Fully Dynamic Maximal Independent Set in Expected Poly-Log Update Time

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Abstract

In the fully dynamic maximal independent set (MIS) problem our goal is to maintain an MIS in a given graph G while edges are inserted and deleted from the graph. The first non-trivial algorithm for this problem was presented by Assadi, Onak, Schieber, and Solomon [STOC 2018] who obtained a deterministic fully dynamic MIS with $O(m^{3/4})$ update time. Later, this was independently improved by Du and Zhang and by Gupta and Khan [arXiv 2018] to $\tilde{O}(m^{2/3})$ update time ¹. Du and Zhang [arXiv 2018] also presented a randomized algorithm against an oblivious adversary with $\tilde{O}(\sqrt{m})$ update time.

The current state of art is by Assadi, Onak, Schieber, and Solomon [SODA 2019] who obtained randomized algorithms against oblivious adversary with $\tilde{O}(\sqrt{n})$ and $\tilde{O}(m^{1/3})$ update times

In this paper, we propose a dynamic randomized algorithm against oblivious adversary with expected worst-case update time of $O(\log^4 n)$. As a direct corollary, one can apply the blackbox reduction from a recent work by Bernstein, Forster, and Henzinger [SODA 2019] to achieve $O(\log^6 n)$ worst-case update time with high probability. This is the first dynamic MIS algorithm with very fast update time of poly-log.

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¹As usual n is the number of vertices, m is the number of edges and $\tilde{O}(\cdot)$ suppresses poly-logarithmic factors.

1 Introduction

A maximal independent set (MIS) of a given graph G = (V, E) is a subset M of vertices such that M does not contain two neighboring vertices and every vertex in $V \setminus M$ has a neighbor vertex in M. In this chapter, we study the maximal independent set (MIS) problem in the dynamic setting, where the graph G is undergoing a sequence of edge insertions and deletions.

MIS is a fundamental problem with both theoretical and practical importance and is used as a fundamental building block in many applications. For instance, MIS has been used for resource scheduling for parallel threads in a multi-core environment, for leader election [7], for resource allocation [13], etc.

The MIS problem has received a lot of attention in the distributed and parallel settings since the influential works of [1, 10, 11]. It is considered a central problem in distributed computing and in particular in the symmetry breaking field. Specifically, attaining a better understanding of MIS in the distributed setting is of particular interest not only because it is a fundamental problem but also because other fundamental problems reduce to it.

Censor-Hillel, Haramaty, and Karnin [6] in their pioneering paper studied the problem of maintaining an MIS in the distributed dynamic setting where the graph changes over time. They gave a randomized algorithm for maintaining an MIS against an oblivious adversary in the distributed dynamic setting; as an open question, the authors asked whether it is possible to maintain an MIS in a dynamic graph with update time faster than recomputing everything from scratch, which triggered a recent line of research.

The first non-trivial algorithm was proposed by Assadi, Onak, Schieber and Solomon [2] who presented a deterministic algorithm with $O(m^{3/4})$ amortized update time. This was the first dynamic algorithm that maintains an MIS with sublinear update time in the sequential model. This upper bound was later improved to $\tilde{O}(m^{2/3})$ independently by Du and Zhang [8] and by Gupta and Khan [9]. In the same paper Du and Zhang [8] overcame the $\tilde{O}(m^{2/3})$ barrier by assuming an oblivious adversary and a randomized algorithm with amortized update time $\tilde{O}(\sqrt{m})$ was proposed. This randomized upper bound was recently improved to $\tilde{O}(\sqrt{n})$ by Assadi et al. [3]. For graphs with bounded arboricity α , a deterministic algorithm with amortized update time of $O(\alpha^2 \log^2 n)$ was proposed in [12].

1.1 Our contribution

In this chapter we present the first dynamic MIS algorithm with very fast update time of polylogarithmic in n. We obtain a randomized dynamic MIS algorithm that works against an oblivious adversary. Moreover, our algorithm can actually maintain a greedy MIS with respect to a random order on the set of vertices; the concept of greedy MIS is defined as follows.

Definition 1. Given any order π on all vertices in V, the greedy MIS M_{π} with respect to π is uniquely defined by the following procedure that gradually builds an MIS: starting with $M_{\pi} = \emptyset$, for each vertex in V under order π , if it is not yet dominated by M_{π} , add it to M_{π} .

We say that an algorithm has worst-case expected update time α if for every update σ , the expected time to process σ is at most α .

Our main result argues that when π is a uniformly random permutation, the corresponding greedy MIS can be maintained under edge updates against an oblivious adversary, which is formalized in the following statement.

Theorem 2. Let π be a random permutation over V. The greedy MIS on G according to order π can be maintained under edge insertions and deletions in worst-case expected $O(\log^4 n)$ time against an oblivious adversary, where the expectation is taken over the random choice of π .

As a corollary, we can apply a black-box reduction from worst-case time dynamic algorithms to expected worst-case time dynamic algorithms that appeared in a recent paper [5].

Theorem 3 ([5]). Let A be an algorithm that maintains a dynamic data structure D with worst-case expected time α , and let n be a parameter such that the maximum number of items stored in the data structure at any point in time is polynomial in n. Then there exists an algorithm A' with the following properties.

- 1. For any sequence of updates $\sigma_1, \sigma_2, \dots, A'$ processes each update σ_i in $O(\alpha \log^2 n)$ time with high probability.
- 2. A' maintains $\Theta(\log n)$ data structures $D_1, D_2, \dots, D_{\Theta(\log n)}$, as well as a pointer to some D_i that is guaranteed to be correct at the current time. Query operations are answered with D_i .

Corollary 4. There is a dynamic MIS algorithm against an oblivious adversary that handles edge updates in worst-case $O(\log^6 n)$ time with high probability, and answers MIS membership queries in constant time.

Independent work: Independent of our work, Behnezhad *et al.* [?] also present a fully dynamic algorithm that maintains a greedy MIS with expected poly-logarithmic running time against oblivious adversaries.

1.2 Technical overview

Our algorithm is a combination of techniques from [6] and [3]. In paper [6], the authors proved a lemma that the expected number of changes made to a greedy MIS by an edge update is bounded by a constant. Unfortunately, they could not achieve an efficient dynamic algorithm since a straightforward implementation of the lemma has a linear dependence on the maximum degree of the graph which could be large.

The issue with the maximum degree was overcome by the algorithm from [3] which relies on what we informally call the degree reduction lemma: if we pick a random subset of k vertices and build a greedy MIS on this subset, then the maximum degree of the induced subgraph on all the rest un-dominated vertices is at most $O(\frac{n \log n}{k})$. Therefore we can do the following to achieve an update time with sub-linear dependence on n. First build an MIS on a randomly selected subset of k vertices and then maintain an MIS on the induced subgraph of all the rest vertices in a brute-force manner. If an edge update lies entirely within the induced subgraph, then it takes time proportional to the maximum degree which is $\tilde{O}(n/k)$; if an edge update lies within the random subset, then we rebuild the whole data structure from scratch. The expected running time of this algorithm is a trade-off between two terms. On the one hand, when the edge update occurs within the induced subgraph, the cost would be proportional to the maximum degree which is $\tilde{O}(n/k)$; on the other hand, when the edge update connects two vertices in the random subset, the cost of rebuilding would be $O(m) = O(n^2)$, and under the assumption of obliviousness, the probability that an edge update lies within the random subset is roughly $O(\frac{k^2}{n^2})$, and so the expected time of rebuilding

would be $\tilde{O}(n^2 \cdot \frac{k^2}{n^2}) = \tilde{O}(k^2)$. Taking $k = \lfloor n^{1/3} \rfloor$ gives a balance of $\tilde{O}(n^{2/3})$ update time. In their paper [3], the authors further refined the running time to $\tilde{O}(\sqrt{n})$ using a hierarchical approach.

We believe the main bottleneck of the above algorithm is that it takes no effort to utilize the lemma from [6]. As a first attempt one could try to look for expensive parts of [3]'s algorithm and try to plug in [6]'s lemma. For example, instead of directly rebuilding, we could try to apply [6]'s lemma when restoring a greedy MIS among the random subset of k vertices if an edge update occurs between them. However, we would again encounter the large degree issue within the random subset.

