# Optimal Streaming Approximations for all Boolean Max-2CSPs and Max- $k$ SAT 

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#### Abstract

We prove tight upper and lower bounds on approximation ratios of all Boolean Max-2CSP problems in the streaming model. Specifically, for every type of Max-2CSP problem, we give an explicit constant $\alpha$, s.t. for any $\varepsilon>0$ (i) there is an $(\alpha-\varepsilon)$-streaming approximation using space $O(\log n)$; and (ii) any $(\alpha+\varepsilon)$-streaming approximation requires space $\Omega(\sqrt{n})$. This generalizes the celebrated work of [Kapralov, Khanna, Sudan SODA 2015; Kapralov, Krachun STOC 2019], who showed that the optimal approximation ratio for Max-CUT was $1 / 2$.

Prior to this work, the problem of determining this ratio was open for all other Max-2CSPs. Our results are quite surprising for some specific Max-2CSPs. For the Max-DICUT problem, there was a gap between an upper bound of $1 / 2$ and a lower bound of $2 / 5$ [Guruswami, Velingker, Velusamy APPROX 2017]. We show that neither of these bounds is tight, and the optimal ratio for Max-DICUT is 4/9. We also establish that the tight approximation for Max-2SAT is $\sqrt{2} / 2$, and for Exact Max-2SAT it is $3 / 4$. As a byproduct, our result gives a separation between space-efficient approximations for Max-2SAT and Exact Max-2SAT. This is in sharp contrast to the setting of polynomial-time algorithms with polynomial space, where the two problems are known to be equally hard to approximate. Finally, we prove that the tight streaming approximation for Max- $k$ SAT is $\sqrt{2} / 2$ for every $k \geq 2$.


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## 1 Introduction

Maximum Boolean Constraint Satisfaction Problems, or Max-CSPs, are a central class of optimization problems, including as special cases problems such as Max-CUT, 3SAT, Graph Coloring, and Vertex Cover [CV08]. Given a set of allowed predicates $\mathcal{F}$, $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ is the optimization problem defined as follows. Every instance $\Psi$ of the problem consists of a set of Boolean variables $\mathcal{X}$, and a set of constraints applied to them. Each constraint is a predicate from $\mathcal{F}$ applied to the variables from $\mathcal{X}$ or their negations. The goal is to compute the maximum number of simultaneously satisfiable constraints. For example, Max- $k$ SAT is $\operatorname{Max}-\mathrm{CSP}\left(\mathcal{F}_{\mathrm{OR}_{\leq k}}\right)$ where $\mathcal{F}_{\mathrm{OR}_{\leq k}}$ is the set of OR predicates on at most $k$ variables.

Schaefer's famous dichotomy theorem [Sch78, TŽ16] states that for any set of allowed predicates $\mathcal{F}$, solving Max-CSP $(\mathcal{F})$ exactly is either in P or NP-hard. However, the landscape of approximation algorithms for Max-CSPs is much more complex (see [MM17] and references therein).

The Max-2CSP problem-Max-CSP where all constraints have length at most 2 -is the most studied case of Max-CSP, and it generalizes many optimization problems on graphs. Starting with the seminal work of Goemans and Williamson [GW95], a series of works [FG95, Zwi00, LLZ02] developed a 0.87401approximation algorithm for all Max-2CSPs, while under the $P \neq N P$ and Unique Games conjectures some Max-2CSPs do not admit 0.9001- and 0.87435-approximations, respectively [Hås01, TSSW00, Aus10].

In this paper, we follow the line of work [KK15, KKS15, KKSV17, GVV17, KK19, GT19] that studies the unconditional hardness of approximating Max-2CSP through the lens of streaming algorithms. Over the last decade, there has been a lot of interest in designing algorithms for processing large streams of data using limited space (see [McG14, Cha15] and references therein). The streaming model was formally defined in [AMS99, HRR98].

A streaming algorithm for a Max-2CSP problem makes one pass through the list of constraints and uses space that is sub-linear (ideally, poly-logarithmic) in the input size. ${ }^{1}$ Since the algorithm is space bounded, it cannot even store an assignment to the input variables. Thus, a streaming algorithm is required to output an estimate of the maximum number of simultaneously satisfiable constraints. Specifically, for $\alpha \in[0,1]$, an $\alpha$-approximate streaming algorithm outputs a value $v$ for which the following two conditions hold with probability $3 / 4$ : (i) there exists an assignment $\sigma$ satisfying at least $v$ constraints, and (ii) $v \geq \alpha \cdot$ val, where val is the maximum number of simultaneously satisfiable constraints.

Prior to this work, the only Max-2CSP for which we knew the optimal streaming approximation factor was Max-CUT. Max-CUT asks us to find a bipartition of the $n$ vertices of an undirected graph that maximizes the number of edges crossing the partition-called the "cut". Note that Max-CUT corresponds to the $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ where $\mathcal{F}$ contains the binary XOR predicate. ${ }^{2}$ [Zel11] shows that exact streaming algorithms for Max-CUT require quadratic space $\Omega\left(n^{2}\right)$. Since a random partition of a graph with $m$ edges has cut of expected size $m / 2$, a trivial streaming algorithm $1 / 2$-approximates Max-CUT with $O(\log m)$ space. It is also easy to see that for every $\varepsilon>0$, it suffices to store $\widetilde{O}(n)$ random edges of the graph to compute a ( $1-\varepsilon$ )-approximation of Max-CUT. A recent line of work [KKS15, KK15, KKSV17, KK19] shows that these two trivial bounds are optimal, i.e., any $(1 / 2+\varepsilon)$-approximation algorithm requires linear space $\Omega(n)$.

However, the case for directed graphs is not nearly so well understood. In the Max-DICUT problem (another special case of Max-2CSP), given a directed graph, one needs to compute the maximum number of edges going from the first to the second part of the graph under any bipartition. While [KK19, KKS15] rules out a $(1 / 2+\varepsilon)$-approximation for Max-DICUT too, the trivial algorithm gives only a $1 / 4$-approximation here. [GVV17] gives a $2 / 5$-approximation for Max-DICUT, still leaving a gap between the upper and lower bounds.

Even the hardness of Max-2SAT is not known in the streaming setting. Recall that in Max-2SAT the only allowed predicates are variables and pairwise ORs. A random assignment gives a $1 / 2$-approximation, and the classical $(\sqrt{5}-1) / 2 \approx 0.61$-approximate algorithm of [LS79] can be implemented in $O(\log n)$ space using $\ell_{1}$-sketching [Ind00, KNW10]. No non-trivial upper bounds are known for Max-2SAT.

