta(x) =$

$\rho(a_1, a_2, a_3, x)$ . By the above fact  $\delta$  is not empty. Since  $\rho$  is a bridge,  $\delta$  is not full. If  $\delta$  contains more than one element, then we get a contradiction with the fact that there are no proper subuniverses of size at least 2.

Then  $\rho$  is a strongly rich relation. By Theorem 8.15, there exist an Abelian group  $(A; +)$  and bijective mappings  $\phi_1, \phi_2, \phi_3, \phi_4: A \rightarrow A$  such that

$$\rho = \{(a_1, a_2, b_1, b_2) \mid \phi_1(a_1) + \phi_2(a_2) + \phi_3(b_1) + \phi_4(b_2) = 0\}.$$

Without loss of generality we can assume that  $\phi_1(x) = x$ . We know that  $(a, a, b, b) \in \rho$  for any  $a, b \in A$ , then  $\phi_1(x) + \phi_2(x) + \phi_3(0) + \phi_4(0) = 0$ , which means that  $\phi_2(x) = -x - \phi_3(0) - \phi_4(0)$ . Since  $(a, b, a, b) \in \rho$  for any  $a, b \in A$ , we have  $\phi_1(x) + \phi_2(0) + \phi_3(x) + \phi_4(0) = 0$ , which means that  $\phi_3(x) = -\phi_1(x) - \phi_2(0) - \phi_4(0) = -x + \phi_3(0)$ . Similarly, since  $\phi_1(0) + \phi_2(0) + \phi_3(x) + \phi_4(x) = 0$ , we have  $\phi_4(x) = x - \phi_3(0) - \phi_2(0) - \phi_1(0) = x + \phi_4(0)$ . Substituting this into the definition of  $\rho$  we obtain

$$\rho = \{(a_1, a_2, b_1, b_2) \mid a_1 - a_2 - a_3 + a_4 = 0\}.$$

It follows from Lemma 8.16 that  $w$  on  $A$  is defined by  $t(x_1 + \dots + x_m)$ , Since  $w$  is special,  $t \cdot (t - 1)$  should be divided by the order of any element of  $A$ . By the idempotency,  $t$  and the order of any element are coprime. Hence,  $t - 1$  should be divided by the order of any element and we may put  $t = 1$ . Therefore, the relation  $\zeta \subseteq A \times A \times A$  defined by  $\zeta = \{(b_1, b_2, b_3) \mid b_1 - b_2 + b_3 = 0\}$  is preserved by  $w$ . If  $(A; +)$  is not simple, then any equivalence class of a congruence is a proper subuniverse of size at least 2, which contradicts our assumption. Therefore,  $(A; +)$  is a simple Abelian group.  $\square$

**Corollary 8.17.1.** *Suppose  $\sigma \subseteq A^2$  is an irreducible congruence and  $\rho$  is a bridge from  $\sigma$  to  $\sigma$  such that  $\tilde{\rho}$  is a full relation. Then there exists a prime number  $p$  and a relation  $\zeta \subseteq A \times A \times \mathbb{Z}_p$  such that  $\text{pr}_{1,2}\zeta = \sigma^*$  and  $(a_1, a_2, b) \in \zeta$  implies that  $(a_1, a_2) \in \sigma \Leftrightarrow (b = 0)$ .*

**Lemma 8.18.** *Suppose  $\rho \subseteq A^4$  is an optimal bridge from  $\sigma_1$  to  $\sigma_2$ , and  $\sigma_1$  and  $\sigma_2$  are different irreducible congruences. Then  $\tilde{\rho} \supsetneq \sigma_2$ .*

*Proof.* Since the first two variables are stable under  $\sigma_1$  and the last two variables are stable under  $\sigma_2$ , we have  $\sigma_1 \subseteq \tilde{\rho}$  and  $\sigma_2 \subseteq \tilde{\rho}$ . Assume that the lemma does not hold, then  $\tilde{\rho} = \sigma_2$ .

Since  $\sigma_1$  and  $\sigma_2$  are different, we obtain  $\sigma_1 \subsetneq \sigma_2$ ,

First, we want  $(a, d)$  to be from  $\sigma_2$  for every  $(a, b, c, d) \in \rho$ . Put  $\rho_1(x_1, x_2, y_1, y_2) = \rho(x_1, x_2, y_1, y_2) \wedge \sigma_2(x_1, y_2)$ . If  $\rho_1$  is a bridge then we replace  $\rho$  by  $\rho_1$ . Assume that  $\rho_1$  is not a bridge, then for every  $(a, b, c, d) \in \rho$  with  $(a, d) \in \sigma_2$  we have  $(a, b) \in \sigma_1$ . Put  $\rho_2(x_1, x_2, y_1, y_2) = \exists z \rho(x_1, x_2, z, y_1) \wedge \sigma_2(x_1, y_2)$ . Let us show that  $\rho_2$  is a bridge. Suppose  $(x_1, x_2) \in \sigma_1$  and  $(x_1, x_2, y_1, y_2) \in \rho_2$ . Then  $(x_1, x_2, z, y_1) \in \rho$ . Since  $\rho$  is a bridge from  $\sigma_1$  to  $\sigma_2$ , this implies that  $(z, y_1) \in \sigma_2$  and  $(x_1, y_1) \in \tilde{\rho}$ . Since  $\tilde{\rho} = \sigma_2$ , we have  $(x_1, y_1) \in \sigma_2$  and therefore  $(y_1, y_2) \in \sigma_2$ . If  $(y_1, y_2) \in \sigma_2$  and  $(x_1, x_2, y_1, y_2) \in \rho_2$ , then  $(x_1, y_1) \in \sigma_2$  and by the above assumption we have  $(x_1, x_2) \in \sigma_1$ . It remains to show that  $\text{pr}_{1,2}(\rho_2) \supsetneq \sigma_1$ . Consider any tuple  $(a_1, a_2, a_3, a_4) \in \rho$  such that  $(a_1, a_2) \notin \sigma_1$ , then  $(a_1, a_2, a_4, a_1) \in \rho_2$ . Thus,  $\rho_2$  is a bridge with the required property, so we replace  $\rho$  by  $\rho_2$ .

Second, we want  $\text{pr}_{1,2}(\rho)$  to be equal to  $\sigma_1^*$ , and  $\text{pr}_{3,4}(\rho)$  to be equal to  $\sigma_2^*$ . To achieve this we replace  $\rho$  by the relation defined by  $\rho(x_1, x_2, y_1, y_2) \wedge \sigma_1^*(x_1, x_2) \wedge \sigma_2^*(y_1, y_2)$ , which has the same properties.

Recall that a *polynomial* is an operation that can be defined by a term over the basic operations of an algebra and constant operations. In our case, a polynomial is an operation defined by a term over the WNU  $w$  and constants. Let  $D$  be a minimal subset (not necessarily a subuniverse) of  $A$  such that



1. there exists a unary polynomial  $h$  such that  $h(h(x)) = h(x)$  and  $h(A) = D$ , and
2.  $(\sigma_2^* \setminus \sigma_2) \cap D^2 \neq \emptyset$ .

Since constants preserve a reflexive bridge  $\rho$  and congruences  $\sigma_1$  and  $\sigma_2$ , the unary polynomial  $h$  also preserves  $\rho$ ,  $\sigma_1$  and  $\sigma_2$ . It is not hard to see that  $h(w(x_1, \dots, x_m))$  is an idempotent WNU on  $D$ , then by  $w^D$  we denote a special WNU on  $D$  that can be derived from the idempotent WNU on  $D$ . For any relation  $\delta$ , by  $\delta^D$  we denote its restriction to  $D$  (that is  $h(\delta)$ ). It is not hard to see that  $\rho^D$  is a bridge from  $\sigma_1^D$  to  $\sigma_2^D$ .

The idea of the proof is to define a bridge  $\epsilon^D$  from  $\sigma_2^D$  to  $\sigma_2^D$  such that  $\widetilde{\epsilon^D} \not\subseteq \sigma_2$ . Then we define a bridge  $\epsilon$  from  $\sigma_2$  to  $\sigma_2$  having the same property and use this bridge to make  $\widetilde{\rho}$  bigger, which gives us a contradiction because  $\rho$  is optimal.

Consider  $(b_1, b_2) \in (\sigma_2^*)^D \setminus \sigma_2^D$  and the unary operation  $g_{b_1}(x) = w^D(b_1, \dots, b_1, x)$ . Since  $w^D$  is a special WNU,  $g_{b_1}(g_{b_1}(x)) = g_{b_1}(x)$ . Let us show that  $(\sigma_2^* \setminus \sigma_2) \cap (D')^2 \neq \emptyset$ , where  $D' = g_{b_1}(D)$ .

Since  $\text{pr}_{3,4}(\rho) = \sigma_2^*$ , there are  $a_1, a_2$  such that  $(a_1, a_2, b_1, b_2) \in \rho$ . Since  $(a_1, b_2) \in \sigma_2$ , and  $(b_1, b_2) \notin \sigma_2$ , we have  $(a_1, b_1) \notin \sigma_2$ . Consider the relation  $\delta(x, y)$  defined by  $\exists x_1 \exists x_2 \exists y_2 \sigma_2(x, x_1) \wedge \rho(x_1, x_2, y, y_2)$ . It follows from the definition that  $\delta$  is stable under  $\sigma_2$ . Also  $(a_1, b_1) \in \delta$ , therefore  $\delta \supsetneq \sigma_2$ . From irreducibility of  $\sigma_2$  we obtain that  $(b_1, b_2) \in \delta$ . Then by the definition of  $\delta$  there exist  $(c_1, c_2, b_2, c_3) \in \rho$  such that  $(b_1, c_1) \in \sigma_2$ . Put  $d_i = h(c_i)$  for  $i = 1, 2, 3$ . Then  $(d_1, d_2, b_2, d_3) \in \rho$  (we have  $h(b_2) = b_2$ ). Since  $h$  preserves  $\sigma_2$  and  $h(b_1) = b_1$ , we have  $(d_1, b_1) \in \sigma_2$ . Therefore,  $(d_1, b_2) \notin \sigma_2$ . Since  $\widetilde{\rho} = \sigma_2$ , we have  $(d_1, d_2) \notin \sigma_1$  and  $(b_2, d_3) \notin \sigma_2$ . Let  $E$  be the equivalence class of  $\sigma_2^D$  containing  $d_1$  and  $d_2$  (they are in one class because  $\text{pr}_{1,2}(\rho) = \sigma_1^* \subseteq \sigma_2$ ). By  $w'$  we denote  $w^D$  restricted to  $E$ , put  $\rho' = \rho \cap (E^2 \times D^2)$  and  $\sigma_1' = \sigma_1 \cap E^2$ .

Since  $(d_1, d_2) \in (\sigma_1^* \setminus \sigma_1) \cap E^2$ , we can find a minimal relation  $\omega \subseteq \sigma_1^* \cap E^2$  stable under  $\sigma_1'$  such that  $\omega \supsetneq \sigma_1'$ . It is not hard to check that the formula

$$\rho''(x_1, x_2, x'_1, x'_2) = \exists y_1 \exists y_2 \rho'(x_1, x_2, y_1, y_2) \wedge \rho'(x'_1, x'_2, y_1, y_2) \wedge \omega(x_1, x_2) \wedge \omega(x'_1, x'_2)$$

defines a reflexive bridge  $\rho''$  from  $\sigma_1'$  to  $\sigma_1'$ . Since  $\widetilde{\rho} = \sigma_2$  and  $E$  is an equivalence class of  $\sigma_2^D$ , we have  $\widetilde{\rho''} = E^2$ . Then by Theorem 8.17, there exists a prime number  $p$  and a relation  $\zeta \subseteq E \times E \times \mathbb{Z}_p$  such that  $\text{pr}_{1,2}(\zeta) = \omega$  and  $(a_1, a_2, b) \in \zeta$  implies  $(a_1, a_2) \in \sigma_1' \Leftrightarrow (b = 0)$ . We want to show that for each  $(e_1, e_2) \in \omega \setminus \sigma_1'$  we have  $(w^D(d_1, \dots, d_1, e_1), w^D(d_1, \dots, d_1, e_2)) \notin \sigma_1$ . In fact, choose  $b \in \mathbb{Z}_p$  such that  $(e_1, e_2, b) \in \zeta$ . Note that  $b \neq 0$  and  $(d_1, d_1, 0) \in \zeta$ . Applying  $w'$  to  $n$  tuples  $(d_1, d_1, 0)$  and one tuple  $(e_1, e_2, b)$  we get, by Lemma 7.23, a tuple  $(w^D(d_1, \dots, d_1, e_1), w^D(d_1, \dots, d_1, e_2), b) \in \zeta$ . Since  $b \neq 0$ , we have

$$(w^D(d_1, \dots, d_1, e_1), w^D(d_1, \dots, d_1, e_2)) \notin \sigma_1.$$

We can find  $(e_3, e_4) \in \sigma_2^* \setminus \sigma_2$  such that  $(e_1, e_2, e_3, e_4) \in \rho$ . Since  $h$  preserves  $\rho$ ,  $w^D$  preserves  $\rho^D$ , and  $\rho^D$  is a bridge, we can derive that  $(w^D(d_1, \dots, d_1, h(e_3)), w^D(d_1, \dots, d_1, h(e_4))) \notin \sigma_2$ . Since  $(d_1, b_1) \in \sigma_2$  we also have  $(w^D(b_1, \dots, b_1, h(e_3)), w^D(b_1, \dots, b_1, h(e_4))) \notin \sigma_2$ . Thus, we proved that  $(\sigma_2^* \setminus \sigma_2) \cap (D')^2 \neq \emptyset$ , where  $D' = g_{b_1}(D)$  and  $g_{b_1}(x) = w^D(b_1, \dots, b_1, x)$ . If  $D' \neq D$ , then we found a smaller set  $D'$  and the corresponding polynomial  $g_{b_1}(h(x))$ , which contradicts the minimality of  $D$ .

Thus, for any  $(b_1, b_2) \in (\sigma_2^*)^D \setminus \sigma_2^D$  we have  $w^D(b_1, \dots, b_1, x) = x$ . Let us show that  $\text{Con}(\rho^D, 1) = \sigma_1^D$  and  $\text{Con}(\rho^D, 3) = \sigma_2^D$ . Choose  $(a_1, a_2, a_3, a_4) \in \rho^D$ . We consider two cases.

Case 1:  $(a_1, a_2) \in \sigma_1$  and  $(a_3, a_4) \in \sigma_2$ . Then for any tuple  $(a'_1, a_2, a_3, a_4) \in \rho^D$  we have  $(a_1, a'_1) \in \sigma_1$ . Similarly, for any tuple  $(a_1, a_2, a'_3, a_4) \in \rho^D$  we have  $(a_3, a'_3) \in \sigma_2$ .

Case 2:  $(a_1, a_2) \notin \sigma_1$  and  $(a_3, a_4) \notin \sigma_2$ . Since  $(a_1, a_4) \in \sigma_2$  and  $(a_1, a_2) \in \sigma_1^* \subseteq \sigma_2$ , we have  $(a_1, a_3), (a_2, a_3), (a_3, a_4) \in (\sigma_2^*)^D \setminus \sigma_2^D$ . Also notice that  $\sigma_2^*$  is symmetric. Assume that  $(a'_1, a_2, a_3, a_4) \in \rho^D$ . Since  $w^D$  preserves  $\rho^D$ , we have

$$(w^D(a'_1, a_1, \dots, a_1), w^D(a_2, \dots, a_2, a_1), w^D(a_3, \dots, a_3, a_1), w^D(a_4, \dots, a_4, a_1)) \in \rho^D.$$

As we showed before this tuple equals  $(a'_1, a_1, a_1, a_1)$ , which means that  $(a_1, a'_1) \in \sigma_1$ . Similarly, if  $(a_1, a_2, a'_3, a_4) \in \rho^D$  we consider a tuple

$$(w^D(a_1, \dots, a_1, a_3), w^D(a_2, \dots, a_2, a_3), w^D(a'_3, a_3, \dots, a_3, a_3), w^D(a_4, \dots, a_4, a_3)) \in \rho^D,$$

that equals  $(a_3, a_3, a'_3, a_3)$ , which means that  $(a_3, a'_3) \in \sigma_2$ . Thus we proved that  $\text{Con}(\rho^D, 1) = \sigma_1^D$  and  $\text{Con}(\rho^D, 3) = \sigma_2^D$ .

Consider  $(a_1, a_2, b_1, b_2) \in \rho^D$  with  $(b_1, b_2) \notin \sigma_2$  and the formula

$$\Theta = \rho(z, x_1, x_2, x_3) \wedge \rho(z', x_1, x'_2, x'_3) \wedge \rho(z, x_4, x_5, x_6) \wedge \rho(z', x_4, x'_5, x'_6).$$

Let  $\epsilon$  be the relation defined by  $\Theta(x_2, x'_2, x_5, x'_5)$ . Since  $h(h(x)) = h(x)$  and  $h$  preserves  $\rho$ ,  $\epsilon^D$  is defined by the same formula but with  $\rho^D$  instead of  $\rho$  everywhere. Let us prove that  $\epsilon^D$  is a bridge from  $\sigma_2^D$  to  $\sigma_2^D$ . Assume that  $(x_2, x'_2) \in \sigma_2^D$ . Since  $(x_1, z), (x_1, z') \in \sigma_1^* \subseteq \sigma_2$ , we have  $(z, z') \in \sigma_2^D$ . Recall that  $(a, d) \in \sigma_2$  whenever  $(a, b, c, d) \in \rho$ , hence  $(x_3, x'_3), (x_6, x'_6) \in \sigma_2^D$ . Since  $\text{Con}(\rho^D, 1) = \sigma_1^D$ , we have  $(z, z') \in \sigma_1$ . Since  $\text{Con}(\rho^D, 3) = \sigma_2^D$ , we have  $(x_5, x'_5) \in \sigma_2$ . In the same way we can show that if  $(x_5, x'_5) \in \sigma_2^D$ , then  $(x_2, x'_2) \in \sigma_2^D$ . Since all the variables of  $\epsilon$  are stable under  $\sigma_2$  and  $\epsilon^D$  is reflexive, we have  $\text{pr}_{1,2}(\epsilon^D) \supseteq \sigma_2^D$ . By sending  $(z, x_1, x_2, x_3)$  to  $(a_1, a_2, b_1, b_2)$ ,  $(z', x_1, x'_2, x'_3)$  to  $(a_2, a_2, a_2, a_2)$ ,  $(z, x_4, x_5, x_6)$  to  $(a_1, a_2, b_1, b_2)$ ,  $(z', x_4, x'_5, x'_6)$  to  $(a_2, a_2, a_2, a_2)$ , we show that  $(b_1, a_2, b_1, a_2) \in \epsilon$ . Since  $(a_1, a_2), (a_1, b_2) \in \sigma_2$  and  $(b_1, b_2) \notin \sigma_2$ , we have  $(b_1, a_2) \notin \sigma_2$ , and therefore  $\text{pr}_{1,2}(\epsilon^D) \not\supseteq \sigma_2^D$ . In the same way we can show that  $\text{pr}_{3,4}(\epsilon^D) \not\supseteq \sigma_2^D$ . Thus  $\epsilon^D$  is a bridge.

Let us show that  $\epsilon$  is a bridge from  $\sigma_2$  to  $\sigma_2$ . Assume the contrary. Then without loss of generality we assume that there exists  $(d_0, d_0, d_1, d_2) \in \epsilon$  such that  $(d_1, d_2) \notin \sigma_2$ . Put  $\delta_0(y, z) = \exists x \epsilon(x, x, y, z)$ . The relation  $\delta_0$  is stable under  $\sigma_2$  and strictly larger than  $\sigma_2$ , hence  $\delta_0 \supseteq \sigma_2^*$  and  $(b_1, b_2) \in \delta_0$ . Then there exists  $d$  such that  $(d, d, b_1, b_2) \in \epsilon$ , which means that  $(h(d), h(d), b_1, b_2) \in \epsilon^D$ . This contradicts the fact that  $\epsilon^D$  is a bridge. Hence,  $\epsilon$  is also a bridge. By sending  $(z, x_1, x_2, x'_2, x_3, x'_3)$  to  $(a_1, a_2, b_1, b_1, b_2, b_2)$  and  $(z', x_4, x_5, x'_5, x_6, x'_6)$  to  $(a_1, a_1, a_1, a_1, a_1, a_1)$  we can show that  $(b_1, a_1) \in \tilde{\epsilon}$ . If we compose bridges  $\rho$  and  $\epsilon$ , then we get a bridge  $\epsilon'$  from  $\sigma_1$  to  $\sigma_2$  containing  $(b_1, b_1, a_1, a_1)$ . Hence  $\tilde{\epsilon}' \supsetneq \tilde{\rho}$ , which contradicts the fact that  $\rho$  is optimal.  $\square$

## 8.4 Existence of a bridge

In this subsection we show four ways to build a bridge: from congruences with an additional property, from a rectangular relation, by composing bridges appearing in the instance, and from a pp-formula.

**Lemma 8.19.** *Suppose  $\sigma, \sigma_1$ , and  $\sigma_2$  are congruences on  $A$ ,  $\sigma \cap \sigma_1 = \sigma \cap \sigma_2$ , and  $\sigma \setminus \sigma_1 \neq \emptyset$ . Then  $\sigma_1$  and  $\sigma_2$  are adjacent.*

*Proof.* Let us define a relation  $\rho$  by

$$\rho(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2 \sigma_1(x_1, z_1) \wedge \sigma_2(z_1, y_1) \wedge \sigma_1(x_2, z_2) \wedge \sigma_2(z_2, y_2) \wedge \sigma(z_1, z_2).$$

It is clear that the first two variables of  $\rho$  are stable under  $\sigma_1$  and the last two variables are stable under  $\sigma_2$ .

Let us show that for any  $(a_1, a_2, a_3, a_4) \in \rho$  that  $(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2$ . In fact, if  $(x_1, x_2) \in \sigma_1$ , then  $(z_1, z_2) \in \sigma_1$ . Since  $\sigma \cap \sigma_1 = \sigma \cap \sigma_2$ , we have  $(z_1, z_2) \in \sigma_2$ . Therefore,  $(y_1, y_2) \in \sigma_2$ .