Our new algorithm is a direct way of combining [6]'s lemma and the degree reduction lemma. The algorithm keeps a random ordering $\pi: V \to [n]$ of all vertices and tries to maintain the random greedy MIS. In order to do so, we explicitly maintain all the induced subgraphs $G_i = (V_i, E_i)$ ($0 \le i \le \log n$) on all vertices which are not dominated by MIS vertices from $\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(2^i)$. For simplicity assume edge (u, v) is inserted where $2^b < \pi(u) < \pi(v) \le 2^{b+1}$ for some integer b. Then, on the one hand, this event happens with probability $O(2^{2b}/n^2)$ when π is uniformly random; on the other hand, all changes to the MIS could only take place in G_b whose maximum degree is bounded by $O(\frac{n \log n}{2^b})$.

Let $S \subseteq V_b$ be the set of all influenced vertices (we will formally define what S is later on; basically S contains all vertices that could possibly enter or leave the MIS during this update). Following similar proofs of [6], we could prove the conditional expectation of S is at most $O(n/2^b)$. As the maximum degree of G_b is bounded by $O(\frac{n \log n}{2^b})$, we could go over all neighbors of S in G_b and maintain memberships of vertices from S in subgraphs G_{b+1}, G_{b+2}, \cdots , which takes $\tilde{O}(n^2/2^{2b})$ time, perfectly canceling out the probability $O(2^{2b}/n^2)$ we just mentioned. However, this is not the end of the story. Not only could vertices from S change their memberships in subgraphs G_{b+1}, G_{b+2}, \cdots , but neighbors of vertices in S as well, which could be as many as $O(n^2/2^{2b})$ in the worst-case. The key to the running time analysis is that π roughly assigns the set S uniform-random positions in $[2^b+1, n]$ even when S is given as prior knowledge. Therefore, on average, the number of neighbors in G_b of a vertex in S is bounded by $\tilde{O}(1)$.

2 Preliminaries

For any subgraph $H \subseteq G$, let $\Delta(H)$ be its maximum vertex degree. For any $U \subseteq V$, define $\Gamma(U)$ to be the set of all neighbors of U in G, and G[U] the induced subgraph of G on U. For any permutation π on V and vertex $u \in V$, define I_u^{π} to be the set of neighbor predecessors of u with respect to π . For any two different vertices $u, v \in V$, we say u has a higher priority than v if $\pi(u) < \pi(v)$. For any pair of indices i, j, define $\pi[i, j] = \{w \mid i \leq \pi(w) \leq j\}$. The following lemma states the basic characterization of a greedy MIS.

Lemma 5 (folklore). An MIS M is the greedy MIS with respect to order π if and only if for all $z \in V$, it satisfies the constraint that either $z \in M$ or $I_z^{\pi} \cap M \neq \emptyset$. For the rest, we will call this constraint the greedy MIS constraint for z.

The following lemma appeared in [3].

Lemma 6 ([3]). Let π be a uniformly random permutation on V and let k be an integer in [n]. Let U be the set of all vertices not dominated by $M_{\pi} \cap \pi[1, k]$, then with high probability of $1 - n^{-4}$, $\Delta(G[U]) \leq O(\frac{n \log n}{k})$.

The next lemma is a slight modification of the previous lemma where we show that even if we fix the position in the permutation of two vertices the lemma still holds.

Lemma 7. Let $u_1, u_2 \in V$ be two different vertices and $k_1, k_2 \in [n]$ be two different indices, and let $1 \leq k \leq n$ be an integer. Let π be a uniformly random permutation on V under the condition that $\pi(u_i) = k_i, i \in \{1, 2\}$. Let U be the set of all vertices not dominated by $M_{\pi} \cap \pi[1, k]$, then with high probability $1 - n^{-2}$, $\Delta(G[U]) \leq O(\frac{n \log n}{k})$.

Proof. Call a permutation π bad if $\Delta(G[U]) \geq \Omega(\frac{n \log n}{k})$. Noticing that $\Pr_{\pi}[\pi(u_i) = k_i, i \in \{1, 2\}] = \frac{1}{n(n-1)/2}$, by Lemma 6 we have:

$$n^{-4} \ge \Pr_{\pi}[\pi \text{ is bad}]$$

$$= \frac{1}{n(n-1)/2} \Pr_{\pi}[\pi \text{ is bad} \mid \forall i, \pi(u_i) = k_i] + (1 - \frac{1}{n(n-1)/2}) \Pr_{\pi}[\pi \text{ is bad} \mid \exists i, \pi(u_i) \ne k_i]$$

$$\ge \frac{1}{n(n-1)/2} \Pr_{\pi}[\pi \text{ is bad} \mid \pi(u_i) = k_i]$$

Hence, $\Pr_{\pi}[\pi \text{ is bad } | \forall i, \pi(u_i) = k_i]$ is at most n^{-2} as well, which concludes the proof.

For the rest of this section, we review the notion of influenced set which was studied in [6]. Given a total order π , an MIS $M=M_{\pi}$, as well as an edge update between u,v, we turn to define v's influenced set S_v^{π} . If v does not violate the greedy MIS constraint after the edge update, then define $S_v^{\pi} = \emptyset$; notice that v always preserves the greedy MIS constraint if $\pi(v) < \pi(u)$. Otherwise, initially set $S_0 = \{v\}$. For any $i \geq 1$, define S_i to be the set of all non-MIS vertices whose MIS predecessors are all in S_{i-1} , plus the set of every MIS vertex who has at least one predecessor in S_{i-1} , namely:

$$S_{i} = \{ w \mid w \in M, S_{i-1} \cap I_{w}^{\pi} \neq \emptyset \} \cup \{ w \mid w \notin M, I_{w}^{\pi} \cap M \subseteq \bigcup_{j=0}^{i-1} S_{j} \}$$

Note that the set M refers to the greedy MIS in the old graph, not in the new graph. Eventually, define v's influenced set to be $S_v^{\pi} = \bigcup_{i=0}^{\infty} S_i$. When $S_v^{\pi} \neq \emptyset$, there is a simple characterization which will be used later.

Lemma 8. Let M be the greedy MIS in the old graph. When $S_v^{\pi} \neq \emptyset$, it is equal to the smallest set S that contains v and satisfies the following two conditions.

- (1) $\forall z \in M, I_z^{\pi} \cap S \neq \emptyset \text{ iff } z \in S.$
- (2) $\forall z \notin M, I_z^{\pi} \cap M \subseteq S \text{ iff } z \in S.$

Proof. Since S_v^{π} satisfies both of (1) and (2), it suffices to prove that any S containing v that satisfies both (1) and (2) would contain S_v^{π} as an subset. This is done by an easy induction on $i \geq 0$ that S contains every S_i .

Lemma 9 ([6]). Let π, σ be two permutations, $S \subseteq V$ a nonempty set, and $v \in V$ be an arbitrary vertex. Suppose an edge update occurs between u, v. Assume $S_v^{\pi} = S$, $\pi(u) < \pi(v), \sigma(u) < \sigma(v), \sigma, \pi$ have the same induced relative order on both S and $V \setminus S$, namely $\pi_S = \sigma_S, \pi_{V \setminus S} = \sigma_{V \setminus S}$, then $M_{\pi} = M_{\sigma}$ in the old graph before the edge update, and $S_v^{\sigma} = S$.

Lemma 10 ([6]). Let π, σ be two permutations, $S \subseteq V$ a vertex subset, and $v \in V$ be an arbitrary vertex. Suppose an edge update occurs between u, v. If $S_v^{\pi} = S \neq \emptyset$, and $\pi(u) < \pi(v), \sigma(u) < \sigma(v), \sigma, \pi$ have the same induced relative order on both $S \setminus \{v\}$ and $V \setminus S$, namely $\pi_{S \setminus \{v\}} = \sigma_{S \setminus \{v\}}, \pi_{V \setminus S} = \sigma_{V \setminus S}$. If $v \neq \arg\min_{z \in S} \{\sigma(z)\}$ then $S_v^{\sigma} = \emptyset$.

It was also shown in [6] that for an edge update (u, v) the expected size of S_v^{π} is constant. In our algorithm we need the following different variants of this claim.

Lemma 11. Suppose an edge update occurs between u, v. Let $1 \le C \le A < B \le n$ be three integers. Then

 $\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \pi(u) = C, \pi(v) \in [A+1, B]] < \frac{n}{B-A}$

Lemma 12. Suppose an edge update occurs between u, v. Let $1 \le A < B \le n$ be two integers. Then

$$\mathbb{E}_{\pi}[|S_{v}^{\pi}| \mid A < \pi(u) < \pi(v) \le B] < \frac{2n}{B - A}$$

3 The Main Algorithm

In this section we describe our fully dynamic MIS algorithm.

