Metric Sublinear Algorithms via Linear Sampling

Hossein Esfandiari* Harvard University Cambridge, MA Michael Mitzenmacher[†] Harvard University Cambridge, MA

Abstract

In this work we provide a new technique to design fast approximation algorithms for graph problems where the points of the graph lie in a metric space. Specifically, we present a sampling approach for such metric graphs that, using a sublinear number of edge weight queries, provides a *linear sampling*, where each edge is (roughly speaking) sampled proportionally to its weight.

For several natural problems, such as densest subgraph and max cut among others, we show that by sparsifying the graph using this sampling process, we can run a suitable approximation algorithm on the sparsified graph and the result remains a good approximation for the original problem. Our results have several interesting implications, such as providing the first sublinear time approximation algorithm for densest subgraph in a metric space, and improving the running time of estimating the average distance.

1 Introduction

In this paper, we aim to design approximation algorithms for several natural graph problems, in the setting where the points in the graph lie in a metric space. Following the seminal work of [34], we aim to provide *sublinear* approximation algorithms; that is, on problems with n points and hence $\binom{n}{2}$ edge distances, we aim to provide randomized algorithms that require $o(n^2)$ time and in fact only consider $o(n^2)$ edges, by making use of sampling. Similar to the previous work, we assume we can query the weight of any single edge in O(1) time; when we use the term "query", we mean an edge weight query throughout.

A well known technique to design sublinear algorithms is uniform sampling; that is, a subset of edges (or vertices) is sampled uniformly at random. Several algorithms use uniform sampling to improve speed, space, or the number of queries [4, 5, 6, 7, 12, 13, 18, 26, 31, 46]. Uniform sampling is very easy to implement, but problematically it is oblivious to the edge weights. When it comes to maximization problems on graphs, a few high weight edges may have a large effect on the solution, and hence the uniform sampling technique may fail to provide a suitable solution because it fails to sample these edges. For example, consider the densest subgraph problem, where the density of a subgraph is the sum of the edges weights divided by the number of vertices. It is known that for general unweighted graphs, the densest subgraph of a uniformly sampled subgraph with $\tilde{O}(\frac{n}{\epsilon^2})$ edges is a $1 - \epsilon$ approximation of the densest subgraph of the original graph [31, 46, 47]. However, as we show in Appendix A this result is not true for weighted graphs, even in a metric space. This problem suggests we should design approaches that sample edges with probabilities proportional to (or otherwise related to) their weight in a metric space.

As our main result, we design a novel sampling approach using a sublinear number of queries for graphs in a metric space, where independently for each edge, the probability the edge is in the sample is proportional

^{*}Supported in part by NSF grants CCF-1320231 and CNS-1228598.

[†]Supported in part by NSF grants CCF-1563710, CCF-1535795, CCF-1320231, and CNS-1228598. Part of this work was done while visiting Microsoft Research New England.

to its weight; we call such a sampling a *linear sampling*. Specifically, for a fixed factor α , we can ensure for an edge e with weight w_e , if $\alpha w_e \leq 1$ then the edge appears in the sample with weight 1 with probability αw_e , and if $\alpha w_e > 1$, then the edge is in the sample with weight αw_e . Hence the edge weights are "downsampled" by a factor of α , in a natural way. We can choose an α to suitably sparsify our sample, graph, run an approximation algorithm on that sample, and use that result to obtain a corresponding, nearlyas-good approximation to the original problem. Interestingly, we only query $\tilde{O}(n + \beta)^1$ edge weights to provide the sample, where β is "almost" the expected weight of the edges in the sampled graph. (See Subsection 1.1 for a formal definition). Our algorithm to construct the sample also runs in $\tilde{O}(n + \beta)$ time.

Utilizing our sampling approach, we show that for several problems a ϕ -approximate solution on a linear sample with expected weight (roughly) $\beta \in o(n^2)$ is a $(\phi - \epsilon)$ -approximate solution on the input graph. From an information theory perspective this says that $\tilde{O}(n + \beta)$ queries are sufficient to find a $1 - \epsilon$ approximate solution for these problems. Moreover, as the sampled graph has a reduced number of edges, if an approximation algorithm on the sampled graph runs in linear time on the sampled edges, the total time is sublinear in the size of the original graph.

In what follows, after describing the related work and a summary of our results, we present our sampling method. Our approach decomposes the graph into a sequence of subgraphs, where the decomposition depends strongly on the fact that the graph lies in a metric space. Using this decomposition, and an estimate of the average edge weight in the graph, we can determine a suitable sampled graph. We then show this sampling approach allows us to find sublinear approximation algorithms for several problems, including densest subgraph and max cut, in the manner described above.

In some applications, such as diversity maximization, it can be beneficial to go slightly beyond metric distances [56]. We can extend our results to more general spaces that satisfy what is commonly referred to as a parametrized triangle inequality [8, 15, 20], in which for every three points a, b and c we have $w_{a,b} + w_{b,c} \ge \lambda w_{c,a}$ for a parameter λ . As an example, if the weight of each edge (u, v) is the squared distance between the two points, the graph satisfies a parametrized triangle inequality with $\lambda = 1/2$. We provide analysis for this more general setting throughout, and refer to a graph satisfying such a parametrized triangle inequality as a λ -metric graph. (Throughout, we take $\lambda \le 1$).

1.1 Our Results

As our main technical contribution we provide an approach to sample a graph $H_{\beta} = (V, E_H)$ from a λ -metric graph $G = (V, E_G)$ with the properties specified below that makes only $\tilde{O}(\frac{n+\beta}{\lambda})$ queries and succeeds with probability at least 1 - O(1/n). It is easy to observe that our algorithm runs in $\tilde{O}(\frac{n+\beta}{\lambda})$ time as well.

- For some fixed factor α (which is a function of β) independently for each edge e we have:
 - If $\alpha w_e \leq 1$, we have edge e with weight 1 in E_H with probability αw_e .
 - If $\alpha w_e > 1$, we have edge e with weight αw_e in E_H .
- We have $\beta \leq \mathbb{E}\left[\sum_{e \in E_H} w'_e\right] \leq 2\beta$, where w'_e is the weight of e in H_{β} .²

As the weight of each edge in E_H is at least 1, $\mathbb{E}\left[\sum_{e \in E_H} w_e\right] \le 2\beta$ implies that $\mathbb{E}\left[|E_H|\right] \le 2\beta$.

 $^{{}^{1}\}tilde{O}(\cdot)$ notation hides logarithmic factors.

²This can be extended to $\beta \leq E\left[\sum_{e \in E_H} w'_e\right] \leq (1 + \gamma)\beta$, for any arbitrary γ (See the footnote on Theorem 10 for details.) In our work, the upper bound only affects the number of queries; we prefer to set $\gamma = 1$ and simplify the argument.

We note that for three points a, b and c in a λ -metric space and any parameter p, $w_{a,b} + w_{b,c} \ge \lambda w_{c,a}$ directly implies $w_{a,b}^p + w_{b,c}^p \ge \frac{\lambda}{2^p} w_{c,a}^p$. Therefore one can use our technique to sample edges proportional to w_e^p (a.k.a. l_p sampling). In the streaming setting, l_p sampling has been extensively studied and appears to have several applications [49]; as far as we are aware, our approach provides the first l_p sampling techniques that uses a sublinear number of edge weight queries.

As previously mentioned, in Section 3 we consider several problems and show that for some $\beta \in o(n^2)$, any ϕ -approximate solution of the problem on H_β is an $(\phi - \epsilon)$ -approximate solution on the original graph with high probability. Specifically, we show that $\beta \in O(\frac{n \log n}{\epsilon^2})$ is sufficient to approximate densest subgraph and max cut, $\beta \in O(\frac{n^2 \log n}{\epsilon^2 k})$ is sufficient to approximate k-hypermatching, and $\beta \in O(\frac{\log n}{\epsilon^2})$ is sufficient to approximate the average distance. Notice that these results directly imply (potentially exponential time) $(1 - \epsilon)$ approximation algorithms with sublinear number of queries for each of the problems. Often our methodology can also yield sublinear time algorithms (since it uses a sublinear number of edges) with possibly worse approximation ratios.

We now briefly describe specific results for the various problems we consider, although we defer the formal problem definitions to Section 3. All of the algorithms discussed below work with high probability. We note that, throughout the paper, we use $\log n$ for $\log_e n$.

For average distance, we provide a $(1 - \epsilon)$ -approximation algorithm that simply finds the sum of the weights of the edges in H_{β} for $\beta \in O(\frac{\log n}{\epsilon^2})$, and hence our algorithm runs in time $\tilde{O}(\frac{n+\frac{1}{\epsilon^2}}{\lambda})$. For a metric graph, this improves the running time of the previous result of Indyk [34] that runs in $O(\frac{n}{\epsilon^{3.5}})$ time, with constant probability.

For densest subgraph, the greedy algorithm yields a 1/2-approximate solution in time quasilinear in the number of edges [21]. The expected number of edges of H_{β} can be bounded by $\tilde{O}(\frac{n}{\lambda\epsilon^2})$ for the densest subgraph on λ -metric graphs. Therefore, our result implies a $(1/2 - \epsilon)$ -approximation algorithm for densest subgraph in λ -metric spaces requiring $\tilde{O}(\frac{n}{\lambda\epsilon^2})$ time.

A sublinear time algorithm for a $(1 - \tilde{\epsilon})$ approximation for metric max cut is already known [35]. The previous result uses $\tilde{O}(\frac{n}{\epsilon^5})$ queries, while we use only $\tilde{O}(\frac{n}{\epsilon^2})$ queries. (We note that this result does not improve the running time, but remains interesting from an information theoretic point of view. Indeed, there are several interesting results on sublinear space algorithms that ignore the computational complexity e.g., max cut [16, 38, 37, 40], set cover [9, 33], vertex cover and hypermatching [24, 22].)

Finally, on the hardness side, in Section 4 we show that $\Omega(n)$ queries are necessary even if one just wants to approximate the size of the solution for densest subgraph, k-hypermatching, max cut, and average distance.

1.2 Other Related Work

Metric spaces are natural in their own right. For example, they represent geographic information, and hence graph problems such as the densest subgraph problem often have a natural interpretation in metric spaces. It also is often reasonable to manage large data sets by embedding objects within a suitable metric space. In networks, for example, the idea of finding network coordinates consistent with latency measurements to predict latency has been widely studied [25, 44, 52, 53, 54, 55].

There are several works on designing sublinear algorithms for different variants of clustering problems in metric spaces due to their application to machine learning [7, 11, 26, 27, 35]. We briefly summarize some of these papers. Alon et al. studies the efficiency of uniform sampling of vertices to check for given parameters k and b if the set of points can be clustered into k subsets each with diameter at most b, ignoring up to an ϵ fraction of the vertices [7]. Czumaj and Sohler studies the efficiency of uniform sampling of vertices for k-median, min-sum k-clustering, and balanced k-median [26]. Badoiu et al. consider the facility location

problem in metric space [11]. They compute the optimal cost of the minimum facility location problem, assuming uniform costs and demands, and assuing every point can open a facility. Moreover, they show that there is no $o(n^2)$ time algorithm that approximates the optimal solution of general case of metric facility location problem to within any factor.

A basic and natural difference between these previous works on clustering problems and the densest subgraph problem that we consider here is that all previous problems aim to decompose the graph into two or more subsets, where each subset consists of points that are close to each other. However, densest subgraph in a metric space aims to pick a diverse, spread out subset of points. (While perhaps counterintuitive, this is clear from the definition, which we provide shortly.) The application of metric densest subgraph in diversity maximization and feature selection is well studied [17, 56].

Sublinear algorithms may also refer to sublinear space algorithms such as streaming algorithms. A related, well-studied setting is semi-streaming [50], often used for graph problems. In the semi-streaming setting the input is a stream of edges and we take one (or a few) passes over the stream, while only using $\tilde{O}(n)$ space. Semi-steaming algorithms have been extensively studied [1, 2, 29, 32, 39, 43].

For the densest subgraph problem, there have been a number of recent papers showing the efficiency of uniform edge sampling in unweighted graphs [18, 31, 46, 47]. Initially, Bhattacharya et al. provided a 0.5 approximation semi-streaming algorithm for this problem [18]. They extended their approach to obtain a 0.25 approximation algorithm for this problem for dynamic streams with $\tilde{O}(1)$ update time and $\tilde{O}(n)$ space. McGregor et al. and Esfandiari et al. independently provide a $(1 - \epsilon)$ -approximation semi-streaming algorithm for this problem [31, 46]. Esfandiari et al. extend the analysis of uniform sampling of edges to several other problem. Mitzenmacher et al. study the efficiency of uniform edge sampling for densest subgraph in hypergraphs [47].