[^1]
### 1.1 Our contribution

In this work, we resolve a natural question about the approximation guarantees of streaming algorithms for every Max-2CSP problem.

Before presenting our results, we need a way to classify Boolean functions of two variables. Let $f:\{0,1\}^{2} \rightarrow\{0,1\}$ be a function, then

- $f$ is of TR-type, or trivial, if $f$ depends on at most one of its inputs (trivial functions are the two constant functions, and the four functions which depend on one of the inputs);
- $f$ is of OR-type if the truth table of $f$ has exactly one 0 and three 1 s ;
- $f$ is of XOR-type if $f$ depends on both inputs, and the truth table of $f$ has exactly two 0 s and two 1 s;
- $f$ is of AND-type if the truth table of $f$ has exactly three 0 s and one 1.

If a set of allowed predicates $\mathcal{F}$ contains only constraints of a type $\Lambda \in\{O R, X O R, A N D\}$, then the corresponding Max-2CSP problem is called Max-2E $\Lambda$ (2-Exact- $\Lambda$, meaning that all constraints have length exactly 2 ). If $\mathcal{F}$ contains only $\Lambda$-type constraints and trivial constraints, then the corresponding Max-2CSP problem is called Max- $2 \Lambda$.

We abuse notation by identifying a set of allowed predicates $\mathcal{F}$ with the set of types of its predicates. Also, for a set $\mathcal{F}=\{\Lambda\}$ containing one element, we write $\mathcal{F}=\Lambda$. Therefore, a $\operatorname{Max} \operatorname{CSP}(\mathcal{F})$ problem is defined by $\mathcal{F} \subseteq\{T R, O R, X O R, A N D\}$. Note that every Max-2CSP problem corresponds to one such $\mathcal{F}$.

For every Max- $\operatorname{CSP}(\mathcal{F})$ problem, we give an explicit constant $\alpha_{\mathcal{F}}$ such that $\left(\alpha_{\mathcal{F}}-\varepsilon\right)$-approximation can be computed in $O(\log n)$ space, while $\left(\alpha_{\mathcal{F}}+\varepsilon\right)$-approximation requires space $\Omega(\sqrt{n})$, for every $\varepsilon>0$.

Theorem 1.1. Let $\mathcal{F} \subseteq\{T R, O R, X O R, A N D\}$ be a set of allowed binary predicates. Let $\alpha_{\mathcal{F}}=\min _{\mathcal{G} \subseteq \mathcal{F}} \alpha_{\mathcal{G}}$, where $\alpha_{\mathcal{G}}$ is given in Table 1.

For every $\varepsilon>0$, there exists an $\left(\alpha_{\mathcal{F}}-\varepsilon\right)$-approximate streaming algorithm for $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ that uses space $O\left(\varepsilon^{-2} \log n\right)$. On the other hand, any $\left(\alpha_{\mathcal{F}}+\varepsilon\right)$-approximate streaming algorithm for Max-CSP $(\mathcal{F})$ requires space $\Omega(\sqrt{n})$.

| Type $\mathcal{G}$ | Tight <br> bound | Previous bound |  |
| :---: | :---: | :---: | :---: |
|  | $\alpha_{\mathcal{G}}$ | $\alpha_{\mathcal{G}}^{\text {pr }}$ | Reference |
| TR | 1 | 1 | Folklore |
| OR | $\frac{3}{4}$ | $\left[\frac{3}{4}, 1\right]$ | Folklore |
| $\{$ TR, OR $\}$ | $\frac{\sqrt{2}}{2}$ | $\left[\frac{\sqrt{5}-1}{2}, 1\right]$ | $[$ LS77 $]$ |
| XOR | $\frac{1}{2}$ | $\frac{1}{2}$ | $[$ KK19] |
| AND | $\frac{4}{9}$ | $\left[\frac{2}{5}, \frac{1}{2}\right]$ | $[$ GVV17 $]$ |

Table 1: Summary of known and new approximation factors $\alpha_{\mathcal{G}}$ for $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$. We have suppressed (1 $1 \pm \varepsilon$ ) multiplicative factors.

Discussion. Interestingly, Theorem 1.1 identifies five Max-2CSP problems which completely characterize the hardness of any Max-2CSP problem. Namely, we show that Max- $\operatorname{CSP}(\mathcal{F})$ is precisely as hard to approximate as the hardest of the problems from Table 1 expressible by predicates from $\mathcal{F}$.

In particular, Theorem 1.1 closes the gap between $2 / 5$ [GVV17] and $1 / 2$ [KKS15] for the streaming approximation ratio of Max-DICUT. We prove that neither of these bounds is tight, and that the correct
bound is $4 / 9 .{ }^{3}$ Similarly, it shows that the $(\sqrt{5}-1) / 2$-approximate algorithm of [LS79] for Max-2SAT can be improved further, and that the optimal approximation ratio is $\sqrt{2} / 2$.

Many streaming problems have space-accuracy tradeoffs allowing for better approximations with more space (e.g., [Cha15, AKL16]). Curiously, Theorem 1.1 shows that every $\operatorname{Max}-2 \operatorname{CSP}(\mathcal{F})$ problem exhibits sharp threshold behavior: it needs only logarithmic space to be approximated up to some constant $\alpha_{\mathcal{F}}$, and it requires polynomial space for every larger approximation factor.

In the classical setting, approximation algorithms for Max-CSPs use space-inefficient techniques including semidefinite and linear programming, and network flow computations [Yan94, GW94, GW95, Hås08, Rag08, RS10, MM17]. On the other hand, the best streaming algorithms for Max-CSPs (except for the work [GVV17]) used only random assignments to the variables of the instance, including Max-CUT, Max-2SAT, and Unique Games problems. We design streaming algorithms for the Max-2AND and Max-2OR problems (i.e., $\mathcal{F}=$ $\{T R$, AND $\}$ and $\mathcal{F}=\{T R, O R\}$ ) which significantly improve on the approximation ratios guaranteed by a random assignment to the variables.