3.1 Data structure

When π is a fixed permutation over V, our algorithm is entirely deterministic. Let $M \subseteq V$ be the greedy MIS with respect to π , and for any $1 \le k \le n$, define $M_k = M \cap \pi[1, k]$. Since M is defined in a greedy manner, M_k dominates the entire set $\pi[1, k]$.

The algorithm explicitly maintains the induced subgraph $G_i = (V_i, E_i), \forall 0 \leq i \leq \log n - 1$, where $V_i = V \setminus (M_{2^i} \cup \Gamma(M_{2^i}))$; by definition $G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{\log n - 1}$. More precisely, given a permutation π , our algorithm maintains at any given point of time the graphs G_i for $0 \leq i \leq \log n - 1$ and the greedy MIS M_{π} . In the following subsection we describe our update algorithm to maintain both the graphs G_i and the MIS M_{π} .

3.2 Update algorithm

Suppose an edge is updated, either inserted or deleted, between $u, v \in V$ with $\pi(u) < \pi(v)$. Suppose $2^a < \pi(u) \le 2^{a+1}$ and $2^b < \pi(v) \le 2^{b+1}$ for integers a and b. There are several easy cases, where $S_v^{\pi} = \emptyset$ and thus we do not need to make changes to M as M stays the greedy MIS with respect to π , and we only need to maintain the subgraphs $G_0, G_1, \dots, G_{\log n-1}$.

- (i) $u \notin M$. In this case, we simply add or remove, depending whether the edge update is an insertion or deletion, the edge (u, v) to/from E_0, E_1, \dots, E_i , where i is the largest index such that $u, v \in V_i$.
- (ii) $u \in M, v \notin M$, the update is a deletion and $I_v^{\pi} \cap M \neq \{u\}$. This case can be handled in the same way as in (i): remove the edge (u, v) in E_0, E_1, \dots, E_i , where i is the largest index such that $u, v \in V_i$, and recompute v's position in the subgraphs $G_a, G_{a+1}, \dots, G_{\log n-1}$.

(iii) $u \in M, v \notin M$ and the update is an insertion. In this case, if $v \in V_a$, then since now v is dominated by $u \in V_a$ we should remove v from all subgraphs $G_k, \forall k > a$. After that, add (u, v) to E_0, E_1, \dots, E_i , where i is the largest index such that $u, v \in V_i$.

For the rest of this section we consider the case where an edge is inserted between $u, v \in M$, or deleted between $u \in M, v \notin M$ with $I_v^{\pi} \cap M = \{u\}$. In both of these cases, $S_v^{\pi} \neq \emptyset$ and thus we need to change v's status in the MIS, and then we must try to fix the greedy MIS M within G_b . We start by computing the nonempty influenced set S_v^{π} with respect to edge update between u, v.

- (1) Initialize an output set $S = \emptyset$ that is promised to be equal to S_v^{π} by the end of the algorithm, as well as a set $T = \{v\}$ that contains a set of candidate vertices that might be included in S during the process.
- (2) In each iteration, extract $z = \arg\min_{z \in T} \{\pi(z)\}$ from T. If $z \in M$, then suppose $2^k < \pi(z) \le 2^{k+1}$; by definition it must be $z \in V_k$. First we add z to S, and scan all neighbors w of z in V_k such that $\pi(w) > \pi(z)$ and add w to T.
 - If $z \notin M$, first scan its adjacency list in G_b ; if all its MIS neighbors with higher priority are in S, then add z to S and add all of its MIS neighbors $w \in V_b$ with $\pi(w) > \pi(z)$ to T.
- (3) When T becomes empty, output S as S_v^{π} .

For convenience we summarize the above procedure as pseudo-code 1.

Algorithm 1: FindInfluencedSet(u, v, b)

```
1 S \leftarrow \emptyset, in easy cases (i)(ii)(iii) T \leftarrow \emptyset, and otherwise T \leftarrow \{v\};
 2 while T \neq \emptyset do
         z \leftarrow \arg\min_{z \in T} \{\pi(z)\}, T \leftarrow T \setminus \{z\};
 3
        if z \in M then
 4
              S \leftarrow S \cup \{z\};
 5
             suppose 2^k < \pi(z) \le 2^{k+1}, and assert z \in V_k;
 6
             for neighbor w \in V_k of z such that \pi(w) > \pi(z) do
 7
                  T \leftarrow T \cup \{w\};
 8
         else
 9
10
              for neighbor w \in V_b \cap M of z such that \pi(w) < \pi(z) do
11
                  if w \notin S then
12
                    flag \leftarrow false and break;
13
             if flag then
14
                  S \leftarrow S \cup \{z\};
15
                  for neighbor w \in V_b \cap M of z with \pi(w) > \pi(z) do
16
17
                      T \leftarrow T \cup \{w\};
18 return S;
```

It will be proved that the output S of Algorithm 1 is equal to S_v^{π} . Once we have found $S = S_v^{\pi}$, we can try to fix the greedy MIS by only looking at G[S]; note that it might be the case that not every vertex in S needs to change its status in the MIS (for example if G[S] is a triangle and v is removed from M due to an insertion, we would not add both vertices in S to M). If the edge update is an insertion, we first remove v from all $V_k, k > a$, and then compute the greedy MIS on $G[S \setminus \{v\}]$ with respect to π ; if the edge update is a deletion, we add v to all $V_k, \forall a < k \le b$, and then compute the greedy MIS on G[S] with respect to π .

Last but not least, we also need to update $G_k, k \geq b+1$. This is done in the straightforward manner: go over every vertex z that has changed its status in MIS in the increasing order with respect to $\pi(z)$. Assuming $2^k < \pi(z) \leq 2^{k+1}$, directly go over all of its neighbors in G_k and recompute their memberships in $G_{b+1}, \dots, G_{\log n-1}$. More specifically, consider the following two cases.

- (1) If z has been added to M, then for every neighbor $w \in \Gamma(z) \cap V_k$, we remove w from all G_l , l > k.
- (2) If z has been removed from M, then z belonged to V_k before the update. Instead of enumerating every neighbor from the current version of $\Gamma(z) \cap V_k$, we go over all of its old neighbors $w \in V_k$ before the update, and compute their memberships in $G_{b+1}, \dots, G_{\log n-1}$.

We also summarize this procedure as pseudo-code 2. After that we can summarize the main update algorithm as pseudo-code 3.

Algorithm 2: FixSubgraphs(S, b)

```
1 for z \in S that has changed its status, in the increasing order in terms of \pi do

2 assume 2^k < \pi(z) \le 2^{k+1};

3 if z has joined M then

4 for w \in V_k \cap \Gamma(z) do

5 else if z has left M then

6 else if z has left M then

7 for neighbor w of z in the old version of V_k before the edge update do

8 compute w's memberships in G_k, G_{k+1}, \cdots, G_{\log n-1};
```

3.3 Correctness

In this section we prove the correctness of our algorithm. We start by proving that the algorithm correctly computes the set S_v^{π} .

Lemma 13. Algorithm 1 correctly outputs the influenced set with respect to v, namely $S = S_v^{\pi}$ when it terminates.

Proof. Let $v=z_1,z_2,\cdots,z_l$ be the sequence of vertices that are added to S sorted in the increasing order with respect to π . We prove inductively that for any $1 \leq i \leq l$, $S_v^{\pi} \cap \pi[\pi(z_1),\pi(z_i)] = \{z_1,z_2,\cdots,z_i\}$. When i=1, the equality holds trivially as $S_v^{\pi} \cap \pi[\pi(z_1),\pi(z_i)] = \{v\} = \{z_1\}$. For the inductive step, suppose we have $S_v^{\pi} \cap \pi[\pi(z_1),\pi(z_i)] = \{z_1,z_2,\cdots,z_i\}$ for some $i \geq 1$. Next we prove $S_v^{\pi} \cap \pi[\pi(z_i)+1,\pi(z_{i+1})] = \{z_{i+1}\}$ in two steps.