Also  $(a, a, a, a) \in \rho$  for any  $a \in A$ . Choose  $(a, b) \in \sigma \setminus \sigma_1$ . Then  $(a, b, a, b) \in \rho$  (put  $z_1 = a$ ,  $z_2 = b$ ), which proves that  $\rho$  is a reflexive bridge.  $\square$

**Lemma 8.20.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a subdirect relation, the first and the last variables of  $\rho$  are rectangular, and there exist  $(b_1, a_2, \dots, a_n), (a_1, \dots, a_{n-1}, b_n) \in \rho$  such that  $(a_1, a_2, \dots, a_n) \notin \rho$ . Then there exists a bridge  $\delta$  from  $\text{Con}(\rho, 1)$  to  $\text{Con}(\rho, n)$  such that  $\tilde{\delta} = \text{pr}_{1,n}(\rho)$ .*

*Proof.* The required bridge can be defined by

$$\delta(x_1, x_2, y_1, y_2) = \exists z_2 \dots \exists z_{n-1} \rho(x_1, z_2, \dots, z_{n-1}, y_1) \wedge \rho(x_2, z_2, \dots, z_{n-1}, y_2).$$

In fact, since the first and the last variables of  $\rho$  are rectangular, we have  $(x_1, x_2) \in \text{Con}(\rho, 1)$  if and only if  $(y_1, y_2) \in \text{Con}(\rho, n)$ . It remains to notice that  $(b_1, a_1, a_n, b_n) \in \delta$  and  $(b_1, a_1) \notin \text{Con}(\rho, 1)$ .  $\square$

Recall that by Lemma 8.11  $\text{Con}(\rho, i)$  is an irreducible congruence for every critical subdirect rectangular relation  $\rho$  and its coordinate  $i$ .

**Lemma 8.21.** *Suppose  $\rho \subseteq A_1 \times \dots \times A_n$  is a critical subdirect rectangular relation. Then*

1. *there exists a bridge  $\delta$  from  $\text{Con}(\rho, 1)$  to  $\text{Con}(\rho, n)$  such that  $\tilde{\delta} = \text{pr}_{1,n}(\rho)$ . Moreover, if  $n = 2$  then  $\text{Con}(\tilde{\delta}, 1) = \text{Con}(\rho, 1)$  and  $\text{Con}(\tilde{\delta}, 2) = \text{Con}(\rho, n)$ ; if  $n > 2$  then  $\text{Con}(\tilde{\delta}, 1) \supsetneq \text{Con}(\rho, 1)$  and  $\text{Con}(\tilde{\delta}, 2) \supsetneq \text{Con}(\rho, n)$ .*
2. *if  $\text{Opt}(\text{Con}(\rho, n)) \neq \text{Con}(\rho, n)$ , then there exists a bridge  $\delta$  from  $\text{Con}(\rho, 1)$  to  $\text{Con}(\rho, n)$  such that  $\tilde{\delta}$  contains the projection of the cover of  $\rho$  onto the first and the last coordinates.*

*Proof.* Using the argument from Lemma 8.10 we find tuples  $(b_1, a_2, \dots, a_n) (a_1, \dots, a_{n-1}, b_n)$  satisfying the conditions of Lemma 8.20. Then, to prove the first part it is sufficient to use the formula from Lemma 8.20 to define a bridge  $\delta$ . If  $n = 2$  then  $\tilde{\delta} = \rho$  and we have the required property. If  $n > 2$  then by Lemma 8.10 we have  $\text{Con}(\tilde{\delta}, 1) = \text{Con}(\text{pr}_{1,n}(\rho), 1) \supsetneq \text{Con}(\rho, 1)$  and  $\text{Con}(\tilde{\delta}, 2) = \text{Con}(\text{pr}_{1,n}(\rho), 2) \supsetneq \text{Con}(\rho, n)$ .

Let us prove the second part of the claim. Let  $\xi$  be an optimal bridge from  $\text{Con}(\rho, n)$  to  $\text{Con}(\rho, 1)$ . Define a bridge  $\delta(x_1, x_2, y_1, y_2)$  by

$$\exists z_2 \dots \exists z_{n-1} \exists u_1 \exists u_2 \rho(x_1, z_2, \dots, z_{n-1}, u_1) \wedge \rho(x_2, z_2, \dots, z_{n-1}, u_2) \wedge \xi(u_1, u_2, y_1, y_2).$$

Note that  $\delta$  is just a composition of the bridge constructed in Lemma 8.20 and  $\xi$ . Then we have  $\tilde{\delta}(x, y) = \exists z_2 \dots \exists z_{n-1} \exists u \rho(x, z_2, \dots, z_{n-1}, u) \wedge \xi(u, y)$ .

Put  $\rho'(x_1, \dots, x_n) = \exists x'_n \rho(x_1, \dots, x_{n-1}, x'_n) \wedge \xi(x'_n, x_n)$ . Since  $\tilde{\xi} \supsetneq \text{Con}(\rho, n)$ , the relation  $\rho'$  contains the cover of  $\rho$ . Since  $\text{pr}_{1,n}(\rho') = \tilde{\delta}$ ,  $\tilde{\delta}$  contains the projection of the cover of  $\rho$  onto the first and the last coordinates, which completes the proof.  $\square$

**Theorem 8.22.** *Suppose  $\Theta$  is a cycle-consistent connected instance. Then for every constraints  $C, C'$  with variables  $x, x'$  there exists a bridge  $\delta$  from  $\text{Con}(C, x)$  to  $\text{Con}(C', x')$  such that  $\tilde{\delta}$  contains all pairs of elements linked in  $\Theta$ . Moreover, if  $\text{Con}(C'', x'') \neq \text{LinkedCon}(\Theta, x'')$  for some constraint  $C'' \in \Theta$  and a variable  $x''$ , then  $\delta$  can be chosen so that  $\tilde{\delta}$  contains all pairs of elements linked in  $\Theta'$ , where  $\Theta'$  is obtained from  $\Theta$  by replacing every constraint relation by its cover.*

*Proof.* Since  $C$  and  $C'$  are connected, there exists a path  $z_0 C_1 z_1 C_2 z_2 \dots C_{t-1} z_{t-1} C_t z_t$ , where  $z_0 = x$ ,  $z_t = x'$ ,  $C_1 = C$ ,  $C_t = C'$ ,  $z_{i-1} \neq z_i$ , and  $C_i$  and  $C_{i+1}$  are adjacent in  $z_i$  for every  $i$ .

By Lemma 8.11, every relation defined by  $\text{Con}(C_0, x_0)$  for some  $C_0$  and  $x_0$  is an irreducible congruence. Suppose  $\zeta_i$  is an optimal bridge from  $\text{Con}(C_i, z_i)$  to  $\text{Con}(C_{i+1}, z_i)$ ,  $\delta_i$  is a bridge from  $\text{Con}(C_i, z_{i-1})$  to  $\text{Con}(C_i, z_i)$  from the first item of Lemma 8.21 for every  $i$ . Then we compose all bridges together and define a new bridge  $\delta(u_0, u'_0, v_t, v'_t)$  from  $\text{Con}(C, x)$  to  $\text{Con}(C', x')$  by

$$\exists u_1 \exists u'_1 \exists v_1 \exists v'_1 \dots \exists u_{t-1} \exists u'_{t-1} \exists v_{t-1} \exists v'_{t-1} \delta_1(u_0, u'_0, v_1, v'_1) \wedge \bigwedge_{i=1}^{t-1} (\zeta_i(v_i, v'_i, u_i, u'_i) \wedge \delta_{i+1}(u_i, u'_i, v_{i+1}, v'_{i+1})). \quad (7)$$

Since  $\tilde{\delta}$  can be defined as a composition of  $\tilde{\zeta}$ 's and  $\tilde{\delta}$ 's, and  $\tilde{\zeta}$ 's are reflexive, it follows that  $\tilde{\delta}$  contains all pairs of elements linked by this path. Since  $\Theta$  is cycle-consistent, if  $x = x'$  then  $\delta$  is a reflexive bridge from  $\text{Con}(C, x)$  to  $\text{Con}(C', x)$ . Thus we proved that any two constraints with a common variable are adjacent.

Using Lemma 6.2, we can show that there exists a path in  $\Theta$  starting at  $x$  and ending at  $x'$  that connects any pair of elements linked in  $\Theta$ . Since any two constraints with a common variable are adjacent, we can assume that the above path  $z_0 C_1 z_1 C_2 z_2 \dots C_{t-1} z_{t-1} C_t z_t$  connects any pair of elements linked in  $\Theta$ . Again, it follows that  $\tilde{\delta}$  contains all pairs of elements linked in  $\Theta$ .

To prove the remaining part of the theorem, assume that  $\text{Con}(C'', x'') \neq \text{LinkedCon}(\Theta, x'')$  for some constraint  $C'' \in \Theta$  and a variable  $x''$ . First, observe that any bridge  $\rho$  from  $\sigma_1$  to  $\sigma_2$  defined by the first item of Lemma 8.21 satisfies one of the following properties:

1.  $\text{Con}(\tilde{\rho}, 1) = \sigma_1$  and  $\text{Con}(\tilde{\rho}, 2) = \sigma_2$ ,
2.  $\text{Con}(\tilde{\rho}, 1) \supsetneq \sigma_1$  and  $\text{Con}(\tilde{\rho}, 2) \supsetneq \sigma_2$ .

If  $\sigma_1 \neq \sigma_2$ , by Lemma 8.18 an optimal bridge from  $\sigma_1$  to  $\sigma_2$  satisfies property (2). If  $\sigma_1 = \sigma_2$ , an optimal bridge from  $\sigma_1$  to  $\sigma_2$  obviously satisfies one of the two properties. Thus, every bridge in (7) satisfies one of the above properties.

Let us show that if a bridge  $\rho$  from  $\sigma_1$  to  $\sigma_2$  satisfies property (1), then for all  $(a_1, b_1), (a_2, b_2) \in \tilde{\rho}$  we have  $(a_1, a_2) \in \sigma_1 \Leftrightarrow (b_1, b_2) \in \sigma_2$ . In fact, if  $(a_1, a_2) \in \sigma_1$  then since the first two variables of  $\rho$  are stable under  $\sigma_1$ , we have  $(a_1, b_2) \in \tilde{\rho}$ , hence  $(b_1, b_2) \in \text{Con}(\tilde{\rho}, 2) = \sigma_2$ . Now we want to show that if we compose bridges  $\rho_1, \dots, \rho_s$  together as in (7) and at least one of the bridges satisfies property (2) then the obtained bridge satisfies property (2). Let  $\rho_j$  be a bridge from  $\sigma_{j-1}$  to  $\sigma_j$  for every  $j$ , then the composition  $\rho$  is a bridge from  $\sigma_0$  to  $\sigma_s$ . Consider the first bridge  $\rho_i$  in the sequence having property (2). Then  $(a_{i-1}, a_i), (b_{i-1}, b_i) \in \tilde{\rho}_i$  for some  $(a_{i-1}, b_{i-1}) \notin \sigma_{i-1}$  and  $a_i = b_i$ . Choose  $a_j$  and  $b_j$  so that  $(a_{j-1}, a_j), (b_{j-1}, b_j) \in \tilde{\rho}_j$  for every  $j$ , and  $a_j = b_j$  for every  $j \geq i$ . Then  $(a_0, b_0) \in \text{Con}(\tilde{\rho}_1, 1) \setminus \sigma_0$  and  $(a_0, a_s), (b_0, b_s) \in \tilde{\rho}$ . Since  $a_s = b_s$ , we get  $(a_0, b_0) \in \text{Con}(\tilde{\rho}, 1)$  and  $\text{Con}(\tilde{\rho}, 1) \supsetneq \sigma_0$ . To prove that  $\text{Con}(\tilde{\rho}, 2) \supsetneq \sigma_s$  we consider the last bridge in the sequence having property (2) and do exactly the same.

By the first part of the theorem  $\text{Opt}(\text{Con}(C'', x'')) \supsetneq \text{Con}(C'', x'')$ , hence an optimal bridge from  $\text{Con}(C'', x'')$  to  $\text{Con}(C'', x'')$  satisfies property (2). We may assume that any path goes through the variable  $x''$  and through the optimal bridge from  $\text{Con}(C'', x'')$  to  $\text{Con}(C'', x'')$ , which guarantees that every bridge we obtain satisfies property (2). Consider a constraint  $C_0 \in \Theta$  and a variable  $x_0$  in it. Considering a path from  $x_0$  to  $x_0$  going through  $x''$  we can build a reflexive bridge having property (2), which means that  $\text{Opt}(\text{Con}(C_0, x_0)) \supsetneq \text{Con}(C_0, x_0)$  for any constraint  $C_0 \in \Theta$  and any variable  $x_0$  in it.

To complete the proof, we replace every  $\delta_i$  in (7) by the corresponding bridge  $\delta'_i$  obtained using the second item of Lemma 8.21, and replace the path by the corresponding path connecting any pair of linked elements in  $\Theta'$ . Since  $\tilde{\delta}'_i$  contains the projection of the cover of  $C_i$  onto the variables  $z_{i-1}$  and  $z_i$ ,  $\tilde{\delta}$  contains all pairs of elements linked in  $\Theta'$ .  $\square$

**Corollary 8.22.1.** *Suppose  $\Theta$  is a cycle-consistent connected instance. Then for every constraints  $C, C'$  with a common variable  $x$  there exists a bridge  $\delta$  from  $\text{Con}(C, x)$  to  $\text{Con}(C', x)$  such that  $\tilde{\delta}$  contains the relation  $\text{LinkedCon}(\Theta, x)$ .*

**Lemma 8.23.** *Suppose  $D^{(1)}$  is a minimal one-of-four reduction for an instance  $\Upsilon$ , the solution set of  $\Upsilon$  is subdirect,  $\Upsilon^{(1)}(x_1, \dots, x_n)$  defines a subdirect key rectangular relation  $\rho$ . For  $i = 1, 2$  the variable  $x_i$  of every constraint of  $\Upsilon$  containing  $x_i$  is stable under an irreducible congruence  $\sigma_i$ , and there exist tuples  $(a_1, a_2, \dots, a_n), (b_1, b_2, b_3, \dots, b_n) \in \rho$ ,  $(a'_1, a_2, \dots, a_n), (b_1, b'_2, b_3, \dots, b_n) \notin \rho$  such that  $(a_1, a'_1) \in \sigma_1^* \setminus \sigma_1$ ,  $(b_2, b'_2) \in \sigma_2^* \setminus \sigma_2$ . Then there exists a bridge  $\delta$  from  $\sigma_1$  to  $\sigma_2$  such that  $\tilde{\delta}$  contains  $\Upsilon(x_1, x_2)$ .*

*Proof.* If  $D^{(1)}$  is a nonlinear reduction then by  $\omega$  we denote the relation defined by  $\Upsilon(x_1, \dots, x_n)$ . If  $D^{(1)}$  is a linear reduction, then by  $\omega$  we denote the relation defined by  $\Omega(x_1, \dots, x_n, u_1, \dots, u_r)$ , where  $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, v_1, \dots, v_r\}$ ,  $\Omega = \Upsilon \wedge \bigwedge_{i=1}^r \delta_i(v_i, u_i)$  and  $\delta_i = \text{ConLin}(D_{v_i})$  (see Lemma 8.14).

We know from Lemmas 8.13 and 8.14 that  $\text{Con}(\omega, j)^{(1)} = \text{Con}(\rho, j)$  for every  $j \in \{1, 2\}$ . Since  $\rho$  is rectangular, we have  $(a_1, a'_1) \notin \text{Con}(\rho, 1)$ , and therefore  $(a_1, a'_1) \notin \text{Con}(\omega, 1)^{(1)}$ . Since  $x_1$  in every constraint containing  $x_1$  is stable under  $\sigma_1$ , the relation  $\text{Con}(\omega, 1)$  is stable under  $\sigma_1$ . Therefore,  $\text{Con}(\omega, 1)$  should be equal  $\sigma_1$ , since otherwise  $\text{Con}(\omega, 1) \supseteq \sigma_1^*$ , which contradicts  $(a_1, a'_1) \notin \text{Con}(\omega, 1)^{(1)}$ . In the same way we can show that  $\text{Con}(\omega, 2) = \sigma_2$ .

Since  $\rho$  is a key relation, there should be a key tuple  $\beta \in (D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}) \setminus \rho$  such that for every  $\alpha \in (D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}) \setminus \rho$  there exists a vector-function  $\Psi$  which preserves  $\rho$  and gives  $\Psi(\alpha) = \beta$ . First, we put  $\alpha_a = (a'_1, a_2, \dots, a_n)$  and apply the corresponding unary vector-function  $\Psi_a$  to  $(a_1, a_2, \dots, a_n)$  to get a tuple  $\beta_a$ . Second, we put  $\alpha_b = (b_1, b'_2, b_3, \dots, b_n)$  and apply the corresponding unary vector-function  $\Psi_b$  to  $(b_1, b_2, b_3, \dots, b_n)$  to get a tuple  $\beta_b$ . As a result we get two tuples  $\beta_a$  and  $\beta_b$  from  $\rho$  that differ from the key tuple  $\beta$  just in the first and second coordinates, respectively. Since  $\text{Con}(\omega, j)^{(1)} = \text{Con}(\rho, j)$  for every  $j \in \{1, 2\}$ , we have  $\beta \notin \omega$  if  $D^{(1)}$  is nonlinear, and  $\beta \notin \text{pr}_{1, \dots, n}(\omega^{(1)})$  if  $D^{(1)}$  is linear.

Then by applying Lemma 8.20 to  $\omega$  and  $\beta_a, \beta_b, \beta$  (if  $D^{(1)}$  is a linear reduction we extend these tuples), we get a bridge  $\delta$  from  $\sigma_1$  to  $\sigma_2$  such that  $\tilde{\delta}$  contains  $\text{pr}_{1,2}(\omega)$ , which is equal to the relation defined by  $\Upsilon(x_1, x_2)$ .  $\square$

## 8.5 Expanded coverings of crucial instances

In this subsection we prove two properties of expanded coverings of crucial instances.

**Lemma 8.24.** *Suppose  $\Theta$  is a crucial instance in  $D^{(1)}$ ,  $\Theta' \in \text{ExpCov}(\Theta)$  via the map  $S: \text{Var}(\Theta') \rightarrow \text{Var}(\Theta)$ , and  $\Theta'$  has no solution in  $D^{(1)}$ . Then for every constraint  $C = \rho(x_1, \dots, x_n)$  in  $\Theta$  there exists a constraint  $C'$  in  $\Theta'$  whose image in  $\Theta$  is  $C$  (i.e.,  $C' = \rho(y_1, \dots, y_n)$  and  $S(y_i) = x_i$  for  $i = 1, 2, \dots, n$ ).*

*Proof.* Let  $\Theta''$  be obtained from  $\Theta$  by replacing every variable  $y$  by  $S(y)$ . Obviously,  $\Theta''$  still does not have a solution in  $D^{(1)}$ . By the definition of expanded coverings every relation in the obtained instance is either unary (and full), or weaker or equivalent to a constraint from  $\Theta'$ . Since  $\Theta$  is crucial in  $D^{(1)}$  and  $\Theta''$  has no solutions in  $D^{(1)}$ , there should be a constraint  $C'$  in  $\Theta'$  such that its image  $C''$  in  $\Theta''$  is weaker or equivalent to  $C$  but not weaker than  $C$ .

Since  $\Theta$  is crucial, all variables of  $C$  are not dummy. Since  $C''$  cannot have more variables than  $C$  we obtain that  $C'' = C$ , which means that  $C'' = \rho(y_1, \dots, y_n)$  and  $S(y_i) = x_i$  for every  $i \in \{1, 2, \dots, n\}$ .  $\square$

**Lemma 8.25.** *Suppose  $\Theta$  is a crucial instance in  $D^{(1)}$ ,  $\Theta' \in \text{ExpCov}(\Theta)$  has no solutions in  $D^{(1)}$ , every constraint relation of  $\Theta$  is a critical rectangular relation, and  $\Theta'$  is connected. Then  $\Theta$  is connected.*

*Proof.* Let  $\Theta''$  be obtained from  $\Theta'$  by replacing every variable  $y$  by  $S(y)$  from the definition of the expanded covering.

Let us show that any two constraints  $C_1$  and  $C_2$  with a common variable  $x$  of  $\Theta$  are adjacent. By Lemma 8.24, there exist constraints  $C'_1$  and  $C'_2$  of  $\Theta'$  whose images in  $\Theta$  are  $C_1$  and  $C_2$ . Since  $\Theta'$  is connected, the instance  $\Theta''$  is also connected. By Corollary 8.22.1 constraints  $C_1$  and  $C_2$  of  $\Theta''$  are adjacent in  $x$ . Therefore,  $C_1$  and  $C_2$  are adjacent in  $\Theta$ . Thus, we proved that any two constraints of  $\Theta$  with a common variable are adjacent. Since  $\Theta$  is crucial in  $D^{(1)}$ , it is not fragmented, which implies that  $\Theta$  is connected.  $\square$

## 8.6 Strategies

**Theorem 8.26.** *Suppose  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$  is a strategy for  $\Omega$ , the solution set of  $\Omega^{(i)}$  is subdirect for every  $i \in \{0, 1, \dots, s\}$ ,  $j < s$ ,  $D^{(s+1)}$  is a one-of-four reduction, at least one of the two reductions  $D^{(j+1)}$ ,  $D^{(s+1)}$  is nonlinear, and  $(\Omega^{(j)}(x_1, \dots, x_n))^{(s+1)}$  defines a nonempty relation. Then  $(\Omega^{(j+1)}(x_1, \dots, x_n))^{(s+1)}$  defines a nonempty relation.*

*Proof.* Let  $\text{Var}(\Omega) = \{x_1, \dots, x_n, y_1, \dots, y_t\}$ ,  $\Omega^{(j)}(x_1, \dots, x_n, y_1, \dots, y_t)$  define a relation  $R$ . Let the reduction  $D^{(j+1)}$  be of type  $\mathcal{T}_1$ , the reduction  $D^{(s+1)}$  be of type  $\mathcal{T}_2$ .

Assume that  $D^{(s+1)}$  is an absorbing reduction. Since  $\Omega^{(s)}(x_1, \dots, x_n, y_1, \dots, y_t)$  defines a subdirect relation, Lemma 7.5 implies that  $\Omega^{(s+1)}(x_1, \dots, x_n, y_1, \dots, y_t)$  defines a nonempty relation. From now on we assume that  $\mathcal{T}_2$  is not the absorbing type.