Additionally, Theorem 1.1 reveals a curious difference between streaming approximation of the cases $\mathcal{G}=\mathrm{OR}$ and $\mathcal{G}=\{T R, \mathrm{OR}\}$ (i.e., Exact Max-2SAT and Max-2SAT). The former problem can be $3 / 4-$ approximated, while the latter does not admit better than $\sqrt{2} / 2$-approximations. This shows that adding trivial constraints to Exact Max-2SAT actually makes the problem harder to approximate. This is in sharp contrast to the classical setting of polynomial-time algorithms with polynomial space, where approximationpreserving reductions between the two problems are known [Yan94]. While 3/4-approximation for Exact Max-2SAT is trivial, many 3/4-approximation algorithms for Max-2SAT use non-efficient (though polynomial) linear programming routines. This led Williamson to pose a question in 1998 whether there exists an algorithm for Max-2SAT which does not use linear programming and at least matches the trivial 3/4approximation guarantee for Exact Max-2SAT [Wil99]. The affirmative answer to this question was given by Poloczek and Schnitger in 2011 [PS11, Pol11, VZ11, PSWVZ17]. Theorem 1.1 complements this result by showing that there is no $\sqrt{2} / 2<3 / 4$-approximation for Max-2SAT in the streaming setting, thus, separating space-efficient approximations for Max-2SAT and Exact Max-2SAT.

Our final contribution is a tight bound on the approximation ratio of streaming algorithms for all Max$k$ SAT problems. We generalize the $\sqrt{2} / 2$-approximation algorithm for Max-2SAT from Theorem 1.1 to an algorithm for Max-SAT, and a matching hardness result trivially follows from the hardness of Max-2SAT.

Theorem 1.2. For every $\varepsilon>0$, there exists an $(\sqrt{2} / 2-\varepsilon)$-approximate streaming algorithm for Max-SAT that uses space $O\left(\varepsilon^{-2} \log n\right)$. On the other hand, for any $k \geq 2, \varepsilon>0$ any $(\sqrt{2} / 2+\varepsilon)$-approximate streaming algorithm for Max-kSAT requires space $\Omega(\sqrt{n})$.

### 1.2 Related Work

Classical setting. For every $\operatorname{Max}-2 \operatorname{CSP}(\mathcal{F})$ problem, a random assignment satisfies in expectation a constant fraction $\alpha_{\mathcal{F}}^{\mathrm{tr}}$ of the constraints (this algorithm can be easily derandomized via the method of conditional expectations). In particular, this algorithm gives $1 / 2$ - and $1 / 4$-approximations for Max-CUT and Max-2CSP. On one hand, Håstad [Hås01] used the PCP theorem to show that some Max-CSP problems, e.g., MAXE3SAT, do not admit better than $\alpha_{\mathcal{F}}^{\text {tr }}$-approximations unless $\mathrm{P}=\mathrm{NP}$. On the other hand, Goemans and Williamson [GW95] used semidefinite programming (SDP) to significantly improve the bounds for Max-CUT and Max-2CSP to 0.87856 and 0.79607 . Håstad [Hås08] proved that there is an SDP-based approximation algorithm with a better than $\alpha_{\mathcal{F}}^{\text {tr }}$ approximation guarantee for every Max-2CSP. Many of the SDP-based approximation algorithms are optimal under the Unique Games Conjecture [KV05, KKMO07]. We refer the reader to [MM17] for an up-to-date overview of the literature.

Streaming setting. While there is a trivial $1 / 2$-approximation for Max-CUT using space $O(\log n)$, Kapralov et al. [KKS15] showed that for any constant $\varepsilon>0$, a $(1 / 2+\varepsilon)$-approximation requires space $\tilde{\Omega}(\sqrt{n})$. Independently, Kogan and Krauthgamer [KK15] showed that (i) ( $1-\varepsilon$ )-approximation requires space $\Omega\left(n^{1-\varepsilon}\right)$

[^2]and (ii) 4/5-approximation requires $\Omega\left(n^{\tau}\right)$ space for some constant $\tau>0$. In a subsequent work, [KKSV17] showed that $(1-\varepsilon)$-approximation requires $\Omega(n)$ space. This line of work culminated in a recent result by Kapralov and Krachun [KK19] showing that any $(1 / 2+\varepsilon)$-approximation for Max-CUT requires $\Omega(n)$ space.

Recently Guruswami et al. [GVV17] gave a $(2 / 5-\varepsilon)$-approximate algorithm for Max-DICUT for any constant $\varepsilon>0$, significantly improving on the trivial $1 / 4$-approximation. For $k$-SAT, Kogan and Krauthgamer [KK15] showed that there is a $(1-\varepsilon)$-approximation using $\tilde{O}\left(\varepsilon^{-2} k n\right)$ space. The hardness side has been widely open prior to this work and, to the best of our knowledge, the only other hardness result is by Guruswami and Tao [GT19] showing that $(1 / p+\varepsilon)$-approximation for Unique Games with alphabet size $p$ requires $\tilde{\Omega}(\sqrt{n})$ space for any constant $\varepsilon>0$.

### 1.3 Techniques

Streaming algorithms. The first step of our proof of Theorem 1.1 is two new algorithms for Max-2OR and Max-2AND that improve on the naive approximations for these problems. For these algorithms, we generalize the notion of bias [GVV17] to all Max-2CSP problems, and prove a series of bounds on the value of Max-2CSP w.r.t. the bias (and the numbers of trivial and non-trivial constraints in the instance). This results in log-space streaming algorithms that sketch the bias (and some additional information about the instance), and compute good estimates of the value of the instance.

It is not hard to see that Max-2AND is the "hardest" Max-2CSP problem, i.e., an $\alpha$-approximation for Max-2AND implies $\alpha$-approximations for all Max-2CSPs (see Section 6). Therefore, the hardness result of [KK19] for Max-CUT holds for Max-2AND as well, ruling out the possibility of $(1 / 2+\varepsilon)$-approximations. On the other hand, a random assignment for Max-2AND formulas only guarantees a $1 / 4$-approximation. A recent work [GVV17] improves the approximation ratio to $(2 / 5-\varepsilon)$ as follows.