Algorithm 3: Update(u, v)

```
1 suppose \pi(u) < \pi(v), and 2^a < \pi(u) \le 2^{a+1}, 2^b < \pi(v) \le 2^{b+1};
 S ← FindInfluencedSet(u, v, b);
3 if S = \emptyset then
    recompute v's memberships among G_a, G_{a+1}, \cdots, G_{\log n-1};
5 else
 6
       if the update is insertion then
           remove v from V_k, k > a;
 7
          run the greedy MIS algorithm on G[S \setminus \{v\}] with respect to order \pi;
 8
9
10
           add v to all V_k, a < k \le b;
           run the greedy MIS algorithm on G[S] with respect to order \pi;
11
       FixSubgraphs(S, b);
12
```

• $z_{i+1} \in S_v^{\pi}$.

This can be verified according to the specification of Algorithm 1 and definition of S_v^{π} in the following way. If z_{i+1} was added to S on line-5, namely $z_{i+1} \in M$, then it must have been introduced to T on line-17 by one of its neighbors that appeared before in π ; this is because this predecessor cannot be in M, and so it was added to S on line-15, and thus z_{i+1} was added to T on line-17. Then according to the definition of S_v^{π} , $z_{i+1} \in S_v^{\pi}$.

If otherwise z_{i+1} was added to S as a non-MIS vertex, then on the one hand z_{i+1} did not have MIS predecessor neighbors not in V_b as $z_{i+1} \in V_b$; on the other hand, z_{i+1} could be added to S only when all of MIS its neighboring predecessors belong to $\{z_1, z_2, \dots, z_i\} \subseteq S_v^{\pi}$. Therefore, according to the definition of S_v^{π} , it should be $z_{i+1} \in S_v^{\pi}$.

• For any $w \in \pi[\pi(z_i) + 1, \pi(z_{i+1}) - 1], w \notin S_v^{\pi}$.

Suppose we choose $w \in S_v^{\pi} \cap \pi[\pi(z_i) + 1, \pi(z_{i+1}) - 1]$ with the smallest order in π . We first rule out the case where $w \in M$. If this should be the case, the w must be adjacent to a vertex $z \in \{z_1, z_2, \dots, z_i\}$; this is not possible because w would have been added to T, when z was added to S on line-15, and then later it would be added to S.

Now we suppose $w \notin M$. By definition of S_v^{π} and the inductive hypothesis, all preceding MIS neighbors of w belong to $\{z_1, z_2, \cdots, z_i\}$. Let $z \in \{z_1, z_2, \cdots, z_i\}$ be the one with the smallest order among its MIS neighbors, and suppose $2^k < \pi(z) \le 2^{k+1}$. Since z is the MIS vertex that dominates w with the smallest order, it must be $w \in V_k$, and therefore when z was added to S on line-5, w would be added to T on line-8, and later to S on line-15, which is a contradiction.

Lemma 14. Algorithm 3 correctly restores the greedy MIS with respect to π .

Proof. We only need to consider the case when $S = S_v^{\pi} \neq \emptyset$ since otherwise no changes are made to the greedy MIS. We first claim that none of the vertices outside S need to change their status

in the greedy MIS. This is because, on the one hand, for any $z \in M \setminus S$, by Lemma 8 we know $I_z^{\pi} \cap S = \emptyset$, and so any changes within S cannot affect z; on the other hand, for any $z \in V \setminus (M \cup S)$, by Lemma 8, there exists a vertex from $M \setminus S$ that dominates z as a predecessor, and therefore z stays a non-MIS vertex, irrespective of changes in S.

Secondly, we claim that recomputing the greedy MIS on $G[S \setminus \{v\}]$ or G[S], depending on whether the update is an insertion or a deletion, has no conflicts with MIS vertices in $M \setminus S$. This is because, again by Lemma 8, for any $z \in S$ that was originally a non-MIS vertex, z is not adjacent to any MIS vertex from $M \setminus S$, and so adding z to M has no conflicts with vertices in $M \setminus S$.

Lemma 15. In each iteration of the outermost loop of Algorithm 2, by the time when line-2 is executed, V_k is already equal to $V \setminus (M_{2^k} \cup \Gamma(M_{2^k}))$.

Proof. We prove the claim by an induction on the value of $\pi(z)$. For the base case where z = v, k = b, note that the only possible change to V_b is v: if the edge update is an insertion, then v would leave V_b ; if the edge update is a deletion, then v might join V_b . In both cases, we have already fixed it right before recomputing the greedy MIS on $G[S \setminus \{v\}]$ or G[S]. Since we turn to fix subgraphs $G_b, G_{b+1}, \dots, G_{\log n-1}$ after we have finished restoring the greedy MIS, it should be $V_b = V \setminus (M_{2^b} \cup \Gamma(M_{2^b}))$ at the beginning of Algorithm 2.

Next we turn to look at the inductive step. We first argue that any vertex w that leaves $V \setminus (M_{2^k} \cup \Gamma(M_{2^k}))$ due to changes in S has already been removed from V_k in previous iterations. This is because we iterate over S in the increasing order with respect to π , and we must have already visited another vertex $z' \in S \cap M$ with $2^l < \pi(z') \le 2^{l+1} \le 2^k$ who is the earliest neighbor of w. By the inductive hypothesis, when z' was enumerated in the for-loop, $V_l = V \setminus (M_{2^l} \cup \Gamma(M_{2^l}))$, and thus w is removed from V_k by then.

Secondly we argue that any vertex w that joins $V \setminus (M_{2^k} \cup \Gamma(M_{2^k}))$ due to changes in S has already been added to V_k in previous iterations. For w to join $V \setminus (M_{2^k} \cup \Gamma(M_{2^k}))$, it must have lost all of its MIS neighbors whose order is less or equal to 2^k . Let $z' \in S \setminus M$ be the one with smallest order and assume $2^l < \pi(z') \le 2^{l+1} \le 2^k$, and so z' must have been removed from M by Algorithm 3. By the inductive hypothesis, by the time when z' was enumerated by Algorithm 2, we fix all old neighbors of z' in V_l , which include w, and hence w's memberships in $G_l, G_{l+1}, \dots, G_{\log n-1}$ were already recomputed from scratch by then.

Corollary 16. The update algorithm correctly maintains subgraphs $G_0, G_1, \dots, G_{\log n-1}$ by the end of its execution.

3.4 Running time analysis

Define \mathcal{B} to be the set of all permutations π such that there exists an index $0 \le k \le \log n - 1$ for which $\Delta(G_k) \ge \Omega(n \log n/2^k)$ either before or after the edge update; we need to emphasize it here that the constant hidden in the $\Omega(\cdot)$ notation is larger that the constant hidden in the notation $O(\cdot)$ in the statement of Lemma 7.

Lemma 17. Let a,b be fixed integers. Denote $\mathcal{E} = \{\pi(u) < \pi(v), \pi(u) \in [2^a + 1, 2^{a+1}], \pi(v) \in [2^b + 1, 2^{b+1}]\}$. Let T_0 be the set of all vertices that have once belonged to T, and let T_1 be the set of all vertices that need to change their memberships among subgraphs $G_{b+1}, \dots, G_{\log n-1}$ during the

execution of Algorithm 2. Note that in the easy cases where $S_v^{\pi} = \emptyset$, we have $T = T_0 = T_1 = \emptyset$. Then we have the following bound on the conditional expectation:

$$\mathbb{E}_{\pi}[|T_0 \cup T_1| \mid \mathcal{E}] = O(n \log^2 n / 2^b + n^2 \cdot \Pr_{\pi}[\pi \in \mathcal{B} \mid \mathcal{E}])$$

We break the proof of the above lemma into several steps.

Lemma 18. $\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}] = O(n/2^b).$

Proof. If a < b, then $\pi(u)$ belongs to $[1,2^b]$. Directly apply Lemma 11 by fixing an arbitrary position $\pi(u) \in [2^a+1,2^{a+1}]$ and setting $C=\pi(u), A=2^b, B=2^{b+1}$, and then we would have $\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}] \leq n/(2^{b+1}-A) = n/2^b$. If a=b, then $\pi(u), \pi(v) \in [2^b+1,2^{b+1}]$. Apply Lemma 12 with $A=2^b, B=2^{b+1}$, and then we also have $\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}] \leq n/2^{b-1}$.