For  $i \in \{j, \dots, s\}$  and  $k \in \{1, \dots, n\}$  put

$$\begin{aligned} B_i &= R \cap (D_{x_1}^{(i)} \times \dots \times D_{x_n}^{(i)} \times D_{y_1}^{(j)} \times \dots \times D_{y_t}^{(j)}), \\ B'_i &= R \cap (D_{x_1}^{(i)} \times \dots \times D_{x_n}^{(i)} \times D_{y_1}^{(j+1)} \times \dots \times D_{y_t}^{(j+1)}), \\ B^k &= R \cap (D_{x_1}^{(s)} \times \dots \times D_{x_{k-1}}^{(s)} \times D_{x_k}^{(s+1)} \times D_{x_{k+1}}^{(s)} \times \dots \times D_{x_n}^{(s)} \times D_{y_1}^{(j)} \times \dots \times D_{y_t}^{(j)}). \end{aligned}$$

By Lemma 7.25  $B_{j+1}$  and  $B'_j$  are one-of-four subuniverses of  $R = B_j$  of type  $\mathcal{T}_1$ . Similarly,  $B_{i+1}$  is a one-of-four subuniverse of  $B_i$  for every  $i$  and  $B^k$  is a one-of-four subuniverse of  $B_s$  of type  $\mathcal{T}_2$  for every  $k$  (here we may need to reduce the domain of the last  $t$  variables to achieve the subdirectness of  $B_i$ ).

Let us show by induction on  $i$  that  $B'_i$  is a one-of-four subuniverse of  $B_i$  of type  $\mathcal{T}_1$ . For  $i = j$  we already know this. Assume that  $B'_i$  is a one-of-four subuniverse of  $B_i$ . By Lemma 7.30,  $B_{i+1} \cap B'_i = B'_{i+1}$  is a one-of-four subuniverse of  $B_{i+1}$  of type  $\mathcal{T}_1$ . Therefore,  $B'_s$  is a one-of-four subuniverse of  $B_s$  of type  $\mathcal{T}_1$ .

We need to prove that  $B^1 \cap \dots \cap B^n \cap B'_s \neq \emptyset$ . Since  $(\Omega^{(j)}(x_1, \dots, x_n))^{(s+1)}$  defines a nonempty relation,  $B^1 \cap \dots \cap B^n \neq \emptyset$ . Since the solution set of  $\Omega^{(s)}$  is subdirect,  $B^k \cap B'_s \neq \emptyset$  for every  $k \in \{1, \dots, n\}$ . Note that  $B^k$  is of type  $\mathcal{T}_2$  and  $B'_s$  is of type  $\mathcal{T}_1$ , and they cannot be both linear. Since  $B^k$  is not a binary absorbing subuniverse, Lemma 7.33 implies that  $B^1 \cap \dots \cap B^n \cap B'_s \neq \emptyset$ .  $\square$

**Corollary 8.26.1.** *Suppose  $\Theta$  is a cycle-consistent CSP instance,  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$  is a strategy for  $\Theta$ ,  $\Upsilon \in \text{ExpCov}(\Theta)$  is a tree-formula,  $x$  is a parent of  $x_1$  and  $x_2$ , and either (i)*

$B$  is a center of  $D_x^{(s)}$ , or (ii)  $B$  is a PC subuniverse of  $D_x^{(s)}$  and  $D_y^{(s)}$  has no nontrivial binary absorbing subuniverse or center for every  $y$ . Then the pp-formula  $\Upsilon^{(s)}(x_1, x_2)$  defines a binary relation with a nonempty intersection with  $B \times B$ .

*Proof.* Since every reduction in a strategy is 1-consistent and  $\Upsilon$  is a tree-formula, the solution set of  $\Upsilon^{(i)}$  is subdirect for every  $i$ . If  $B = D_x^{(s)}$  then the claim follows from the definition of a strategy (every reduction is 1-consistent). Otherwise, let us define a reduction  $D^{(s+1)}$  by  $D_x^{(s+1)} = D_{x_1}^{(s+1)} = D_{x_2}^{(s+1)} = B$ ,  $D_y^{(s+1)} = D_y^{(s)}$  for the remaining variables. Thus, we have a nonlinear reduction  $D^{(s+1)}$ . Since the instance  $\Theta$  is cycle-consistent, and  $x$  is a parent of  $x_1$  and  $x_2$ ,  $\Upsilon(x_1, x_2)$  defines a reflexive relation. Hence,  $(\Upsilon(x_1, x_2))^{(s+1)}$  defines a nonempty relation. By Theorem 8.26, we obtain that  $(\Upsilon^{(1)}(x_1, x_2))^{(s+1)}$  defines a nonempty relation. Repeatedly applying Theorem 8.26, we show that  $(\Upsilon^{(2)}(x_1, x_2))^{(s+1)}, (\Upsilon^{(3)}(x_1, x_2))^{(s+1)}, \dots, (\Upsilon^{(s)}(x_1, x_2))^{(s+1)}$  define nonempty relations, which means that  $\Upsilon^{(s)}(x_1, x_2)$  has a nonempty intersection with  $B \times B$ .  $\square$

**Lemma 8.27.** Suppose  $R \subseteq A_0 \times B_0$  is a subdirect relation, and

1.  $A_0 \supseteq A_1 \supseteq \dots \supseteq A_{s+1}$  and  $A_{i+1}$  is a one-of-four subuniverse of  $A_i$  for  $i \in \{0, 1, 2, \dots, s\}$ ;
2.  $B_0 \supseteq B_1 \supseteq \dots \supseteq B_{t+1}$  and  $B_{i+1}$  is a one-of-four subuniverse of  $B_i$  for  $i \in \{0, 1, 2, \dots, t\}$ ;
3.  $A_{s+1}$  and  $B_{t+1}$  are linear subuniverses of  $A_s$  and  $B_t$ , respectively;
4. there exist  $a \in A_{s+1}$ ,  $b \in B_{s+1}$ ,  $a' \in A_0$ ,  $b' \in B_0$  such that  $(a, b'), (a', b), (a', b') \in R$ ;
5.  $R \cap (A_s \times B_t) \neq \emptyset$ .

Then  $R \cap (A_{s+1} \times B_{t+1}) \neq \emptyset$ .

*Proof.* Denote  $a'' = w(a, a', \dots, a')$ ,  $b'' = w(b, b', \dots, b') = w(b', \dots, b', b)$ .

We prove by induction on  $s + t$ . Assume that  $s + t = 0$ , which implies  $s = t = 0$ . By Lemma 7.23, we have  $a'' \in A_1$  and  $b'' \in B_1$ . Since  $w$  preserves  $R$ ,  $(a'', b'') \in R$ , which completes this case.

Let us prove the induction step. Assume that  $s + t > 0$ . Without loss of generality we assume that  $s > 0$ . Put

$$R'(x_1, x_2, y_1, y_2) = R(x_1, y_2) \wedge R(y_1, x_2) \wedge R(y_1, y_2).$$

Put  $P_i = R' \cap (A_i \times B_0 \times A_0 \times B_0)$ ,  $Q_i = R' \cap (A_0 \times B_i \times A_0 \times B_0)$ ,  $T = R' \cap (A_0 \times B_0 \times A_1 \times B_0)$ . Since  $R'$  is subdirect, by Lemma 7.25  $P_{i+1}$  is a one-of-four subuniverse of  $P_i$ ,  $Q_{i+1}$  is a one-of-four subuniverse of  $Q_i$  for every  $i$ , and  $T$  is a one-of-four subuniverse of  $R' = P_0 = Q_0$ . We want to prove that  $P_{s+1} \cap Q_{t+1} \cap T \neq \emptyset$ . Since  $(a, b, a', b') \in P_{s+1} \cap Q_{t+1}$ , Lemma 7.31, implies that  $P_{s+1} \cap Q_t$  and  $P_s \cap Q_{t+1}$  are one-of-four subuniverses of  $P_s \cap Q_t$ . Since  $R \cap (A_s \times B_t) \neq \emptyset$ , we have  $T \cap P_s \cap Q_t \neq \emptyset$ . Lemma 7.31 implies that  $P_0 \supseteq P_1 \supseteq \dots \supseteq P_s \supseteq P_s \cap Q_1 \supseteq \dots \supseteq P_s \cap Q_t$  (here all inclusions mean one-of-four subuniverses), which by the same lemma implies that  $T \cap P_s \cap Q_t$  is a one-of-four subuniverse of  $P_s \cap Q_t$ .

If  $A_1$  is not a linear subuniverse of  $A_0$ , then by Theorem 7.33 the intersection of one-of-four subuniverses  $P_{s+1} \cap Q_t$ ,  $P_s \cap Q_{t+1}$ , and  $T \cap P_s \cap Q_t$  of different types cannot be empty, that is,  $P_{s+1} \cap Q_{t+1} \cap T \neq \emptyset$ , which completes this case.

Assume that  $A_1$  is linear. Since  $(a, b, a', b'), (a', b', a', b'), (a', b', a, b') \in R'$  and  $w$  preserves  $R'$ , we obtain  $(a, b, a', b'), (a'', b'', a', b'), (a'', b'', a'', b') \in R'$ . Note that by Lemma 7.23  $a'' \in A_1$  and  $b'' \in B_1$ . We look at  $R'$  as a binary relation  $R' \subseteq (A_0 \times B_0) \times (A_0 \times B_0)$  to apply the inductive assumption. Put  $\mathcal{A}_i = \text{pr}_{1,2}(R') \cap (A_{i+1} \times B_1)$  for  $0 \leq i \leq s$  and  $\mathcal{A}_i = \text{pr}_{1,2}(R') \cap$

$(A_{s+1} \times B_{i-s+1})$  for  $s+1 \leq i \leq s+t$ . Combining Lemma 7.25 and Lemma 7.31 we derive that  $\mathcal{A}_{i+1}$  is a one-of-four subuniverse of  $\mathcal{A}_i$  for  $i = 0, 1, \dots, s+t-1$ . Put  $\mathcal{B}_i = \text{pr}_{3,4}(R' \cap (A_1 \times B_1 \times A_i \times B_0))$  for  $i = 0, 1$ . Combining Lemmas 7.25, 7.31, and 7.32, we derive that  $\mathcal{B}_1$  is a linear subuniverse of  $\mathcal{B}_0$ . Then we apply the inductive assumption to  $R' \cap (\mathcal{A}_0 \times \mathcal{B}_0)$ ,  $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots \supseteq \mathcal{A}_{s+t}$  and  $\mathcal{B}_0 \supseteq \mathcal{B}_1$ , and show that  $R' \cap (A_{s+1} \times B_{t+1} \times A_1 \times B_0) = P_{s+1} \cap Q_{t+1} \cap T \neq \emptyset$ .

Put  $B'_i = \text{pr}_2(R \cap (A_1 \times B_i))$  for  $i = 0, 1, \dots, s+1$ . By Lemma 7.32,  $B'_{i+1}$  is a one-of-four subuniverse of  $B'_i$  for every  $i$ . Applying the inductive assumption to  $R \cap (A_1 \times B_0)$ ,  $A_1 \supseteq A_2 \supseteq \dots \supseteq A_{s+1}$ , and  $B'_0 \supseteq B'_1 \supseteq \dots \supseteq B'_{t+1}$ , we obtain that  $A_{s+1} \cap B_{t+1} \neq \emptyset$ .  $\square$

**Lemma 8.28.** *Suppose  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$  is a strategy for a subdirect constraint  $\rho(x_1, \dots, x_n)$ ,  $D^{(s+1)}$  is a linear reduction, and*

$$\begin{aligned} (b_1, \dots, b_t, a_{t+1}, \dots, a_n) &\in \rho, \\ (a_1, \dots, a_t, b_{t+1}, \dots, b_n) &\in \rho, \\ (b_1, \dots, b_t, b_{t+1}, \dots, b_n) &\in \rho, \\ (a_1, \dots, a_t, a_{t+1}, \dots, a_n) &\in D^{(s+1)}. \end{aligned}$$

*Then there exists  $(d_1, d_2, \dots, d_n) \in \rho^{(s+1)}$ .*

*Proof.* For  $i = 0, 1, 2, \dots, s+1$  put

$$A_i = \text{pr}_{1,2,\dots,t}(\rho \cap (D_{x_1}^{(i)} \times \dots \times D_{x_t}^{(i)} \times D_{x_{t+1}} \times \dots \times D_{x_n})),$$

$$B_i = \text{pr}_{1,2,\dots,t}(\rho \cap (D_{x_1} \times \dots \times D_{x_t} \times D_{x_{t+1}}^{(i)} \times \dots \times D_{x_n}^{(i)})).$$

Since  $\rho^{(i)}$  is subdirect for every  $i \in \{0, 1, \dots, s\}$ , Lemma 7.25 implies that  $A_{i+1}$  is a one-of-four subuniverse of  $A_i$  and  $B_{i+1}$  is a one-of-four subuniverse of  $B_i$  for every  $i \in \{0, 1, \dots, s\}$ . Since  $(a_1, \dots, a_t) \in A_{s+1}$  and  $(a_{t+1}, \dots, a_n) \in B_{s+1}$ , Lemma 8.27 implies that  $A_{s+1} \cap B_{s+1} \neq \emptyset$ , which completes the proof.  $\square$

## 8.7 Growing population divides into colonies.

In this section we prove a theorem that clarifies the inductive strategy used in the proof of Theorem 9.6. To simplify explanation we decided to avoid our usual terminology. Instead, we argue in terms of organisms, reproduction, and friendship.

We consider a set  $X$  whose elements we call *organisms*. At the moment 1 we had a set of organisms  $X_1$ . At every moment some organisms give a birth to new organisms, as a result we get a sequence of organisms  $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ , where  $\bigcup_i X_i = X$ ,  $X_i \subseteq X$ , and  $|X_i| < \infty$  for every  $i$ . We assume that each organism from  $X \setminus X_1$  has exactly one parent.

Every organism has a characteristic that we call *strength*. Thus we have a mapping  $\xi : X \rightarrow \{1, 2, \dots, S\}$  that assigns a characteristic to every organism. Also we have a binary reflexive symmetric relation  $F$  on the set  $X$ , which we call *friendship*. For an organism  $x$  by  $\text{BirthDate}(x)$  we denote the minimal  $i$  such that  $x \in X_i$ . A sequence of organisms  $x_1, \dots, x_n$  such that  $x_i$  is a friend of  $x_{i+1}$  for every  $i$  is called a *path*.

**Theorem 8.29.** *Suppose  $X_1, X_2, X_3, \dots$ ,  $\xi$ , and  $F$  satisfy the following conditions:*

1. **A child is always weaker than its parent.** *If  $y$  is the parent of  $x$ , then  $\xi(y) > \xi(x)$ .*
2. **Older friends are parents' friends.** *If  $\text{BirthDate}(y) < \text{BirthDate}(x)$  and  $x$  is a friend of  $y$ , then the parent of  $x$  is a friend of  $y$  (or the parent of  $x$  is  $y$ ).*



3. **Only friends' kids can be friends.** If  $\text{BirthDate}(x) = \text{BirthDate}(y)$  and  $x$  is a friend of  $y$ , then the parents of  $x$  and  $y$  are friends.
4. **No one can have infinitely many friends.**  $|\{y \in X \mid (x, y) \in F\}| < \infty$  for every  $x \in X$ .
5. **Reproduction never stops.**  $|\bigcup_i X_i| = \infty$ .

Then there exists  $N$  such that  $X_N$  can be divided into two nonempty disjoint sets  $X'_N$  and  $X''_N$  such that there is no friendship between  $X'_N$  and  $X''_N$ .

*Proof.* Assume the contrary. Then there exists a path between any two organisms.

For every moment  $t$  and every organism  $x$  by  $x^t$  we denote the predecessor of  $x$  from  $X_t$  with the maximal  $\text{BirthDate}$ , that is a closest predecessor who already lived at the moment  $t$ . For example  $x^t = x$  for  $t \geq \text{BirthDate}(x)$ , and  $x^{\text{BirthDate}(x)-1}$  is the parent of  $x$ .

Suppose we have a path of organisms  $x_1, \dots, x_n$ . We claim that  $x_1^t, \dots, x_n^t$  is also a path for any  $t$ . We will prove by induction starting with sufficiently large  $t$  such that  $x_1, \dots, x_n \in X_t$  and therefore  $(x_1^t, \dots, x_n^t) = (x_1, \dots, x_n)$ . As the inductive step, we assume that this is a path for  $t = t_0$  and show that this is a path for  $t = t_0 - 1$ . The induction step follows from hypotheses (2) and (3). The path  $x_1^t, \dots, x_n^t$  will be called *a path at the moment  $t$* . Note that organisms of the path at the moment  $t$  are not weaker than the corresponding organisms of the original path.

Choose a maximal strength  $s$  such that we have infinitely many organisms of strength  $s$ . Then infinitely many of them have the same parent, hence, there exists a parent reproducing infinitely many times.

For every  $x$  and a strength  $s$  by  $\text{KIDS}(x, s)$  we denote the set of all children  $y$  of  $x$  such that there exists a path from  $x$  to  $y$  with all the organisms in the path stronger than  $s$ . We consider the maximal  $s_0$  such that  $\text{KIDS}(x, s_0)$  is infinite for some organism  $x$ . Since we can always put  $s_0 = 0$  for a parent reproducing infinitely many times,  $s_0$  exists. Note that this implies that  $x$  is stronger than  $s_0 + 1$ .

By  $Y$  we denote the set of all organisms  $y$  such that there exists a path from  $x$  to  $y$  with all the organisms in the path stronger than  $s_0 + 1$ . Note that  $Y$  includes  $x$ . Let us show that  $Y$  is finite. Assume the opposite. Let  $s$  be the maximal strength such that we have infinitely many organisms of this strength in  $Y$ . Consider an organism  $v$  from  $Y$  with strength  $s$  such that  $\text{BirthDate}(v) > \text{BirthDate}(x)$  (we still have infinitely many of them). Considering the path from  $x$  to  $v$  at the moment  $\text{BirthDate}(v) - 1$ , we get a path from  $x$  to the parent of  $v$ , which means that the parent of  $v$  is also in  $Y$ . Since parents are stronger than children and we may have only finitely many organisms stronger than  $s$  in  $Y$ , we have only finitely many such parents in  $Y$ . Therefore, there exists a parent  $z \in Y$  with infinitely many children from  $Y$ . Since we can glue a path from the parent to  $x$  and a path from  $x$  to its kid, this implies that  $\text{KIDS}(z, s_0 + 1)$  is infinite. This contradicts the maximality of  $s_0$  and proves that  $Y$  is finite.

Let  $t$  be the first moment such that  $X_t$  contains all friends of friends of organisms from  $Y$ . Consider an organism  $y$  from  $\text{KIDS}(x, s_0)$  with  $\text{BirthDate}(y) > t$ . Choose a path from  $x$  to  $y$  with all organisms stronger than  $s_0$ . We consider the last organism  $u$  in the path such that  $\text{BirthDate}(u) < \text{BirthDate}(y)$ . Taking the fragment of this path from  $y$  to  $u$  at the moment  $\text{BirthDate}(y) - 1$  we obtain a path from  $x$  to  $u$  with all organisms but  $u$  stronger than  $s_0 + 1$ . This means that all the organisms but  $u$  in this path are from  $Y$ . Thus  $u$  has a friend from  $Y$ , which means that all friends of  $u$  were born before the moment  $t$ . This contradicts the fact that an organism next to  $u$  in the original path from  $x$  to  $y$  was born after the moment  $\text{BirthDate}(y) - 1$ .  $\square$

## 9 Proof of the Main Theorems

### 9.1 Existence of a next reduction

The next Lemma has its roots in Theorem 20 from [29], where the authors proved that bounded width 1 is equivalent to tree duality.

**Lemma 9.1.** *Suppose  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$  is a strategy for a 1-consistent CSP instance  $\Theta$ , and  $D^{(\top)}$  is a reduction of  $\Theta^{(s)}$ .*

1. *If there exists a 1-consistent reduction contained in  $D^{(\top)}$  and  $D^{(s+1)}$  is maximal among such reductions, then for every variable  $y$  of  $\Theta$  there exists a tree-formula  $\Upsilon_y \in \text{Coverings}(\Theta)$  such that  $\Upsilon_y^{(\top)}(y)$  defines  $D_y^{(s+1)}$ .*
2. *Otherwise, there exists a tree-formula  $\Upsilon \in \text{Coverings}(\Theta)$  such that  $\Upsilon^{(\top)}$  has no solutions.*

*Proof.* The proof is based on the constraint propagation procedure. We consider the instance  $\Theta^{(s)}$ . We start with an empty set  $\Upsilon_y$  (empty tree-formula) for every  $y$ , these tree-formulas define the reduction  $D^{(\top)}$ .

Then we introduce a recursive algorithm that gives a correct tree-formula  $\Upsilon_y$  for every variable  $y$ . If at some step the reduction defined by these tree-formulas is 1-consistent, then we are done. Otherwise, we consider a constraint  $C$  that breaks 1-consistency. Then the current restrictions of the variables  $z_1, \dots, z_l$  in the constraint  $C = \rho(z_1, \dots, z_l)$  imply a stronger restriction of some variable  $z_i$  and the corresponding domain  $D_{z_i}^{(s)}$ . We change the tree-formula  $\Upsilon_{z_i}$  describing the reduction of the variable  $z_i$  in the following way  $\Upsilon_{z_i} := C \wedge \Upsilon_{z_1} \wedge \dots \wedge \Upsilon_{z_l}$ .

Note that we have to be careful with all the variables appearing in different  $\Upsilon_y$  to avoid collisions. Every time we join  $\Upsilon_u$  and  $\Upsilon_v$  we rename the variables so that they do not have common variables.