Let $\Psi$ be a Max-2EAND instance with $m$ constraints, and val be the maximum number of simultaneously satisfiable constraints in $\Psi$. [GVV17] defines the bias of a variable $x$ as the absolute difference between the number of positive and negative occurrences of $x$, and the bias of the instance as the sum of biases of its variables. It is easy to see that for every instance, val $\leq(m+$ bias $) / 2$. [GVV17] proves that the assignment of the input variables according to their biases satisfies at least bias constraints (see Lemma 3.2). Then they conclude that $\max ($ bias, $m / 4$ ) is a $2 / 5$-approximation of val:

$$
\frac{\max (\text { bias }, m / 4)}{\text { val }} \geq \frac{\operatorname{bias} / 5+(m / 4)(4 / 5)}{(m+\text { bias }) / 2}=2 / 5
$$

The upper and lower bounds of [GVV17] are shown in red and blue in Figure 1, and the gap between the bounds indeed achieves $2 / 5$ when bias $=m / 4$. While both lower bounds val $\geq \max$ (bias, $m / 4$ ) are tight as functions of bias and $m$, we show that in the important regime of low bias $\in[0, m / 3]$, these bounds can be improved to

$$
\begin{equation*}
\text { val } \geq \frac{m}{4}+\frac{\text { bias }^{2}}{4(m-2 \text { bias })} \tag{1.3}
\end{equation*}
$$

Unlike the lower bound of val $\geq$ bias from [GVV17], our lower bound cannot be achieved by a greedy assignment to the input variables. Instead, we design a distribution of assignments, whose expected value is at least (1.3). This improved lower bound on val (shown in green in Figure 1) leads to a $4 / 9$-approximation by a sketch for the expression (1.3). Namely, we give a $O(\log n)$-space streaming algorithm that approximates the green and red bounds in Figure 1, and returns their maximum.

Perhaps surprisingly, the optimal approximation ratio for Max-2OR significantly differs from both the 3/4-approximation for Max-2EOR, and the trivial 1/2-approximation. The classical algorithm of [LS79] can be implemented in the streaming setting, but it only provides a $(\sqrt{5}-1) / 2 \approx 0.61$-approximation. We prove that the tight bound for Max-2OR is even larger- $\sqrt{2} / 2$. Proofs of these upper and lower bounds are perhaps the most technical parts of this work. It can be shown that various naive random assignments to the variables used in $1 / 2$ - and $(\sqrt{5}-1) / 2$-approximations cannot lead to better bounds. Instead we construct a family of distributions of assignments which depend on individual biases of the variables. We use these distributions


Figure 1: Upper and lower bounds on the maximum number val of simultaneously satisfiable constraints of Max-2AND as a function of bias. The blue line is the upper bound $\frac{m+\text { bias }}{2}$, and the red line is the lower bound $\max \left(\frac{m}{4}\right.$, bias) from [GVV17] (see Lemma 3.2). The green line is the new lower bound $\frac{m}{4}+\frac{\text { bias }^{2}}{4(m-2 \text { bias })}$ from Lemma 3.3 in the interval bias $\in[0, m / 3]$.
to prove the existence of assignments of some high value $v$, and finally we show a way to approximate $v$ in logarithmic space. We remark that it is not always possible to satisfy $m \sqrt{2} / 2$ constraints, thus, we also prove non-trivial upper bounds on val for the case when our estimate $v$ is low $v<m \sqrt{2} / 2$. (See Lemma 3.6 and Lemma 3.7 for formal statements of these results.)

Hardness results. We develop a framework for proving hardness results for various Max-2CSP problems, and use it to establish tight bounds for every Max-2CSP. This framework is based on the communication complexity lower bound of [KKS15] for the Distributional Boolean Hidden Partition problem (DBHP) (which, in turn, extends the results of [GKK+ 07, VY11] for Boolean Hidden Matching and Boolean Hidden Hypermatching). In DBHP, Alice holds a random bipartition of $[n]$, and Bob has a (random) graph $G$ on $n$ vertices with some edges marked. Their goal is to use minimal communication to distinguish between the following two cases: in the YES case, the set of Bob's marked edges is exactly the edges of $G$ that cross Alice's bipartition; while in the NO case, a random subset of the edges is marked. [KKS15] proved a lower bound of $\Omega(\sqrt{n})$ on the randomized one-way communication complexity of DBHP.

For a set of allowed predicates $\mathcal{G}$, we construct a reduction from DBHP to Max- $\operatorname{CSP}(\mathcal{G})$, which naturally induces distributions $\mathcal{D}^{Y}$ and $\mathcal{D}^{N}$ of $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ instances. Then by a careful analysis we show that the gap between the optimal solutions of instances from $\mathcal{D}^{Y}$ and $\mathcal{D}^{N}$ achieves $\alpha_{\mathcal{G}}+\varepsilon$ with high probability. This, amplified by a series of repetitions, lets us conclude that a space-efficient $\left(\alpha_{\mathcal{G}}+\varepsilon\right)$-approximate algorithm would contradict the lower bound on the communication complexity of DBHP.

In our framework, we give separate reductions from DBHP to Max-2EAND and Max-2OR with approximation ratios $4 / 9+\varepsilon$ and $\sqrt{2} / 2+\varepsilon$, respectively. For the Max-2EOR problem, we give an efficient streaming reduction from Max-CUT to Max-2EOR which asserts that an $\alpha$-approximation for Max-2EOR implies an $\alpha /(3-2 \alpha)$-approximation for Max-CUT. This, equipped with the lower bound from [KK19], proves a linear lower bound $\Omega(n)$ on the space complexity of $(3 / 4+\varepsilon)$-approximations of Max-2EOR.

Putting it all together. Finally, we show that our algorithms for the five problems from Table 1 can be combined together to handle every Max-2CSP problem. Similarly, we prove that the established lower bounds for these five problems cover all possible Max-2CSPs. This implies that every Max-2CSP problem
$\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ is precisely as hard to approximate as the hardest problem from Max-TR, Max-2EOR, Max-2OR, Max-2EXOR, Max-2EAND expressible in $\mathcal{F}$, and finishes the proof of Theorem 1.1.

### 1.4 Structure

In Section 2, we review some necessary background knowledge. In Section 3, we provide streaming algorithms with optimal approximation ratios for all Max-2CSP problems. Sections 4 and 5 are devoted to proving tight bounds on the approximation ratios of streaming algorithms from Section 3. In particular, Section 4 contains the general framework for our lower bounds, and the reductions from Distributional Boolean Hidden Partition to Max-2CSP problems. Section 5 provides a tight analysis of the approximation ratios resulting from these reductions. In Section 6, we combine the results of the previous sections to prove Theorem 1.1. Finally, Section 7 gives an optimal streaming approximation algorithm for Max-SAT and finishes the proof of Theorem 1.2.