Fix any set S such that $v \in S \subseteq V$, as well as any relative order σ_+ on S and any relative order σ_- on $V \setminus S$, such that there exists a permutation π with $S_v^{\pi} = S$, $\pi_S = \sigma_+$, $\pi_{V \setminus S} = \sigma_-$. Therefore, we can further decompose the conditional expectations as follows:

$$\mathbb{E}_{\pi}[|T_0 \cup T_1| \mid \mathcal{E}] = \sum_{S, \sigma_+, \sigma_-} \Pr_{\pi}[S_v^{\pi} = S, \pi_S = \sigma_+, \pi_{V \setminus S} = \sigma_- \mid \mathcal{E}]$$

$$\cdot \mathbb{E}_{\pi}[|T_0 \cup T_1| \mid \mathcal{E}, S_v^{\pi} = S, \pi_S = \sigma_+, \pi_{V \setminus S} = \sigma_-]$$

Therefore, it suffices to study the upper bound on $\mathbb{E}_{\pi}[|T_0 \cup T_1| \mid \mathcal{E}, S_v^{\pi} = S, \pi_S = \sigma_+, \pi_{V \setminus S} = \sigma_-]$. For notational convenience, define $\Omega = \{\pi \mid \mathcal{E}, \pi_S = \sigma_+, \pi_{V \setminus S} = \sigma_-\}$. By Lemma 9, if there exists one $\pi \in \Omega$ such that $S_v^{\pi} = S$, then all $\pi \in \Omega$ would satisfy $S_v^{\pi} = S$; plus $\forall \pi \in \Omega$, all M_{π} 's are the same in the old graph before the edge update, which we can safely denote as a common MIS M.

First we study the conditional expectation $\mathbb{E}_{\pi}[|T_0| \mid \pi \in \Omega]$. As can be seen from Algorithm 1, any vertex, which belonged to M before the edge update, that has once been added to T must have eventually joined S. So we only need to bound the total number of vertices in $T_0 \setminus M$. Again by Algorithm 1, any $w \in T_0 \setminus M$ was added to T by an MIS predecessor $z \in S$ on line-8. Therefore, $|T_0 \setminus M|$ is bounded by the sum of (lower priority) neighbors of all $z \in S \cap M$. So it suffices to study individual contribution of all $z \in S \cap M$ to $T_0 \setminus M$. Formally, $\forall z \in S \setminus M, w \in T_0 \setminus M$, we say z contributes w to T_0 if w was added to T on line-8 when z is being processed. First we notice a basic property of T_0 .

Lemma 19. $v = \arg\min_{z \in T_0} \{\pi(z)\}, \text{ for any } \pi \in \Omega.$

Proof. This property is directly guaranteed by Algorithm 1: on line-8 or line-17, it only adds vertices w to T whose order is strictly larger than z who has just entered S. Since v is the first vertex that has been added to S, all vertices that join T should have larger order than v.

Lemma 20. For any k > b, $\mathbb{E}_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]|] < \frac{2^k |S|}{n}$.

Proof. Decompose the expectation as following:

$$\mathbb{E}_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]|]$$

$$= \sum_{j=2^{b+1}}^{2^{b+1}} \Pr_{\pi \in \Omega}[\pi(v) = j] \cdot \mathbb{E}_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]| \mid \pi(v) = j]$$

When $\pi(v) = j$, the rest of $S \setminus \{v\}$ are free to choose positions on [j+1,n], as v always takes the smallest order among S, which is guaranteed by Lemma 19 as $S \subseteq T_0$. Hence, for any $l \in [1, \min\{2^k - j, |S| - 1\}]$, conditioned on $\pi(v) = j$, the probability that $|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]| = l$ is equal to $\frac{\binom{2^k - j}{l} \cdot \binom{n - 2^k}{|S| - 1 - l}}{\binom{n - j}{|S| - 1}}$. Therefore, the expectation is computed as follows:

$$\mathbb{E}_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]| \mid \pi(v) = j]$$

$$= \sum_{l=1}^{\min\{2^k - j, |S| - 1\}} l \cdot \Pr_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]| = l \mid \pi(v) = j]$$

$$= \sum_{l=1}^{\min\{2^k - j, |S| - 1\}} l \cdot \frac{\binom{2^k - j}{l} \cdot \binom{n - 2^k}{|S| - 1 - l}}{\binom{n - j}{|S| - 1}}$$

$$= \sum_{l=1}^{\min\{2^k - j, |S| - 1\}} (2^k - j) \cdot \binom{2^k - j - 1}{l - 1} \cdot \binom{n - 2^k}{|S| - 1 - l} / \binom{n - j}{|S| - 1}$$

$$= (2^k - j) \cdot \binom{n - j - 1}{|S| - 2} / \binom{n - j}{|S| - 1} = \frac{(2^k - j)(|S| - 1)}{n - j} < \frac{2^k |S|}{n}$$

Finally, we have:

$$\mathbb{E}_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]|] = \sum_{j=2^b+1}^{2^{b+1}} \Pr_{\pi \in \Omega}[\pi(v) = j]$$

$$\cdot \mathbb{E}_{\pi \in \Omega}[|(S \setminus \{v\}) \cap \pi[2^b + 1, 2^k]| \mid \pi(v) = j]$$

$$< \sum_{j=2^b+1}^{2^{b+1}} \Pr_{\pi \in \Omega}[\pi(v) = j] \cdot \frac{2^k |S|}{n} = \frac{2^k |S|}{n}$$

Lemma 21. The expected contribution of all $z \in S \cap M \setminus \{v\}$ to T_0 is at most $O(|S| \log^2 n + |S| n \cdot \Pr_{\pi \in \Omega}[\pi \in \mathcal{B}])$.

Proof. Consider any index $b \leq k \leq \log n - 1$. When $2^k < \pi(z) \leq 2^{k+1}$, the total number of neighbors of z in V_k is at most $O(n \log n/2^k + n \cdot \mathbf{1}[\pi \in \mathcal{B}])$, by definition of \mathcal{B} . Therefore, by Lemma 20 the expected total contribution of $z \in S \cap M \setminus \{v\}$ to T_0 that lies in $[2^k + 1, 2^{k+1}]$ is bounded by $O(|S| \log n + 2^k |S| \cdot \mathbf{1}[\pi \in \mathcal{B}])$. Taking a summation over all k we can finalize the proof.

By Lemma 21, we have an upper bound on conditional expectation:

$$\mathbb{E}_{\pi}[|T_0| \mid \pi \in \Omega] \le O(|S| \log^2 n + n \log n / 2^b + |S| n \cdot \Pr_{\pi \in \Omega}[\pi \in \mathcal{B}])$$

Here we have an extra additive term as an upper bound on the contribution of v to T_0 .

Next we try to study $\mathbb{E}_{\pi}[|T_1| \mid \pi \in \Omega]$. By Algorithm 2, for any $z \in S$ that has changed its status in M, we go over some of the neighbors of z and update their memberships in $G_{k+1}, \dots, G_{\log n-1}$

using brute force, and by definition these neighbors would belong to T_1 . Similar to what we did with T_0 , we say z contributes these neighbors to T_1 . Next we need to carefully analyze the total number of these neighbors.

Lemma 22. The expected contribution of all $z \in S \setminus \{v\}$ to T_1 is at most $O(|S| \log^2 n + |S| n \cdot \Pr_{\pi \in \Omega}[\pi \in \mathcal{B}])$.

Proof. Let $k \in [b, \log n - 1]$ be any index. Assume $2^k < \pi(z) \le 2^{k+1}$. Consider the following two possibilities.

- z has joined M during the update algorithm. In this case, z must belong to V_k and thus the total number of its neighbors in $V \setminus (M_{2^k} \cup \Gamma(M_{2^k}))$ is at most $O(n \log n/2^k + n \cdot \mathbf{1}[\pi \in \mathcal{B}])$, and by Lemma 15 we know $V_k = V \setminus (M_{2^k} \cup \Gamma(M_{2^k}))$ by the time z is processed by Algorithm 2, and thus the total number of its neighbors in V_k is at most $O(n \log n/2^k + n \cdot \mathbf{1}[\pi \in \mathcal{B}])$.
- z has just left M during the update algorithm. In this case, z was selected by M and thus belonged to V_k before the update. As Algorithm 2 only iterates over z's old neighbors in V_k , the total number of these neighbors is also bounded by $O(n \log n/2^k + n \cdot \mathbf{1}[\pi \in \mathcal{B}])$.