For the max cut problem, Kapralov, Khanna, and Sudan [37] and independently Kogan and Krauthgamer [40] showed that a streaming $(1 - \epsilon)$ -approximation algorithm to estimate the size of max cut requires $n^{1-O(\epsilon)}$ space. Later, Kapralov, Khanna and Sudan [38] show that for some small ϵ any streaming $(1 - \epsilon)$ -approximation algorithm to estimate the size of max cut requires $\Omega(n)$ space. Very recently, Bhaskara et al. [16] provide a 2-pass $(1 - \epsilon)$ -approximation streaming algorithm using $\tilde{O}(n^{1-\delta})$ space for graphs with average degree n^{δ} .

Finally, when considering matching algorithms, there are numerous works on maximum matching in streaming and semi-streaming setting [10, 14, 23, 24, 22, 42, 41, 30, 36]. Note that a maximal matching is a 0.5 approximation to the maximum matching, and it is easy to provide one in the semi-streaming setting. However, improving this approximation factor in one pass is yet open. There are several works that improve this approximation factor in a few passes [3, 14, 42, 45]. Maximum matching in hypergraphs has also been considered in the streaming setting [22].

While a 0.5-approximation for unweighted matching in the semi-streaming setting is trivial, such an approximation for weighted matching appears nontrivial. There is a sequence of works improving the approximation factor of weighted matching in the semi-streaming setting [19, 29, 51], and just recently Paz and Schwartzman provide a semi-streaming $(0.5 - \epsilon)$ -approximation algorithm.

There are, of course, many, many other related problems; see [28], for example, for a survey on sublinear algorithms.

2 Providing a Linear Sampling

In this section we provide a technique to construct the desired sampled graph $H_{\beta} = (V, E_H)$ from a metric graph $G = (V, E_G)$. We first provide a useful decomposition of the graph. We show this decomposition allows us to obtain a graph H^{α} that satisfies the first property of H_{β} , namely that edge weights are scaled down (in expectation, for edges with scaled weights less than 1) by a factor of α . We then show how to

determine a proper value α so that expected sum of the edge weight is between β and 2β as desired.

2.1 A Graph Decomposition

We start with a decomposition for a metric graph G, assuming an upper bound L on the weight of the edges. For an suitable number t determined later we define the following sequences.

- A sequence of graphs $G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_t$.
- A sequence of vertex sets ν_1, \ldots, ν_t .
- A sequence of weights, $L_1 = L, L_2 = \frac{L_1}{2}, \ldots, L_t = \frac{L_{t-1}}{2}$.

We denote the vertex set and edge set of G_i by V_i and E_i respectively. We begin with $G_1 = G$, and G_i is constructed from G_{i-1} by removing vertices in ν_{i-1} , i.e. $G_i = G_{i-1} \setminus \nu_{i-1}$. However, defining ν_i , which depends on G_i , requires the following additional definitions. For any $i \in \{1, \ldots, t\}$ and $\Lambda \in [0, 1]$, define $G_i^{\Lambda} = (V_i, E_i^{\Lambda})$ to be the graph obtained by removing all edges with weight less than ΛL_i from G_i , i.e., $e \in E_i^{\Lambda}$ if and only if $e \in E_i$ and $w_e \ge \Lambda L_i$.

We now define G_i and ν_i iteratively as follows. We define $\nu_{i,\xi}$ to be the set of vertices in $G_i^{\lambda/4}$ with degree at least $\xi|V_i|$. We let ν_i be an arbitrary subset such that $\nu_{i,1/2} \subseteq \nu_i \subseteq \nu_{i,1/4}$. As mentioned, $G_1 = G$ and $G_i = G_{i-1} \setminus \nu_{i-1}$. We define $E_{\nu_i} = E_i \setminus E_{i+1}$. Note that E_{ν_i} is the set of edges neighboring ν_i in G_i .

Lemma 1 For any $i \in \{1, ..., t\}$, the set of vertices in $G_i^{\lambda/4}$ with degree at least $\frac{|V_i|}{2}$ (i.e., $\nu_{i,1/2}$) is a vertex cover for $G_i^{1/2}$.

Proof : Let $(u, v) \in E_i^{1/2}$ be an edge in $G_i^{1/2}$, and let d_u and d_v be the degrees of u and v in $G_i^{\lambda/4}$ respectively. Next we show that $d_u + d_v \ge |V_i|$. Hence we have $d_u \ge \frac{|V_i|}{2}$ or $d_v \ge \frac{|V_i|}{2}$. This means that $\nu_{i,1/2}$ covers (u, v) as desired.

Notice that, $(u, v) \in E_i^{1/2}$ means that $w_{u,v} \ge \frac{L_i}{2}$. Hence, by the λ -triangle inequality, for any $v' \in V_i$ we have $w_{(u,v')} + w_{(v',v)} \ge \lambda w_{(u,v)} \ge \frac{\lambda L_i}{2}$. Now we are ready to bound $d_u + d_v$.

$$\begin{split} d_{u} + d_{v} &= \left(1 + \sum_{v' \in V_{i} \setminus \{u,v\}} 1_{(u,v') \in E_{i}^{\lambda/4}}\right) + \left(1 + \sum_{v' \in V_{i} \setminus \{u,v\}} 1_{(v,v') \in E_{i}^{\lambda/4}}\right) & \text{Extra 1 is for } (u,v) \\ &= 2 + \sum_{v' \in V_{i} \setminus \{u,v\}} \left(1_{(u,v') \in E_{i}^{\lambda/4}} + 1_{(v,v') \in E_{i}^{\lambda/4}}\right) \\ &= 2 + \sum_{v' \in V_{i} \setminus \{u,v\}} \left(1_{w_{(u,v')} \ge \lambda L_{i}/4} + 1_{w_{(v,v')} \ge \lambda L_{i}/4}\right) \\ &\ge 2 + \sum_{v' \in V_{i} \setminus \{u,v\}} 1 & \text{Since } w_{(u,v')} + w_{(v',v)} \ge \frac{\lambda L_{i}}{2} \\ &= 2 + |V_{i}| - 2 = |V_{i}|, \end{split}$$

which completes the proof.

Lemma 2 For any $i \in \{1, ..., t\}$, L_i is an upper bound on weight of the edges in G_i , i.e., we have $\max_{e \in E_i} w_e \leq L_i$.

Proof : For i = 1, $L_1 = L$ which is an upper bound on weight of the edges in $G_1 = G$. For i > 1, $\nu_{i-1,1/2}$ is a vertex cover of $G_{i-1}^{1/2}$, by Lemma 1. Moreover, by definition we have $\nu_{i-1,1/2} \subseteq \nu_{i-1}$. Hence ν_{i-1} is a vertex cover of $G_{i-1}^{1/2}$. This means that every edge with weight at least $\frac{L_{i-1}}{2}$ has a neighbor in ν_{i-1} . Recall that $G_i = G_{i-1} \setminus \nu_{i-1}$, and hence, G_i has no edge with weight at least $\frac{L_{i-1}}{2} = L_i$.

The following theorem compares the average weight of the edges in E_{ν_i} with L_i . We later use this in Theorem 7 to bound the number of queries.

Lemma 3 For any $i \in \{1, \ldots, t\}$, we have

$$\frac{\lambda}{32}L_i|V_i||\nu_i| \le \sum_{e \in E_{\nu_i}} w_e \le L_i|V_i||\nu_i|.$$

Proof : We start by proving the upper bound. Recall that E_{ν_i} is the set of edges neighboring ν_i in G_i . Hence the number of edges in E_{ν_i} is upper bounded by sum of the degrees of the vertices of ν_i in G_i . The degree of each vertex in G_i is $|V_i| - 1 < |V_i|$, and there are $|\nu_i|$ vertices in ν_i . Thus, we have $|E_{\nu_i}| \le |V_i| |\nu_i|$. Moreover, by Lemma 2, for each $e \in E_{\nu_i} \subseteq E_i$ we have $w_e \le L_i$. Therefore we have

$$\sum_{e \in E_{\nu_i}} w_e \le \sum_{e \in E_{\nu_i}} L_i \le L_i |V_i| |\nu_i|$$

Next we prove the lower bound. Recall that we have $\nu_i \subseteq \nu_{i,1/4}$. Thus, for each $v \in \nu_i$, the degree of v in $G_i^{\lambda/4}$ is at least $\frac{|V_i|}{4}$. Thus, for any *fixed* $v \in \nu_i$ we have

$$\sum_{(u,v)\in E_{\nu_i}} w_{(u,v)} = \sum_{(u,v)\in E_i} w_{(u,v)}$$
 Definition of E_{ν_i} for $v \in \nu_i$ (1)

$$\geq \sum_{(u,v)\in E_i^{\lambda/4}} w_{(u,v)}$$

$$\geq \sum_{(u,v)\in E_i^{\lambda/4}} \frac{\lambda L_i}{4}$$
 Definition of $G_i^{\lambda/4}$

$$\geq \frac{|V_i|}{4} \frac{\lambda L_i}{4} = \frac{\lambda}{16} |V_i| L_i.$$
 $v \in \nu_i \subseteq \nu_{i,1/4}$ (2)

Note that each edge in E_{ν_i} intersects at most two vertices in ν_i . Therefore, we have

$$\sum_{e \in E_{\nu_i}} w_e \ge \frac{1}{2} \sum_{v \in \nu_i} \sum_{(u,v) \in E_{\nu_i}} w_{(u,v)}$$
$$\ge \frac{1}{2} \sum_{v \in \nu_i} \frac{\lambda}{16} |V_i| L_i$$
By Inequality 1
$$\ge \frac{\lambda}{32} L_i |V_i| |\nu_i|,$$

which completes the proof of the lemma.

Lemma 5 provides a technique to construct ν_i using O(n) queries, with high probability. This to prove this lemma we sample some edges. Notice that these sampled edges are different from the edges that we sample to keep in H_{β} . We use the following standard version of the Chernoff bound (see e.g. [48]) in Lemma 5 as well as the rest of the paper.

Lemma 4 (Chernoff Bound) Let $x_1, x_2, ..., x_r$ be a sequence of independent binary (i.e., 0 or 1) random variables, and let $X = \sum_{i=1}^r x_i$. For any $\epsilon \in [0, 1]$, we have

$$\Pr\left(|X - \mathbf{E}[X]| \ge \epsilon \mathbf{E}[X]\right) \le 2\exp(-\epsilon^2 \mathbf{E}[X]/3).$$

As we are now moving to doing sampling, we briefly remark on some noteworthy points. First, there is some probability of failure in our results. We therefore refer to the success probability in our results, and note that our algorithms may fail "silently"; that is, we may not realize the algorithm has failed (because of a low probability event in the sampling). Also, we emphasize that in general, in what follows, when referring to the number of queries required, we mean the expected number of queries. However, using expectations is for convenience; all of our results throughout the paper could instead be turned into results bounding the number of queries required with high probability (say probability 1 - O(1/n) using Chernoff bounds at the cost of at most constant factors in the standard way. Finally, in some places we may sample which edges we decide to query from a set of edges with a fixed probability p. In such situations, instead of iterating through each edge (which could take time quadratic in the number of vertices) we can generate the number of samples from a binomial distribution and then generate the samples without replacement; alternatively, we could determine which sample is the next sample at each step using by calculating a geometrically distributed random variable. We assume this work can be done in constant time per sample. For this reason, our time depends on the number of queries, and not the total number of edges.

Lemma 5 For any $i \in \{1, ..., t\}$, given G_i and L_i , one can construct ν_i using $192(\log n + \log t)|V_i| \in \tilde{O}(n)$ expected queries, succeeding with probability at least $1 - \frac{1}{nt}$.