## 2 Preliminaries

Let $\mathbb{N}=\{1,2, \ldots$,$\} be the set of natural numbers, and [n]=\{1,2, \ldots, n\}$ for any $n \in \mathbb{N}$. We use $\sqcup$ for the disjoint union of two sets. For an $0<\varepsilon<1, B \in(1 \pm \varepsilon)$ is shorthand for $1-\varepsilon \leq B \leq 1+\varepsilon$. For ease of exposition we will abuse notation and associate a vector $X \in\{0,1\}^{n}$ with the set $X \subseteq[n], X=\left\{i: X_{i}=1\right\}$.

As we explained in Section 1, we will primarily consider Max- $\operatorname{CSP}(\mathcal{G})$ where $\mathcal{G} \in$ $\{T R, O R,\{T R, O R\}, X O R, A N D\}$. In order to get familiar with these problems, we provide several examples in Table 2.

| type $\mathcal{G}$ | OR | $\{$ TR, OR $\}$ | XOR | AND |
| :---: | :---: | :---: | :---: | :---: |
| problem name | Max-2EOR | Max-2OR | Max-2EXOR | Max-2EAND |
| special case | Exact Max-2SAT | Max-2SAT | Max-CUT | Max-DICUT |

Table 2: For each case $\mathcal{G} \in\{O R,\{T R, O R\}, X O R, A N D\}$, we give the name of the $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ problem, as well as one well-studied special case/alternative name of the problem.

For an instance $\Psi$ of a Max-2CSP problem, we denote the number of clauses (constraints) in $\Psi$ by $m=|\Psi|$. We denote the set of Boolean variables of $\Psi$ by $\mathcal{X}=\left\{x_{1}, \ldots, x_{n}\right\}$. A literal $\ell$ is called positive if $\ell=x_{i}$, and negative if $\ell=\neg x_{i}$ for some variable $x_{i}$. A 1-clause is a clause (constraint) which depends only on one variable. We use $\operatorname{pos}_{i}^{(1)}(\Psi)$ and $\operatorname{pos}_{i}^{(2)}(\Psi)$ for the number of 1- and 2-clauses where the variable $x_{i}$ appears positively. Similarly, $\operatorname{neg}_{i}^{(1)}(\Psi)$ and $\operatorname{neg}_{i}^{(2)}(\Psi)$ denote the number of 1- and 2-clauses containing $\neg x_{i}$.

For an assignment $\sigma: \mathcal{X} \rightarrow\{0,1\}$ of the variables of $\Psi$, we denote the number of clauses of $\Psi$ satisfied by $\sigma$ as $\operatorname{val}_{\Psi}(\sigma)$. We denote the maximum number of simultaneously satisfiable clauses in $\Psi$ as $\mathrm{val}_{\Psi}$ :

$$
\operatorname{val}_{\Psi}=\max _{\sigma} \operatorname{val}_{\Psi}(\sigma)
$$

For $\alpha \in[0,1]$ and a set of allowed predicates $\mathcal{F}$, an algorithm $\mathcal{A}$ is an $\alpha$-approximation to the $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ problem if on any input $\Psi, \mathcal{A}$ outputs $v$, such that with probability $3 / 4$, it holds that $\mathrm{val}_{\Psi} \geq v \geq \alpha \cdot \mathrm{val}_{\Psi}$. For example, when $\alpha=1$, the algorithm solves $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ exactly (with probability $3 / 4$ ).

We will use the following definition of the bias of $\Psi$, which generalizes the definition from [GVV17] to all Max-2CSPs with clauses of length 1 or $2 .{ }^{4}$

[^3]Definition 2.1 (Bias). The bias of a variable $x_{i}$ of an instance $\Psi$ is defined as

$$
\operatorname{bias}_{i}(\Psi)=\frac{1}{2} \cdot\left|2 \operatorname{pos}_{i}^{(1)}(\Psi)+\operatorname{pos}_{i}^{(2)}(\Psi)-2 \operatorname{neg}_{i}^{(1)}(\Psi)-\operatorname{neg}_{i}^{(2)}(\Psi)\right|
$$

The bias vector of $\Psi$ is a vector $\boldsymbol{b} \in \mathbb{R}^{n}$, where $\boldsymbol{b}_{i}=$ bias $_{i}(\Psi)$. Finally, the bias of the formula $\Psi$ is defined as the sum of biases of its variables:

$$
\operatorname{bias}(\Psi)=\sum_{i=1}^{n} \operatorname{bias}_{i}(\Psi)=\frac{1}{2} \sum_{i=1}^{n}\left|2 \operatorname{pos}_{i}^{(1)}(\Psi)+\operatorname{pos}_{i}^{(2)}(\Psi)-2 \operatorname{neg}_{i}^{(1)}(\Psi)-\operatorname{neg}_{i}^{(2)}(\Psi)\right|
$$

Note that for every formula $\Psi$ with $|\Psi|=m$ clauses, $0 \leq \operatorname{bias}(\Psi) \leq m$.
In order to approximate the bias of a formula $\Psi$, we will use a streaming algorithm for approximating the $\ell_{1}$ norm of the bias vector of $\Psi$.

Theorem 2.2 ([Ind00, KNW10]). Given a stream $S$ of poly $(n)$ updates $(i, v) \in[n] \times\{1,-1\}$, let $x_{i}=$ $\sum_{(i, v) \in S} v$ for $i \in[n]$. There exists a 1-pass streaming algorithm, which uses $O\left(\log n / \varepsilon^{2}\right)$ bits of memory and outputs a $(1 \pm \varepsilon)$-approximation to the value $\ell_{1}(x)=\sum_{i}\left|x_{i}\right|$ with probability $3 / 4$.

We will need the following concentration inequality [KK19].
Lemma 2.3 ([KK19, Lemma 2.5]). Let $X=\sum_{i \in[N]} X_{i}$ where $X_{i}$ are Bernoulli random variables such that for any $k \in[N], \mathbb{E}\left[X_{k} \mid X_{1}, \ldots, X_{k-1}\right] \leq p$ for some $p \in(0,1)$. Let $\mu=N p$. For any $\Delta>0$,

$$
\operatorname{Pr}[X \geq \mu+\Delta] \leq \exp \left(-\frac{\Delta^{2}}{2 \mu+2 \Delta}\right)
$$

Finally, we will use the lower bound on the space complexity of streaming algorithms for approximate Max-CUT from [KK19].

Theorem 2.4. For any constant $\varepsilon>0$, any streaming algorithm that $(1 / 2+\varepsilon)$-approximates Max-CUT with success probability at least $3 / 4$ requires $\Omega(n)$ space.