In any case, the contribution of z to T_1 is at most $O(n \log n/2^k + n \cdot \mathbf{1}[\pi \in \mathcal{B}])$. Therefore, by Lemma 20 the expected total contribution of $z \in S \cap M \setminus \{v\}$ to T_1 that lies in $[2^k + 1, 2^{k+1}]$ is bounded by $O(|S| \log n + 2^k |S| \cdot \mathbf{1}[\pi \in \mathcal{B}])$. Taking a summation over all k we can finalize the proof.

Taking a summation over all $z \in S \setminus \{v\}$ that has changed its status in the MIS we have:

$$\mathbb{E}_{\pi}[|T_1| \mid \pi \in \Omega] \le O(|S| \log^2 n + n \log n / 2^b + |S| n \cdot \Pr_{\pi \in \Omega}[\pi \in \mathcal{B}])$$

Here the extra additive term also stands for v's contribution to T_1 .

Proof of Lemma 17. To summarize, by Lemma 21 and Lemma 22, we have proved:

$$\mathbb{E}_{\pi}[|T_0 \cup T_1| \mid \pi \in \Omega] \le O(|S| \log^2 n + n \log n / 2^b + |S| n \cdot \Pr_{\pi \in \Omega}[\pi \in \mathcal{B}])$$

Recall a previous decomposition we would then have:

$$\mathbb{E}_{\pi}[|T_{0} \cup T_{1}| \mid \mathcal{E}] = \sum_{S,\sigma_{+},\sigma_{-}} \Pr_{\pi}[S_{v}^{\pi} = S, \pi_{S} = \sigma_{+}, \pi_{V \setminus S} = \sigma_{-} \mid \mathcal{E}]$$

$$\cdot \mathbb{E}_{\pi}[|T_{0} \cup T_{1}| \mid \mathcal{E}, S_{v}^{\pi} = S, \pi_{S} = \sigma_{+}, \pi_{V \setminus S} = \sigma_{-}]$$

$$\leq \sum_{S,\sigma_{+},\sigma_{-}} O(|S| \log^{2} n + n \log n / 2^{b}$$

$$+ |S|n \cdot \Pr_{\pi \in \Omega}[\pi \in \mathcal{B}]) \cdot \Pr_{\pi}[S_{v}^{\pi} = S, \pi_{S} = \sigma_{+}, \pi_{V \setminus S} = \sigma_{-} \mid \mathcal{E}]$$

$$= \sum_{S} O(|S| \log^{2} n \cdot \Pr_{\pi}[S_{v}^{\pi} = S \mid \mathcal{E}]) + O(n \log n / 2^{b})$$

$$+ \sum_{S,\sigma_{+},\sigma_{-}} |S|n \cdot \Pr_{\pi}[\pi \in \mathcal{B} \mid \mathcal{E}, S_{v}^{\pi} = S, \pi_{S} = \sigma_{+}, \pi_{V \setminus S} = \sigma_{-}]$$

$$\cdot \Pr_{\pi}[S_{v}^{\pi} = S, \pi_{S} = \sigma_{+}, \pi_{V \setminus S} = \sigma_{-} \mid \mathcal{E}]$$

$$\leq O(\mathbb{E}_{\pi}[|S_{v}^{\pi}| \log^{2} n \mid \mathcal{E}] + n \log n / 2^{b} + n^{2} \Pr_{\pi}[\pi \in \mathcal{B} \mid \mathcal{E}])$$

$$\leq O(n \log^{2} n / 2^{b} + n^{2} \Pr_{\pi}[\pi \in \mathcal{B} \mid \mathcal{E}])$$

The last inequality holds by Lemma 18.

To remove the extra term $\Pr_{\pi}[\pi \in \mathcal{B} \mid \mathcal{E}]$, apply Lemma 7 by fixing values of $\pi(u), \pi(v)$ and taking union bound over all k equal to powers of 2, we would know that $\pi \notin \mathcal{B}$ with high probability, namely $\Pr_{\pi}[\pi \in \mathcal{B} \mid \mathcal{E}] \leq n^{-2} \log n$, and thus $\mathbb{E}_{\pi}[|T_0 \cup T_1| \mid \mathcal{E}] \leq O(n \log^2 n/2^b + \log n) = O(n \log^2 n/2^b)$.

By definition of T_0 and T_1 , the total update time is proportional to $\Delta(G_b) \cdot (|T_0| + |T_1|)$ whose expectation is then bounded by $O(n^2 \log^3 n/2^{2b})$. Since fixing the memberships of v takes time at most $O(n \log^2 n/2^a)$, it immediately says that the expected update time is $O(n^2 \log^3 n/2^{2b} + n \log^2 n/2^a)$. Since the adversary is oblivious to the randomness used in the algorithm, the probability of inserting an edge between $\pi[2^a + 1, 2^{a+1}]$ and $\pi[2^b + 1, 2^{b+1}]$ is $O(2^{a+b}/n^2)$. Hence, the expected running time would be $O(2^{a+b}/n^2 \cdot (n^2 \log^3 n/2^{2b} + n \log^2 n/2^a)) = O(2^{a-b} \log^3 n + \log^2 n)$. Summing over all different indices a, b, the total time would be $O(\log^4 n)$.

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References

- [1] Noga Alon, László Babai, and Alon Itai. A fast and simple randomized parallel algorithm for the maximal independent set problem. *J. Algorithms*, 7(4):567–583, December 1986.
- [2] Sepehr Assadi, Krzysztof Onak, Baruch Schieber, and Shay Solomon. Fully dynamic maximal independent set with sublinear update time. In *Proceedings of the 50th Annual ACM SIGACT*

- Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018, pages 815–826, 2018.
- [3] Sepehr Assadi, Krzysztof Onak, Baruch Schieber, and Shay Solomon. Fully dynamic maximal independent set with sublinear in n update time. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1919–1936. SIAM, 2019.
- [4] S. Behnezhad, M. Derakhshan, M. Hajiaghayi, C. Stein, and M. Sudan. Fully dynamic maximal independent set with polylogarithmic update time. *To appear in FOCS*, 2019.
- [5] Aaron Bernstein, Sebastian Forster, and Monika Henzinger. A deamortization approach for dynamic spanner and dynamic maximal matching. In *Proceedings of the Thirtieth Annual* ACM-SIAM Symposium on Discrete Algorithms, pages 1899–1918. SIAM, 2019.
- [6] Keren Censor-Hillel, Elad Haramaty, and Zohar Karnin. Optimal dynamic distributed mis. In Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing, pages 217–226. ACM, 2016.
- [7] Sebastian Daum, Seth Gilbert, Fabian Kuhn, and Calvin Newport. Leader election in shared spectrum radio networks. In *Proceedings of the 2012 ACM Symposium on Principles of Distributed Computing*, PODC '12, pages 215–224. ACM, 2012.
- [8] Yuhao Du and Hengjie Zhang. Improved algorithms for fully dynamic maximal independent set. arXiv preprint arXiv:1804.08908, 2018.
- [9] Manoj Gupta and Shahbaz Khan. Simple dynamic algorithms for maximal independent set and other problems. arXiv preprint arXiv:1804.01823, 2018.
- [10] Nathan Linial. Distributive graph algorithms global solutions from local data. In *Proceedings* of the 28th Annual Symposium on Foundations of Computer Science, SFCS '87, pages 331–335, Washington, DC, USA, 1987. IEEE Computer Society.
- [11] Michael Luby. A simple parallel algorithm for the maximal independent set problem. SIAM J. Comput., 15(4):1036–1055, November 1986.
- [12] Krzysztof Onak, Baruch Schieber, Shay Solomon, and Nicole Wein. Fully dynamic mis in uniformly sparse graphs. arXiv preprint arXiv:1808.10316, 2018.
- [13] Dongxiao Yu, Yuexuan Wang, Qiang-Sheng Hua, and Francis C.M. Lau. Distributed $(\delta + 1)$ -coloring in the physical model. *Theor. Comput. Sci.*, 553(C):37–56, December 2014.

A Missing Proofs

A.1 Proof of Lemma 9

Proof. Here we present an conceptually simpler proof than the one presented in [6]. Notice that it suffices to consider the case where $\sigma(z) = \pi(z), \forall z \notin \{x,y\}$ and $\sigma(x) = \pi(y), \sigma(y) = \pi(x)$, where x,y is an arbitrary pair of consecutive vertices in π such that $x \in S$ and $y \notin S$. As $\sigma(u) < \sigma(v), \pi(u) < \pi(v)$, it can never be x = v and y = u. Let $M = M_{\pi}$ be the greedy MIS in the old graph. The proof follows from the two statements below.