Obviously, this procedure will eventually stop. If  $\Upsilon_y^{(\top)}(y)$  defines an empty set for some  $y$ , then  $\Upsilon_y$  can be taken as  $\Upsilon$  to witness condition (2). Otherwise, these tree-formulas define a 1-consistent reduction, which is a maximal 1-consistent reduction since it is defined by tree-formulas.  $\square$

**Theorem 9.2.** *Suppose  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$  is a strategy for a cycle-consistent CSP instance  $\Theta$ .*

- *If  $D_x^{(s)}$  has a nontrivial binary absorbing subuniverse  $B$  then there exists a 1-consistent absorbing reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  with  $D_x^{(s+1)} \subseteq B$ .*
- *If  $D_x^{(s)}$  has a nontrivial center  $B$  then there exists a 1-consistent central reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  with  $D_x^{(s+1)} \subseteq B$ .*
- *If  $D_y^{(s)}$  has no nontrivial binary absorbing subuniverse or center for every  $y$  but there exists a nontrivial PC subuniverse  $B$  in  $D_x^{(s)}$  for some  $x$ , then there exists a 1-consistent PC reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  with  $D_x^{(s+1)} \subseteq B$ .*

*Proof.* Without loss of generality we assume that  $B$  is a minimal one-of-four subuniverse of this type. Let us define a reduction  $D^{(\top)}$  by  $D_x^{(\top)} = B$  and  $D_y^{(\top)} = D_x^{(s)}$  for  $y \neq x$ , and apply Lemma 9.1. We consider two cases corresponding to two cases of Lemma 9.1.

Case 1. There exists a 1-consistent reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  such that  $D_y^{(s+1)}$  is defined by  $\Upsilon_y(y)$  for a tree-formula  $\Upsilon_y$  for every variable  $y$ . Let  $R$  be the solution set of  $\Upsilon_y^{(s)}$ . Since  $\Upsilon_y$  is a tree-formula and  $\Theta^{(s)}$  is 1-consistent, the solution set  $R$  is subdirect. Applying Corollaries 7.1.2, 7.6.2, 7.13.2 to  $R$  we derive that  $D_y^{(s+1)}$  is a one-of-four subuniverse of the corresponding type.

Case 2. There exists a tree-formula  $\Upsilon \in \text{Coverings}(\Theta)$  such that  $\Upsilon^{(\top)}$  has no solutions. We consider the minimal set of variables  $\{x_1, \dots, x_k\}$  from  $\Upsilon$  whose parent is  $x$  such that  $\Upsilon^{(s)}(x_1, \dots, x_k)$  does not have any tuple in  $B^k$ . Since  $\Theta^{(s)}$  is 1-consistent and  $\Upsilon$  is a tree-formula,  $k \geq 2$ . If  $B$  is a binary absorbing subuniverse, then we get a contradiction with Lemma 7.5. For other cases with  $k = 2$  we get a contradiction from Corollary 8.26.1. If  $k \geq 3$  and  $B$  is a center then we get a contradiction with Corollary 7.10.3. If  $k \geq 3$  and  $B$  is a PC subuniverse then we get a contradiction with Corollary 7.13.3.  $\square$

As a corollary we can derive that cycle-consistency is a sufficient condition to guarantee the existence of a solution of an instance whose domains avoid linear algebras (so called bounded width case). Note that this corollary follows from the result by Marcin Kozik in [43].

**Corollary 9.2.1.** *Suppose  $\Theta$  is a cycle-consistent CSP instance, for every domain  $D_x$  there is no  $B \subseteq D_x$  and a congruence  $\sigma$  on  $B$  such that  $B/\sigma$  is a nontrivial linear algebra. Then  $\Theta$  has a solution.*

*Proof.* We recursively build a strategy  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ . We start with  $s = 0$ . If every domain  $D_x^{(s)}$  is of size 1, then we already have a solution because  $\Theta^{(s)}$  is 1-consistent. Otherwise, by Theorem 5.1 on every domain  $D_x^{(s)}$  of size greater than 1 there exists a nontrivial one-of-four subuniverse. Note that this subuniverse cannot be linear because this contradicts the assumption that there is no  $B \subseteq D_x$  and a congruence  $\sigma$  on  $B$  such that  $B/\sigma$  is linear. If we found a binary absorbing subuniverse or a center, then by Theorem 9.2 we can always find a next 1-consistent absorbing or central reduction  $D^{(s+1)}$ . Otherwise, by the same theorem we can find a 1-consistent PC reduction. Since the strategy cannot be infinite, we eventually stop with the instance whose variable domains are of size 1.  $\square$

**Theorem 9.3.** *Suppose  $D^{(0)}, D^{(1)}, \dots, D^{(s)}$  is a strategy for a cycle-consistent CSP instance  $\Theta$ , and  $D^{(\top)}$  is a nonlinear 1-consistent reduction of  $\Theta^{(s)}$ . Then there exists a 1-consistent minimal reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  of the same type such that  $D_x^{(s+1)} \subseteq D_x^{(\top)}$  for every variable  $x$ .*

*Proof.* Let the reduction  $D^{(\top)}$  be of type  $\mathcal{T}$ . Let us consider a minimal by inclusion 1-consistent reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  of type  $\mathcal{T}$  such that  $D_x^{(s+1)} \subseteq D_x^{(\top)}$  for every variable  $x$ .

Assume that for some  $z$  the domain  $D_z^{(s+1)}$  is not a minimal one-of-four subuniverse of type  $\mathcal{T}$ . Then choose a minimal one-of-four subuniverse  $B$  of  $D_z^{(s)}$  of this type contained in  $D_z^{(s+1)}$ . We define a reduction  $D^{(\perp)}$  of  $\Theta^{(s)}$  by  $D_z^{(\perp)} = B$ ,  $D_y^{(\perp)} = D_y^{(s+1)}$  if  $y \neq z$ , and apply Lemma 9.1. Since  $D_y^{(s+1)}$  is a minimal by inclusion reduction, there exists a tree-formula  $\Upsilon \in \text{Coverings}(\Theta)$  such that  $\Upsilon^{(\perp)}$  has no solutions. Again, we consider a minimal set of variables  $\{z_1, \dots, z_k\}$  from  $\Upsilon$  whose parent is  $z$  such that  $\Upsilon^{(s+1)}(z_1, \dots, z_k)$  does not have any tuple in  $B^k$ . Since the reduction  $D^{(s+1)}$  is 1-consistent,  $B \subsetneq D_z^{(s+1)}$ , and  $\Upsilon$  is a tree-formula, we have  $k \geq 2$ . If  $D^{(\top)}$  is an absorbing or central reduction of  $\Theta^{(s)}$ , then it is also an absorbing or central reduction of  $\Theta^{(s+1)}$ . Then we can get a contradiction just as we did in the proof of Theorem 9.2 using Lemma 7.5, Corollary 8.26.1 or Corollary 7.10.3.

It remains to consider the case when  $B$  is a PC subuniverse. Choose a minimal set of variables  $y_1, \dots, y_t$  of  $\Upsilon$  different from  $z_1, \dots, z_k$  such that  $(\Upsilon^{(s)}(z_1, \dots, z_k, y_1, \dots, y_t))^{(s+1)}$  does not have tuples with the first  $k$  elements from  $B$ . If  $t = 0$  and  $k = 2$  then  $\Upsilon^{(s)}(z_1, z_2)$  has an empty intersection with  $B \times B$ , which contradicts Corollary 8.26.1. If  $t + k \geq 3$  then the relation defined by  $\Upsilon^{(s)}(z_1, \dots, z_k, y_1, \dots, y_t)$  is  $(B, \dots, B, D_{y_1}^{(s+1)}, \dots, D_{y_t}^{(s+1)})$ -essential relation, which contradicts Corollary 7.13.3.  $\square$

**Theorem 9.4.** *Suppose  $D^{(\top)}$  is a 1-consistent PC reduction for a cycle-consistent irreducible CSP instance  $\Theta$ , and  $\Theta$  is not linked and not fragmented. Then there exist a reduction  $D^{(1)}$  of*

$\Theta$  and a minimal strategy  $D^{(1)}, \dots, D^{(s)}$  for  $\Theta^{(1)}$  such that the solution set of  $\Theta^{(1)}$  is subdirect, the reductions  $D^{(2)}, \dots, D^{(s)}$  are nonlinear,  $D_x^{(s)} \subseteq D_x^{(\top)}$  for every variable  $x$ .

*Proof.* Since  $\Theta$  is not linked, there exists a maximal congruence  $\sigma_x$  on  $D_x$  for a variable  $x$  of  $\Theta$  such that  $\text{LinkedCon}(\Theta, x) \subseteq \sigma_x$ . Choose an equivalence class  $D_x^{(1)}$  of  $\sigma_x$  with a nonempty intersection with  $D_x^{(\top)}$ . For every variable  $y$  by  $D_y^{(1)}$  we denote the set of all elements of  $D_y$  linked to an element of  $D_x^{(1)}$ . Note that for every  $y$  there is a congruence  $\sigma_y$  on  $D_y$  such that  $D_x/\sigma_x \cong D_y/\sigma_y$ . Then  $D_y^{(1)}$  is an equivalence class of  $\sigma_y$ . By Corollaries 7.1.1 and 7.6.1, there is no nontrivial binary absorbing subuniverse or center on  $D_x/\sigma_x$ . Then by Theorem 5.1,  $\sigma_x$  is either PC congruence, or linear congruence, which means that  $D^{(1)}$  is a PC reduction or linear reduction.

Let us show that  $D_y^{(1)} \cap D_y^{(\top)} \neq \emptyset$  for every  $y$ . Since  $\Theta$  is not fragmented, we may consider a path starting at  $x$  and ending at  $y$ . Since the reduction  $D^{(\top)}$  is 1-consistent, this path connects an element of  $D_x^{(1)} \cap D_x^{(\top)}$  with some element of  $D_y^{(\top)}$ , which is also in  $D_y^{(1)}$ .

Since  $\Theta$  is irreducible, the solution set of  $\Theta^{(1)}$  is subdirect. We build the remaining part of the strategy in the following way. Suppose we already have  $D^{(0)}, D^{(1)}, \dots, D^{(t)}$ , where the reductions  $D^{(2)}, \dots, D^{(t)}$  are absorbing or central. If there exists a nontrivial binary absorbing subuniverse or a nontrivial center on  $D_y^{(t)}$  for some  $y$ , then by Theorems 9.2, 9.3 we can find the next minimal 1-consistent absorbing or central reduction  $D^{(t+1)}$ .

Suppose there is no binary absorbing subuniverse or center on  $D_y^{(t)}$  for every  $y$ . Put  $D_y^{(\perp)} = D_y^{(\top)} \cap D_y^{(t)}$  for every variable  $y$ . By Lemma 7.31  $D_y^{(\perp)}$  is a PC subuniverse of  $D_y^{(t)}$  for every variable  $y$ . Hence,  $D^{(\perp)}$  is a PC reduction of  $\Theta^{(t)}$ .

Then we apply Lemma 9.1 to find a 1-consistent reduction of  $\Theta^{(t)}$  smaller than  $D^{(\perp)}$ . If we cannot find it, then there exists a tree-formula  $\Upsilon$  such that  $\Upsilon^{(\perp)}$  has no solutions. Let  $R$  be the solution set of  $\Upsilon$ . Note that  $R^{(i)}$  is a subdirect relation for  $i = 0, 1, \dots, t$  because  $\Upsilon$  is a tree-formula and  $D^{(i)}$  is a 1-consistent reduction. By Lemma 7.25,  $R^{(\top)}$  is a PC subuniverse of  $R$ . Since  $D^{(\top)}$  is 1-consistent, the intersection  $R^{(1)} \cap R^{(\top)}$  is not empty.

Let us prove by induction on  $i$  that  $R^{(i)} \cap R^{(\top)}$  is a nonempty PC subuniverse of  $R^{(i)}$  for  $i = 1, 2, \dots, t$ . By the inductive assumption, we assume that  $R^{(i-1)} \cap R^{(\top)}$  is a nonempty PC subuniverse of  $R^{(i-1)}$  (for  $i = 1$  it follows from the definition). By Lemma 7.25,  $R^{(i)}$  is a one-of-four subuniverse of  $R^{(i-1)}$ . For  $i \geq 2$  it is not a PC subuniverse, then by Theorem 7.33, the intersection of  $R^{(i-1)} \cap R^{(\top)}$  and  $R^{(i)}$ , that is  $R^{(i)} \cap R^{(\top)}$ , cannot be empty. For  $i = 1$  we already know that  $R^{(1)} \cap R^{(\top)} \neq \emptyset$ . Applying Theorem 7.30 to  $R^{(i-1)} \cap R^{(\top)} \subseteq R^{(i-1)}$  and  $R^{(i)} \subseteq R^{(i-1)}$  we derive that  $R^{(i)} \cap R^{(\top)}$  is a nonempty PC subuniverse of  $R^{(i)}$ . Thus, we proved that  $R^{(t)} \cap R^{(\top)}$  is not empty, which contradicts the assumption about the tree-formula  $\Upsilon$ .

Hence, there exists a 1-consistent reduction  $D^{(\Delta)}$  of  $\Theta^{(t)}$  smaller than  $D^{(\perp)}$  such that for every variable  $y$  the new domain  $D_y^{(\Delta)}$  can be defined by a tree-formula  $\Upsilon_y^{(\perp)}$ . Since the solution set of  $\Upsilon_y^{(\perp)}$  is subdirect, by Corollary 7.13.2, the domain  $D_y^{(\Delta)}$  is a PC subuniverse of  $D_y^{(t)}$ . Hence  $D^{(\Delta)}$  is a PC reduction of  $\Theta^{(t)}$ . It remains to apply Theorem 9.3 to find a minimal PC reduction  $D^{(t+1)}$  smaller than  $D_y^{(\Delta)}$ , put  $s = t + 1$ , and finish the strategy.  $\square$

## 9.2 Existence of a linked connected component

In this subsection we prove that all constraints in a crucial instance have the parallelogram property, show that we can always find a linked connected component with required properties, and prove that we cannot pass from an instance having solutions to an instance having no solutions while applying a nonlinear reduction.

**Theorem 9.5.** Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Omega(x_1, \dots, x_n)$  is a subconstraint of  $\Theta$ , the solution set of  $\Omega^{(1)}$  is subdirect,  $\Theta \setminus \Omega$  has a solution in  $D^{(1)}$ , and  $\Theta$  has no solutions in  $D^{(1)}$ . Then there exist instances  $\Upsilon_1, \dots, \Upsilon_t \in \text{Coverings}(\Omega)$  such that  $\Phi = (\Theta \setminus \Omega) \cup \Upsilon_1 \cup \dots \cup \Upsilon_t$  has no solutions in  $D^{(1)}$ , each  $\Upsilon_i(x_1, \dots, x_n)$  is a subconstraint of  $\Phi$ , and  $\Upsilon_i^{(1)}(x_1, \dots, x_n)$  defines a subdirect key relation with the parallelogram property for every  $i$ .

**Theorem 9.6.** Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Theta$  is crucial in  $D^{(1)}$  and not connected. Then there exists an instance  $\Theta' \in \text{ExpCov}(\Theta)$  that is crucial in  $D^{(1)}$  and contains a linked connected component whose solution set is not subdirect.

**Theorem 9.7.** Suppose  $D^{(1)}$  is a 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance  $\Theta$ . If  $\Theta$  has a solution then it has a solution in  $D^{(1)}$ .

**Theorem 9.8.** Suppose  $D^{(0)}, \dots, D^{(s)}$  is a minimal strategy for a cycle-consistent irreducible CSP instance  $\Theta$ , and a constraint  $\rho(x_1, \dots, x_n)$  of  $\Theta$  is crucial in  $D^{(s)}$ . Then  $\rho$  is a critical relation with the parallelogram property.

**Theorem 9.9.** Suppose  $D^{(0)}, \dots, D^{(s)}$  is a minimal strategy for a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Upsilon(x_1, \dots, x_n)$  is a subconstraint of  $\Theta$ , the solution set of  $\Upsilon^{(s)}$  is subdirect,  $k \in \{1, 2, \dots, n-1\}$ ,  $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, u_1, \dots, u_t\}$ ,

$$\Omega = \Upsilon_{x_1, \dots, x_k, u_1, \dots, u_t}^{y_1, \dots, y_k, v_1, \dots, v_t} \wedge \Upsilon_{x_{k+1}, \dots, x_n, u_1, \dots, u_t}^{y_{k+1}, \dots, y_n, v_{t+1}, \dots, v_{2t}} \wedge \Upsilon_{x_1, \dots, x_n, u_1, \dots, u_t}^{y_1, \dots, y_n, v_{2t+1}, \dots, v_{3t}},$$

and  $\Theta^{(s)}$  has no solutions. Then  $(\Theta \setminus \Upsilon) \cup \Omega$  has no solutions in  $D^{(s)}$ .

To prove these theorems we need to introduce a partial order on domain sets. To every domain set  $D^{(\top)}$  we assign a tuple of integers  $\text{Size}(D^{(\top)}) = (|D_1|, |D_2|, \dots, |D_s|)$ , where  $D_1, D_2, \dots, D_s$  is the set of all different domains of  $D^{(\top)}$  ordered by their size starting from the large one. Then the lexicographic order on tuples of integers induces a partial order on domain sets, that is we say that  $(a_1, \dots, a_k) < (b_1, \dots, b_l)$  if there exists  $j \in \{1, 2, \dots, \min(k+1, l)\}$  such that  $a_i = b_i$  for every  $i < j$ , and  $a_j < b_j$  or  $j = k+1$ .

It follows from the definition that  $\leq$  is transitive and there does not exist an infinite descending chain of reductions. Note that duplicating domains does not affect this partial order, that is why we do not make the size of a domain set larger if we consider an expanded covering. At the same time, for every minimal (proper) one-of-four reduction  $D^{(1)}$  of the instance with a domain set  $D^{(0)}$  we have  $\text{Size}(D^{(1)}) < \text{Size}(D^{(0)})$ . Let us show this for a central reduction. We replace every domain having a nontrivial center by a smaller domain and we do not change other domains. Let  $D_y^{(0)}$  be a domain of the maximal size having a nontrivial center. Then  $|D_y^{(0)}|$  will be replaced by smaller numbers in the sequence  $\text{Size}(D^{(0)})$  making the sequence smaller.

We prove theorems of this subsection simultaneously by the induction on the size of the domain sets. Let  $D^{(\perp)}$  be a domain set. Assume that Theorems 9.5, 9.6, and 9.7 hold on instances  $\Theta$  with a domain set  $D^{(0)}$  if  $\text{Size}(D^{(0)}) < \text{Size}(D^{(\perp)})$ , and Theorems 9.8 and 9.9 hold if  $\text{Size}(D^{(s)}) < \text{Size}(D^{(\perp)})$ . Let us prove Theorems 9.5, 9.6, and 9.7 on instances  $\Theta$  with a domain set  $D^{(0)}$  if  $\text{Size}(D^{(0)}) = \text{Size}(D^{(\perp)})$ , and Theorems 9.8 and 9.9 for  $\text{Size}(D^{(s)}) = \text{Size}(D^{(\perp)})$ .

**Theorem 9.5.** Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Omega(x_1, \dots, x_n)$  is a subconstraint of  $\Theta$ , the solution set of  $\Omega^{(1)}$  is subdirect,  $\Theta \setminus \Omega$  has a solution in  $D^{(1)}$ , and  $\Theta$  has no solutions in  $D^{(1)}$ . Then there exist instances  $\Upsilon_1, \dots, \Upsilon_t \in \text{Coverings}(\Omega)$  such that  $\Phi = (\Theta \setminus \Omega) \cup \Upsilon_1 \cup \dots \cup \Upsilon_t$  has no solutions in  $D^{(1)}$ , each  $\Upsilon_i(x_1, \dots, x_n)$  is a subconstraint of  $\Phi$ , and  $\Upsilon_i^{(1)}(x_1, \dots, x_n)$  defines a subdirect key relation with the parallelogram property for every  $i$ .

*Proof.* Let  $\Sigma$  be the set of all relations defined by  $\Upsilon^{(1)}(x_1, \dots, x_n)$  where  $\Upsilon \in \text{Coverings}(\Omega)$ . To every relation  $\rho \in \Sigma$  we assign a constraint  $((x_1, \dots, x_n); \rho)$ , which we denote by  $C(\rho)$ . We can find  $\Sigma_0 \subseteq \Sigma$  such that the instance  $(\Theta^{(1)} \setminus \Omega^{(1)}) \cup C(\Sigma_0)$  has no solutions, but if we replace any relation of  $\Sigma_0$  by all bigger relations from  $\Sigma$  (weaker in terms of constraints) then we get an instance with a solution.

Let  $\Sigma_0 = \{\rho_1, \dots, \rho_t\}$ . For each  $\rho_i$  and each  $\alpha \notin \rho_i$  we consider an inclusion-maximal relation  $\rho_{i,\alpha} \supseteq \rho_i$  from  $\Sigma$  such that  $\alpha \notin \rho_{i,\alpha}$ . Since  $\rho_i = \bigcap_{\alpha \notin \rho_i} \rho_{i,\alpha}$ , if  $\rho_i \neq \rho_{i,\alpha}$  for each  $\alpha$  then  $\rho_i$  could be replaced by bigger relations that are still in  $\Sigma$ , which contradicts our assumptions. Then for each  $\rho_i$  there exists a tuple  $\alpha_i$  such that  $\rho_i$  is an inclusion-maximal relation without  $\alpha_i$  in  $\Sigma$ .

By Corollary 8.12.1,  $\rho_i$  is a key relation for every  $i$ . Therefore we get a sequence of instances  $\Upsilon_1, \dots, \Upsilon_t \in \text{Coverings}(\Omega)$  such that  $\Upsilon_i^{(1)}$  defines  $\rho_i$  for every  $i$ . Put  $\Phi = (\Theta \setminus \Omega) \cup \Upsilon_1 \cup \dots \cup \Upsilon_t$ . We choose variables in the instance so that the only common variables of  $\Upsilon_1, \dots, \Upsilon_t$  are  $x_1, \dots, x_n$ , which guarantees that  $\Upsilon_i(x_1, \dots, x_n)$  is a subconstraint of  $\Phi$ .

Since  $\Phi$  is a covering of  $\Theta$ , by Lemma 6.1,  $\Phi$  is cycle-consistent and irreducible. Assume that  $\rho_i$  does not have the parallelogram property. Without loss of generality we assume that the failing partition is  $\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_n\}$ . Define the instance  $\Omega_i$  from  $\Upsilon_i$  using the construction from Theorem 9.9. Then the relation defined by  $\Omega_i^{(1)}(x_1, \dots, x_n)$  is bigger than  $\rho_i$  and  $\Omega_i \in \text{Coverings}(\Omega)$ , which means that  $(\Phi \setminus \Upsilon_i) \cup \Omega_i$  has a solution in  $D^{(1)}$  and contradicts the inductive assumption for Theorem 9.9. Hence,  $\rho_i$  has the parallelogram property for every  $i$ .  $\square$

To prove the next theorem we will need additional definitions and few auxiliary lemmas. First, we assign a characteristic to every variable of an instance, then we introduce a partial order on the set of characteristics. After that, we define three transformations of the instance giving an expanded covering of the original instance. We will prove that these transformations change the characteristics in a good way, so they can be used to generate an instance required in Theorem 9.6.