### 2.1 Total variation distance

Definition 2.5 (Total variation distance of discrete random variables). Let $\Omega$ be a finite probability space and $X, Y$ be random variables with support $\Omega$. The total variation distance between $X$ and $Y$ is defined as follows.

$$
\|X-Y\|_{t v d}:=\frac{1}{2} \sum_{\omega \in \Omega}|\operatorname{Pr}[X=\omega]-\operatorname{Pr}[Y=\omega]|
$$

We will use the two following properties of the total variation distance.
Proposition 2.6. Let $\Omega$ be a finite probability space and $X, Y$ be random variables with support $\Omega$.

1. (Triangle inequality) Let $W$ be an arbitrary random variable, then we have $\|X-Y\|_{t v d} \geq\|X-W\|_{t v d}-$ $\|Y-W\|_{t v d}$.
2. (Data processing inequality) Let $W$ be a random variable that is independent of both $X$ and $Y$, and $f$ be a function, then we have $\|f(X, W)-f(Y, W)\|_{t v d} \leq\|X-Y\|_{t v d}$.

The triangle inequality for the total variation distance is a standard fact; and the proof of the data processing inequality can be found in [KKS15, Claim 6.5].

## 3 Streaming Algorithms

In this section, we present optimal approximation algorithms for Max-2CSPs using $O(\log n)$ space. In Theorem 1.1 in Section 6 we will prove that it is actually sufficient to design optimal algorithms for Max-CSP $(\mathcal{G})$ in the following five cases $\mathcal{G} \in\{T \mathrm{R}, \mathrm{OR},\{\mathrm{TR}, \mathrm{OR}\}, \mathrm{XOR}, \mathrm{AND}\}$. In Section 3.1, we present the trivial algorithm for Max-2CSPs, this algorithm turns our to be optimal for $\mathcal{G} \in\{T R, O R, X O R\}$. Then we develop and analyze optimal algorithms for the cases $\mathcal{G}=$ AND and $\mathcal{G}=\{$ TR, OR $\}$ in Sections 3.2 and 3.3, respectively.

For ease of exposition, we will assume that input instances never contain unsatisfiable and tautological clauses $(e . g .,(x \wedge \neg x),(x \vee \neg x))$. This assumption is without loss of generality, because a streaming algorithm can ignore unsatisfiable clauses and have a separate counter for tautological clauses.

### 3.1 Trivial Algorithm

First we present the trivial algorithm: this algorithm takes a Max-2CSP instance $\Psi$, counts the number of clauses $m=|\Psi|$ in it, and outputs the expected number of clauses satisfied by a uniform random assignment to the variables of $\Psi$. In Section 4 we will show that this algorithm gives the best streaming approximation not only in the case of Max-2XOR (the Max-CUT problem), but also in the case of Max-2EOR.

Proposition 3.1 (Folklore). For a function $f:\{0,1\}^{2} \rightarrow\{0,1\}$, let $\alpha_{f} \in[0,1]$ denote the fraction of $1 s$ in its truth table. Then for a set of allowed predicates $\mathcal{F}$, we define $\alpha_{\mathcal{F}}^{t r}=\min _{f \in \mathcal{F}} \alpha_{f}$. There exists a streaming algorithm that uses $O(\log n)$ space, and computes $\alpha_{\mathcal{F}}^{t r}$-approximation for $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ with success probability 1.

For example, for the problem Max-2EOR (i.e., $\mathcal{F}=\{\mathrm{OR}\}$ ), we have $\alpha_{\mathrm{OR}}=3 / 4$, as every clause is satisfied by 3 out of 4 possible assignments to its variables. Since the problem Max-2OR (i.e., $\mathcal{F}=\{T R, O R\}$ ) also allows clauses of length 1 (which are satisfied by 1 out of 2 possible assignments to the variable), we have $\alpha_{\{\mathrm{TR}, \mathrm{OR}\}}=1 / 2$.

Proof of Proposition 3.1.

```
Algorithm \(1 \alpha_{\mathcal{F}}^{\mathrm{tr}}\)-approximation streaming algorithm for \(\operatorname{Max}-\operatorname{CSP}(\mathcal{F})\)
Input: \(\Psi\)-an instance of \(\operatorname{Max}-\operatorname{CSP}(\mathcal{F})\).
    1: Use \(O(\log n)\) bits to compute \(m=|\Psi|\).
Output: \(v=\alpha_{\mathcal{F}}^{\mathrm{tr}} \cdot m\).
```

To prove that Algorithm 1 computes an $\alpha_{\mathcal{F}}^{\mathrm{tr}}$-approximation, we need to show that (i) there exists an assignment $\sigma$ such that $\operatorname{val}_{\Psi}(\sigma) \geq v=\alpha_{\mathcal{F}}^{\operatorname{tr}} \cdot m$, and (ii) $v=\alpha_{\mathcal{F}}^{\mathrm{tr}} \cdot m \geq \alpha_{\mathcal{F}}^{\mathrm{tr}} \cdot \operatorname{val}_{\Psi}$.

Note that since $\mathrm{val}_{\Psi} \leq|\Psi|=m$, (ii) holds trivially. The existence of an assignment $\sigma$ satisfying (i) is guaranteed by the following bound on the expected number of clauses satisfied by a uniform random assignment $\sigma$ :

$$
\underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right]=\sum_{C \in \Psi} \operatorname{Pr}_{\sigma}[C \text { is satisfied by } \sigma]=\sum_{C \in \Psi} \alpha_{C} \geq \alpha_{\mathcal{F}}^{\operatorname{tr}} \cdot m
$$

Remark. For an $\left(\alpha_{\mathcal{F}}^{t r}-\varepsilon\right)$-approximation, one can reduce the space usage of Algorithm 1 to $O(\log \log n+\log (1 / \varepsilon))$ bits by using the approximate counting algorithm of Morris [Mor78, GS09].