Proof : If $|V_i| \leq 384(\log n + \log t)$, we have $E_i = \binom{|V_i|}{2} = \frac{1}{2}384(\log n + \log t)(|V_i| - 1) \leq 192(\log n + \log t)|V_i|$. Hence in this case we query all the edges and construct ν_i . In what follows we assume $|V_i| \geq 384(\log n + \log t)$. To construct ν_i we sample each edge in E_i with probability $p = \frac{384(\log n + \log t)}{|V_i|}$. We add a vertex v to ν_i if and only if at least $\frac{3}{8}384(\log n + \log t)$ of its sampled neighbors has weight $\frac{\lambda L_i}{4}$. The number of sampled edges is $p\binom{|V_i|}{2} = \frac{1}{2}384(\log n + \log t)(|V_i| - 1) \leq 192(\log n + \log t)|V_i|$. We denote the degree of a vertex $v \in V_i$ in $G_i^{\lambda/4}$ by d_v . Let Y_e be a binary random variable that is 1 if

We denote the degree of a vertex $v \in V_i$ in $G_i^{A/4}$ by d_v . Let Y_e be a binary random variable that is 1 if we sample e and 0 otherwise. Let us define $Z_e = Y_e \mathbb{1}_{w_e \ge \lambda L_i/4}$ and $Z_v = \sum_{u \in V_i} Z_{(u,v)}$. Recall that we add $v \in V_i$ to ν_i if and only if $Z_v \ge \frac{3}{8}384(\log n + \log t)$. Notice that, for any $v \in V_i$ we have

$$E[Z_v] = E\left[\sum_{u \in V_i} Z_{(u,v)}\right] = \sum_{u \in V_i} E[Y_{(u,v)}] 1_{w_{u,v} \ge \lambda L_i/4} = p \sum_{u \in V_i} 1_{w_{u,v} \ge \lambda L_i/4} = p d_v.$$
(3)

As Z_v is the sum of independent binary random variables, by the Chernoff bound we have

$$\begin{aligned} \Pr\left[|Z_v - \operatorname{E}\left[Z_v\right]| &\geq \frac{384(\log n + \log t)}{8}\right] &\leq 2\exp\left(-\frac{1}{3}\left(\frac{384(\log n + \log t)}{8\operatorname{E}\left[Z_v\right]}\right)^2\operatorname{E}\left[Z_v\right]\right) & \text{Chernoff bound} \\ &= 2\exp\left(-\frac{768(\log n + \log t)^2}{\operatorname{E}\left[Z_v\right]}\right) \\ &= 2\exp\left(-\frac{768(\log n + \log t)^2}{pd_v}\right) & \operatorname{E}\left[Z_v\right] = pd_v \\ &= 2\exp\left(-\frac{2(\log n + \log t)|V_i|}{d_v}\right) & p = \frac{384(\log n + \log t)}{|V_i|} \\ &\leq 2\exp\left(-2(\log n + \log t)\right) & d_v \leq |V_i| \\ &= \frac{2}{n^2t^2} \leq \frac{1}{n^2t}. & \text{Assuming } t \geq 2 \end{aligned}$$

By applying the union bound we have

$$\Pr\left[\exists_{v \in V_i} | Z_v - \mathbb{E}\left[Z_v\right] | \ge 48(\log n + \log t)\right] \le \sum_{v \in V_i} \Pr\left[|Z_v - \mathbb{E}\left[Z_v\right]| \ge 48(\log n + \log t)\right] = |V_i| \frac{1}{n^2 t} \le \frac{1}{nt}$$

This means that with probability at least $1 - \frac{1}{nt}$, simultaneously for all vertices $v \in V_i$ we have

$$|Z_v - \operatorname{E}[Z_v]| \le 48(\log n + \log t).$$
(4)

Next assuming that for all vertices $v \in V_i$ we have $|Z_v - E[Z_v]| \le 48(\log n + \log t)$ we show that the ν_i that we pick satisfy the property $\nu_{i,1/2} \subseteq \nu_i \subseteq \nu_{i,1/4}$.

Applying Equality 3 to Inequality 4 gives us $|Z_v - pd_v| \ge 48(\log n + \log t)$. By replacing p with $\frac{384(\log n + \log t)}{|V_i|}$ and rearranging the inequality we have

$$Z_v \leq \frac{384(\log n + \log t)}{|V_i|} d_v + 48(\log n + \log t) < \frac{3}{8}384(\log n + \log t) \qquad \text{assuming } d_v < \frac{|V_i|}{4} + \frac{1}{4} + \frac{1}$$

This means that if $d_v < \frac{|V_i|}{4}$ we have $Z_v < \frac{3}{8}384(\log n + \log t)$ and hence $v \notin \nu_i$. Therefore we have $\nu_i \subseteq \nu_{i,1/4}$. Similarly, we have

$$Z_v \ge \frac{384(\log n + \log t)}{|V_i|} d_v - 48(\log n + \log t) \ge \frac{3}{8}384(\log n + \log t) \qquad \text{assuming } d_v \ge \frac{|V_i|}{2}.$$

This means that if $d_v \ge \frac{|V_i|}{2}$ we have $Z_v \ge \frac{3}{8}384(\log n + \log t)$ and hence $d_v \in \nu_i$. Therefore we have $\nu_{i,1/2} \subseteq \nu_i$.

Finally, for completeness we use the following lemma to find a good upper bound L on $\max_{e \in E} w_e$ in order to start our construction of the graph decomposition (which required an upper bound on the weight of the edges).

Lemma 6 For any λ -metric graph G = (V, E), one can compute a number L such that $\max_{e \in E} w_e \leq L \leq \frac{2}{\lambda} \max_{e \in E} w_e$ using n - 1 queries.

Proof: Let $v' \in V$ be an arbitrary vertex. We set $L = \frac{2}{\lambda} \max_{u' \in V} w_{u',v'}$. Note that, one can simply query all the n-1 neighbors of v' and calculate L. Clearly, we have $L = \frac{2}{\lambda} \max_{u' \in V} w_{u',v'} \leq \frac{2}{\lambda} \max_{e \in E} w_e$. Next, we show that $\max_{e \in E} w_e \leq L$.

Let (u, v) be an edge such that $w_{(u,v)} = \max_{e \in E} w_e$. If $v' \in \{u, v\}$ we have $\max_{u' \in V} w_{u',v'} = \max_{e \in E} w_e$ which directly implies $L \leq \frac{2}{\lambda} \max_{e \in E} w_e$ as desired. Otherwise, note that by the λ -triangle inequality we have $w_{(u,v')} + w_{(v,v')} \geq \lambda w_{(u,v)}$. Thus, we have $\max(w_{(u,v')}, w_{(v,v')}) \geq \frac{\lambda}{2} w_{(u,v)}$. Therefore, we have

$$L = \frac{2}{\lambda} \max_{u' \in V} w_{u',v'} \ge \frac{2}{\lambda} \max(w_{(u,v')}, w_{(v,v')}) \ge w_{(u,v)} = \max_{e \in E} w_e,$$

as desired.

2.2 Constricting H^{α}

We know show how to construct what we call H^{α} , which is derived from our original metric graph G. Recall H^{α} has the property that for each original edge e of weight w_e , independently, if $\alpha w_e > 1$, then H^{α} contains edge e with weight αw_e , and if $\alpha w_e < 1$, then H^{α} contains edge e with weight 1 with probability αw_e .

We define \overline{w} to be the average of the weight of edges in G. We use this notion in the following lemma as well as Lemma 8 and Theorem 10.

The following theorem constructs H^{α} using an expected $O(n \log^2 n + n \log^2 \max_{e \in E} \alpha w_e + \alpha \overline{w} {n \choose 2})$ queries.

Theorem 7 For any α one can construct H^{α} using $O(n \log^2 n + n \log^2 \max_{e \in E} \alpha w_e + \frac{1}{\lambda} \alpha \overline{w} {n \choose 2} + \frac{n}{\lambda})$ queries in expectation, succeeding with probability at least $1 - \frac{1}{n}$.

Proof : By Lemma 6 we find an upper bound L on the weight of the edges, using n - 1 queries. Recall that t is the number of graphs in our decomposition. We set $t = \log_2 n + \log_2 \max_{e \in E} \alpha w_e$. Given $L_1 = L$, by definition we have

$$L_{t} = \frac{L}{2^{t}} = \frac{L}{n \max_{e \in E} \alpha w_{e}} \qquad t = \log_{2} n + \log_{2} \max_{e \in E} \alpha w_{e}$$
$$\leq \frac{2}{\lambda \alpha n}. \qquad L \leq \frac{2}{\lambda} \max_{e \in E} w_{e} \qquad (5)$$

Recall that, using Lemma 5, one can construct ν_i and thus V_{i+1} using $192(\log n + \log t)n$ queries, succeeding with probability at least $1 - \frac{1}{nt}$. We start with $G_1 = G$ and iteratively apply Lemma 5 to construct the sequence G_1, \ldots, G_t and ν_1, \ldots, ν_t . We apply Lemma 5 t times, and hence using a union bound, all of the G_i were successfully constructed with probability at least $1 - t\frac{1}{nt} = 1 - \frac{1}{n}$. Next, we show how to construct H^{α} assuming the sequences G_1, \ldots, G_t and ν_1, \ldots, ν_t are valid. Note that constructing the graph decomposition we use at most $192(\log n + \log t)n \times t \in O(n \log^2 n + n \log^2 \max_{e \in E} \alpha w_e)$ queries.

Recall that $E_{\nu_i} = E_i \setminus E_{i+1}$. Also, we have $E_1 = E$. Thus, the sequence $E_{\nu_1}, \ldots, E_{\nu_{t-1}}$ is a decomposition of $E \setminus E_t$. Also, note that $E_{\nu_i} = \{(u, v) | u \in \nu_i \text{ and } v \in V_i\}$. Therefore, given G_1, \ldots, G_t and ν_1, \ldots, ν_t we can decompose the edge set E into $E_{\nu_1}, \ldots, E_{\nu_t}$.

Let j be the smallest index such that $L_j \leq \frac{1}{\alpha}$. Notice that $j \leq t$ by Inequality 5. For each $i \in \{1, \ldots, j-1\}$ we query each edge $e \in E_{\nu_i}$. If $\alpha w_e > 1$, add edge e with weight αw_e to E_H . If $\alpha w_e \leq 1$, we add edge e with weight 1 to E_H with probability αw_e independently.

For each $i \in \{j, \ldots, t-1\}$ we query each edge $e \in E_{\nu_i}$ with probability αL_i . We add a queried edge e to E_H with probability $\frac{w_e}{L_i}$ and withdraw it otherwise. Note that $\alpha L_i \leq \alpha L_j \leq \alpha \frac{1}{\alpha} = 1$. Also L_i is an upper bound on the edge weights in $E_i \supseteq E_{\nu_i}$, and thus $\frac{w_e}{L_i} \leq 1$. Therefore, the probabilities αL_i and $\frac{w_e}{L_i}$ are valid. Also, notice that we add each edge to E_H with probability $\alpha L_i \times \frac{w_e}{L_i} = \alpha w_e$ as desired.

For each edge $e \in E_t$ we query e with probability $\frac{2}{\lambda n}$. We add a queried edge e to E_H with probability $\frac{\lambda n}{2} \alpha w_e$ and withdraw it otherwise. Recall L_t is an upper bound on the edges edges weights in E_t , and by Inequality 5 we have $L_t \leq \frac{2}{\lambda \alpha n}$. Thus, we have

$$\frac{\lambda n}{2}\alpha w_e \le \frac{\lambda n}{2}\alpha L_t \le \frac{\lambda n}{2}\alpha \frac{2}{\lambda\alpha n} = 1.$$

Therefore, $\frac{\lambda n}{2} \alpha w_e$ is a valid probability. Again, notice that we add each edge to E_H with probability $\frac{2}{\lambda n} \times \frac{\lambda n}{2} \alpha w_e = \alpha w_e$ as desired. Next we bound the total number of edges that we query.

Let Y_e be a random variable that is 1 if we query e and 0 otherwise. We bound the expected number of edges that we query by

$$\begin{split} \mathbf{E}\left[\sum_{e \in E} Y_e\right] &= \sum_{e \in E} \mathbf{E}\left[Y_e\right] \\ &= \sum_{i=1}^{j-1} \sum_{e \in E_{\nu_i}} \mathbf{E}\left[Y_e\right] + \sum_{i=j}^{t-1} \sum_{e \in E_{\nu_i}} \mathbf{E}\left[Y_e\right] + \sum_{e \in E_t} \mathbf{E}\left[Y_e\right] - E_{\nu_1}, \dots, E_{\nu_{t-1}}, E_t \text{ is a decomposition of } E \\ &= \sum_{i=1}^{j-1} \sum_{e \in E_{\nu_i}} 1 + \sum_{i=j}^{t-1} \sum_{e \in E_{\nu_i}} \alpha L_i + \sum_{e \in E_t} \frac{2}{\lambda n} \\ &\leq \sum_{i=1}^t \sum_{e \in E_{\nu_i}} \alpha L_i + \sum_{e \in E_t} \frac{2}{\lambda n} \\ &\leq \sum_{i=1}^t \sum_{e \in E_{\nu_i}} \alpha L_i + \frac{n}{\lambda} \\ &\leq \alpha \sum_{i=1}^t \sum_{e \in E_{\nu_i}} \alpha L_i + \frac{n}{\lambda} \\ &\leq \alpha \sum_{i=1}^t \sum_{e \in E_{\nu_i}} \frac{32}{\lambda} w_e + \frac{n}{\lambda} \\ &\leq \frac{32}{\lambda} \alpha \sum_{e \in E} w_e + \frac{n}{\lambda} \\ &\in O(\frac{1}{\lambda} \alpha \binom{n}{2} \overline{w} + \frac{n}{\lambda}). \end{split}$$

We used $O(n \log^2 n + n \log^2 \max_{e \in E} \alpha w_e)$ queries to construct the sequences G_1, \ldots, G_t and ν_1, \ldots, ν_t , and used $O(\frac{1}{\lambda}\alpha\binom{n}{2}\overline{w} + \frac{n}{\lambda})$ queries to construct H^{α} based on these sequences. Therefore, in total we used $O(n \log^2 n + n \log^2 \max_{e \in E} \alpha w_e + \frac{1}{\lambda}\alpha \overline{w}\binom{n}{2} + \frac{n}{\lambda})$ queries in expectation. \Box

2.3 Constructing H_{β}

The following lemma relates β with α . We use this to construct H_{β} using H^{α} .