Let us assign a characteristic to every variable of an instance  $\Phi$  whose constraints are critical and rectangular. For a variable  $x$  let  $\mathfrak{C}_1$  be the set of all minimal congruences among the set  $\text{Con}(\Phi, x)$ . Then let  $\mathfrak{C}_2$  be the set of all minimal congruences among the congruences of  $\text{Con}(\Phi, x)$  that are not adjacent with any congruence from  $\mathfrak{C}_1$ . Thus, we assign a pair  $(\mathfrak{C}_1, \mathfrak{C}_2)$  to every variable  $x$ , which we denote  $\xi(\Phi, x)$  and call *characteristic*.

Let us introduce a partial order on the set of all characteristics. For two sets of irreducible congruences  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  we write  $\mathfrak{C}_1 \leq \mathfrak{C}_2$  if for every  $\sigma \in \mathfrak{C}_1$  there exists  $\delta \in \mathfrak{C}_2$  such that  $\delta \subseteq \sigma$ . We write  $\mathfrak{C}_1 < \mathfrak{C}_2$  if  $\mathfrak{C}_1 \leq \mathfrak{C}_2$  and  $\mathfrak{C}_2 \not\leq \mathfrak{C}_1$ . It is easy to see that  $\leq$  is a transitive relation.

By  $\uparrow \text{Opt}(\mathfrak{C})$  we denote the set of all congruences  $\sigma$  such that  $\sigma \supseteq \delta$  for some  $\delta \in \text{Opt}(\mathfrak{C})$ . We write  $(\mathfrak{C}_1, \mathfrak{C}_2) \lesssim (\mathfrak{C}'_1, \mathfrak{C}'_2)$  if one of the following conditions holds:

1.  $\mathfrak{C}_1 < \mathfrak{C}'_1$ ;
2.  $\mathfrak{C}_1 = \mathfrak{C}'_1$  and  $\mathfrak{C}_2 \leq \mathfrak{C}'_2$ ;
3.  $\mathfrak{C}_1 = \mathfrak{C}'_1$ ,  $\mathfrak{C}_2 \not\leq \mathfrak{C}'_2$ ,  $\mathfrak{C}'_2 \not\leq \mathfrak{C}_2$ ,  $\mathfrak{C}_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1)) < \mathfrak{C}'_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1))$ .

**Lemma 9.10.**  $\lesssim$  is a transitive relation.

*Proof.* Assume that  $(\mathfrak{C}_1, \mathfrak{C}_2) \lesssim (\mathfrak{C}'_1, \mathfrak{C}'_2)$  and  $(\mathfrak{C}'_1, \mathfrak{C}'_2) \lesssim (\mathfrak{C}''_1, \mathfrak{C}''_2)$ .

If  $\mathfrak{C}_1 < \mathfrak{C}'_1$  or  $\mathfrak{C}'_1 < \mathfrak{C}''_1$ , then  $\mathfrak{C}_1 < \mathfrak{C}''_1$ , which completes this case.

Thus, we assume that  $\mathfrak{C}_1 = \mathfrak{C}'_1 = \mathfrak{C}''_1$ . It follows from (2) and (3) that

$$\mathfrak{C}_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1)) \leq \mathfrak{C}'_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1)) \leq \mathfrak{C}''_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1)).$$

If  $\mathfrak{C}_2 \leq \mathfrak{C}'_2 \leq \mathfrak{C}''_2$ , then  $\mathfrak{C}_2 \leq \mathfrak{C}''_2$ , which completes this case. Thus, we assume that at least one of the above comparisons is strict (comes from (3)). Hence,  $\mathfrak{C}_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1)) < \mathfrak{C}''_2 \setminus (\uparrow \text{Opt}(\mathfrak{C}_1))$ . Therefore,  $\mathfrak{C}''_2 \not\leq \mathfrak{C}_2$  and (2) or (3) holds for  $(\mathfrak{C}_1, \mathfrak{C}_2)$  and  $(\mathfrak{C}'_1, \mathfrak{C}''_2)$ , which completes the proof.  $\square$

**Remark 5.** Note that  $\leq$  is not a partial order in general, but it is a partial order on sets of mutually non-inclusive congruences. Similarly,  $\lesssim$  is not a partial order in general, but it is a partial order if we consider only pairs  $(\mathfrak{C}_1, \mathfrak{C}_2)$  such that all the congruences of  $\mathfrak{C}_i$  are not included into each other for  $i = 1, 2$ . Thus, as it follows from the definition of the characteristic, we defined a partial order on the set of all characteristics.

A variable  $x$  of an instance  $\Theta$  is called *stable* if all the congruences in  $\text{Con}(\Theta, x)$  are adjacent. We say that variables  $y_1$  and  $y_2$  are *friends* in  $\Theta$  if they appear in the scope of some constraint of  $\Theta$ .

**Transformation  $T_1(\Theta)$ : make an instance crucial in  $D^{(1)}$ .** Using Remark 3, we replace constraints by all weaker constraints until we get a CSP instance that is crucial in  $D^{(1)}$ .

Note that  $T_1(\Theta) \in \text{ExpCov}(\Theta)$ .

Below we assume that the instance  $\Theta$  is crucial in  $D^{(1)}$ , which by the inductive assumption for Theorem 9.8 means that every constraint in  $\Theta$  has the parallelogram property and is critical.

**Transformation  $T_2(\Theta, \sigma_1, \sigma_2, x)$ : split a variable.** Let  $\Omega_i$  be the set of all constraints  $C \in \Theta$  such that  $\text{Con}(C, x) = \sigma_i$  for  $i \in \{1, 2\}$ . Let  $\Omega_0$  be the set of all constraints  $C \in \Theta \setminus (\Omega_1 \cup \Omega_2)$  containing  $x$ . We transform our instance in the following way:

1. Choose 2 new variables  $x_1$  and  $x_2$ ;
2. Rename  $x$  by  $x_1$  in all constraints from  $\Omega_0$  and  $\Omega_1$ ;
3. Rename  $x$  by  $x_2$  in all constraints from  $\Omega_2$ ;
4. Add the constraints  $\sigma_1^*(x_1, x_2)$  and  $\sigma_2^*(x_1, x_2)$ ;
5. For every  $\sigma \in \text{Con}(\Omega_0, x)$  add the constraint  $\sigma(x_1, x_2)$ .

Note that  $T_2(\Theta, \sigma_1, \sigma_2, x)$  is an expanded covering of  $\Theta$ , where the parent of  $x_1$  and  $x_2$  is  $x$ .

**Lemma 9.11.** Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Theta$  is crucial in  $D^{(1)}$ , and congruences  $\sigma_1, \sigma_2 \in \text{Con}(\Theta, x)$  are not adjacent. Then the instance  $T_2(\Theta, \sigma_1, \sigma_2, x)$  has no solutions in  $D^{(1)}$ .

*Proof.* Let  $\Theta' = T_2(\Theta, \sigma_1, \sigma_2, x)$ ,  $\sigma$  be the intersection of all congruences from  $\text{Con}(\Omega_0, x)$ .

Assume that  $\Theta'$  has a solution in  $D^{(1)}$ . Suppose  $(x_1, x_2) = (a_1, a_2)$  in this solution. Put  $\Upsilon = \sigma_1(x_1, x) \wedge \sigma_2(x_2, x) \wedge \sigma(x_2, x)$ . Consider the instance  $\Theta' \wedge \Upsilon$ . Since  $(x_1, x_2) \in \sigma$  (by the definition of the transformation) and  $(x_2, x) \in \sigma$ , we have  $(x, x_1) \in \sigma$ . Then each solution of  $\Theta' \wedge \Upsilon$  can be taken as a solution of  $\Theta$  (we just ignore  $x_1$  and  $x_2$ ). Hence, the instance  $\Theta' \wedge \Upsilon$  has no solutions in  $D^{(1)}$ . We apply Theorem 9.5 to the subconstraint  $\Upsilon(x_1, x_2)$  to obtain a sequence of formulas  $\Omega_1, \dots, \Omega_t \in \text{Coverings}(\Upsilon)$  such that  $\Theta' \cup \Omega_1 \cup \dots \cup \Omega_t$  has no solutions in  $D^{(1)}$ , and  $\Omega_i^{(1)}(x_1, x_2)$  defines a subdirect key relation  $\rho_i$  with the parallelogram property for every  $i$ . Note that the relation  $\rho_i$  is reflexive, therefore,  $\rho_i$  is a congruence on  $D_{x_1}^{(1)}$ . If the reduction  $D^{(1)}$  is nonlinear then by  $\omega_i$  we denote the relation defined by  $\Omega_i(x_1, x_2)$ . If the reduction  $D^{(1)}$  is linear then by  $\omega_i$  we denote the relation defined by  $\Omega'_i(x_1, x_2, u_1, \dots, u_r)$

from Lemma 8.14. We know from Lemmas 8.14 and 8.13 that  $\text{Con}(\omega_i, 1)^{(1)} = \text{Con}(\rho_i, 1) = \rho_i$ . Every constraint in  $\Omega_i$ , which contains  $x_1$  must have  $\sigma_1$  for its constraint relation; thus the first variable of  $\omega_i$  is stable under  $\sigma_1$  and  $\text{Con}(\omega_i, 1) \supseteq \sigma_1$ . Consider two cases:

Case 1. Assume that  $\rho_i \neq \sigma_1^{(1)}$  for every  $i$ , then  $\text{Con}(\omega_i, 1) \supseteq \sigma_1^*$ . Hence  $\rho_i \supseteq (\sigma_1^*)^{(1)}$  for every  $i$ . Then we may put  $x_1 = a_1$  and  $x = x_2 = a_2$  to get a solution of  $\Theta' \cup \Omega_1 \cup \dots \cup \Omega_t$  in  $D^{(1)}$ , which contradicts the properties of the sequence  $\Omega_1, \dots, \Omega_t$ .

Case 2. Assume that  $\rho_i = \sigma_1^{(1)}$  for some  $i$ . Since  $(a_1, a_2) \in (\sigma_1^*)^{(1)} \setminus \sigma_1$  and  $\text{Con}(\omega_i, 1)^{(1)} = \rho_i$ , we have  $\text{Con}(\omega_i, 1) \not\supseteq \sigma_1^*$ . Hence  $\text{Con}(\omega_i, 1) = \sigma_1$ . Suppose  $D^{(1)}$  is a nonlinear reduction.  $\Upsilon(x_1, x_2)$  contains  $\sigma_2 \cap \sigma$ , and therefore  $\sigma_2 \cap \sigma \subseteq \text{Con}(\omega_i, 1) = \sigma_1$ . The symmetric conclusion  $\sigma_1 \cap \sigma \subseteq \sigma_2$  can be obtained by a symmetric argument, switching the roles of  $\sigma_1$  and  $\sigma_2$ . Since,  $(a_1, a_2) \in \sigma \setminus \sigma_1$ , by Lemma 8.19  $\sigma_1$  and  $\sigma_2$  are adjacent, which contradicts our assumptions. Similarly, if  $D^{(1)}$  is a linear reduction, we can show that  $\sigma_2 \cap \sigma \cap \text{ConLin}(D_x) \subseteq \text{Con}(\omega_i, 1) = \sigma_1$ . Indeed, suppose  $(c, d) \in \sigma_2 \cap \sigma \cap \text{ConLin}(D_x)$ . To witness that  $(c, d) \in \text{Con}(\omega_i, 1)$  we need to define two tuples from  $\omega_i$  that differ only in the first component. To obtain the first tuple we assign  $c$  to every variable of  $\Omega'_i$ . To obtain the second tuple we assign  $d$  to all variables whose parent is  $x_1$  or  $x$ , and  $c$  to the remaining variables (including  $u_1, \dots, u_r$ ). Thus, we can show that  $\sigma_2 \cap \sigma \cap \text{ConLin}(D_x) \subseteq \sigma_1$  and  $\sigma_1 \cap \sigma \cap \text{ConLin}(D_x) \subseteq \sigma_2$ . Since  $(a_1, a_2) \in (\sigma \cap \text{ConLin}(D_x)) \setminus \sigma_1$ , Lemma 8.19 implies that  $\sigma_1$  and  $\sigma_2$  are adjacent, which contradicts our assumptions.  $\square$

Informally speaking, the following lemma states that when we apply  $T_1(T_2(\Theta, \sigma_1, \sigma_2, x))$  the characteristic of every new variable is less than the characteristic of its parent, the characteristic of old variables does not change, and if a stable variable gets a new friend then the friend's parent is not its friend anymore.

**Lemma 9.12.** *Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Theta$  is crucial in  $D^{(1)}$ , congruences  $\sigma_1, \sigma_2$  are minimal congruences among  $\text{Con}(\Theta, x)$ ,  $\sigma_1$  and  $\sigma_2$  are not adjacent,  $\Theta' = T_1(T_2(\Theta))$ . Then*

1.  $\xi(\Theta', y') < \xi(\Theta, y)$ , if  $y$  is a parent of  $y'$  and  $y' \neq y$ ;
2.  $\xi(\Theta', y) = \xi(\Theta, y)$  if  $y \in \text{Var}(\Theta) \cap \text{Var}(\Theta')$ ;
3. if  $y$  is stable in  $\Theta$ ,  $y' \in \text{Var}(\Theta') \setminus \text{Var}(\Theta)$ , then  $y$  cannot be a friend of both  $y'$  and the parent of  $y'$  in  $\Theta'$ ;
4.  $\Theta$  and  $\Theta'$  have a common variable.

*Proof.* By Lemma 9.11 and the definition of  $T_1$ ,  $\Theta'$  is crucial, then by Lemma 8.24 for every constraint  $C$  in  $\Theta$  there exists a constraint  $C'$  in  $\Theta'$  whose image in  $\Theta$  is  $C$ . Therefore, when we apply  $T_1$  we weaken only binary constraints we added in  $T_2$  but not the constraints from  $\Theta$ . Then Claim (2) follows from the definition of the transformation.

$\text{Con}(\Theta', x_1)$  has all the congruences of  $\text{Con}(\Theta, x)$  but  $\sigma_2$ . Additionally, it may contain congruences  $\delta$  such that  $\delta \supseteq \sigma_1^*$ ,  $\delta \supseteq \sigma_2^*$ , or  $\delta \not\supseteq \sigma$  for  $\sigma \in \text{Con}(\Omega_0, x)$ . None of these congruences are minimal, so they cannot affect the first coordinate of  $\xi(\Theta', x_1)$ . Thus,  $\xi(\Theta', x_1) < \xi(\Theta, x)$ . Similarly, we can show that  $\xi(\Theta', x_2) < \xi(\Theta, x)$ , which completes Claim (1).

Claim (3) follows from the fact that  $x$ , which is the only parent of variables from  $\text{Var}(\Theta') \setminus \text{Var}(\Theta)$ , is not in  $\Theta'$ .

Since a crucial instance cannot have just one variable,  $\Theta$  and  $\Theta'$  have a common variable, which is Claim (4).  $\square$



For an instance  $\Omega \subseteq \Theta$  by  $\text{MinVar}(\Omega, \Theta)$  we denote the set of all variables  $x$  such that there exists  $\sigma \in \text{Con}(\Omega, x)$  that is minimal among  $\text{Con}(\Theta, x)$ .

**Transformation  $T_3(\Theta, \Omega)$  for a connected component  $\Omega$ .** Let  $\text{MinVar}(\Omega, \Theta) = \{x_1, \dots, x_s\}$ , where  $s \geq 1$ . Let us define the new instance in the following way:

1. Choose new variables  $x'_1, \dots, x'_s$ ;
2. Rename the variables  $x_1, \dots, x_s$  by  $x'_1, \dots, x'_s$  in  $\Theta \setminus \Omega$ ;
3. Add the covers of all constraints from  $\Omega$  with  $x'_1, \dots, x'_s$  instead of  $x_1, \dots, x_s$ ;
4. For every  $j$  and every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$  add the constraint  $\sigma^*(x_j, x'_j)$ ;
5. For every  $j$  and  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$  such that  $\text{LinkedCon}(\Omega, x_j) \not\subseteq \sigma$  add the constraint  $\delta_j(x_j, x'_j)$ , where  $\{\delta_j\} = \text{Opt}(\text{Con}(\Omega, x_j))$ .

Note that by Corollary 8.22.1 all congruences of  $\text{Con}(\Omega, x_j)$  are adjacent. Then by Lemma 6.4  $\text{Opt}(\text{Con}(\Omega, x_j))$  contains just one element and  $\delta_j$  is well-defined in (5). Also,  $T_3(\Theta, \Omega)$  is an expanded covering of  $\Theta$ , where the parent of every  $x'_i$  is  $x_i$ .

**Lemma 9.13.** *Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Theta$  is crucial in  $D^{(1)}$ ,  $\Omega$  is a connected component of  $\Theta$ , the solution set of  $\Omega$  is subdirect,  $\Omega$  has a solution in  $D^{(1)}$ , and for every  $x \in \text{Var}(\Omega)$  any two congruences that are minimal among  $\text{Con}(\Theta, x)$  are adjacent. Then the instance  $T_3(\Theta, \Omega)$  has no solutions in  $D^{(1)}$ .*

*Proof.* Suppose  $\text{Var}(\Omega) \setminus \text{MinVar}(\Omega, \Theta) = \{z_1, \dots, z_n\}$ . Since the solution set of  $\Omega$  is subdirect, by Lemma 8.5 we know that the solution set of  $\Omega^{(1)}$  is subdirect. We consider two cases:

Case 1:  $\text{LinkedCon}(\Omega, y) = \text{Con}(C, y)$  for every variable  $y$  and every constraint  $C \in \Omega$  having  $y$  in the scope. This means that  $\text{Con}(\Omega, y)$  contains exactly one congruence for every variable  $y \in \text{Var}(\Omega)$ . Since  $\text{Con}(C, x_j)$  is minimal among  $\text{Con}(\Theta, x_j)$  for every  $j$  and every constraint  $C \in \Omega$  containing  $x_j$ , we have  $\text{LinkedCon}(\Omega, x_j) \subsetneq \sigma$  for every  $j$  and every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ . Notice that for any constraint  $C \in \Omega$  having  $y_1$  and  $y_2$  in the scope we have  $\text{LinkedCon}(\Omega, y_1) \supseteq \text{Con}(\text{pr}_{y_1, y_2}(C), y_1)$ . Since all constraints of  $\Omega$  are rectangular and critical, Lemma 8.10 together with  $\text{LinkedCon}(\Omega, y_1) = \text{Con}(C, y_1)$  imply that the constraint  $C$  should be binary. Thus, all the constraint relations are binary.

Assume that  $n = 0$ . Since  $\Theta$  is crucial in  $D^{(1)}$ , the instance  $\Omega$ , viewed as a graph whose vertexes are variables, cannot have a cycle (otherwise, removing a constraint(edge) from the cycle would not affect the solution set, which contradicts the fact that  $\Theta$  is crucial). Hence, we can choose a constraint  $C \in \Omega$  with a variable  $x_j$  that appears just once in  $\Omega$ . We replace the variable  $x_j$  in  $\Theta \setminus \{C\}$  by  $x'_j$  and add the constraint  $\sigma_0^*(x_j, x'_j)$ , where  $\sigma_0 = \text{Con}(C, x_j)$ . The obtained instance we denote by  $\Theta'$ . Since the constraint  $C$  is crucial in  $D^{(1)}$ ,  $\Theta'$  has a solution in  $D^{(1)}$ . Since  $\sigma_0^* \subseteq \sigma$  for every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ , the solution of  $\Theta'$  gives a solution of  $\Theta$  in  $D^{(1)}$ , which contradicts our assumptions.

Suppose  $n > 0$ . By  $\Omega'$  we denote the copy of  $\Omega$  with covers instead of constraints we introduced in (3). For every variable  $y$  of  $\Omega$  by  $\sigma_y$  we denote the minimal congruence such that  $\sigma_y \supsetneq \text{LinkedCon}(\Omega, y)$ . Since  $\text{LinkedCon}(\Omega, y)$  is an irreducible congruence,  $\sigma_y$  is well-defined. For every constraint  $C = \rho(u, v)$  of  $\Omega$ , by  $C'$  we denote the constraint  $\rho'(u, v)$ , where  $\rho'(u, v) = \exists u' \exists v' \rho(u', v') \wedge \sigma_u(u, u') \wedge \sigma_v(v, v')$ . Let us show that  $\rho'$  is a rectangular relation such that  $\text{Con}(\rho', 1) = \sigma_u$  and  $\text{Con}(\rho', 2) = \sigma_v$ , that is a bijective mapping between equivalence classes of  $\sigma_u$  and  $\sigma_v$ . Since  $\rho$  is rectangular, the congruence  $\sigma_u$  generates a congruence on  $D_v$  that is strictly greater than  $\text{Con}(\rho, 2)$ , and therefore containing  $\sigma_v$ . Therefore,  $\sigma_v$  has at least

as many equivalence classes as  $\sigma_u$ . The same is true for  $\sigma_u$ , which means that the congruence generated on  $D_v$  from  $\sigma_u$  using  $\rho$  is equal to  $\sigma_v$ . Therefore,  $\rho'$  is a rectangular relation such that  $\text{Con}(\rho', 1) = \sigma_u$  and  $\text{Con}(\rho', 2) = \sigma_v$ . Note that  $\rho' \supsetneq \rho$ .

Since  $n > 0$  and  $\Omega$  is not fragmented, there exists a path in  $\Omega$  connecting a variable  $z_i$  with a variable  $x_j$  for every  $i$  and  $j$ . Then we glue a path going from  $x_j$  to  $z_i$  in  $\Omega$  with a path going from  $z_i$  to  $x'_j$  in  $\Omega'$ . For every constraint  $C \in \Omega$  the constraint  $C'$  (defined above) is weaker or equivalent to its cover in  $\Omega'$ . Therefore, every constraint  $C$  in the obtained path from  $x_j$  to  $x'_j$  is not weaker than  $C'$ , which means (by the properties of  $C'$ ) that  $x_j$  and  $x'_j$  should be equivalent modulo  $\sigma_{x_j}$  for every  $j$  in any solution of  $T_3(\Theta, \Omega)$ .