Claim 23. M was also the greedy MIS on the old version of G with respect to σ .

Proof of claim. Recall from Lemma 5, M is the greedy MIS with respect to σ in the old graph if M is an MIS and for all $z \in V \setminus M$, $I_z^{\sigma} \cap M \neq \emptyset$. The first half is easy: M was the greedy MIS in the old version of G with respect to π , so M is certainly an MIS in the old graph. Now we turn to verify the greedy MIS constraints.

Since σ agrees with π on every vertex except for $\{x,y\}$, we only need to verify $\forall z \in \{x,y\} \setminus M$, $I_z^{\sigma} \cap M \neq \emptyset$. We can assume x,y are neighbors in the updated graph G; otherwise switching the orders between x,y in π does not affect the greedy MIS constraint. Consider several cases.

- $x \in M, y \notin M$. In this case, if $\pi(y) > \pi(x)$, then $\sigma(x) > \sigma(y)$, and thus $x \in I_y^{\sigma} \cap M \neq \emptyset$. If $\pi(y) < \pi(x)$, then since $y \notin S$, $I_y^{\pi} \cap M \setminus S$ must be nonempty, and so there exists $z \in I_y^{\pi} \cap M \setminus S$ that dominates y. As $\sigma(z) = \pi(z) < \pi(x) = \sigma(y)$, $I_y^{\sigma} \cap M$ is also nonempty.
- $x, y \notin M$. Since x, y are consecutive in π , switching their positions in σ does not affect the invariant that $I_z^{\sigma} \cap M \neq \emptyset, \forall z \in \{x, y\}$.
- $x \notin M, y \in M$. By definition of S, $\pi(y) < \pi(x)$ as otherwise y would belong to S, and so $\sigma(y) > \sigma(x)$. If $x \neq v$, then x cannot belong to S by definition since x is dominated by some MIS vertices outside of S. If x = v, then $y \neq u$ as $\sigma(v) > \sigma(u)$. Right after the edge update x is still dominated by a vertex in M, namely y, which is also a predecessor in π , so $S = \emptyset$ which is a contradiction.

Claim 24. $S_v^{\sigma} = S$.

Proof of claim. By the previous claim, M was also the greedy MIS on G with respect to order σ . We first argue that $S_v^{\sigma} \supseteq S$. To do this, we prove by an induction that for every $i \ge 0$, $S_i \subseteq S_v^{\sigma}$; we refer readers to the definition of influenced sets for the meaning of S_i , where S_i 's are defined with respect to permutation π , not σ .

- Basis. For i=0, to argue $v \in S_v^{\sigma}$ we only need to prove $S_v^{\sigma} \neq \emptyset$. As $S_v^{\pi} \neq \emptyset$, the edge update can only be an insertion (u,v) and $u,v \in M$, or an edge deletion (u,v) and $u \in M, v \notin M$ plus that u is the only MIS predecessor that dominates v. Since σ and π agree on all vertices whose orders are $\leq \pi(v)$, v would also violate its greedy MIS constraint with respect to σ , and so $S_v^{\sigma} \neq \emptyset$.
- Induction. Suppose we already have $S_{i-1} \subseteq S_v^{\sigma}$. Then, by Lemma 8, any $z \in M$ such that $S_{i-1} \cap I_z^{\sigma} \neq \emptyset$ should belong to S_v^{σ} . Since π and σ have the same relative order on S, $S_{i-1} \cap I_z^{\sigma}$ would be the same as $S_{i-1} \cap I_z^{\pi}$ for any $z \in S_i \cap M$. On the other hand, for any $z \in S_i \setminus (M \cup \{v\})$, we claim $I_z^{\sigma} \cap M$ is also equal to $I_z^{\pi} \cap M$. The only possible violation comes from the case that z = x and $y \in M$. However this is also not possible: if $\pi(y) > \pi(x)$, then as $y \notin S$, by definition when $x \neq u$, it would have been excluded from S, and otherwise if x = v we would have $S_v^{\pi} = \emptyset$; if $\pi(y) < \pi(x)$, then y would have been added to S; both lead to contradictions.

Therefore, by definition of S_i , we also have $S_i \subseteq S_v^{\sigma}$.

To prove $S_v^{\sigma} \subseteq S$, by Lemma 8 it suffices to verify that (1) $\forall z \in M, \ I_z^{\sigma} \cap S \neq \emptyset$ iff $z \in S$; (2) $\forall z \notin M, \ I_z^{\sigma} \cap M \subseteq S$ iff $z \in S$. As σ is equal to π except for x, y, we only need to consider $z \in \{x, y\}$ in (1)(2). We can assume x, y are adjacent; otherwise switching the orders between x, y in π does not affect the invariant. Then it can never be the case where $x \notin M, y \in M$ as it would contradict the definition of S. So it is either $x \in M, y \notin M$ or $x, y \notin M$. Consider two cases.

- $x \in M, y \notin M$. In this case, $I_x^{\sigma} \cap S = \emptyset$ always holds as switching the positions between x, y does not affect the equality $I_x^{\sigma} \cap S = I_x^{\pi} \cap S \neq \emptyset$.
 - If $\pi(y) < \pi(x)$, then since $y \notin M$, it must be $I_z^{\pi} \cap M \neq \emptyset$, and because $y \notin S$, there exists $z \in I_z^{\pi} \cap M \setminus S$. So $\sigma(z) = \pi(z) < \pi(y) = \sigma(x)$. By the previous claim we already know $M_{\sigma} = M$, and so $I_z^{\sigma} \cap M \nsubseteq S$. If $\pi(y) > \pi(x)$, then $I_z^{\sigma} \cap S \subseteq I_z^{\pi} \cap S = \emptyset$.
- $x, y \notin M$. Since x, y are consecutive in π , switching their positions in σ does not change $I_z^{\sigma} \cap M, \forall z \in \{x, y\}.$
- $x \notin M, y \in M$. By definition of S, $\pi(y) < \pi(x)$ as otherwise y would belong to S, and so $\sigma(y) > \sigma(x)$. If $x \neq u$, then x cannot belong S by definition since x is dominated by some MIS vertices outside of S. If x = v, then $y \neq u$ as $\sigma(v) > \sigma(u)$. Right after the edge update x is still dominated by a vertex in M, namely y, which is also a predecessor in π , so $S = \emptyset$ which is a contradiction.

A.2 Proof of Lemma 10

Proof. As the lemma is stated in a slightly different way from [6], for completeness we also present a proof here. Define an intermediate permutation τ by this operation: remove v from order σ and reinsert it back right after u. Then $\tau(u) < \tau(v), \tau_S = \pi_S, \tau_{V \setminus S} = \pi_{V \setminus S}$, and thus by Lemma 9 we have $S_v^{\tau} = S$. Namely, τ and π satisfy the same condition in the statement of the lemma.

Let $w = \arg\min_{x \in S \setminus \{v\}} \{\tau(x)\}$. First we argue that w and v are neighbors. If w was in M_{τ} , then by the inductive definition of S_v^{τ} , there exists $z \in S \setminus M_{\tau}$ such that z is a predecessor neighbor of w. By minimality of w, z must be equal to v, and hence w and v are adjacent. If w was not in M_{τ} , then it has an MIS predecessor $z \in S \cap M_{\tau}$, similarly by minimality of w, z must be equal to v, and hence w and v are adjacent.

Recalling the relation between τ and σ , we can view σ as a permutation derived from τ by first removing v from τ and then reinsert v back to τ at a certain position somewhere behind w. We claim that right after we remove v from τ before reinsertion, w belongs to the greedy MIS M_{τ} with respect to the current τ (which is without v). Consider the only two cases where S_v^{τ} could be nonempty.

- The edge update is an insertion and both of u, v were in M_{τ} . After the removal of v, w is no longer dominated by any MIS predecessor in M_{τ} , hence w must join M_{τ} .
- The edge update is a deletion, and u was in M_{τ} while v was not in M_{τ} , plus that u is the only MIS predecessor that dominates v. Since v was not in M_{τ} , then by minimality of $\tau(w)$ among $S \setminus \{v\}$, the only predecessor of w in S was v, and thus $w \in M_{\tau}$ before and after v's removal.