Lemma 8 Let $\gamma \in [1, \infty)$ be an arbitrary number. Let $\frac{1}{\gamma}\overline{w} \leq \hat{w} \leq \overline{w}$, $\alpha = \frac{\beta}{\binom{n}{2}\hat{w}}$, and $H^{\alpha} = (V, E_H)$. We have

$$\beta \leq \mathbf{E} \left[\sum_{e \in E_H} w'_e \right] \leq \gamma \beta,$$

where w'_e is the weight of e in H^{α} .

Proof : We have

$$\mathbf{E}\left[\sum_{e \in E_{H}} w'_{e}\right] = \sum_{e \in E} \mathbf{E}\left[w'_{e}\right] = \sum_{e \in E} \alpha w_{e}$$
$$= \sum_{e \in E} \frac{\beta}{\binom{n}{2} \hat{w}} w_{e} = \frac{\beta}{\hat{w}} \frac{\sum_{e \in E} w_{e}}{\binom{n}{2}}$$
Definition of α
$$= \frac{\overline{w}}{\hat{w}} \beta.$$
Definition of \overline{w}

This together with $\frac{1}{\gamma}\overline{w} \leq \hat{w}$ gives us

$$\mathbf{E}\left[\sum_{e\in E_H} w'_e\right] = \frac{\overline{w}}{\hat{w}}\beta \le \gamma\beta$$

Similarly, by applying $\hat{w} \leq \overline{w}$ we have

$$\mathbf{E}\left[\sum_{e\in E_H} w'_e\right] = \frac{\overline{w}}{\hat{w}}\beta \ge \beta$$

Lemma 9 shows how to estimate \overline{w} . We use this lemma together with Lemma 8 to find a proper α based on the desired β to construct H_{β} . We note that in a metric space, i.e. $\lambda = 1$, the following lemma gives a $1 - \epsilon$ approximation of the average weight of the edges using $\tilde{O}(\frac{n}{\epsilon^2})$ queries, while the previous algorithm of Indyk [34] uses $O(\frac{n}{\epsilon^{3.5}})$ queries³. In the next section, using H_{β} we improve this lemma and estimate the average weight of the edges using only $\tilde{O}(n + \frac{1}{\epsilon^2})$ queries.

Lemma 9 For $\epsilon \in (0,1]$, one can find an estimator \hat{w} of the average weight of the edges \overline{w} such that $(1-\epsilon)\overline{w} \leq \hat{w} \leq (1+\epsilon)\overline{w}$, with probability $1-\frac{2}{n}$, using $O(n\log^2 n + \frac{n\log n}{\epsilon^2\lambda}) \in \tilde{O}(\frac{n}{\epsilon^2\lambda})$ queries.

³Note that the algorithm in [34] works with a constant probability while our algorithm works with probability $1 - \frac{1}{n}$. The previous algorithm requires an extra logarithmic factor to work with probability $1 - \frac{1}{n}$.

Proof : We first use $O(\frac{n}{\lambda})$ queries to provide an estimate \hat{w}' such that $\frac{1}{2n}\overline{w} \leq \hat{w}' \leq \overline{w}$. Next we set $\alpha = \frac{\beta}{\binom{n}{2}\hat{w}}$ and construct a corresponding H^{α} . We use Lemma 8 and Theorem 7 to lower bound the total weight of sampled edges by $\frac{3\log(2n)}{\epsilon^2}$ and upper bound the number of queries by $O(n\log^2 n + \frac{n\log n}{\epsilon^2\lambda})$. At the end we use the lower bound on the total weight of sampled edges to show that the average weight of edges in H^{α} is concentrated around \overline{w} .

Let v be an arbitrary vertex. We have

$$\sum_{u \in V \setminus \{v\}} w_{u,v} = \frac{1}{n} \sum_{u \in V \setminus \{v\}} w_{u,v} + \frac{n-1}{n} \sum_{u \in V \setminus \{v\}} w_{u,v}$$

$$\geq \frac{1}{n} \sum_{u \in V \setminus \{v\}} w_{u,v} + \frac{n-1}{n} \frac{\lambda}{n-2} \sum_{u,u' \in V \setminus \{v\}} w_{u,u'} \qquad \lambda \text{-triangle inequality}$$

$$\geq \frac{\lambda}{n} \sum_{e \in E} w_e$$

Hence for a set $S \subseteq V$ with $|S| = \lceil \frac{1}{\lambda} \rceil$ we have

$$\sum_{v \in S} \sum_{u \in V \setminus \{v\}} w_{u,v} > \sum_{v \in S} \frac{\lambda}{n} \sum_{e \in E} w_e \ge \frac{1}{n} \sum_{e \in E} w_e$$

On the other hand every edge appears at most twice in $\sum_{v \in S} \sum_{u \in V \setminus \{v\}} w_{u,v}$ and hence we have $\sum_{v \in S} \sum_{u \in V \setminus \{v\}} w_{u,v} \leq 2 \sum_{e \in E} w_e$. Therefore, by setting $\hat{w}' = \frac{1}{2\binom{n}{2}} \sum_{v \in S} \sum_{u \in V \setminus \{v\}} w_{u,v}$ we have $\frac{1}{2n}\overline{w} \leq \hat{w}' \leq \overline{w}$. Hence, one can query at most $\frac{n}{|\lambda|}$ edges to find a number \hat{w}' such that $\frac{1}{2n}\overline{w} \leq \hat{w}' \leq \overline{w}$. Next, we set $\alpha = \frac{3\log(2n)}{\epsilon^2\binom{n}{2}\hat{w}'}$. By Lemma 8 we have

$$\frac{3\log(2n)}{\epsilon^2} \le \mathbf{E}\left[\sum_{e \in E_H} w'_e\right] \le \frac{6n\log(2n)}{\epsilon^2},\tag{6}$$

where w'_e is the weight of e in H^{α} . By Lemma 7, with probability $1 - \frac{1}{n}$, the expected number of queries we need to construct H^{α} is at most

$$O\left(n\log^2 n + n\log^2 \max_{e \in E} \alpha w_e + \frac{1}{\lambda} \alpha \overline{w} \binom{n}{2} + \frac{n}{\lambda}\right) \in O\left(n\log^2 n + n\log^2\left(\frac{3n\log(2n)}{\epsilon^2}\right) + \frac{1}{\lambda} \frac{6n\log(2n)}{\epsilon^2} + \frac{n}{\lambda}\right) \in O\left(n\log^2 n + \frac{n\log n}{\epsilon^2\lambda}\right).$$
 Assuming $\epsilon \ge \frac{1}{n}$ w.l.o.g.

Now we set $\hat{w} = \frac{1}{\binom{n}{2}} \sum_{e \in E_H} \frac{w'_e}{\alpha}$, where w'_e is the weight of e in H^{α} . To complete the proof we show that $(1 - \epsilon)\overline{w} \le \hat{w} \le (1 + \epsilon)\overline{w}$, with probability $1 - \frac{1}{n}$. Notice that

$$\mathbf{E}\left[\hat{w}\right] = \mathbf{E}\left[\frac{1}{\binom{n}{2}}\sum_{e\in E_H}\frac{w'_e}{\alpha}\right] = \frac{1}{\binom{n}{2}}\sum_{e\in E}w_e = \overline{w}.$$
(7)

Let χ_e be a binary random variable that indicates whether χ_e is sampled in H^{α} or not. Note that

$$\hat{w} - \mathbf{E}\left[\hat{w}\right] = \frac{1}{\binom{n}{2}} \sum_{e \in E_H} \frac{w'_e}{\alpha} - \frac{1}{\binom{n}{2}} \sum_{e \in E} \mathbf{E}\left[\frac{w'_e}{\alpha}\right]$$
$$= \frac{1}{\binom{n}{2}\alpha} \left(\sum_{e \in E_H} w'_e - \sum_{e \in E} \mathbf{E}\left[w'_e\right]\right)$$
$$= \frac{1}{\binom{n}{2}\alpha} \left(\sum_{w_e \leq \frac{1}{\alpha}} \chi_e - \sum_{w_e \leq \frac{1}{\alpha}} \mathbf{E}\left[w'_e\right]\right). \qquad w'_e = \mathbf{E}\left[w'_e\right] \text{ when } w_e > \frac{1}{\alpha}$$
(8)

Therefore, we have

$$\begin{split} \Pr\left[|\hat{w} - \overline{w}| \leq \epsilon \overline{w}\right] &= \Pr\left[|\hat{w} - \operatorname{E}\left[\hat{w}\right]| \leq \epsilon \overline{w}\right] & \text{By Equality 7} \\ &= \Pr\left[\left|\frac{1}{\binom{n}{2}\alpha} \left(\sum_{w_e \leq \frac{1}{\alpha}} \chi_e - \sum_{w_e \leq \frac{1}{\alpha}} \operatorname{E}\left[w'_e\right]\right)\right| \leq \epsilon \overline{w}\right] & \text{By Equality 8} \\ &= \Pr\left[\left|\sum_{w_e \leq \frac{1}{\alpha}} \chi_e - \sum_{w_e \leq \frac{1}{\alpha}} \operatorname{E}\left[w'_e\right]\right| \leq \epsilon \alpha \binom{n}{2} \overline{w}\right] \\ &\leq 2 \exp\left(-\frac{1}{3} \left(\frac{\epsilon \alpha \binom{n}{2} \overline{w}}{\sum_{w_e \leq \frac{1}{\alpha}} \operatorname{E}\left[w'_e\right]}\right)^2 \sum_{w_e \leq \frac{1}{\alpha}} \operatorname{E}\left[w'_e\right]\right) & \text{Chernoff Bound} \\ &\leq 2 \exp\left(-\frac{1}{3} \frac{\epsilon^2 \alpha^2 \binom{n}{2}^2 \overline{w}^2}{\sum_{w_e \leq \frac{1}{\alpha}} \operatorname{E}\left[w'_e\right]}\right) \\ &= 2 \exp\left(-\frac{1}{3} \frac{\epsilon^2 (\sum_{e \in E} \operatorname{E}\left[w'_e\right])^2}{\sum_{w_e \leq \frac{1}{\alpha}} \operatorname{E}\left[w'_e\right]}\right) \\ &\leq 2 \exp\left(-\frac{1}{3} \epsilon^2 \sum_{e \in E} \operatorname{E}\left[w'_e\right]\right) \\ &\leq 2 \exp\left(-\frac{1}{3} \epsilon^2 \sum_{e \in E} \operatorname{E}\left[w'_e\right]\right) \\ &\leq 2 \exp\left(-\frac{1}{3} \epsilon^2 \frac{3 \log(2n)}{\epsilon^2}\right) & \text{By Inequality 6} \\ &= 2 \exp\left(-\log(2n)\right) = \frac{1}{n}. \end{split}$$

This means that with probability $1 - \frac{1}{n}$ we have $(1 - \epsilon)\overline{w} \le \hat{w} \le (1 + \epsilon)\overline{w}$ as desired.

The following theorem constructs H_{β} using $\tilde{O}(\frac{n+\beta}{\lambda})$ queries, with high probability.

Theorem 10 For any β one can construct H_{β} using expected $O(n \log^2 n + n \log^2 \beta + \frac{\beta}{\lambda} + \frac{n \log n}{\lambda}) \in \tilde{O}(\frac{n+\beta}{\lambda})$ expected queries, with probability of success at least $1 - \frac{3}{n}$.