Assume that  $T_3(\Theta, \Omega)$  has a solution in  $D^{(1)}$  with

$$(x_1, \dots, x_s, x'_1, \dots, x'_s) = (b_1, \dots, b_s, b'_1, \dots, b'_s).$$

Since  $\sigma_{x_j} \subseteq \sigma$  for every  $\sigma \in \text{Con}(\Theta \setminus \Omega)$ , we have  $(b_i, b'_i) \in \sigma$ . Therefore, we can assign

$$(x_1, \dots, x_s, x'_1, \dots, x'_s) = (b_1, \dots, b_s, b_1, \dots, b_s).$$

to get a solution of  $\Theta^{(1)}$  (the remaining variables take on the same values). This contradiction completes this case.

Case 2:  $\text{LinkedCon}(\Omega, z) \neq \text{Con}(C_z, z)$  for some variable  $z$  and some constraint  $C_z \in \Omega$

Assume that  $n = 0$  and  $\text{LinkedCon}(\Omega, x_j) \subseteq \sigma$  for every  $j$  and every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ . We rename the variable  $z$  in  $C_z$  by  $z'$  and add the constraint  $\sigma_L(z, z')$ , where  $\sigma_L = \text{LinkedCon}(\Omega, z)$ . Since  $\Theta$  is crucial in  $D^{(1)}$ , the new instance has a solution  $\beta$  in  $D^{(1)}$ . Let  $z$  be equal to  $c$  in  $\beta$ . Since the solution set of  $\Omega^{(1)}$  is subdirect, there exists a solution  $\gamma$  of  $\Omega^{(1)}$  with  $z = c$ . Note that the corresponding elements of  $\beta$  and  $\gamma$  are linked in  $\Omega$ . Since  $\text{LinkedCon}(\Omega, x_j) \subseteq \sigma$  for every  $j$  and every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ , we can build a solution of  $\Theta^{(1)}$  with the values for  $x_j$  from  $\gamma$  and the values for the remaining variables from  $\beta$ , which gives us a contradiction.

Thus, we assume that  $n > 0$  or  $\text{LinkedCon}(\Omega, x_h) \not\subseteq \sigma$  for some  $h$  and  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_h)$ . In this case we consider a different transformation defined as follows:

1. Choose new variables  $x'_1, \dots, x'_s$  and  $x''_1, \dots, x''_s$ .
2. Add a copy of  $\Omega$  to  $\Theta$  with all the variables  $x_1, \dots, x_s$  replaced by  $x'_1, \dots, x'_s$ . We denote the copy by  $\Omega'$ .
3. Rename  $x_1, \dots, x_s$  in  $\Theta \setminus \Omega$  by  $x''_1, \dots, x''_s$ .
4. For every  $i$  and every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$  add a new variable  $y$  and add the constraints  $\sigma(x'_i, y)$  and  $\sigma(x''_i, y)$ .
5. For every  $i$  and every  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$  add the constraint  $\sigma^*(x_i, x''_i)$ .
6. For every  $j$  and  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$  such that  $\text{LinkedCon}(\Omega, x_j) \not\subseteq \sigma$  add the constraint  $\delta_j(x_j, x'_j)$ , where  $\{\delta_j\} = \text{Opt}(\text{Con}(\Omega, x_j))$ .

Since here we just copied  $\Omega$ , any solution of the obtained instance would give a solution to  $\Theta$  (we use values of  $x'_1, \dots, x'_s, z_1, \dots, z_n$  to generate a solution), hence the obtained instance has no solutions in  $D^{(1)}$ . We replace constraints from  $\Omega'$  containing at least one of the variables  $x'_1, \dots, x'_s$  by their covers step by step. Thus, in one step we replace just one constraint from  $\Omega'$ . We consider two cases.

Assume that after all replacements we get an instance  $\Theta_0$  without solutions in  $D^{(1)}$ . Any solution of  $T_3(\Theta, \Omega)$  gives a solution of  $\Theta_0$ : if  $x'_i = a_i$  in the solution of  $T_3(\Theta, \Omega)$ , then we put

$x'_i = x''_i = y = a_i$  in  $\Theta_0$  for every  $i$  and the corresponding  $y$ 's (the remaining variables take the same values). Therefore,  $T_3(\Theta, \Omega)$  has no solutions in  $D^{(1)}$ , which completes this case.

Assume that after some replacement the instance gets a solution in  $D^{(1)}$ . Suppose the instance before this replacement is  $\Theta'$  and the corresponding constraint to be replaced is  $C$ . Choose a variable  $x'_l \in \text{Var}(C)$ .

Let  $\delta = \text{Con}(C, x'_l)$ ,  $\rho$  be an optimal bridge from  $\delta$  to  $\delta$ . Let us define a new bridge by

$$\rho'(u_1, u_2, u_3, u_4) = \exists v_1 \exists v_2 \rho(u_1, u_2, v_1, v_2) \wedge \rho(u_3, u_4, v_1, v_2) \wedge \rho(u_1, u_1, u_3, u_3) \wedge \delta^*(u_3, u_4).$$

Since  $\rho'(x, x, y, y) = \rho(x, x, y, y)$ ,  $\rho'$  is also an optimal bridge. Additionally,  $\rho'$  has the following property: if  $(a, b, c, d) \in \rho'$  then  $(a, c) \in \tilde{\rho}$  and  $(c, d) \in \delta^*$ .

Then we change  $\Theta'$  in the following way. We add three new variables  $u_1, u_2, x'''_l$ , replace  $x'_l$  in  $C$  by  $x'''_l$ , add the constraint  $\rho'(x'_l, x'''_l, u_1, u_2)$  and the constraint  $\delta(u_1, u_2)$ . We denote the new instance by  $\Theta''$ . By the definition of a bridge,  $\Theta''$  has no solutions in  $D^{(1)}$ .

By  $\Upsilon$  we denote all constraints of  $\Theta''$  containing  $x'_j$  for some  $j$  or  $x'''_l$ . Let  $\{y_1, \dots, y_t\}$  be the set of all variables of  $\Upsilon$  except for  $z_1, \dots, z_n, x_1, \dots, x_s, x'_1, \dots, x'_s, u_1, u_2$ , and  $x'''_l$ . Suppose that the variable  $x_{i_j}$  is the corresponding variable and  $\sigma_j$  is the corresponding congruence for  $y_j$  (see step (4) of the transformation).

If we remove the constraint  $\delta(u_1, u_2)$  from  $\Theta''$ , then it is equivalent to making a constraint  $C$  of  $\Theta'$  weaker, which means that we get a solution of  $\Theta''$  in  $D^{(1)}$  after the removal. Let

$$(x_1, \dots, x_s, x'_1, \dots, x'_s, x''_1, \dots, x''_s, y_1, \dots, y_t, z_1, \dots, z_n, u_1, u_2) = (a_1, \dots, a_s, a'_1, \dots, a'_s, a''_1, \dots, a''_s, d_1, \dots, d_t, b_1, \dots, b_n, c_1, c_2)$$

in this solution.

First, we want to show that  $(a_l, a_l, c_1, c_1) \in \rho'$ . By the definition of  $\rho'$  and  $\Theta''$ , we have  $(a'_l, c_1) \in \tilde{\rho}$ . We consider two subcases. Case 2A. Suppose  $n > 0$ . Gluing a path from  $x_l$  to  $z_1$  in  $\Omega$  and a path from  $z_1$  to  $x'_1$  in  $\Omega'$ , we show that  $a_l$  and  $a'_l$  are linked in  $\Omega'$ . We apply Theorem 8.22 to get a bridge from  $\delta$  to  $\delta$  containing  $(a_l, a_l, a'_l, a'_l)$ . Then we compose this bridge with the bridge  $\rho$  to obtain a bridge from  $\delta$  to  $\delta$  containing  $(a_l, a_l, c_1, c_1)$ . Since the bridge  $\rho'$  is optimal, we have  $(a_l, a_l, c_1, c_1) \in \rho'$ .

Case 2B.  $\text{LinkedCon}(\Omega, x_h) \not\subseteq \sigma$  for some  $h$  and  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_h)$ . Let  $\zeta \in \text{Con}(\Omega, x_h)$  and  $\xi_0$  be an optimal bridge from  $\zeta$  to  $\zeta$ . Note that by step (6) of the new transformation  $(a_h, a'_h) \in \zeta$ . We know that  $a_l$  and  $a_h$  are linked in  $\Omega$ ,  $a'_h$  and  $a'_l$  are linked in  $\Omega'$ . We apply Theorem 8.22 to get a bridge  $\xi_1$  from  $\delta$  to  $\zeta$  containing  $(a_l, a_l, a_h, a_h)$  and a bridge  $\xi_2$  from  $\zeta$  to  $\delta$  containing  $(a'_h, a'_h, a'_l, a'_l)$ . Then we compose  $\xi_1, \xi_0, \xi_2$  and  $\rho$  (in this order) to obtain a bridge from  $\delta$  to  $\delta$  containing  $(a_l, a_l, c_1, c_1)$ . Since the bridge  $\rho'$  is optimal, we have  $(a_l, a_l, c_1, c_1) \in \rho'$ .

Consider a subconstraint  $\Upsilon(y_1, \dots, y_t, x_1, \dots, x_s, z_1, \dots, z_n, u_1, u_2)$ . The constraint  $\delta(u_1, u_2)$  is isolated in  $\Theta'' \setminus \Upsilon$ , hence  $\Theta'' \setminus \Upsilon$  has a solution in  $D^{(1)}$ . Using Theorem 9.5, we find  $\Upsilon_1, \dots, \Upsilon_v \in \text{Coverings}(\Upsilon)$  such that  $\Upsilon_i^{(1)}(y_1, \dots, y_t, x_1, \dots, x_s, z_1, \dots, z_n, u_1, u_2)$  defines a key relation  $\rho_i$  with the parallelogram property for every  $i$ . Since  $(\Theta'' \setminus \Upsilon) \cup \Upsilon_1 \cup \dots \cup \Upsilon_v$  has no solutions in  $D^{(1)}$ , we can choose  $k$  such that  $\rho_k$  omits the tuple  $(d_1, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1)$ . By the definition of  $\Upsilon$ , we can substitute the value  $c_1$  instead of  $u_1$  and  $u_2$  and put  $x_i = x'_i = x''_i = a_i$  and  $y_j = x_{i_j}$  for every  $i$  and  $j$  to get a solution of  $\Upsilon$ . Precisely, for every  $j$  we put  $d'_j = a_{i_j}$ , then  $(d'_1, \dots, d'_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1) \in \rho_k$  (here we used that  $(a_l, a_l, c_1, c_1) \in \rho'$ ). Also we know that

$$(d_1, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1) \notin \rho_k,$$

$$(d_1, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_2) \in \rho_k.$$

Then we consider the minimal  $j$  such that  $(d'_1, \dots, d'_j, d_{j+1}, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1) \in \rho_k$ . It follows from the definition of  $\Theta'$  that  $(d_j, d'_j) \in \sigma_j^*$ , and from the definition of  $\rho'$  that  $(c_1, c_2) \in \delta^*$ . Then by Lemma 8.23 there exists a bridge  $\zeta_1$  from  $\delta$  to  $\sigma_j$  such that  $\tilde{\zeta}_1$  contains  $\Upsilon(u_2, y_j)$ , and therefore it contains  $\Omega(x_l, x_{i_j})$ .

Suppose  $\delta_0 \in \text{Con}(\Omega, x_{i_j})$ . Applying Theorem 8.22 to  $\Omega$  and the variables  $x_{i_j}$  and  $x_l$ , we get a bridge  $\zeta_2$  from  $\delta_0$  to  $\delta$  such that  $\tilde{\zeta}_2$  contains all elements linked in  $\Omega$ , and therefore it contains  $\Omega(x_{i_j}, x_l)$ . Composing the bridges  $\zeta_2$  and  $\zeta_1$  we get a bridge from  $\delta_0$  to  $\sigma_j$ . Since  $\Omega(x_l, x_{i_j})$  is subdirect, the obtained bridge is reflexive. Hence  $\delta_0$  and  $\sigma_j$  are adjacent, which contradicts the fact that  $\sigma_j \in \text{Con}(\Theta \setminus \Omega, x_{i_j})$ .  $\square$

Below we prove a property of the transformation  $T_3$  similar to the property of  $T_2$  proved in Lemma 9.12.

**Lemma 9.14.** *Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Theta$  is crucial in  $D^{(1)}$ ,  $\Omega$  is a connected component, the solution set of  $\Omega$  is subdirect,  $\Omega$  has a solution in  $D^{(1)}$ , for every  $x \in \text{Var}(\Omega)$  any two congruences that are minimal among  $\text{Con}(\Theta, x)$  are adjacent, and  $\Theta' = T_1(T_3(\Theta, \Omega))$ . Then*

1.  $\xi(\Theta', y') < \xi(\Theta, y)$ , if  $y$  is a parent of  $y'$  and  $y' \neq y$ ;
2.  $\xi(\Theta', y) \leq \xi(\Theta, y)$ , if  $y \in \text{Var}(\Theta) \cap \text{Var}(\Theta')$ ;
3.  $\xi(\Theta', y) < \xi(\Theta, y)$ , if  $y \in \text{MinVar}(\Omega, \Theta)$  and  $y$  not stable in  $\Theta$ ;
4. if  $y$  is stable in  $\Theta$ ,  $y' \in \text{Var}(\Theta') \setminus \text{Var}(\Theta)$ , then  $y$  cannot be a friend of both  $y'$  and the parent of  $y'$  in  $\Theta'$ ;
5.  $\Theta$  and  $\Theta'$  have a common variable.

*Proof.* By Lemma 9.13  $\Theta'$  is crucial, then by Lemma 8.24 for every constraint  $C$  in  $\Theta$  there exists a constraint  $C'$  in  $\Theta'$  whose image in  $\Theta$  is  $C$ . Therefore, when we apply  $T_1$  we weaken only binary constraints and covers we added in  $T_3$  but not the constraints from  $\Theta$ .

First, let us show that  $\xi(\Theta, z) = \xi(\Theta', z)$  for every  $z \in \text{Var}(\Theta) \setminus \{x_1, \dots, x_s\}$ . The only constraints with  $z$  we added are the constraints we obtained from the covers of constraints from  $\Omega$  using  $T_1$ . Let  $C''$  be the cover of a constraint  $C'$  from  $\Omega$ . Let  $C'''$  be a constraint obtained from  $C''$  using  $T_1$ . By Lemma 8.10,  $\text{Con}(C'', z) \supsetneq \text{Con}(C', z)$ . By the definition of  $T_1$ ,  $\text{Con}(C''', z) \supseteq \text{Con}(C'', z)$ . Hence,  $\text{Con}(C''', z) \supsetneq \text{Con}(C', z)$ . Since  $\text{Con}(C', z)$  is not adjacent with a minimal congruence among  $\text{Con}(\Theta, z)$ ,  $\text{Con}(C''', z)$  cannot affect the characteristic of  $z$ . Therefore,  $\xi(\Theta, z) = \xi(\Theta', z)$ .

Second, let us show that  $\xi(\Theta', x_i) \leq \xi(\Theta, x_i)$  for every  $i$ . Since any two congruences that are minimal among  $\text{Con}(\Theta, x_i)$  are adjacent and the congruence we add in (5) cannot be a new minimal congruence among  $\text{Con}(\Theta, x_i)$ , the first components of  $\xi(\Theta', x_i)$  and  $\xi(\Theta, x_i)$  are equal. If the variable  $x_i$  is stable then we do not add anything in (4) and (5), hence the second components of  $\xi(\Theta', x_i)$  and  $\xi(\Theta, x_i)$  are empty, which completes Claim (2) for this case. Assume that  $x_i$  is not stable. Then the second component of  $\xi(\Theta', x_i)$  has congruences appeared in (4) and (5) instead of congruences from  $\text{Con}(\Theta \setminus \Omega, x_i)$ . The congruences we added in (4) are bigger than the corresponding congruences from  $\text{Con}(\Theta \setminus \Omega, x_i)$ . Hence if we added nothing in (5) then  $\xi(\Theta', x_i) < \xi(\Theta, x_i)$  because of the second components. Otherwise, consider a minimal congruence  $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$  such that  $\text{LinkedCon}(\Omega, x_i) \not\subseteq \sigma$ . By Corollary 8.22.1, congruences we obtain using (5) are greater than  $\text{LinkedCon}(\Omega, x_i)$ , hence they cannot be smaller than  $\sigma$ . Therefore,  $\sigma$  belongs to the second component of  $\xi(\Theta, x_i)$  and the second component of  $\xi(\Theta', x_i)$  does not have a congruence that is equal to or smaller than

$\sigma$ . We conclude that either second components of  $\xi(\Theta, x_i)$  and  $\xi(\Theta', x_i)$  are incomparable, or the second component of  $\xi(\Theta', x_i)$  is smaller. Note that all the congruences we obtain in (5) are from  $\uparrow \text{Opt}(\text{Con}(\Omega, x_i))$ , which means that  $\xi(\Theta', x_i) < \xi(\Theta, x_i)$  in this case. Thus we proved Claims (2) and (3).

To prove Claim (1) we need to show that  $\xi(\Theta', x'_i) < \xi(\Theta, x_i)$  for every  $i$ . Every congruence from  $\text{Con}(\Theta \setminus \Omega, x_i)$  is bigger than some congruence from  $\text{Con}(\Omega, x_i)$ . Hence,  $\xi(\Theta', x'_i) < \xi(\Theta, x_i)$  because of the first components.

To prove Claim (4) consider two cases. Case 1. Suppose  $x_i$  is a stable variable. Since  $x_i$  cannot be a friend of  $x'_j$ , we obtain the necessary condition. Case 2. Suppose  $z \in \text{Var}(\Theta) \setminus \{x_1, \dots, x_s\}$  is a stable variable. By the definition of being stable we conclude that  $z \notin \text{Var}(\Omega)$ . Hence  $z$  cannot be a friend of  $x_j$ , which proves Claim (4).

The Claim (5) follows from the fact that  $x_1$  should be in both  $\Theta$  and  $\Theta'$ .  $\square$

**Theorem 9.6.** *Suppose  $D^{(1)}$  is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Theta$  is crucial in  $D^{(1)}$  and not connected. Then there exists an instance  $\Theta' \in \text{ExpCov}(\Theta)$  that is crucial in  $D^{(1)}$  and contains a linked connected component whose solution set is not subdirect.*

*Proof.* We build a sequence of instances  $\Theta_1, \Theta_2, \Theta_3, \dots$  such that  $\Theta_{i+1} \in \text{ExpCov}(\Theta_i)$ , and every  $\Theta_i$  is crucial in  $D^{(1)}$ . Recall that by the inductive assumption for Theorem 9.8 all constraint relations of each  $\Theta_i$  are critical relations with the parallelogram property. We start with  $\Theta_1 = \Theta$ . We want the final element of this sequence to contain a linked connected component whose solution set is not subdirect. Suppose we already defined  $\Theta_i$ .

If there exist congruences  $\sigma_1, \sigma_2 \in \text{Con}(\Theta_i, x)$  for some variable  $x$  that are not adjacent and minimal among  $\text{Con}(\Theta, x)$ , then put  $\Theta_{i+1} := T_1(T_2(\Theta_i, \sigma_1, \sigma_2, x))$ . By Lemma 9.11,  $\Theta_{i+1}$  is crucial in  $D^{(1)}$ .

Otherwise, we know that any two minimal congruences in  $\text{Con}(\Theta, x)$  for every variable  $x$  are adjacent. By Lemma 8.25,  $\Theta_i$  is not connected. Since  $\Theta_i$  is crucial, it is also not fragmented. Then there exist a variable  $x$  that is not stable. Choose “an oldest” nonstable variable in  $\Theta_i$ , that is a variable  $x$  with the minimal number  $j$  such that  $x \in \text{Var}(\Theta_j)$ . Choose the connected component  $\Omega$  containing a minimal congruence of  $\text{Con}(\Theta, x)$ . Put  $\Theta_{i+1} := T_1(T_3(\Theta_i, \Omega))$ .

If  $\Omega$  is not linked, then irreducibility of  $\Theta$  implies that the solution set of  $\Omega$  is subdirect. If  $\Omega$  is linked and the solution set of  $\Omega$  is not subdirect, then the theorem is proved and we stop the process. Thus, we assume that the solution set of  $\Omega$  is subdirect. Since  $\Theta_i$  is crucial in  $D^{(1)}$  and not connected,  $\Omega^{(1)}$  has a solution. Then by Lemma 9.13,  $\Theta_{i+1}$  is crucial in  $D^{(1)}$ .

Thus, the next element of the sequence is defined either by  $T_1(T_2(\Theta_i, \sigma_1, \sigma_2, x))$ , or by  $T_1(T_3(\Theta_i, \Omega))$ . Now, we want to prove that the sequence  $\Theta_1, \Theta_2, \Theta_3, \dots$  cannot be infinite, which means that the last element with the required property exists. To prove this we are going to use Theorem 8.29. First, we extend a partial order  $\lesssim$  on characteristics to a linear order  $\leq$  such that  $(\Omega_1, \Omega_2) \lesssim (\Omega'_1, \Omega'_2)$  implies  $(\Omega_1, \Omega_2) \leq (\Omega'_1, \Omega'_2)$ . Second, we consider the set of all pairs  $(x, \xi(\Theta_i, x))$ , where  $x \in \text{Var}(\Theta_i)$ , as the set of organisms. Two organisms are friends if the corresponding variables were friends in  $\Theta_i$  for some  $i$  (if they’ve ever been friends). If  $x \in \text{Var}(\Theta_i) \cap \text{Var}(\Theta_{i+1})$  and  $\xi(\Theta_{i+1}, x) < \xi(\Theta_i, x)$ , then we say that  $(x, \xi(\Theta_i, x))$  is the parent of  $(x, \xi(\Theta_{i+1}, x))$ . Also, if  $x \in \text{Var}(\Theta_{i+1}) \setminus \text{Var}(\Theta_i)$ , and  $x'$  is a parent of  $x$  in  $\Theta_i$ , then  $(x', \xi(\Theta_i, x'))$  is the parent of  $(x, \xi(\Theta_{i+1}, x))$ . The characteristic  $\xi(\Theta_i, x)$  is considered as the strength of the organism  $(x, \xi(\Theta_i, x))$ . Then the set of organisms  $X_i$  is the set of all pairs  $(x, \xi(\Theta_j, x))$  for  $j \leq i$ .