Remark. Formally, Algorithm 1 only guarantees a 1/2-approximation for the problem Max-CSP $(T R)$, i.e., the problem where all clauses have length 1 . In this case, in order to achieve a $(1-\varepsilon)$-approximation using $O(\log n)$ space for arbitrary constant $\varepsilon>0$, one can use an $\ell_{1}$-sketch (Theorem 2.2) to approximate the bias vector of the input formula. Indeed, it is easy to see that for an instance $\Psi$ of Max-CSP(TR) with $m$ clauses, $v a l_{\Psi}=(m+\operatorname{bias}(\Psi)) / 2$.

| Type $\mathcal{G}$ | TR | OR | $\{$ TR, OR $\}$ | XOR | AND |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha_{\mathcal{G}}^{\text {tr }}$ | 1 | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| $\alpha_{\mathcal{G}}^{\text {opt }}$ | 1 | $\frac{3}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $\frac{4}{9}$ |

Table 3: For various sets of predicates $\mathcal{G}$, the table presents (i) $\alpha_{\mathcal{G}}^{\text {tr }}$-the approximation ratio guaranteed by the trivial algorithm for $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$, and (ii) $\alpha_{\mathcal{G}}^{\mathrm{opt}}$ - the optimal approximation ratio of streaming algorithms, proven in Sections 3 and 4 for Max- $\operatorname{CSP}(\mathcal{G})$. We have suppressed $(1-\varepsilon)$ multiplicative factors for the case $\mathcal{G}=\mathrm{TR}$.

We give $\alpha_{\mathcal{G}}^{\mathrm{tr}}$ for relevant sets of predicates in Table 3.
As we show in the following sections, this trivial approximation algorithm can be improved for the Max-2AND and Max-2OR problems.

### 3.2 Algorithm for Max-2AND and Max-2EAND

Consider a Max-2AND instance $\Psi^{\prime}$ where all clauses are of length 1 or 2 . Note that $\Psi^{\prime}$ can be written as an equivalent Max-2AND instance $\Psi$, where 1-clauses of $\Psi^{\prime}$ are replaced with 2-clauses containing the same literal twice. ${ }^{5}$ In this section, we will consider such representation of every instance of Max-2AND, i.e., we will assume that all clauses have exactly 2 (not necessarily distinct) literals. Note that in this case, the bias (see Definition 2.1) of $\Psi$ is simply

$$
\operatorname{bias}(\Psi)=\frac{1}{2} \sum_{i=1}^{n}\left|\operatorname{pos}_{i}^{(2)}(\Psi)-\operatorname{neg}_{i}^{(2)}(\Psi)\right|
$$

where $\operatorname{pos}_{i}^{(2)}(\Psi)$ and $\operatorname{neg}_{i}^{(2)}(\Psi)$ are the numbers of occurrences of $x_{i}$ and $\neg x_{i}$ in 2-clauses.
[GVV17] gave lower and upper bounds for the maximum number of satisfied clauses val ${ }_{\Psi}$ in terms of $\operatorname{bias}(\Psi)$ and $m$ (the number of clauses in $\Psi)$. For the sake of being self contained, and to verify that these bounds hold for our slightly more general case where 2-clauses may contain repeated literals, we present the proofs of these bounds in Lemma 3.2 in Section 3.2.1.

Lemma 3.2 ([GVV17]). Let $\Psi$ be a Max-2AND instance with $m$ clauses. Then

$$
\operatorname{bias}(\Psi) \leq v a l_{\Psi} \leq \frac{m+\operatorname{bias}(\Psi)}{2}
$$

We improve the lower bound of [GVV17] in the important regime of $\operatorname{bias}(\Psi) \leq m / 3$ in the following lemma.

Lemma 3.3. Let $\Psi$ be a Max-2AND instance with $m$ clauses and bias $(\Psi) \leq m / 3$. Then

$$
\operatorname{val}_{\Psi} \geq \frac{m}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))} \geq \frac{2(m+\operatorname{bias}(\Psi))}{9}
$$

The proof of Lemma 3.3 is based on biased random sampling, and is postponed to Section 3.2.1. For a pictorial view of this improvement, see Figure 1.

We are now ready to present a streaming algorithm that (4/9)-approximates Max-2AND and Max-2EAND.

[^4]Theorem 3.4 ( $\frac{4}{9}$-approximation for Max-2AND and Max-2EAND). For any $\varepsilon \in(0,0.01)$, there exists a streaming algorithm that uses space $O\left(\varepsilon^{-2} \log n\right)$ and computes $\left(\frac{4}{9}-\varepsilon\right)$-approximation for Max-2AND and Max-2EAND with success probability at least 3/4.

Proof. The algorithm uses the bounds from Lemmas 3.2 and 3.3 to approximate the value of a given instance of Max-2AND.

```
Algorithm \(2\left(\frac{4}{9}-\varepsilon\right)\)-approximation streaming algorithm for Max-2AND
Input: \(\Psi\)-an instance of Max-2AND. Error parameter \(\varepsilon \in(0,0.01)\).
    1: Approximate the \(\ell_{1}\)-norm of the bias vector with error \(\delta=\varepsilon / 2\) (Theorem 2.2):
    Compute \(B \in(1 \pm \delta) \operatorname{bias}(\Psi)\).
    2: Count the number of clauses \(m=|\Psi|\).
    3: if \(B \in\left[0, \frac{m}{3}(1-\delta)\right]\) then
    Output: \(v=\frac{2(m+B)}{9(1+\delta)}\).
    else
    Output: \(v=\frac{B}{(1+\delta)}\).
```

To prove the correctness of Algorithm 2, we show that (i) val $l_{\Psi} \geq v$ and (ii) $v \geq\left(\frac{4}{9}-\varepsilon\right) \cdot$ val $_{\Psi}$, where $v$ is the output of Algorithm 2.
(i) $\boldsymbol{v} \leq \boldsymbol{v a l}_{\Psi}$. Since $B$ is an $(1 \pm \delta)$-approximation of the bias, with probability at least $3 / 4$ we have that $(1-\delta) \cdot \operatorname{bias}(\Psi) \leq B \leq(1+\delta) \cdot \operatorname{bias}(\Psi)$.