Proof : First, using Lemma 9 we find an estimator \hat{w} of the average weight of the edges \overline{w} such that $\frac{1}{2}\overline{w} \leq \hat{w} \leq \overline{w}$, with probability $1 - \frac{2}{n}$, using $O(n \log^2 n + \frac{n \log n}{\lambda})$ expected queries. Lemma 8 says that by picking $\alpha = \frac{\beta}{\binom{n}{2}\hat{w}}$, we have $\beta \leq \mathbb{E}\left[\sum_{e \in E_H} w'_e\right] \leq 2\beta$, where w'_e is the weight of e in $H^{\alpha} = (V, E_H)$.

⁴ By Theorem 7 one can construct H^{α} using $O(n \log^2 n + n \log^2 \max_{e \in E} \alpha w_e + \frac{1}{\lambda} \alpha \overline{w} {n \choose 2} + \frac{n}{\lambda})$ expected queries, with probability $1 - \frac{1}{n}$. Note that, we have

$$\max_{e \in E} \alpha w_e = \alpha \max_{e \in E} w_e$$

$$\leq \alpha \binom{n}{2} \overline{w} \qquad \overline{w} \geq \frac{\max_{e \in E} w_e}{\binom{n}{2}}$$

$$= \frac{4\beta}{3\binom{n}{2}\hat{w}} \binom{n}{2} \overline{w} \qquad \alpha = \frac{4\beta}{3\binom{n}{2}\hat{w}}$$

$$\leq 2\beta \qquad \qquad \frac{2}{3} \overline{w} \leq \hat{w}$$

Also, we have

$$\alpha \overline{w} n^2 = \frac{4\beta}{3\binom{n}{2}\hat{w}} \overline{w} \binom{n}{2} \qquad \qquad \text{By } \alpha = \frac{4\beta}{3\binom{n}{2}\hat{w}}$$
$$= \frac{4\overline{w}}{3\hat{w}}\beta$$
$$\leq 2\beta. \qquad \qquad \qquad \text{By } \frac{2}{3}\overline{w} \leq \hat{w}$$

By $\alpha \overline{w} n^2 \leq 2\beta$ and $\max_{e \in E} \alpha w_e \leq 2\beta$ we have

$$n\log^{2} n + n\log^{2} \max_{e \in E} \alpha w_{e} + \frac{1}{\lambda} \alpha \overline{w} \binom{n}{2} + \frac{n}{\lambda} \leq n\log^{2} n + n\log^{2}(2\beta) + \frac{2\beta}{\lambda} + \frac{n}{\lambda}$$
$$\in O(n\log^{2} n + n\log^{2}\beta + \frac{\beta + n}{\lambda}).$$

Therefore, the total number of expected queries is $O(\frac{n \log n}{\lambda} + n \log^2 n + n \log^2 \beta + \frac{\beta + n}{\lambda}) \in \tilde{O}(\frac{\beta + n}{\lambda})$. We properly estimate \hat{w} with probability at least $1 - \frac{2}{n}$ and Theorem 7 holds with probability at least $1 - \frac{1}{n}$. Therefore, by the union bound, the statement of this theorem holds with probability at least $1 - \frac{3}{n}$.

3 Applications of Linear Sampling

In this section we use the sketch H_{β} to develop approximation algorithms for densest subgraph, maximum k-hypermatching, and maximum cut, as well as estimating the average distance. We first define the problems and provide relevant notation. The densest subgraph of a graph G = (V, E) is an induced subgraph of G, indicated by its set of vertices $S^* \subseteq V$, that maximizes $\frac{\sum_{u,v \in S^*} w_{u,v}}{|S^*|}$. We indicate the value of the densest subgraph by opt_D . The max cut of a graph G = (V, E) is a decomposition of the vertex set of G into two sets $S^*, V \setminus S^* \subseteq V$, that maximizes $\sum_{u \in S^*, v \in V \setminus S^*} w_{u,v}$. We indicate the value of the max cut by opt_C . A k-hypermatching of a set of points V is a decomposition of V into a collection of n/k sets $\mathbb{S}^* = \{S_1^*, S_2^*, \dots, S_{n/k}^*\}$, each of size k. One can also see this as covering a graph G = (V, E) with clusters

⁴Note that, for any $\eta \in (0, 1]$, one can use lemma 9 to find \hat{w} such that $\frac{1}{1+\eta}\overline{w} \leq \hat{w} \leq \overline{w}$, with probability $1 - \frac{2}{n}$, using $O(n \log^2 n + \frac{n \log n}{\eta^2 \lambda})$ expected queries, and then apply Lemma 8 to show that by picking $\alpha = \frac{\beta}{\binom{n}{2}\hat{w}}$, we have $\beta \leq E\left[\sum_{e \in E_H} w'_e\right] \leq (1+\eta)\beta$. We use $\eta = 1$ throughout for convenience.

of size k. A maximum k-hypermatching is a k-hypermatching that maximizes $\sum_{i=1}^{n/k} \sum_{u,v \in S_i^*} w_{u,v}$. We use opt_M to indicate the value of the maximum k-hypermatching.

For a sketch $H_{\beta} = (V, E_H)$ we define random variables $X_{u,v}$ and $Y_{u,v}$. $Y_{u,v}$ is 0 if $(u, v) \notin E_H$, and is equal to the weight of the edge (u, v) in H_{β} otherwise. $X_{u,v} = 1$ if $Y_{u,v} = 1$ and $X_{u,v} = 0$ otherwise. Recall that if we sample an edge e with $\alpha w_e \leq 1$, weight of e in H_{β} is 1. Note that $\mathbf{E}[Y_{u,v}] = \alpha w_{u,v}$.

We first start with a simple application, using H_{β} to estimate the average weight of the edges using $\beta = O(\frac{\log n}{\varepsilon^2})$. This together with Theorem 10 allows us to find the average weight of the edges in a λ -metric space with probability $1 - \frac{4}{n}$ using $O(n \log^2 n + \frac{\log n}{\lambda \varepsilon^2} + \frac{n \log n}{\lambda}) \in \tilde{O}(\frac{n+1/\varepsilon^2}{\lambda})$ expected queries.⁵ In particular for a metric space this gives a $1 - \varepsilon$ approximation of the average weight of the edges using $\tilde{O}(n + \frac{1}{\varepsilon^2})$ queries.

In what follows (throughout this section), when considering the failure probability of the approximation algorithms, we assume that H_{β} has been constructed successfully. That is, we provide for a failure probability in this stage of at most 1/n, which when combined with Theorem 10 allows for our success probability of at least $1 - \frac{4}{n}$ overall.

Theorem 11 Take $\beta = \frac{3 \log(2n)}{\epsilon^2}$. We have

$$(1-\varepsilon)\overline{w} \le \frac{1}{\alpha\binom{n}{2}} \sum_{e \in E} Y_e \le (1+\varepsilon)\overline{w}$$

with probability at least $1 - \frac{1}{n}$.

Proof : We define $\hat{w} = \frac{1}{\alpha\binom{n}{2}} \sum_{e \in E} Y_e$ Notice that

$$\mathbf{E}\left[\hat{w}\right] = \mathbf{E}\left[\frac{1}{\binom{n}{2}}\sum_{e\in E}\frac{Y_e}{\alpha}\right] = \frac{1}{\binom{n}{2}}\sum_{e\in E}w_e = \overline{w}.$$
(9)

⁵Again, we emphasize that we can turn these results into bounds with a corresponding upper bound on the queries, with a small increase in the failure probability.

We have

$$\begin{split} \Pr\left[|\hat{w} - \overline{w}| \leq \epsilon \overline{w}\right] &= \Pr\left[|\hat{w} - \mathbb{E}\left[\hat{w}\right]| \leq \epsilon \overline{w}\right] & \text{By Equality 9} \\ &= \Pr\left[\left|\frac{1}{\binom{n}{2}\alpha}\left(\sum_{e \in E} Y_e - \sum_{e \in E} \mathbb{E}\left[Y_e\right]\right)\right| \leq \epsilon \overline{w}\right] & \text{If } Y_e \neq X_e, Y_e = \mathbb{E}[Y_e] \\ &= \Pr\left[\left|\sum_{e \in E} X_e - \sum_{e \in E} \mathbb{E}\left[X_e\right]\right| \leq \epsilon \alpha \binom{n}{2}\overline{w}\right] & \text{Chernoff Bound} \\ &\leq 2 \exp\left(-\frac{1}{3}\left(\frac{\epsilon \alpha \binom{n}{2}\overline{w}}{\sum_{e \in E} \mathbb{E}\left[X_e\right]}\right)^2 \sum_{e \in E} \mathbb{E}\left[X_e\right]\right) & \text{Chernoff Bound} \\ &\leq 2 \exp\left(-\frac{1}{3}\frac{\epsilon^2 \alpha^2 \binom{n}{2}\overline{w^2}}{\sum_{e \in E} \mathbb{E}\left[X_e\right]}\right) & \leq 2 \exp\left(-\frac{1}{3}\frac{\epsilon^2 (\sum_{e \in E} \mathbb{E}\left[Y_e\right])^2}{\sum_{e \in E} \mathbb{E}\left[X_e\right]}\right) & \leq 2 \exp\left(-\frac{1}{3}\epsilon^2 \frac{\sum_{e \in E} \mathbb{E}\left[Y_e\right]}{\sum_{e \in E} \mathbb{E}\left[X_e\right]}\right) & \leq 2 \exp\left(-\frac{1}{3}\epsilon^2 \frac{\sum_{e \in E} \mathbb{E}\left[Y_e\right]}{\sum_{e \in E} \mathbb{E}\left[X_e\right]}\right) & \leq 2 \exp\left(-\frac{1}{3}\epsilon^2 \frac{2 \log(2n)}{\epsilon^2}\right) & \beta = \frac{3 \log(2n)}{\epsilon^2} \\ &\leq 2 \exp\left(-\log(2n)\right) = \frac{1}{n}. \end{split}$$

This means that with probability $1 - \frac{1}{n}$ we have $(1 - \epsilon)\overline{w} \le \hat{w} \le (1 + \epsilon)\overline{w}$ as desired.

Next we provide our results for the densest subgraph problem.

Theorem 12 Take $\beta = \frac{9 \log n}{\varepsilon^2} n$. Let S be a ϕ -approximation solution to the densest subgraph problem on H_{β} . S is a $\phi - 2\varepsilon$ approximation solution to the densest subgraph on G, with probability at least $1 - \frac{1}{n}$.

Proof: We start by lower bounding opt_D.

$$\mathsf{opt}_{\mathsf{D}} \ge \frac{\sum_{u,v \in V} w_{u,v}}{|V|} = \frac{\frac{1}{\alpha} \sum_{u,v \in V} \mathbf{E}[Y_{u,v}]}{n} \ge \frac{1}{\alpha} \frac{\beta}{n} = \frac{1}{\alpha} \frac{9\log n}{\varepsilon^2}.$$
 (10)

Let S' be a subset of V. We define $X_{S'} = \sum_{u,v \in S'} X_{u,v}$, and $Y_{S'} = \sum_{u,v \in S'} Y_{u,v}$. Note that we have $X_{S'} \leq Y_{S'}$. We have $\mathbf{E}[Y_{S'}] = \sum_{u,v \in S'} \mathbf{E}[Y_{u,v}] = \alpha \sum_{u,v \in S'} w_{u,v}$. Hence, we have

$$\mathsf{opt}_{\mathsf{D}} \ge \frac{\sum_{u,v \in S'} w_{u,v}}{|S'|} = \frac{\mathbf{E}[Y_{S'}]}{\alpha |S'|} \ge \frac{\mathbf{E}[X_{S'}]}{\alpha |S'|} \tag{11}$$

Note that $X_{u,v}$'s are chosen independently, and hence by applying the Chernoff bound to $X_{S'}$ for $\epsilon =$

$$\begin{split} \varepsilon \frac{\operatorname{copt_{D}}[S']}{\mathbf{E}[X_{S'}]} & \text{ we have} \\ & \Pr\left[|Y_{S'} - \mathbf{E}[Y_{S'}]| \ge \varepsilon \alpha \operatorname{opt_{D}}|S'|\right] = \Pr\left[|X_{S'} - \mathbf{E}[X_{S'}]| \ge \varepsilon \alpha \operatorname{opt_{D}}|S'|\right] & \text{ If } Y_{e} \neq X_{e}, Y_{e} = \mathbf{E}[Y_{e}] \\ & \le 2 \exp\left(-\frac{1}{3}\left(\varepsilon \frac{\alpha \operatorname{opt_{D}}|S'|}{\mathbf{E}[X_{S'}]}\right)^{2} \mathbf{E}[X_{S'}]\right) & \text{ Chernoff bound} \\ & = 2 \exp\left(-\frac{1}{3}\varepsilon^{2} \frac{\alpha^{2} \operatorname{opt_{D}}^{2}|S'|^{2}}{\mathbf{E}[X_{S'}]}\right) \\ & \le 2 \exp\left(-\frac{1}{3}\varepsilon^{2} \alpha \operatorname{opt_{D}}|S'|\right) & \text{ By Inequality 11} \\ & \le 2 \exp\left(-\frac{1}{3}\varepsilon^{2} \frac{9 \log n}{\varepsilon^{2}}|S'|\right) & \text{ By Inequality 10} \\ & = 2 \exp\left(-3|S'|\log n\right). \end{split}$$