Let us check all the assumptions we have in Theorem 8.29. Condition (1) follows from Lemma 9.12 (claims 1,2) and Lemma 9.14 (claims 1,2). Conditions (2) and (3) follow from the fact that each  $\Theta_{i+1}$  is from  $\text{ExpCov}(\Theta_i)$ .

Since the transformation  $T_2$  (followed by  $T_1$ ) replace a variable by two variables with smaller characteristic and does not change the characteristic of other variables, for the sequence to be infinite, we need to apply the transformation  $T_3$  infinitely many times. By Lemma 9.14 (claim 3) we always reduce the characteristic of the chosen nonstable variable when we apply  $T_3$ , which means that every variable will be stable at some moment.

It remains to show that condition (4) holds. As we noticed above, every variable will be stable at some moment. It remains to show that a variable  $z$  stable in  $\Theta_i$  cannot get infinitely many friends (here we care only about the variables but not about the organisms). By Lemma 9.12 (claim 3) and Lemma 9.14 (claim 4), the variable  $z$  cannot be a friend of some variable  $y$  appeared in  $\Theta_j$  for  $j > i$  and the parent of  $y$ . Thus, if we consider the set of friends of  $z$  in  $\Theta_j$  for  $j > i$ , then we see that going from  $\Theta_j$  to  $\Theta_{j+1}$  we can replace an old friend by new friends (that are weaker) but we cannot add a new friend keeping its parent. Therefore, after getting stable a variable cannot get infinitely many friends and condition (4) holds.

Since  $\Theta_i$  is crucial in  $D^{(1)}$ , it is not fragmented. By Lemma 9.12 (claim 4) and Lemma 9.14 (claim 5),  $\Theta_i$  and  $\Theta_{i+1}$  have at least one common variable for every  $i$ . Therefore, the set of all organisms cannot be divided into two disjoint sets with no friendship between them. Thus, condition (5) of Theorem 8.29 cannot hold, which proves that the process will stop at some  $\Theta_i$  having a linked connected component whose solution set is not subdirect.  $\square$

**Theorem 9.7.** *Suppose  $D^{(1)}$  is a 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance  $\Theta$ . If  $\Theta$  has a solution then it has a solution in  $D^{(1)}$ .*

*Proof.* Assume the contrary, that is,  $\Theta$  has a solution but  $\Theta^{(1)}$  has no solutions. By Theorem 9.3, there exists a minimal 1-consistent nonlinear reduction such that  $\Theta$  has no solutions in it.

First, we consider the set of all minimal 1-consistent nonlinear reductions of  $\Theta$ , which we denote by  $\mathfrak{R}$ . Then we consider an instance  $\Theta' \in \text{ExpCov}(\Theta)$  with the minimal positive number of reductions  $D^{(\Delta)} \in \mathfrak{R}$  such that  $\Theta'$  has no solutions in  $D^{(\Delta)}$ . Note that this transformation of  $\Theta$  to  $\Theta'$  can be omitted if  $D^{(1)}$  is not a PC reduction. Then we weaken the instance  $\Theta'$  (replace any constraint by all weaker constraints) while we still have a reduction  $D^{(\Delta)} \in \mathfrak{R}$  such that  $\Theta'$  has no solutions in  $D^{(\Delta)}$ . After that we remove all dummy variables from constraints and denote the obtained instance by  $\Theta''$ . Note that  $\Theta''$  is not fragmented (since it is crucial in some  $D^{(\Delta)}$ ),  $\Theta'' \in \text{ExpCov}(\Theta)$ , and for any reduction  $D^{(\Delta)} \in \mathfrak{R}$  the instance  $\Theta''$  is either crucial in  $D^{(\Delta)}$ , or has a solution in  $D^{(\Delta)}$ . The last property also holds for any expanded covering if  $\Theta''$  which is crucial in some reduction  $D^{(\Delta)}$ . Choose a reduction  $D^{(\Delta)}$  from  $\mathfrak{R}$  such that  $\Theta''$  is crucial in it.

Assume that  $\Theta''$  is not linked. If  $D^{(\Delta)}$  is a PC reduction, then we apply Theorem 9.4 to find a reduction  $D^{(1)}$  (it is a different reduction  $D^{(1)}$ ) and a strategy  $D^{(1)}, \dots, D^{(s)}$  for  $\Theta''^{(1)}$  such that the solution set of  $\Theta''^{(1)}$  is subdirect, the strategy has only nonlinear reductions,  $D_y^{(s)} \subseteq D_y^{(\Delta)}$  for every  $y$ . Then  $\Theta''^{(1)}$  is cycle-consistent and irreducible. By the inductive assumption  $\Theta''^{(2)}$  has a solution, then by Lemma 8.6  $\Theta''^{(2)}$  is cycle-consistent and irreducible, by the inductive assumption  $\Theta''^{(3)}$  has a solution, and so on. Thus we can prove that  $\Theta''^{(s)}$  has a solution, which means that  $\Theta''^{(\Delta)}$  has a solution and contradicts our assumption.

If  $D^{(\Delta)}$  is an absorbing or central reduction, then we choose a variable  $x$  of  $\Theta''$  and an element  $c \in D_x^{(\Delta)}$ , and for every variable  $y$  by  $D_y^{(\top)}$  we denote the set of all elements of  $D_y$  linked to  $c$ . Since  $\Theta''$  is irreducible, the solution set of  $\Theta''^{(\top)}$  is subdirect. Therefore,  $\Theta''^{(\top)}$  is irreducible and cycle-consistent. By Lemmas 7.1, 7.6 the reduction  $D^{(\perp)}$ , defined by  $D_y^{(\perp)} = D_y^{(\top)} \cap D_y^{(\Delta)}$  for every variable  $y$ , is an absorbing or central reduction for  $\Theta''^{(\top)}$ . Since  $D^{(\Delta)}$  is a 1-consistent reduction and  $D^{(\top)}$  is just a linked component, the reduction  $D^{(\perp)}$  is also 1-consistent. By the inductive assumption,  $\Theta''^{(\perp)}$  has a solution, which gives a contradiction.

Thus, we assume that  $\Theta''$  is linked. Recall that by the inductive assumption for Theorem 9.8, every constraint of  $\Theta''$  is critical and has the parallelogram property. If  $\Theta''$  is not connected, then by Theorem 9.6, there exists an instance  $\Upsilon \in \text{ExpCov}(\Theta'')$  that is crucial in  $D^{(\Delta)}$  and contains a linked connected subinstance  $\Omega$ . If  $\Theta''$  is connected, then  $\Theta''$  is a linked connected component itself and we put  $\Upsilon = \Omega = \Theta''$ . At the moment we have  $\Upsilon \in \text{ExpCov}(\Theta'')$  that is crucial in  $D^{(\Delta)}$  and a linked connected subinstance  $\Omega$ .

Let  $x_1$  be the first variable in a constraint  $C \in \Omega$ . By Lemma 8.11,  $\text{Con}(C, x_1)$  is irreducible. By Corollary 8.22.1, there exists a bridge  $\delta$  from  $\text{Con}(C, x_1)$  to  $\text{Con}(C, x_1)$  such that  $\delta(x, x, y, y)$  is a full relation. By Corollary 8.17.1, there exists a relation  $\zeta \subseteq D_{x_1} \times D_{x_1} \times \mathbb{Z}_p$  such that  $(y_1, y_2, 0) \in \zeta \Leftrightarrow (y_1, y_2) \in \text{Con}(C, x_1)$  and  $\text{pr}_{1,2}(\zeta) = \text{Con}(C, x_1)^*$ . Let us replace the variable  $x_1$  of  $C$  in  $\Upsilon$  by  $x'_1$  and add the constraint  $\zeta(x_1, x'_1, z)$ . The obtained instance we denote by  $\Upsilon'$ . Let  $\text{Var}(\Upsilon) = \{x_1, \dots, x_n\}$ ,  $\Upsilon'(x_1, \dots, x_n, z)$  define the relation  $S$ , which is the projection of the solution set of  $\Upsilon'$  onto all variables but  $x'_1$ . Let  $C = R(x_1, x_{i_1}, \dots, x_{i_s})$ ,  $R'(x_1, x_{i_1}, \dots, x_{i_s}) = \exists x'_1 R(x'_1, x_{i_1}, \dots, x_{i_s}) \wedge (x_1, x'_1) \in \text{Con}(C, x_1)^*$ . The projection of  $S$  onto the first  $n$  variables is the solution set of the instance  $\Upsilon$  whose constraint  $C$  is replaced by the weaker constraint  $R'(x_1, x_{i_1}, \dots, x_{i_s})$ . Since  $\Upsilon$  is crucial in  $D^{(\Delta)}$ , the solution set  $S$  contains a tuple whose first  $n$  elements are from  $D^{(\Delta)}$ . Moreover, the last element of all such tuples is not equal to 0, since otherwise this would imply that  $\Upsilon$  has a solution in  $D^{(\Delta)}$ .

By the assumption,  $\Theta$  has a solution, and therefore  $\Upsilon$  has a solution, which means that  $\Upsilon'$  has a solution with  $z = 0$  and, equivalently,  $S$  has a tuple whose last element is 0. Since  $\mathbb{Z}_p$  does not have proper subalgebras of size greater than 1, we have  $\text{pr}_{n+1}(S) = \mathbb{Z}_p$ .

Let us show for  $i \in \{1, 2, \dots, n\}$  that  $(\text{pr}_i(S))^{(\Delta)}$  is a one-of-four subuniverse of  $\text{pr}_i(S)$  of the same type as  $D^{(\Delta)}$ . For absorbing and central reductions it follows from Lemma 7.28. For the PC type we consider a PC congruence  $\sigma$  on  $D_{x_i}$ . By Theorems 9.2, 9.3, for every equivalence class  $U$  of  $\sigma$  there exists a minimal 1-consistent PC reduction  $D^{(\nabla)} \in \mathfrak{R}$  such that  $D_{x_i}^{(\nabla)} \subseteq U$ . As we assumed earlier, for any reduction from  $\mathfrak{R}$  the instance  $\Upsilon$  is either crucial in it, or has a solution in it. Therefore,  $\Upsilon'$  has a solution in any reduction from  $\mathfrak{R}$ , and  $\Upsilon'$  has a solution with  $x_i \in U$ . Hence,  $\sigma$  restricted to  $\text{pr}_i(S)$  is still a PC congruence. Moreover,  $(\text{pr}_i(S))^{(\Delta)}$  is an intersection of equivalence classes of the corresponding PC congruences on  $\text{pr}_i(S)$ . Thus, we showed that  $(\text{pr}_i(S))^{(\Delta)}$  is a one-of-four subuniverse of  $\text{pr}_i(S)$  of the same type as  $D^{(\Delta)}$ .

By Lemma 7.25,  $S^{(\Delta)}$  is a nonlinear one-of-four subuniverse of  $S$  (here we do not reduce the last variable). Also, by Lemma 7.25, the set of all tuples from  $S$  whose last element is 0 is a linear subuniverse of  $S$ , we denote this subuniverse by  $S_0$ . By Lemma 7.29, the intersection  $S^{(\Delta)} \cap S_0$  is not empty, which means that  $\Upsilon$  has a solution in  $D^{(\Delta)}$  and contradicts our assumptions.  $\square$

Note that Theorem 9.8 could be derived from Theorem 9.9, but we decided to keep the original proof of Theorem 9.8 because it demonstrates the idea for both theorems in an easier way.

**Theorem 9.8.** *Suppose  $D^{(0)}, \dots, D^{(s)}$  is a minimal strategy for a cycle-consistent irreducible CSP instance  $\Theta$ , and a constraint  $\rho(x_1, \dots, x_n)$  of  $\Theta$  is crucial in  $D^{(s)}$ . Then  $\rho$  is a critical relation with the parallelogram property.*

*Proof.* Since  $\rho(x_1, \dots, x_n)$  is crucial,  $\rho$  is a critical relation. Let  $\Theta'$  be obtained from  $\Theta$  by replacement of  $\rho(x_1, \dots, x_n)$  by all weaker constraints. Since  $\Theta$  is crucial in  $D^{(s)}$ ,  $\Theta'$  has a solution in  $D^{(s)}$ . By Lemma 6.1,  $\Theta'$  is cycle-consistent and irreducible.

Assume that  $|D_x^{(s)}| = 1$  for every variable  $x$ . Since the reduction  $D^{(s)}$  is 1-consistent, we get a solution, which contradicts the fact that  $\Theta$  has no solutions in  $D^{(s)}$ .

If we have a nontrivial binary absorbing subuniverse, or a nontrivial center, or a nontrivial PC subuniverse on some domain  $D_x^{(s)}$ , then by Theorems 9.2, 9.3, there exists a minimal non-linear 1-consistent reduction  $D^{(s+1)}$  for  $\Theta$ . As we explained before,  $\text{Size}(D^{(s+1)}) < \text{Size}(D^{(s)})$ .

Then, by Lemma 8.6,  $\Theta'^{(s)}$  is cycle-consistent and irreducible. By Theorem 9.7,  $\Theta'$  has a solution in  $D^{(s+1)}$ . Hence,  $\rho(x_1, \dots, x_n)$  is crucial in  $D^{(s+1)}$ . By the inductive assumption  $\rho$  has the parallelogram property.

It remains to consider the case when  $\text{ConLin}(D_x^{(s)})$  is proper for every  $x$  such that  $|D_x^{(s)}| > 1$ . Let  $\alpha$  be a solution of  $\Theta'$  in  $D^{(s)}$ . Let the projection of  $\alpha$  onto the variables  $x_1, \dots, x_n$  be  $(a_1, \dots, a_n)$ .

Assume that  $\rho$  does not have the parallelogram property. Without loss of generality we can assume that there exist  $c_1, \dots, c_n$  and  $d_1, \dots, d_n$  such that

$$\begin{aligned} (c_1, \dots, c_k, c_{k+1}, \dots, c_n) &\notin \rho, \\ (c_1, \dots, c_k, d_{k+1}, \dots, d_n) &\in \rho, \\ (d_1, \dots, d_k, c_{k+1}, \dots, c_n) &\in \rho, \\ (d_1, \dots, d_k, d_{k+1}, \dots, d_n) &\in \rho. \end{aligned}$$

Put

$$\begin{aligned} \rho'(x_1, \dots, x_n) &= \exists y_1 \dots \exists y_n \rho(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \wedge \\ &\rho(y_1, \dots, y_k, x_{k+1}, \dots, x_n) \wedge \rho(y_1, \dots, y_k, y_{k+1}, \dots, y_n). \end{aligned}$$

Obviously,  $\rho \subsetneq \rho'$  and  $\rho' \in \Gamma$ , therefore  $(a_1, \dots, a_n) \in \rho'$ . Hence, there exist  $b_1, \dots, b_n$  such that

$$\begin{aligned} (a_1, \dots, a_k, b_{k+1}, \dots, b_n) &\in \rho, \\ (b_1, \dots, b_k, a_{k+1}, \dots, a_n) &\in \rho, \\ (b_1, \dots, b_k, b_{k+1}, \dots, b_n) &\in \rho. \end{aligned}$$

By Lemma 8.28, there exists a tuple  $(e_1, \dots, e_n) \in \rho$  such that  $(a_i, e_i) \in \text{ConLin}(D_{x_i}^{(s)})$  for every  $i$ .

Consider the minimal linear reduction  $D^{(s+1)}$  of  $\Theta^{(s)}$  such that  $\alpha \in D^{(s+1)}$ . Then we have  $(e_1, \dots, e_n) \in \rho^{(s+1)}$ , and by Lemma 8.5,  $D^{(s+1)}$  is a 1-consistent reduction of  $\Theta^{(s)}$ . Since  $\Theta'$  has a solution in  $D^{(s+1)}$ ,  $\rho(x_1, \dots, x_n)$  is crucial in  $D^{(s+1)}$ . We get a longer minimal strategy with smaller  $\text{Size}(D^{(s+1)})$ , hence by the inductive assumption the relation  $\rho$  is a critical relation with the parallelogram property.  $\square$

**Theorem 9.9.** *Suppose  $D^{(0)}, \dots, D^{(s)}$  is a minimal strategy for a cycle-consistent irreducible CSP instance  $\Theta$ ,  $\Upsilon(x_1, \dots, x_n)$  is a subconstraint of  $\Theta$ , the solution set of  $\Upsilon^{(s)}$  is subdirect,  $k \in \{1, 2, \dots, n-1\}$ ,  $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, u_1, \dots, u_t\}$ ,*

$$\Omega = \Upsilon_{x_1, \dots, x_k, u_1, \dots, u_t}^{y_1, \dots, y_k, v_1, \dots, v_t} \wedge \Upsilon_{x_{k+1}, \dots, x_n, u_1, \dots, u_t}^{y_{k+1}, \dots, y_n, v_{t+1}, \dots, v_{2t}} \wedge \Upsilon_{x_1, \dots, x_n, u_1, \dots, u_t}^{y_1, \dots, y_n, v_{2t+1}, \dots, v_{3t}},$$

*and  $\Theta^{(s)}$  has no solutions. Then  $(\Theta \setminus \Upsilon) \cup \Omega$  has no solutions in  $D^{(s)}$ .*

*Proof.* Put  $\Theta' = (\Theta \setminus \Upsilon) \cup \Omega$ . Since  $\Omega$  is a covering of  $\Upsilon$ , Lemma 6.1 implies that  $\Theta \cup \Omega$  is cycle-consistent and irreducible. Assume that  $\Theta'$  has a solution in  $D^{(s)}$ .

We recursively build a strategy  $D^{(s)}, D^{(s+1)}, \dots, D^{(q)}$  for  $\Theta \cup \Omega = \Theta' \cup \Upsilon$  satisfying the following conditions:

1.  $D^{(s)}, D^{(s+1)}, \dots, D^{(q)}$  is a minimal strategy for  $\Theta'^{(s)}$ ;



2. if  $s \leq j < q$  and  $D^{(j+1)}$  is a linear reduction, then for each  $i \in \{1, 2, \dots, t\}$

$$D_{u_i}^{(j+1)} = \text{pr}_{n+i}(\rho' \cap (D_{x_1}^{(j+1)} \times \dots \times D_{x_n}^{(j+1)} \times D_{u_1}^{(j)} \times \dots \times D_{u_t}^{(j)})),$$

where  $\rho'$  is the relation defined by  $\Upsilon(x_1, \dots, x_n, u_1, \dots, u_t)$ ;

3. the solution set of  $\Upsilon^{(j)}$  is subdirect for  $s \leq j \leq q$ ;
4.  $\Theta'$  has a solution in  $D^{(q)}$ .

Note that here we allow  $D^{(j)}$  to be equal to  $D^{(j+1)}$  in a strategy, which can happen if  $D^{(j+1)}$  is a proper reduction for  $\Upsilon^{(j)}$  but not proper for  $\Theta^{(j)}$ .

We will prove that we can make this sequence longer while  $|D_{x_i}^{(q)}| > 1$  for some  $i$ . By Theorem 5.1, there exists a nontrivial one-of-four subuniverse on  $D_x^{(q)}$  if  $|D_x^{(q)}| > 1$ . We consider two cases:

Case 1. There exists a nontrivial binary absorbing subuniverse, or a nontrivial center, or a nontrivial PC congruence on some domain  $D_x^{(q)}$ . Then applying Theorems 9.2, 9.3 to the strategy  $D^{(0)}, D^{(1)}, \dots, D^{(q)}$  of  $\Theta \cup \Omega$ , we conclude that there exists a minimal 1-consistent nonlinear reduction  $D^{(q+1)}$  for  $(\Theta \cup \Omega)^{(q)}$ . By Lemma 8.6,  $\Theta^{(q)}$  and  $\Upsilon^{(q)}$  are cycle-consistent and irreducible. By Theorem 9.7,  $\Theta'$  has a solution in  $D^{(q+1)}$  and  $\Upsilon$  has a solution in  $D^{(q+1)}$ . By Lemma 8.5, the solution set of  $\Upsilon^{(q+1)}$  is subdirect. Thus, we made the sequence longer.

Case 2.  $\text{ConLin}(D_x^{(q)})$  is proper for every  $x$  such that  $|D_x^{(q)}| > 1$ . Let  $\alpha$  be a solution of  $\Theta'$  in  $D^{(q)}$ . We define the new linear reduction  $D^{(q+1)}$  as follows. For all variables but  $u_1, \dots, u_t$ , we choose an equivalence class of  $\text{ConLin}(D_x^{(q)})$  containing the corresponding element of the solution  $\alpha$ . For the variable  $u_i$  we define  $D_{u_i}^{(q+1)}$  by the formula in (2) from the above list for  $j = q$ . By Lemma 8.5,  $D^{(q+1)}$  is 1-consistent for  $\Theta'$ . Note that it does not follow from the definition that  $D_{u_i}^{(q+1)}$  is not empty and we will prove this later.

Let the projection of  $\alpha$  onto the variables  $x_1, \dots, x_n$  be  $(a_1, \dots, a_n)$ . Suppose  $\Upsilon^{(s)}(x_1, \dots, x_n)$  defines a relation  $\rho$ . Since  $\alpha$  is a solution of  $\Theta^{(s)}$ , there exist  $b_1, \dots, b_n$  such that

$$\begin{aligned} (a_1, \dots, a_k, b_{k+1}, \dots, b_n) &\in \rho, \\ (b_1, \dots, b_k, a_{k+1}, \dots, a_n) &\in \rho, \\ (b_1, \dots, b_k, b_{k+1}, \dots, b_n) &\in \rho. \end{aligned}$$

Since the solution set of  $\Upsilon^{(j)}$  is subdirect for  $s \leq j \leq q$ , we can apply Lemma 8.28 to  $\rho$  and the strategy  $D^{(s)}, \dots, D^{(q)}$ . Hence, there exists a tuple  $(d_1, \dots, d_n) \in \rho$  such that  $(a_i, d_i) \in \text{ConLin}(D_{x_i}^{(q)})$  for every  $i$ . Therefore,  $(\Upsilon^{(s)}(x_1, \dots, x_n))^{(q+1)}$  is not empty.