When we insert v back to τ at some position after w, which produces permutation σ , since w is now an MIS predecessor of v, v does not belong to M_{σ} . If the edge update is insertion then no changes would be made to M_{σ} and thus $S_v^{\sigma} = \emptyset$; if the edge update is deletion, then since v has a neighboring MIS predecessor other than u, which is w, M_{σ} would also stay unchanged, and thus $S_v^{\sigma} = \emptyset$.

A.3 Proof of Lemma 11

Proof. For notational convenience, define $\mathcal{E} = \{\pi(u) = C, \pi(v) \in [A+1, B]\}$. For any vertex set S containing v, and partial orders σ_+, σ_- on $S \setminus \{v\}$ and $V \setminus S$, with the property that there exists at least one permutation π that satisfies event \mathcal{E} , as well as $S_v^{\pi} = S, \pi_{S \setminus \{v\}} = \sigma_+, \pi_{V \setminus S} = \sigma_-$, define a set of permutations

$$\Omega_{S,\sigma_+,\sigma_-} = \{ \pi \mid \mathcal{E}, \pi_{S \setminus \{v\}} = \sigma_+, \pi_{V \setminus S} = \sigma_-, \min_{z \in S} \{\pi(z)\} > A \}$$

By Lemma 9 and Lemma 10, for any $\pi \in \Omega_{S,\sigma_+,\sigma_-}$, we have $S_v^{\pi} = S$ when $\pi(v) = \min_{z \in S} \{\pi(z)\}$, and $S_v^{\pi} = \emptyset$ otherwise. Here is a basic property of $\Omega_{S,\sigma_+,\sigma_-}$.

Claim 25. For any two different sets

$$\Omega_{S,\sigma_+,\sigma_-} = \{\pi \mid \mathcal{E}, \pi_{S \setminus \{v\}} = \sigma_+, \pi_{V \setminus S} = \sigma_-, \min_{z \in S} \{\pi(z)\} > A\}$$

and

$$\Omega_{S',\sigma'_{+},\sigma'_{-}} = \{ \pi \mid \mathcal{E}, \pi_{S' \setminus \{v\}} = \sigma'_{+}, \pi_{V \setminus S'} = \sigma'_{-}, \min_{z \in S} \{\pi(z)\} > A \}$$

we claim $\Omega_{S,\sigma_+,\sigma_-}$ and $\Omega_{S',\sigma'_+,\sigma'_-}$ are disjoint.

Proof of claim. Suppose otherwise there exists $\tau \in \Omega_{S,\sigma_+,\sigma_-} \cap \Omega_{S',\sigma'_+,\sigma'_-}$. By definition, there exists $\pi \in \Omega_{S,\sigma_+,\sigma_-}$ that satisfies event \mathcal{E} , as well as $S_v^{\pi} = S, \pi_{S\setminus\{v\}} = \sigma_+, \pi_{V\setminus S} = \sigma_-$. By Lemma 10, v takes the minimum in π among S.

Remove v from τ and reinsert v back to τ right at position A+1. We claim τ stays in $\Omega_{S,\sigma_+,\sigma_-} \cap \Omega_{S',\sigma'_+,\sigma'_-}$; this is because removal and reinsertion of v preserves τ 's induced order on $S \setminus \{v\}, V \setminus S$ and $S' \setminus \{v\}, V \setminus S'$. Now, since v takes the minimum among S in τ , we have $\tau_S = \pi_S, \tau_{V \setminus S} = \pi_{V \setminus S}$. By Lemma 9, $S_v^{\tau} = S_v^{\pi} = S$. Similarly we can also have $S_v^{\tau} = S'$. Therefore, S = S'. As $\tau \in \Omega_{S',\sigma'_+,\sigma'_-}$, we know immediately $\sigma_+ = \tau_S = \tau_{S'} = \sigma'_+, \sigma_- = \tau_{V \setminus S} = \tau_{V \setminus S'} = \sigma'_-$, which is a contradiction that Ω and Ω' are different.

By this claim, we can decompose the expectation as a sum of conditional ones:

$$\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}] = \sum_{S, \sigma_+, \sigma_-} \Pr_{\pi}[\pi \in \Omega_{S, \sigma_+, \sigma_-} \mid \mathcal{E}] \cdot \mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}, \pi \in \Omega_{S, \sigma_+, \sigma_-}]$$

So it suffices to compute each term in the summation. Fix any S, σ_+, σ_- and $\Omega = \Omega_{S,\sigma_+,\sigma_-}$ Notice that by Lemma 9 and Lemma 10 we have:

$$\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}, \pi \in \Omega] = |S| \cdot \Pr_{\pi}[\pi(v) = \min_{z \in S} \{\pi(z)\} \mid \mathcal{E}, \pi \in \Omega]$$

To bound the probability $\Pr_{\pi}[\pi(v) = \min_{z \in S} \{\pi(z)\} \mid \mathcal{E}, \pi \in \Omega]$, on the one hand, any permutation $\pi \in \Omega$ can be constructed by picking an arbitrary position for v among [A+1, B], and then

assign arbitrary positions for $S \setminus \{v\}$, so $|\Omega| = (B - A) \cdot \binom{n - A - 1}{|S| - 1}$. On the other hand, the total number of permutations such that v takes the minimum among S is $\binom{n - A}{|S|} - \binom{n - B}{|S|}$. Therefore, as π is uniformly drawn from Ω , we have:

$$\Pr_{\pi}[\pi(v) = \min_{z \in S} \{\pi(z)\} \mid \mathcal{E}, \pi \in \Omega] = \frac{\binom{n-A}{|S|} - \binom{n-B}{|S|}}{(B-A) \cdot \binom{n-A-1}{|S|-1}}$$
$$= \frac{\binom{n-A}{|S|} - \binom{n-B}{|S|}}{(B-A) \cdot \binom{n-A}{|S|} \cdot \frac{|S|}{n-A}} < \frac{n-A}{(B-A)|S|}$$

Hence, $\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}, \pi \in \Omega] = |S| \cdot \Pr_{\pi}[\pi(v) = \min_{z \in S} \{\pi(z)\} \mid \mathcal{E}, \pi \in \Omega] < \frac{n-A}{B-A}$. Since all Ω are disjoint, ranging over all different choices for S, σ_+, σ_- , we have

$$\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \pi(u) = C, \pi(v) \in [A+1, B]] < \frac{n-A}{B-A} < \frac{n}{B-A}$$

A.4 Proof of Lemma 12

Proof. For u, v to both lie in [A+1, B], B must be larger than A+1. For notational convenience, define $\mathcal{E} = \{A < \pi(u) < \pi(v) \leq B\}$. We decompose the expectation as:

$$\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}] = \sum_{k=A+1}^{B-1} \Pr_{\pi}[\pi(u) = k \mid \mathcal{E}] \cdot \mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}, \pi(u) = k]$$
$$= \sum_{k=A+1}^{B-1} \frac{B-k}{\binom{B-A}{2}} \cdot \mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}, \pi(u) = k]$$

The second equality holds as $\Pr_{\pi}[\pi(u) = k \mid \mathcal{E}] = \frac{B-k}{\binom{B-A}{2}}$; this is because, conditioned on $\pi(u) = k$ as well as event \mathcal{E} , there are $(B-k) \cdot (n-A-2)!$ permutations π , while there are $\binom{B-A}{2} \cdot (n-A-2)!$ permutations π that satisfy event \mathcal{E} . Since π is drawn uniformly at random from the set of all permutations that satisfy event \mathcal{E} , we have $\Pr_{\pi}[\pi(u) = k \mid \mathcal{E}] = \frac{B-k}{\binom{B-A}{2}}$.

Using Lemma 11, we have:

$$\mathbb{E}_{\pi}[|S_v^{\pi}| \mid \mathcal{E}, \pi(u) = k] \le \frac{n}{B - k}$$

Therefore,

$$\mathbb{E}_{\pi}[|S_{v}^{\pi}| \mid \mathcal{E}] = \sum_{k=A+1}^{B-1} \frac{B-k}{\binom{B-A}{2}} \cdot \mathbb{E}_{\pi}[|S_{v}^{\pi}| \mid \mathcal{E}, \pi(u) = k]$$

$$< \sum_{k=A+1}^{B-1} \frac{B-k}{(B-A-1)(B-A)/2} \cdot \frac{n}{B-k} < \frac{2n}{B-A}$$