Next we union bound over all choices of S'.

$$\begin{split} \Pr\left[\exists_{S'} \left| Y_{S'} - \mathbf{E}[Y_{S'}] \right| &\geq \varepsilon \alpha \mathsf{opt}_{\mathsf{D}}[S'|] = \Pr\left[\exists_k \exists_{|S'|=k} \left| Y_{S'} - \mathbf{E}[Y_{S'}] \right| \geq \varepsilon \alpha \mathsf{opt}_{\mathsf{D}}k\right] \quad \text{Union bound} \\ &\leq \sum_{k=2}^n \Pr\left[\exists_{|S'|=k} \Pr\left[|Y_{S'} - \mathbf{E}[Y_{S'}]\right| \geq \varepsilon \alpha \mathsf{opt}_{\mathsf{D}}k\right] \quad \text{Union bound} \\ &\leq \sum_{k=2}^n \sum_{|S'|=k} \Pr\left[|Y_{S'} - \mathbf{E}[Y_{S'}]\right| \geq \varepsilon \alpha \mathsf{opt}_{\mathsf{D}}k\right] \quad \text{Union bound} \\ &\leq \sum_{k=2}^n \sum_{|S'|=k} 2\exp\left(-3k\log n\right) \\ &= \sum_{k=2}^n 2\left(\binom{n}{k}\exp\left(-3k\log n\right)\right) \\ &\leq \sum_{k=2}^n 2\exp\left(-3k\log n\right) \\ &\leq \sum_{k=2}^n 2\exp\left(-4\log n\right) \\ &\leq 2\exp\left(-3\log n\right) \\ &\leq 2\exp\left(-3\log n\right) \\ &= \frac{2}{n^3} < \frac{1}{n}. \qquad n \geq 2 \end{split}$$

Therefore, with probability at least $1 - \frac{1}{n}$ simultaneously for all $S' \subseteq V$ we have

$$\left|Y_{S'} - \mathbf{E}[Y_{S'}]\right| \le \varepsilon \alpha \mathsf{opt}_{\mathsf{D}}|S'|. \tag{12}$$

Next we prove the statement of the theorem in the cases where Inequality 12 holds. Let S^* be a densest

subgraph of G. We have

$$\begin{split} \frac{\sum_{u,v \in S} w_{u,v}}{|S|} &= \frac{\frac{1}{\alpha} \mathbf{E}[\sum_{u,v \in S} Y_{u,v}]}{|S|} & \mathbf{E}[Y_{u,v}] = \alpha w_{u,v} \\ &= \frac{1}{\alpha} \frac{\mathbf{E}[Y_S]}{|S|} & \text{Definition of } Y_S \\ &\geq \frac{1}{\alpha} \frac{Y_S}{|S|} - \varepsilon \mathsf{opt}_{\mathsf{D}} & \text{By Inequality 12} \\ &\geq \frac{1}{\alpha} \phi \max_{S''} \frac{Y_{S''}}{|S''|} - \varepsilon \mathsf{opt}_{\mathsf{D}} & S \text{ is a } \phi \text{ approximation on } H_{\beta} \\ &\geq \frac{1}{\alpha} \phi \frac{\mathbf{Y}_{S^*}}{|S^*|} - \varepsilon \mathsf{opt}_{\mathsf{D}} & \\ &\geq \frac{1}{\alpha} \phi \frac{\mathbf{E}[Y_{S^*}]}{|S^*|} - 2\varepsilon \mathsf{opt}_{\mathsf{D}} & \text{By Inequality 12} \\ &\geq \phi \frac{\sum_{u,v \in S^*} w_{u,v}}{|S^*|} - 2\varepsilon \mathsf{opt}_{\mathsf{D}} & \\ &= (\phi - 2\varepsilon)\mathsf{opt}_{\mathsf{D}}. & \text{Definition of } S^* \end{split}$$

Recall that, as stated in the introduction, this result implies a $(1/2 - \epsilon)$ -approximation algorithm for densest subgraph in λ -metric spaces requiring $\tilde{O}(\frac{n}{\lambda\epsilon^2})$ time.

The following theorem shows the efficiency of our technique for k-hypermatching.

Theorem 13 Choose $\beta = \frac{6 \log n}{\varepsilon^2} \frac{n^2}{k-1} \in \tilde{O}(\frac{n^2}{\epsilon^2 k})$. Let $\mathbb{S} = \{S_1, S_2, \dots, S_{n/k}\}$ be a ϕ -approximation solution to the k-hypermatching on unweighted graph H_{β} . \mathbb{S} is a $\phi - 2\varepsilon$ approximation solution to the k-hypermatching on G, with probability at least $1 - \frac{1}{n}$.

Proof: Let $\mathbb{S}'' = \{S_1'', S_2'', \dots, S_{n/k}''\}$ be a k-hypermatching chosen uniformly at random among all k-hypermatchings. Note that the number of edges that fall in \mathbb{S}'' is $\frac{n}{k} {k \choose 2} = \frac{n(k-1)}{2}$, while there are ${n \choose 2} = \frac{n(n-1)}{2}$ edges in G in total. Hence, due to symmetry each edge falls in \mathbb{S}'' with probability $\frac{k-1}{n-1} \leq \frac{k-1}{n}$. Now, we give a lower bound on $\operatorname{opt}_{\mathsf{M}}$. We later use this bound in our concentration bound.

$$\mathsf{opt}_{\mathsf{M}} \ge \mathbf{E}[\sum_{i=1}^{n/k} \sum_{u,v \in S_i''} w_{u,v}] = \frac{k-1}{n} \sum_{u,v \in V} w_{u,v} = \frac{k-1}{n} \frac{1}{\alpha} \sum_{u,v \in V} \mathbf{E}[Y_{u,v}] \ge \frac{k-1}{n} \frac{1}{\alpha} \beta = \frac{1}{\alpha} \frac{6n \log n}{\varepsilon^2}.$$
(13)

Let $\mathbb{S}' = \{S'_1, S'_2, \dots, S'_{n/k}\}$ be a k-hypermatching of G (i.e., a decomposition of V into n/k distinct subsets of size k). We define $X_{\mathbb{S}'} = \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} X_{u,v}$ and $Y_{\mathbb{S}'} = \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} Y_{u,v}$. We have $\mathbf{E}[X_{\mathbb{S}'}] = \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} \mathbf{E}[X_{u,v}] = \alpha \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} w_{u,v}$. Hence we have

$$\mathsf{opt}_{\mathsf{M}} \ge \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} w_{u,v} = \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} \frac{1}{\alpha} \mathbf{E}[Y_{\mathbb{S}'}] \ge \sum_{i=1}^{n/k} \sum_{u,v \in S'_i} \frac{1}{\alpha} \mathbf{E}[X_{\mathbb{S}'}].$$
(14)

Note that $X_{u,v}$'s are chosen independently, and hence by applying the Chernoff bound to $X_{\mathbb{S}'}$ for $\epsilon = \varepsilon \frac{\alpha \operatorname{opt}_{\mathsf{M}}}{\mathbf{E}[X_{\mathbb{S}'}]}$ we have

$$\begin{split} \Pr\left[|Y_{\mathbb{S}'} - \mathbf{E}[Y_{\mathbb{S}'}]| \geq \varepsilon \alpha \mathsf{opt}_{\mathsf{M}}\right] &= \Pr\left[|X_{\mathbb{S}'} - \mathbf{E}[X_{\mathbb{S}'}]| \geq \varepsilon \alpha \mathsf{opt}_{\mathsf{M}}\right] & \text{If } Y_e \neq X_e, Y_e = \mathbf{E}[Y_e] \\ &\leq 2 \exp\left(-\frac{1}{3} \left(\varepsilon \frac{\alpha \mathsf{opt}_{\mathsf{M}}}{\mathbf{E}[X_{\mathbb{S}'}]}\right)^2 \mathbf{E}[X_{\mathbb{S}'}]\right) & \text{Chernoff bound} \\ &= 2 \exp\left(-\frac{1}{3} \varepsilon^2 \frac{\alpha^2 \mathsf{opt}_{\mathsf{M}}^2}{\mathbf{E}[X_{\mathbb{S}'}]}\right) \\ &\leq 2 \exp\left(-\frac{1}{3} \varepsilon^2 \alpha \mathsf{opt}_{\mathsf{M}}\right) & \text{By Inequality 14} \\ &\leq 2 \exp\left(-\frac{1}{3} \varepsilon^2 \frac{6n \log n}{\varepsilon^2}\right) & \text{By Inequality 13} \\ &= 2 \exp\left(-2n \log n\right) \end{split}$$

Next we union bound over all choices of S'.

$$\Pr\left[\exists_{\mathbb{S}'} |Y_{\mathbb{S}'} - \mathbf{E}[Y_{\mathbb{S}'}]\right] \ge \varepsilon \alpha \mathsf{opt}_{\mathsf{M}}] \le \sum_{\mathbb{S}'} \Pr\left[|Y_{\mathbb{S}'} - \mathbf{E}[Y_{\mathbb{S}'}]\right| \ge \varepsilon \alpha \mathsf{opt}_{\mathsf{M}}] \qquad \text{Union bound}$$
$$\le \sum_{\mathbb{S}'} 2 \exp\left(-2n \log n\right)$$
$$\le 2n^n \exp\left(-2n \log n\right)$$
$$\le 2\exp\left(-n \log n\right) \le \frac{1}{n}. \qquad n \ge 2$$

Therefore, with probability $1 - \frac{1}{n}$ simultaneously for all $\mathbb{S}' \subseteq V$ we have

$$\left|Y_{\mathbb{S}'} - \mathbf{E}[Y_{\mathbb{S}'}]\right| \le \varepsilon \alpha \mathsf{opt}_{\mathsf{M}}.\tag{15}$$

Next we prove the statement of the theorem in the cases where Inequality 15 holds. Let \mathbb{S}^* =

 $\{S_1^*,S_2^*,\ldots,S_{n/k}^*\}$ be a maximum k-hypermatching of G. We have

$$\begin{split} \sum_{i=1}^{n/k} \sum_{u,v \in S_i} w_{u,v} &= \frac{1}{\alpha} \mathbf{E} \Big[\sum_{i=1}^{n/k} \sum_{u,v \in S_i} Y_{u,v} \Big] & \mathbf{E}[Y_{u,v}] = \alpha w_{u,v} \\ &= \frac{\mathbf{E}[Y_{\mathbb{S}}]}{\alpha} & \text{Definition of } Y_{\mathbb{S}} \\ &\geq \frac{Y_{\mathbb{S}}}{\alpha} - \varepsilon \operatorname{opt}_{\mathbb{M}} & \text{By Inequality 15} \\ &\geq \phi \frac{\max_{\mathbb{S}''} Y_{\mathbb{S}''}}{\alpha} - \varepsilon \operatorname{opt}_{\mathbb{M}} & \mathbb{S} \text{ is a } \phi \text{ approximation on } H_{\beta} \\ &\geq \phi \frac{\mathbf{E}[Y_{\mathbb{S}^*}]}{\alpha} - \varepsilon \operatorname{opt}_{\mathbb{M}} & \text{By Inequality 15} \\ &\geq \phi \frac{\mathbf{E}[Y_{\mathbb{S}^*}]}{\alpha} - 2\varepsilon \operatorname{opt}_{\mathbb{M}} & \text{By Inequality 15} \\ &\geq \phi \sum_{i=1}^{n/k} \sum_{u,v \in \mathbb{S}^*_i} w_{u,v} - 2\varepsilon \operatorname{opt}_{\mathbb{M}} & \mathbf{E}[Y_{u,v}] = \alpha w_{u,v} \\ &= (\phi - 2\varepsilon) \operatorname{opt}_{\mathbb{M}}. & \text{Definition of } S^* \end{split}$$

Finally we show the efficiency of our sketch for finding the maximum cut, again following the same basic proof outline. Here, we indicate a cut by the set of vertices of its smaller side, breaking ties arbitrarily.