Let us show by induction on  $j = s, s+1, \dots, q$  that  $(\Upsilon^{(j)}(x_1, \dots, x_n))^{(q+1)}$  is not empty. For  $j = s$  we already know this. Assume that  $(\Upsilon^{(j)}(x_1, \dots, x_n))^{(q+1)}$  is not empty. If the reduction  $D^{(j+1)}$  is not linear then we apply Theorem 8.26 to  $(\Upsilon^{(j)}(x_1, \dots, x_n))^{(q+1)}$  and the strategy  $D^{(s)}, \dots, D^{(q)}$ , and obtain that  $(\Upsilon^{(j+1)}(x_1, \dots, x_n))^{(q+1)}$  is not empty. If the reduction  $D^{(j+1)}$  is linear then it follows from the definition of  $D_{u_i}^{(j+1)}$  that  $(\Upsilon^{(j+1)}(x_1, \dots, x_n))^{(q+1)}$  is not empty. Thus, we can prove that  $(\Upsilon^{(q)}(x_1, \dots, x_n))^{(q+1)}$  is not empty, and therefore  $D_{u_i}^{(q+1)}$  is not empty for every  $i$ . Considering the solution set of  $\Upsilon$  and applying Corollary 7.24.1, we derive that  $D_{u_i}^{(q+1)}$  is a linear subuniverse of  $D_{u_i}^{(q)}$ . Hence, the reduction  $D^{(q+1)}$  is a 1-consistent linear reduction for  $\Upsilon^{(q)}$ .

By Lemma 8.5,  $(\Upsilon^{(q)}(x_1, \dots, x_n))^{(q+1)}$  is subdirect. From the definition of  $D_{u_i}^{(q+1)}$  we derive that the projection of the solution set of  $\Upsilon^{(q+1)}$  onto  $u_i$  is  $D_{u_i}^{(q+1)}$  for every  $i$ , which means that the solution set of  $\Upsilon^{(q+1)}$  is subdirect. Hence, we get a longer strategy having all the necessary properties.

Thus, we showed that we can make the sequence longer until  $|D_{x_i}^{(q)}| = 1$  for every  $i$ . Assume that we reached this final state. Since both  $\Upsilon$  and  $\Theta'$  have a solution in  $D^{(q)}$  and  $x_1, \dots, x_n$  are their only common variables,  $\Theta$  has a solution in  $D^{(q)}$ , which contradicts the fact that  $\Theta$  has no solutions in  $D^{(s)}$ .  $\square$

### 9.3 Theorems from Section 5

In this subsection we assume that the variables of the instance  $\Theta$  are  $x_1, \dots, x_n$ , and the domain of  $x_i$  is  $D_i$  for every  $i$ . The first two theorems are proved together.

**Theorem 5.5.** *Suppose  $\Theta$  is a cycle-consistent irreducible CSP instance, and  $B$  is a nontrivial binary absorbing subuniverse or a nontrivial center of  $D_i$ . Then  $\Theta$  has a solution if and only if  $\Theta$  has a solution with  $x_i \in B$ .*

**Theorem 5.6.** *Suppose  $\Theta$  is a cycle-consistent irreducible CSP instance, there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on  $D_j$  for every  $j$ ,  $(D_i; w)/\sigma$  is a polynomially complete algebra, and  $E$  is an equivalence class of  $\sigma$ . Then  $\Theta$  has a solution if and only if  $\Theta$  has a solution with  $x_i \in E$ .*

*Proof.* By Theorems 9.2, 9.3, there exists a minimal 1-consistent nonlinear reduction  $D^{(1)}$  such that  $D_{x_i}^{(1)} \subseteq B$  for Theorem 5.5, and  $D_{x_i}^{(1)} \subseteq E$  for Theorem 5.6. By Theorem 9.7, there exists a solution in  $D^{(1)}$ .  $\square$

The next theorem will be used in the proof of Theorem 5.7 from Section 5.

**Theorem 9.15.** *Suppose the following conditions hold:*

1.  $\Theta$  is a linked cycle-consistent irreducible CSP instance;
2. there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on  $D_j$  for every  $j$ ;
3. if we replace every constraint of  $\Theta$  by all weaker constraints then the obtained instance has a solution with  $x_i = b$  for every  $i$  and  $b \in D_i$  (the obtained instance has a subdirect solution set);
4.  $D^{(1)}$  is a minimal linear reduction for  $\Theta$ ;
5.  $\Theta$  is crucial in  $D^{(1)}$ .

*Then there exists a constraint  $\rho(x_{i_1}, \dots, x_{i_s})$  in  $\Theta$  and a subuniverse  $\zeta$  of  $\mathbf{D}_{i_1} \times \dots \times \mathbf{D}_{i_s} \times \mathbb{Z}_p$  such that the projection of  $\zeta$  onto the first  $s$  coordinates is bigger than  $\rho$  but the projection of  $\zeta \cap (D_{i_1} \times \dots \times D_{i_s} \times \{0\})$  onto the first  $s$  coordinates is equal to  $\rho$ .*

*Proof.* We consider two cases. Case 1. Assume that  $\Theta$  contains just one constraint  $\rho(x_1, \dots, x_n)$ . By Corollary 7.24.1,  $D'_n = \text{pr}_n(\rho \cap (D_1^{(1)} \times \dots \times D_{n-1}^{(1)} \times D_n))$  is a linear subuniverse of  $D_n$ . By Lemma 7.20,  $D_n^{(1)}$  and  $D'_n$  can be viewed as products of affine subspaces and can be defined by linear equations. Since  $D_n^{(1)} \cap D'_n = \emptyset$  and  $D_n^{(1)}$  is a minimal reduction, we can take an equation defining  $D'_n$  that does not hold on  $D_n^{(1)}$  to get a maximal linear congruence  $\sigma$  on  $D_n$  such that  $D_n^{(1)}$  and  $D'_n$  are in different equivalence classes of  $\sigma$ . Note that  $D_n/\sigma \cong \mathbb{Z}_p$  for some  $p$ . Let  $\psi$  be the corresponding homomorphism from  $D_n$  to  $\mathbb{Z}_p$ . Put

$$\zeta(x_1, \dots, x_n, z) = \exists x'_n \rho(x_1, \dots, x_{n-1}, x'_n) \wedge (\psi(x_n) = \psi(x'_n) + z),$$

where the expression  $(\psi(x_n) = \psi(x'_n) + z)$  defines ternary subalgebra of  $D_n \times D_n \times \mathbb{Z}_p$ . Thus, we have  $\rho$  and  $\zeta$  with the required properties.

Case 2.  $\Theta$  contains more than one constraint. Then by condition (5), every constraint  $C^{(1)}$  is not empty, which by Lemma 8.5 implies that  $C^{(1)}$  is subdirect. Then  $D^{(1)}$  is a minimal 1-consistent linear reduction. By Theorem 9.8, every constraint in  $\Theta$  is critical and has the parallelogram property. If  $\Theta$  is not connected, then by Theorem 9.6 there exists an instance  $\Theta' \in \text{ExpCov}(\Theta)$  that is crucial in  $D^{(1)}$  and contains a linked connected component  $\Omega$  such that the solution set of  $\Omega$  is not subdirect. By condition (3), since the solution set of  $\Omega$  is not subdirect,  $\Omega$  should contain a constraint relation from the original instance  $\Theta$ . If  $\Theta$  is connected, then  $\Theta$  is a linked connected component itself and we put  $\Omega = \Theta$ . Thus, in both cases we have a linked connected instance  $\Omega$  having a constraint relation  $\rho$  from  $\Theta$ . Let  $\rho(x_{i_1}, \dots, x_{i_s})$  be a constraint of  $\Theta$ .

By Lemma 8.11,  $\text{Con}(\rho, 1)$  is an irreducible congruence. By Corollary 8.22.1, there exists a bridge  $\delta$  from  $\text{Con}(\rho, 1)$  to  $\text{Con}(\rho, 1)$  such that  $\tilde{\delta}$  is a full relation. By Corollary 8.17.1, there exists a relation  $\xi \subseteq D_{i_1} \times D_{i_1} \times \mathbb{Z}_p$  such that  $(x_1, x_2, 0) \in \xi \Leftrightarrow (x_1, x_2) \in \text{Con}(\rho, 1)$  and  $\text{pr}_{1,2}(\xi) = \text{Con}(\rho, 1)^*$ .

It remains to put  $\zeta(x_{i_1}, \dots, x_{i_s}, z) = \exists x'_{i_1} \rho(x'_{i_1}, x_{i_2}, \dots, x_{i_s}) \wedge \xi(x_{i_1}, x'_{i_1}, z)$ .  $\square$

**Theorem 5.7.** *Suppose the following conditions hold:*

1.  $\Theta$  is a linked cycle-consistent irreducible CSP instance with domain set  $(D_1, \dots, D_n)$ ;
2. there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on  $D_j$  for every  $j$ ;
3. if we replace every constraint of  $\Theta$  by all weaker constraints then the obtained instance has a solution with  $x_i = b$  for every  $i$  and  $b \in D_i$  (the obtained instance has a subdirect solution set);
4.  $L_i = D_i / \sigma_i$  for every  $i$ , where  $\sigma_i$  is the minimal linear congruence on  $D_i$ ;
5.  $\phi : \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k} \rightarrow L_1 \times \dots \times L_n$  is a homomorphism, where  $q_1, \dots, q_k$  are prime numbers;
6. if we replace any constraint of  $\Theta$  by all weaker constraints then for every  $(a_1, \dots, a_k) \in \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$  there exists a solution of the obtained instance in  $\phi(a_1, \dots, a_k)$ .

Then  $\{(a_1, \dots, a_k) \mid \Theta \text{ has a solution in } \phi(a_1, \dots, a_k)\}$  is either empty, or is full, or is an affine subspace of  $\mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$  of codimension 1 (the solution set of a single linear equation).

*Proof.* Put  $B = \{(a_1, \dots, a_k) \mid \Theta \text{ has a solution in } \phi(a_1, \dots, a_k)\}$ . If  $B$  is full then there is nothing to prove. Assume that  $B$  is not full, then consider  $(b_1, \dots, b_k) \notin B$ . It follows from condition (6) that  $\Theta$  is crucial in  $\phi(b_1, \dots, b_k)$ . Note that  $\phi(b_1, \dots, b_k)$  defines a minimal linear reduction for  $\Theta$ .

By Theorem 9.15 there exists a constraint  $\rho(x_{i_1}, \dots, x_{i_s})$  in  $\Theta$  and a subuniverse  $\zeta$  of  $\mathbf{D}_{i_1} \times \dots \times \mathbf{D}_{i_s} \times \mathbb{Z}_p$  such that the projection of  $\zeta$  onto the first  $s$  coordinates is bigger than  $\rho$  but the projection of  $\zeta \cap (D_{i_1} \times \dots \times D_{i_s} \times \{0\})$  onto the first  $s$  coordinates is equal to  $\rho$ .

Then we add a new variable  $z$  with domain  $\mathbb{Z}_p$  and replace  $\rho(x_{i_1}, \dots, x_{i_s})$  by  $\zeta(x_{i_1}, \dots, x_{i_s}, z)$ . We denote the obtained instance by  $\Upsilon$ . Let  $L$  be the set of all tuples  $(a_1, \dots, a_k, b) \in \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k} \times \mathbb{Z}_p$  such that  $\Upsilon$  has a solution with  $z = b$  in  $\phi(a_1, \dots, a_k)$ . We know that the projection of  $L$  onto the first  $k$  coordinates is a full relation and  $(b_1, \dots, b_k, 0) \notin L$ . Therefore  $L$  is defined by one linear equation. If this equation is  $z = b$  for some  $b \neq 0$ , then  $B$  is empty. Otherwise, we put  $z = 0$  in this equation and get an equation describing all  $(a_1, \dots, a_k)$  such that  $\Theta$  has a solution in  $\phi(a_1, \dots, a_k)$ .  $\square$

## 10 Conclusions

Even though the main problem has been resolved, there are many important questions that are still open. In this section we will discuss some consequences of this result, as well as some open questions and generalizations of the CSP.

### 10.1 A general algorithm for the CSP

The algorithm presented in the paper, as well as the algorithm of Andrei Bulatov [20, 21], uses detailed knowledge of the algebra and depends exponentially on the size of the domain. Is there a “truly polynomial algorithm”? By CSP-WNU we denote the following decision problem: given a formula

$$\rho_1(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \rho_s(v_{s,1}, \dots, v_{1,n_s}),$$

where all relations  $\rho_1, \dots, \rho_s$  are preserved by a WNU (we just know it exists); decide whether this formula is satisfiable.

**Problem 1.** *Does there exist a polynomial algorithm for CSP-WNU?*

If the domain is fixed then CSP-WNU can be solved by the algorithm presented in this paper. In fact, we know from [2, Theorem 4.2] that from a WNU on a domain of size  $k$  we can always derive a WNU (and also a cyclic operation) of any prime arity greater than  $k$ . Thus, we can find finitely many WNU operations on domain of size  $k$  such that any constraint language preserved by a WNU is preserved by one of them. It remains to apply the algorithm for each WNU and return a solution if one of them gave a solution.

### 10.2 A simplification of the algorithm.

We believe that the algorithm presented in the paper can be simplified. For instance, we strongly believe that the function WEAKENEVERYCONSTRAINT can be removed from the main function SOLVE without any consequences.

**Problem 2.** *Would the algorithm still work if the function WEAKENEVERYCONSTRAINT was removed from the function SOLVE?*

This would reduce the complexity of the algorithm significantly (the depth of the recursion would be  $|A|$  instead of  $|A| + |\Gamma|$ , see Lemma 5.2).

### 10.3 A generalization for the nonWNU case.

Another important question is whether some results and ideas introduced in this paper can be applied for constraint languages not preserved by a WNU. For example, it is not clear what assumptions are sufficient to reduce safely a domain to a binary absorbing subuniverse.

**Problem 3.** *What are the weakest assumptions for Theorems 5.5 and 5.6 to hold.*

### 10.4 Infinite domain CSP

If we allow the domain to be infinite, the situation is changing significantly. As it was shown in [10] every computational problem is equivalent (under polynomial-time Turing reductions) to a problem of the form CSP( $\Gamma$ ). In [12] the authors gave a nice example of a constraint

language  $\Gamma$  such that  $\text{CSP}(\Gamma)$  is undecidable. Let  $\Gamma$  consists of three relations (predicates)  $x + y = z$ ,  $x \cdot y = z$  and  $x = 1$  over the set of all integers  $\mathbb{Z}$ . Then the Hilbert's 10-th problem can be expressed as  $\text{CSP}(\Gamma)$ , which proves undecidability of  $\text{CSP}(\Gamma)$ .

A reasonable assumption on  $\Gamma$  which sends the CSP back to the class NP is that  $\Gamma$  is a reduct of a finitely bounded homogeneous structure. A nice result for such constraint languages is the full complexity classification of the CSPs over the reducts of  $(\mathbb{Q}; <)$  [11]. This additional assumption allows to formulate a statement of the algebraic dichotomy conjecture for the complexity of the infinite domain CSP [7]. For more information about the infinite domain CSP and the algebraic approach see [9, 12]. For a method of reducing an infinite domain CSP to CSPs over finite domains see [51].

## 10.5 Valued CSP

A natural generalization of the Constraint Satisfaction Problem is the *Valued Constraint Satisfaction Problem* (VCSP), where constraint relations are replaced by mappings to the set of rational numbers, and conjunctions are replaced by sum [55]. For a finite set  $A$  and a set  $\Gamma$  of mappings  $A \rightarrow \mathbb{Q} \cup \{\infty\}$  by  $\text{VCSP}(\Gamma)$  we denote the following problem: given a formula

$$f(x_1, \dots, x_n) = f_1(v_{1,1}, \dots, v_{1,n_1}) + \dots + f_s(v_{s,1}, \dots, v_{s,n_s}),$$

where all the mappings  $f_1, \dots, f_s$  are from  $\Gamma$  and  $v_{i,j} \in \{x_1, \dots, x_n\}$  for every  $i, j$ ; find an assignment  $(a_1, \dots, a_n)$  that minimizes  $f(x_1, \dots, x_n)$ .

In [42, Theorem 21], the authors proved that the dichotomy conjecture for CSP would imply the dichotomy conjecture for the Valued CSP, and described all sets of mappings  $\Gamma$  such that  $\text{VCSP}(\Gamma)$  is tractable (modulo the CSP Dichotomy Conjecture). Thus, the result obtained in this paper implies the characterization of the complexity of  $\text{VCSP}(\Gamma)$  for all  $\Gamma$ .

## 10.6 Quantified CSP

An equivalent definition of  $\text{CSP}(\Gamma)$  is to evaluate a sentence  $\exists x_1 \dots \exists x_n (\rho_1(\dots) \wedge \dots \wedge \rho_s(\dots))$ , where  $\rho_1, \dots, \rho_s$  are from the constraint language  $\Gamma$ . Then a natural generalization of CSP is the *Quantified Constraint Satisfaction Problem* (QCSP), where we allow to use both existential and universal quantifiers. For a constraint language  $\Gamma$ ,  $\text{QCSP}(\Gamma)$  is the problem to evaluate a sentence of the form  $\forall x_1 \exists y_1 \dots \forall x_n \exists y_n (\rho_1(\dots) \wedge \dots \wedge \rho_s(\dots))$ , where  $\rho_1, \dots, \rho_s$  are relations from the constraint language  $\Gamma$  (see [15, 25, 26, 49]).

It was conjectured by Hubie Chen [26, 24] that for any constraint language  $\Gamma$  the problem  $\text{QCSP}(\Gamma)$  is either solvable in polynomial time, or NP-complete, or PSpace-complete. Recently, this conjecture was disproved in [64], where the authors found constraint languages  $\Gamma$  such that  $\text{QCSP}(\Gamma)$  is coNP-complete (on 3-element domain), DP-complete (on 4-element domain),  $\Theta_2^P$ -complete (on 10-element domain). Also the authors classified the complexity of the Quantified Constraint Satisfaction Problem for constraint languages on 3-element domain containing all unary singleton relations (so called idempotent case), that is, they showed that for such languages  $\text{QCSP}(\Gamma)$  is either tractable, or NP-complete, or coNP-complete, or PSpace-complete. Nevertheless, for higher domain as well as for the nonidempotent case the complexity is not known.

**Problem 4.** *What can be the complexity of  $\text{QCSP}(\Gamma)$ ?*

Now it is hard to believe that there will be a simple answer to this question, that is why it is interesting to start with 3-element domain (nonidempotent case) and 4-element domain. Another natural question is how many complexity classes can be expressed by  $\text{QCSP}(\Gamma)$  up to polynomial equivalence. Probably more important problem is to describe all tractable cases.

**Problem 5.** Describe all constraint languages  $\Gamma$  such that  $\text{QCSP}(\Gamma)$  is tractable.

## 10.7 Promise CSP

Another natural generalization of the CSP is *the Promise Constraint Satisfaction Problem*, where a promise about the input is given (see [16, 23]). Let  $\Gamma = \{(\rho_1, \sigma_1), \dots, (\rho_t, \sigma_t)\}$ , where  $\rho_i$  and  $\sigma_i$  are relations of the same arity over the domains  $A$  and  $B$ , respectively. Then  $\text{PCSP}(\Gamma)$  is the following decision problem: given two formulas

$$\begin{aligned} &\rho_{i_1}(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \rho_{i_s}(v_{s,1}, \dots, v_{s,n_s}), \\ &\sigma_{i_1}(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \sigma_{i_s}(v_{s,1}, \dots, v_{s,n_s}), \end{aligned}$$

where  $(\rho_{i_j}, \sigma_{i_j})$  are from  $\Gamma$  for every  $i$  and  $v_{i,j} \in \{x_1, \dots, x_n\}$  for every  $i, j$ ; distinguish between the case when both of them are satisfiable, and when both of them are not satisfiable. Thus, we are given two CSP instances and a promise that if one has a solution then another has a solution. Usually it is also assumed that there exists a mapping (homomorphism)  $h: A \rightarrow B$  such that  $h(\rho_i) \subseteq \sigma_i$  for every  $i$ . In this case, the satisfiability of the first formula implies the satisfiability of the second one. To make sure that the promise can actually make an NP-hard problem tractable, see example 2.8 in [23].

The most popular example of the Promise CSP is graph  $(k, l)$ -colorability, where we need to distinguish between  $k$ -colorable graphs and not even  $l$ -colorable, where  $k \leq l$ . This problem can be written as follows.

**Problem 6.** Let  $|A| = k$ ,  $|B| = l$ ,  $\Gamma = \{(\neq_A, \neq_B)\}$ . What is the complexity of  $\text{PCSP}(\Gamma)$ ?

Recently, it was proved [23] that  $(k, l)$ -colorability is NP-hard for  $l = 2k - 1$  and  $k \geq 3$  but even the complexity of  $(3, 6)$ -colorability is still not known.

Even for two element domain the problem is widely open, but recently a dichotomy for symmetric Boolean PCSP was proved [30].

**Problem 7.** Let  $A = B = \{0, 1\}$ . Describe the complexity of  $\text{PCSP}(\Gamma)$  for all  $\Gamma$ .

## 10.8 Surjective CSP

Another modification of the CSP is *the Surjective Constraint Satisfaction Problem*. For a constraint language  $\Gamma$  over a domain  $A$ ,  $\text{SurjCSP}(\Gamma)$  is the following decision problem: given a formula

$$\rho_1(\dots) \wedge \dots \wedge \rho_s(\dots),$$

where all relations  $\rho_1, \dots, \rho_s$  are from  $\Gamma$ ; decide whether there exists a surjective solution, that is a solution with  $\{x_1, \dots, x_n\} = A$ . Only few results are known about the complexity of the Surjective CSP [27]. That is why, we suggest to start studying this question with a very concrete constraint language on a 3-element domain.

**Problem 8.** Suppose  $A = \{a, b, c\}$ ,  $R = \{(x, y, z) \mid \{x, y, z\} \neq A\}$ . What is the complexity of  $\text{SurjCSP}(\{R\})$ ?

After this problem (called *no-rainbow problem*) we can move to the general question.

**Problem 9.** Describe the complexity of  $\text{SurjCSP}(\Gamma)$  for all constraint languages  $\Gamma$ .

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