Theorem 14 Choose $\beta = \frac{18 \log n}{\varepsilon^2} n$. Let S be a ϕ -approximation solution to the maximum cut on H_{β} . S is $a \phi - 2\varepsilon$ approximation solution to the maximum cut on G, with probability at least $1 - \frac{1}{n}$.

Proof : First we lower bound opt_C . Note that in the optimum solution moving a vertex from one side to the other does not increase the value of the cut. Thus, for each vertex $v \in V$ the total weight of the edges neighboring v in the cut is at least half of the total weight of all edges neighboring v. Hence we have

$$\mathsf{opt}_{\mathsf{C}} \ge \frac{1}{2} \sum_{v \in V} \sum_{u \in V} w_{u,v} = \frac{1}{2\alpha} \sum_{v \in V} \sum_{u \in V} \mathbf{E}[Y_{u,v}] \ge \frac{1}{2\alpha} \beta = \frac{1}{\alpha} \frac{9 \log n}{\varepsilon^2} n.$$
(16)

Let S' be a subset of V. We define $X_{S'} = \sum_{v \in S'} \sum_{u \in V \setminus S'} X_{u,v}$, and $Y_{S'} = \sum_{v \in S'} \sum_{u \in V \setminus S'} Y_{u,v}$. Note that we have $X_{S'} \leq Y_{S'}$. We have $\mathbf{E}[Y_{S'}] = \sum_{v \in S'} \sum_{u \in V \setminus S'} \mathbf{E}[Y_{u,v}] = \alpha \sum_{v \in S'} \sum_{u \in V \setminus S'} w_{u,v}$. Hence, we have

$$\mathsf{opt}_{\mathsf{C}} \ge \sum_{v \in S'} \sum_{u \in V \setminus S'} w_{u,v} = \frac{\mathbf{E}[Y_{S'}]}{\alpha} \ge \frac{\mathbf{E}[X_{S'}]}{\alpha}.$$
(17)

Note that the $X_{u,v}$'s are independent, and hence by applying the Chernoff bound to $X_{S'}$ for $\epsilon = \varepsilon \frac{\alpha \operatorname{opt}_{\mathsf{C}}}{\mathbf{E}[X_{S'}]}$ we

have

$$\begin{aligned} \Pr\left[|Y_{S'} - \mathbf{E}[Y_{S'}]| &\geq \varepsilon \alpha \mathsf{opt}_{\mathsf{C}}\right] &= \Pr\left[|X_{S'} - \mathbf{E}[X_{S'}]| &\geq \varepsilon \alpha \mathsf{opt}_{\mathsf{C}}\right] & \text{If } Y_e \neq X_e, Y_e = \mathbf{E}[Y_e] \\ &\leq 2 \exp\left(-\frac{1}{3}\left(\varepsilon \frac{\alpha \mathsf{opt}_{\mathsf{C}}}{\mathbf{E}[X_{S'}]}\right)^2 \mathbf{E}[X_{S'}]\right) & \text{Chernoff bound} \\ &= 2 \exp\left(-\frac{1}{3}\varepsilon^2 \frac{\alpha^2 \mathsf{opt}_{\mathsf{C}}^2}{\mathbf{E}[X_{S'}]}\right) \\ &\leq 2 \exp\left(-\frac{1}{3}\varepsilon^2 \alpha \mathsf{opt}_{\mathsf{C}}\right) & \text{By Inequality 17} \\ &\leq 2 \exp\left(-\frac{1}{3}\varepsilon^2 \frac{9 \log n}{\varepsilon^2}n\right) & \text{By Inequality 16} \\ &= 2 \exp\left(-3n \log n\right) \end{aligned}$$

Next we union bound over all choices of S'.

$$\Pr\left[\exists_{S'} | Y_{S'} - \mathbf{E}[Y_{S'}] | \ge \varepsilon \alpha \mathsf{opt}_{\mathsf{C}}\right] \le \sum_{S' \subseteq V} \Pr\left[|Y_{S'} - \mathbf{E}[Y_{S'}]| \ge \varepsilon \alpha \mathsf{opt}_{\mathsf{C}}k \right] \qquad \text{Union bound}$$
$$\le \sum_{S' \subseteq V} 2 \exp\left(-3n \log n\right)$$
$$= 2^{n+1} \exp\left(-3n \log n\right)$$
$$\le \frac{1}{n^n} < \frac{1}{n}. \qquad n \ge 2$$

Therefore, with probability at least $1-\frac{1}{n}$ simultaneously for all $S'\subseteq V$ we have

$$\left|Y_{S'} - \mathbf{E}[Y_{S'}]\right| \le \varepsilon \alpha \mathsf{opt}_{\mathsf{C}}.\tag{18}$$

Next we prove the statement of the theorem in the cases where Inequality 18 holds. Let S^* be a maximum cut of G. We have

$$\begin{split} \sum_{v \in S^*} \sum_{u \in V \setminus S^*} w_{u,v} &= \sum_{v \in S^*} \sum_{u \in V \setminus S^*} \frac{1}{\alpha} \mathbf{E}[Y_{u,v}] & \mathbf{E}[Y_{u,v}] = \alpha w_{u,v} \\ &= \frac{1}{\alpha} \mathbf{E}[Y_S] & \text{Definition of } Y_S \\ &\geq \frac{1}{\alpha} Y_S - \varepsilon \mathsf{opt}_{\mathsf{C}} & \text{By Inequality 18} \\ &\geq \frac{1}{\alpha} \phi \max_{S''} Y_{S''} - \varepsilon \mathsf{opt}_{\mathsf{C}} & S \text{ is a } \phi \text{ approximation on } H_\beta \\ &\geq \frac{1}{\alpha} \phi \mathbf{Y}_{S^*} - \varepsilon \mathsf{opt}_{\mathsf{C}} \\ &\geq \frac{1}{\alpha} \phi \mathbf{E}[Y_{S^*}] - 2\varepsilon \mathsf{opt}_{\mathsf{C}} & \text{By Inequality 18} \\ &\geq \phi \sum_{v \in S^*} \sum_{u \in V \setminus S^*} w_{u,v} - 2\varepsilon \mathsf{opt}_{\mathsf{C}} & \mathbf{E}[Y_{u,v}] = \alpha w_{u,v} \\ &= (\phi - 2\varepsilon)\mathsf{opt}_{\mathsf{C}}. & \text{Definition of } S^* \end{split}$$

4 Impossibility Results

In this section we consider all of the problems of the previous section and show that it is necessary to use $\Omega(n)$ queries even if we just want to estimate the value of the solutions. In particular, we show that $\Omega(n)$ queries are required to distinguish the following two graphs.

- In G_1 we have n vertices $\{v_1, \ldots, v_n\}$ and the weight of all edges are 0.
- In G_2 again we have *n* vertices. Pick an index $r \in \{1, ..., n\}$ uniformly at random. The weight of each edge neighboring v_r is 1. The weight of all other edges is 0.

The following lemma shows the hardness of distinguishing G_1 and G_2 .

Lemma 15 For any $\delta \in (0, 0.5]$, it is impossible to distinguish G_1 and G_2 using $\delta n - 1$ queries with probability $0.5 + \delta$.

Proof : Let Alg be a (possibly randomized) algorithm that distinguishes G_1 and G_2 using at most $\delta n - 1$ queries. For simplicity, and without loss of generality, we assume that Alg makes exactly $\delta n - 1$ queries. Let $(u_1, u_2), (u_3, u_4), \ldots, (u_{k-1}, u_k)$ be the sequence of edges probed by Alg, where the u_i 's may be random variables and $k = 2\delta n - 2$. Notice that v_r is chosen uniformly at random. Hence, in case that the input is G_2 , for any arbitrary $j \in \{1, \ldots, k\}$ we have $\Pr[u_j = v_r] = \frac{1}{n}$. Therefore, we have

$$\Pr\left[\exists_{i \in \{1, \dots, \frac{k}{2}\}} w_{u_{2i-1}, u_{2i}} \neq 1\right] = \Pr\left[\exists_{j \in \{1, \dots, k\}} u_j = v_r\right]$$

$$\leq \sum_{j=1}^k \Pr\left[u_i = v_r\right]$$

$$= \frac{k}{n}$$

$$< 2\delta.$$
Since $\Pr\left[u_j = v_r\right] = \frac{1}{n}$
Since $k = 2\delta n - 2$

Hence, in the case that the input is G_2 , with probability at least $1 - 2\delta$ all the edges that Alg queries have weight 0. Trivially, in the case that the input is G_1 all the queried edges have weight 0. Therefore the probability that Alg distinguishes G_1 and G_2 is less than $2\delta + \frac{1-2\delta}{2} = 0.5 + \delta$.

Note that the weight of the edges of G_1 is 0, while average weight of the edges of G_2 is $\frac{n-1}{\binom{n}{2}} = \frac{2}{n}$. Therefore any algorithm that estimates the average weight of the edges within any multiplicative factor distinguishes G_1 and G_2 . This together with Lemma 15 proves the following corollary.

Corollary 16 Any approximation algorithm that estimates the average distance a in metric graphs within any multiplicative factor with probability 0.51 requires $\Omega(n)$ queries.

Note that the density of the densest subgraph of G_1 is 0, while the density of the densest subgraph of G_2 is $\frac{n-1}{n} \ge \frac{1}{2}$. Therefore any algorithm that estimates the density of the densest subgraph within any multiplicative factor distinguishes G_1 and G_2 . This together with Lemma 15 proves the following corollary.

Corollary 17 Any approximation algorithm that estimates the density of the densest subgraph in a metric graphs within any multiplicative factor with probability 0.51 requires $\Omega(n)$ queries.

Notice that the value of the maximum matching of G_1 is 0 while the value of the maximum matching of G_2 is 1. Therefore any algorithm that estimates the value of the maximum matching within any multiplicative factor distinguishes G_1 and G_2 . This together with Lemma 15 proves the following corollary.

Corollary 18 Any approximation algorithm that estimates the value of the maximum matching in a metric graphs within any multiplicative factor with probability 0.51 requires $\Omega(n)$ queries.

Notice that the value of the maximum cut of G_1 is 0 while the value of the maximum cut of G_2 is n-1. Therefore any algorithm that estimates the value of the maximum cut distinguishes G_1 and G_2 . This together with Lemma 15 proves the following corollary.

Corollary 19 Any approximation algorithm that estimates the value of the maximum cut in a metric graphs within any multiplicative factor with probability 0.51 requires $\Omega(n)$ queries.

5 Conclusion

We have show that in metric graphs one can efficiently obtain a linear sampling with a sublinear number of edge queries, allowing efficient sparsification that leads to efficient approximation algorithms. We believe this technique may be useful in generating approximation algorithms for other problems beyond those considered here. Open questions include possibly improving the lower bounds, or otherwise bridging the gap between the upper and lower bounds on required queries.

References

- [1] Kook Jin Ahn and Sudipto Guha. Graph sparsification in the semi-streaming model. In *Proceedings of the 36th International Colloquium on Automata, Languages and Programming*, pages 328–338, 2009.
- [2] Kook Jin Ahn and Sudipto Guha. Laminar families and metric embeddings: Nonbipartite maximum matching problem in the semi-streaming model. *Manuscript, available at* http://arxiv.org/abs/1104.4058, 2011.
- [3] Kook Jin Ahn and Sudipto Guha. Linear programming in the semi-streaming model with application to the maximum matching problem. In *Proceedings of the 38th International Colloquium on Automata, Languages, and Programming*, pages 526–538, 2011.
- [4] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Analyzing graph structure via linear measurements. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 459–467, 2012.
- [5] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Graph sketches: sparsification, spanners, and subgraphs. In Proceedings of the 31st ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, pages 5–14, 2012.
- [6] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Spectral sparsification in dynamic graph streams. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pages 1–10. 2013.
- [7] Noga Alon, Seannie Dar, Michal Parnas, and Dana Ron. Testing of clustering. *SIAM Journal on Discrete Mathematics*, 16(3):393–417, 2003.

- [8] Thomas Andreae and Hans-Jürgen Bandelt. Performance guarantees for approximation algorithms depending on parametrized triangle inequalities. SIAM Journal on Discrete Mathematics, 8(1):1–16, 1995.
- [9] Sepehr Assadi. Tight space-approximation tradeoff for the multi-pass streaming set cover problem. In Proceedings of the 36th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, pages 321–335, 2017.
- [10] Sepehr Assadi, Sanjeev Khanna, Yang Li, and Grigory Yaroslavtsev. Tight bounds for linear sketches of approximate matchings. *arXiv preprint arXiv:1505.01467*, 2015.
- [11] Mihai Bădoiu, Artur Czumaj, Piotr Indyk, and Christian Sohler. Facility location in sublinear time. In Proceedings of the 32nd International Colloquium on Automata, Languages, and Programming, pages 866–877, 2005.
- [12] MohammadHossein Bateni, Hossein Esfandiari, and Vahab Mirrokni. Almost optimal streaming algorithms for coverage problems. *arXiv preprint arXiv:1610.08096*, 2016.
- [13] MohammadHossein Bateni, Hossein Esfandiari, and Vahab Mirrokni. Distributed coverage maximization via sketching. arXiv preprint arXiv:1612.02327, 2016.
- [14] Soheil Behnezhad, Mahsa Derakhshan, Hossein Esfandiari, Elif Tan, and Hadi Yami. Brief announcement: Graph matching in massive datasets. In *Proceedings of the 29th ACM Symposium on Parallelism in Algorithms and Architectures*, pages 133–136, 2017.
- [15] Michael A Bender and Chandra Chekuri. Performance guarantees for the tsp with a parameterized triangle inequality. *Information Processing Letters*, 73(1-2):17–21, 2000.
- [16] Aditya Bhaskara, Samira Daruki, and Suresh Venkatasubramanian. Sublinear algorithms for maxcut and correlation clustering. arXiv preprint arXiv:1802.06992, 2018.
- [17] Aditya Bhaskara, Mehrdad Ghadiri, Vahab Mirrokni, and Ola Svensson. Linear relaxations for finding diverse elements in metric spaces. In *Advances in Neural Information Processing Systems*, pages 4098–4106, 2016.
- [18] Sayan Bhattacharya, Monika Henzinger, Danupon Nanongkai, and Charalampos Tsourakakis. Spaceand time-efficient algorithm for maintaining dense subgraphs on one-pass dynamic streams. In *Proceedings of the 47th Annual ACM Symposium on Theory of Computing*, pages 173–182, 2015.
- [19] Marc Bury and Chris Schwiegelshohn. Sublinear estimation of weighted matchings in dynamic data streams. *arXiv preprint arXiv:1505.02019*, 2015.
- [20] L Sunil Chandran and L Shankar Ram. Approximations for atsp with parametrized triangle inequality. In Proceedings of the 19th Annual Symposium on Theoretical Aspects of Computer Science, pages 227–237, 2002.
- [21] Moses Charikar. Greedy approximation algorithms for finding dense components in a graph. In *International Workshop on Approximation Algorithms for Combinatorial Optimization*, pages 84–95, 2000.

- [22] Rajesh Chitnis, Graham Cormode, Hossein Esfandiari, MohammadTaghi Hajiaghayi, Andrew McGregor, Morteza Monemizadeh, and Sofya Vorotnikova. Kernelization via sampling with applications to finding matchings and related problems in dynamic graph streams. In *Proceedings of the 27th Annual* ACM-SIAM Symposium on Discrete Algorithms, pages 1326–1344, 2016.
- [23] Rajesh Chitnis, Graham Cormode, Hossein Esfandiari, MohammadTaghi Hajiaghayi, and Morteza Monemizadeh. Brief announcement: New streaming algorithms for parameterized maximal matching & beyond. In *Proceedings of the 27th ACM Symposium on Parallelism in Algorithms and Architectures*, pages 56–58, 2015.
- [24] Rajesh Chitnis, Graham Cormode, MohammadTaghi Hajiaghayi, and Morteza Monemizadeh. Parameterized streaming: Maximal matching and vertex cover. In *Proceedings of the 26th Annual ACM-SIAM Aymposium on Discrete Algorithms*, pages 1234–1251, 2015.
- [25] Russ Cox, Frank Dabek, Frans Kaashoek, Jinyang Li, and Robert Morris. Practical, distributed network coordinates. ACM SIGCOMM Computer Communication Review, 34(1):113–118, 2004.
- [26] Artur Czumaj and Christian Sohler. Sublinear-time approximation for clustering via random sampling. In Proceedings of the 31st International Colloquium on Automata, Languages, and Programming, pages 396–407, 2004.
- [27] Artur Czumaj and Christian Sohler. Small space representations for metric min-sum k-clustering and their applications. In *Proceedings of the 24th Annual Symposium on Theoretical Aspects of Computer Science*, pages 536–548, 2007.
- [28] Artur Czumaj and Christian Sohler. Sublinear-time algorithms. In *Property testing*, pages 41–64. 2010.
- [29] Leah Epstein, Asaf Levin, Julián Mestre, and Danny Segev. Improved approximation guarantees for weighted matching in the semi-streaming model. SIAM Journal on Discrete Mathematics, 25(3):1251– 1265, 2011.
- [30] Hossein Esfandiari, Mohammad T Hajiaghayi, Vahid Liaghat, Morteza Monemizadeh, and Krzysztof Onak. Streaming algorithms for estimating the matching size in planar graphs and beyond. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1217–1233, 2015.
- [31] Hossein Esfandiari, MohammadTaghi Hajiaghayi, and David P Woodruff. Applications of uniform sampling: Densest subgraph and beyond. *arXiv preprint arXiv:1506.04505*, 2015.
- [32] Joan Feigenbaum, Sampath Kannan, Andrew McGregor, Siddharth Suri, and Jian Zhang. On graph problems in a semi-streaming model. *Theoretical Computer Science*, 348(2):207–216, 2005.
- [33] Sariel Har-Peled, Piotr Indyk, Sepideh Mahabadi, and Ali Vakilian. Towards tight bounds for the streaming set cover problem. In *Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium* on *Principles of Database Systems*, pages 371–383, 2016.
- [34] Piotr Indyk. Sublinear time algorithms for metric space problems. In *Proceedings of the 31st Annual ACM Symposium on Theory of Computing*, pages 428–434, 1999.

- [35] Piotr Indyk. A sublinear time approximation scheme for clustering in metric spaces. In *Proceedings* of the 40th Annual IEEE Symposium on Foundations of Computer Science, pages 154–159, 1999.
- [36] Michael Kapralov, Sanjeev Khanna, and Madhu Sudan. Approximating matching size from random streams. In *Proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 734–751, 2014.
- [37] Michael Kapralov, Sanjeev Khanna, and Madhu Sudan. Streaming lower bounds for approximating max-cut. In *Proceedings of the 26h Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1263–1282, 2015.
- [38] Michael Kapralov, Sanjeev Khanna, Madhu Sudan, and Ameya Velingker. $(1 + \omega (1))$ -approximation to max-cut requires linear space. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1703–1722, 2017.
- [39] Jonathan A. Kelner and Alex Levin. Spectral sparsification in the semi-streaming setting. In Proceedings of the 28th Annual Symposium on Theoretical Aspects of Computer Science, pages 440–451, 2011.
- [40] Dmitry Kogan and Robert Krauthgamer. Sketching cuts in graphs and hypergraphs. In Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science, pages 367–376, 2015.
- [41] Christian Konrad. Maximum matching in turnstile streams. In Proceedings of the 23rd Annual European Symposium on Algorithms, pages 840–852. 2015.
- [42] Christian Konrad, Frédéric Magniez, and Claire Mathieu. Maximum matching in semi-streaming with few passes. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pages 231–242. 2012.
- [43] Christian Konrad and Adi Rosén. Approximating semi-matchings in streaming and in two-party communication. In *Proceedings of the 40th Annual International Colloquium on Automata, Languages* and Programming, pages 637–649, 2013.
- [44] Jonathan Ledlie, Paul Gardner, and Margo I Seltzer. Network coordinates in the wild. In USENIX Symposium on Networked Systems Design and Implementation, volume 7, pages 299–311, 2007.
- [45] Andrew McGregor. Finding graph matchings in data streams. In Approximation, Randomization and Combinatorial Optimization. Algorithms and Techniques, pages 170–181. 2005.
- [46] Andrew McGregor, David Tench, Sofya Vorotnikova, and Hoa T Vu. Densest subgraph in dynamic graph streams. In *International Symposium on Mathematical Foundations of Computer Science*, pages 472–482, 2015.
- [47] Michael Mitzenmacher, Jakub Pachocki, Richard Peng, Charalampos Tsourakakis, and Shen Chen Xu. Scalable large near-clique detection in large-scale networks via sampling. In *Proceedings of the 21th* ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, pages 815–824, 2015.
- [48] Michael Mitzenmacher and Eli Upfal. *Probability and computing: Randomized algorithms and probabilistic analysis*. Cambridge University Press, 2005.

- [49] Morteza Monemizadeh and David P Woodruff. 1-pass relative-error l_p -sampling with applications. In *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1143–1160, 2010.
- [50] Shanmugavelayutham Muthukrishnan. *Data streams: Algorithms and applications*. Now Publishers Inc, 2005.
- [51] Ami Paz and Gregory Schwartzman. A $(2 + \epsilon)$ -approximation for maximum weight matching in the semi-streaming model. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2153–2161, 2017.
- [52] Peter Pietzuch, Jonathan Ledlie, Michael Mitzenmacher, and Margo Seltzer. Network-aware overlays with network coordinates. In *Proceedings of the 26th IEEE International Conference on Distributed Computing Systems Workshops.* 2006.
- [53] Yuval Shavitt and Tomer Tankel. Big-bang simulation for embedding network distances in euclidean space. IEEE/ACM Transactions on Networking (TON), 12(6):993–1006, 2004.
- [54] Jian Tang, Meng Qu, Mingzhe Wang, Ming Zhang, Jun Yan, and Qiaozhu Mei. Line: Large-scale information network embedding. In *Proceedings of the 24th International Conference on the World Wide Web*, pages 1067–1077, 2015.
- [55] Kevin Verbeek and Subhash Suri. Metric embedding, hyperbolic space, and social networks. *Computational Geometry*, 59:1–12, 2016.
- [56] Sepehr Abbasi Zadeh, Mehrdad Ghadiri, Vahab S Mirrokni, and Morteza Zadimoghaddam. Scalable feature selection via distributed diversity maximization. In AAAI Conference on Artificial Intelligence, pages 2876–2883, 2017.

A Uniform edge sampling fails to find the densest subgraph

It is known that for general unweighted graphs, if we sample each edge with a small probability $p \in \tilde{\Omega}(\frac{1}{\epsilon^2 n})$, the densest subgraph of the sampled subgraph is a $(1 - \epsilon)$ -approximation of the densest subgraph of the original graph [31]. Here with a simple example we show that this result is not true for weighted graphs in a metric space even when p is a small constant.

Consider a graph G with vertex set $V = \{v_1, v_2, \dots, v_n\}$, where the weight of each each intersecting v_1 is $\frac{n}{2} + 1$ and the weight of each other edge is 1. The densest subgraph of G contains the whole graph, and its density is $\frac{\binom{n-1}{2} + (n-1)(\frac{n}{2}+1)}{n} = \frac{(n-1)n}{n} = n-1$.

its density is $\frac{\binom{n-1}{2} + (n-1)(\frac{n}{2}+1)}{n} = \frac{(n-1)n}{n} = n-1$. Let G_p be a subgraph of G obtained by sampling each edge with probability p. Using a simple Chernoff bound it is easy to show that with high probability G_p has at most $2p\binom{n-1}{2}$ edges of weight 1. Similarly, with high probability the number of edges of weight $\frac{n}{2} + 1$ in G_p is between 1 and 2p(n-1).

Let $H = (V_H, E_H)$ be the densest subgraph of G_p . We have

$$\frac{\sum_{e \in E_H} w_e}{|V_H|} \le \frac{2p\binom{n-1}{2} + 2p(n-1)(n/2+1)}{|V_H|} = \frac{2pn(n-1)}{|V_H|}.$$

On the other hand the density of one single edge with weight $\frac{n}{2} + 1$ is $\frac{n}{4} + \frac{1}{2}$. Thus we have $\frac{n}{4} + \frac{1}{2} \le \frac{2pn(n-1)}{|V_H|}$ which implies $|V_H| \le 8pn$. Therefore the density of the densest subgraph induced by $|V_H|$ is at most

$$\frac{\binom{|V_H|-1}{2} + |V_H|(\frac{n}{2}+1)}{|V_H|} \le \frac{|V_H|-1}{2} + \frac{n}{2} + 1 \le 4pn - \frac{1}{2} + \frac{n}{2} + 1 = (\frac{1}{2} + 4p)n + \frac{1}{2}.$$

Therefore, the subgraph of G induced by V_H is not better than a $\frac{(\frac{1}{2}+4p)n+\frac{1}{2}}{n-1} \simeq 0.5 + 4p$ approximate solution.