First, consider the case where $B \in\left[0, \frac{m}{3}(1-\delta)\right]$ :

$$
v=\frac{2(m+B)}{9(1+\delta)} \leq \frac{2(1+\delta)(m+\operatorname{bias}(\Psi))}{9(1+\delta)}=\frac{2(m+\operatorname{bias}(\Psi))}{9} \leq \operatorname{val}_{\Psi}
$$

where the last inequality uses the bound from Lemma 3.3.
Now consider the case where $B>\frac{m}{3}(1-\delta)$ :

$$
v=\frac{B}{(1+\delta)} \leq \operatorname{bias}(\Psi) \leq \operatorname{val}_{\Psi}
$$

where the last inequality follows from the bound $\operatorname{val}_{\Psi} \geq \operatorname{bias}(\Psi)$ from Lemma 3.2.
(ii) $\boldsymbol{v} \geq\left(\frac{4}{9}-\boldsymbol{\varepsilon}\right) \cdot \mathbf{v a l}_{\Psi}$. First, consider the case where $B \in\left[0, \frac{m}{3}(1-\delta)\right]:$,

$$
v=\frac{2(m+B)}{9(1+\delta)} \geq \frac{2(1-\delta)(m+\operatorname{bias}(\Psi))}{9(1+\delta)} \geq \frac{2(1-2 \delta)(m+\operatorname{bias}(\Psi))}{9} \geq\left(\frac{4}{9}-\varepsilon\right) \cdot \mathrm{val}_{\Psi}
$$

where the last inequality follows from the bound $\operatorname{val}_{\Psi} \leq \frac{m+\operatorname{bias}(\Psi)}{2}$ of Lemma 3.2 and $\delta=\varepsilon / 2$.
Now consider the case where $B>\frac{m}{3}(1-\delta)$. From Lemma 3.2, val ${ }_{\Psi} \leq \frac{m+\operatorname{bias}(\Psi)}{2}$. Then

$$
\begin{aligned}
\frac{v}{\operatorname{val}_{\Psi}} & \geq \frac{2 v}{m+\operatorname{bias}(\Psi)}=\frac{2 B}{(1+\delta)(m+\operatorname{bias}(\Psi))} \geq \frac{2 B}{(1+\delta)\left(m+\frac{B}{1-\delta}\right)} \geq \frac{2 B}{(1+3 \delta)(m+B)} \\
& \geq \frac{\frac{2 m(1-\delta)}{3}}{(1+3 \delta) \frac{4 m}{3}}=\frac{1}{2} \cdot \frac{1-\delta}{1+3 \delta}>\frac{4}{9}
\end{aligned}
$$

for every $\delta<0.01$.
We conclude that Algorithm 2 outputs a $(4 / 9-\varepsilon)$-approximation for Max-2AND and Max-2EAND.

### 3.2.1 Proofs of Lemma 3.2 and Lemma 3.3

Lemma 3.2 ([GVV17]). Let $\Psi$ be a Max-2AND instance with $m$ clauses. Then

$$
\operatorname{bias}(\Psi) \leq \operatorname{va}_{\Psi} \leq \frac{m+\operatorname{bias}(\Psi)}{2}
$$

Proof. In order to prove the lower bound $\operatorname{val}_{\Psi} \geq \operatorname{bias}(\Psi)$, we give an assignment $\sigma$ to the input variables which satisfies at least $\operatorname{bias}(\Psi)$ clauses. This assignment $\sigma$ will greedily assign the value of each variable according to its bias: the variables which appear positively more often than negatively will be assigned 1 , and the remaining variables will be assigned 0 .

Recall that $\operatorname{pos}_{i}^{(2)}(\Psi)$ and $\operatorname{neg}_{i}^{(2)}(\Psi)$ denote the number of clauses where $x_{i}$ appears positively and negatively. For every variable $x_{i}$ with $\operatorname{pos}_{i}^{(2)}(\Psi) \geq \operatorname{neg}_{i}^{(2)}(\Psi)$, we set $\sigma\left(x_{i}\right)=1$, and we set $\sigma\left(x_{i}\right)=0$ otherwise. Note that the number of unsatisfied literals in this case is $\sum_{i} \min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}$. Thus, the number of unsatisfied clauses is also bounded from above by $\sum_{i} \min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}$.

From

$$
\begin{aligned}
2 \operatorname{bias}(\Psi) & =\sum_{i} \max \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}-\min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\} \\
2 m & =\sum_{i} \max \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}+\min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}
\end{aligned}
$$

we have that

$$
\begin{equation*}
\sum_{i} \min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}=m-\operatorname{bias}(\Psi) \tag{3.5}
\end{equation*}
$$

Thus,

$$
\operatorname{val}_{\Psi}(\sigma) \geq m-(m-\operatorname{bias}(\Psi))=\operatorname{bias}(\Psi)
$$

For the upper bound of $\operatorname{val}_{\Psi} \leq \frac{m}{2}+\frac{\operatorname{bias}(\Psi)}{2}$ we note that for every assignment $\sigma$, the number of unsatisfied literals is at least $\sum_{i} \min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}$. (Since $x_{i}=1$ produces $\operatorname{pos}_{i}^{(2)}(\Psi)$ unsatisfied literals, while $x_{i}=0$ produces $\operatorname{neg}_{i}^{(2)}(\Psi)$ unsatisfied literals.) Thus, the number of unsatisfied clauses is at least $\frac{1}{2} \sum_{i} \min \left\{\operatorname{pos}_{i}^{(2)}(\Psi), \operatorname{neg}_{i}^{(2)}(\Psi)\right\}$. From (3.5), we have that for every assignment $\sigma$,

$$
\operatorname{val}_{\Psi}(\sigma) \leq m-\frac{1}{2}(m-\operatorname{bias}(\Psi))=\frac{m+\operatorname{bias}(\Psi)}{2}
$$

Lemma 3.3. Let $\Psi$ be a Max-2AND instance with $m$ clauses and bias $(\Psi) \leq m / 3$. Then

$$
\operatorname{val}_{\Psi} \geq \frac{m}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))} \geq \frac{2(m+\operatorname{bias}(\Psi))}{9}
$$

Proof. First we show that for every Max-2AND instance $\Psi$ with $m$ clauses and bias $(\Psi) \leq m / 3$, there exists an assignment $\sigma$ s.t.

$$
\operatorname{val}_{\Psi}(\sigma) \geq \frac{m}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))}
$$

Without loss of generality we can assume that every variable appears in $\Psi$ positively at least as many times as it appears negatively, i.e., $\operatorname{pos}_{i}^{(2)}(\Psi) \geq \operatorname{neg}_{i}^{(2)}(\Psi)$ for every $i \in[n] .{ }^{6}$ We prove the existence of such

[^5]an assignment $\sigma$ by giving a distribution of assignments whose expected number of satisfied clauses is at least $\frac{2(m+\operatorname{bias}(\Psi))}{9}$. Let $\gamma \in[0,0.5]$ be a parameter to be assigned later. For each variable $x_{i}$, we assign $x_{i}=1$ with probability $\frac{1}{2}+\gamma$, and $x_{i}=0$ with probability $\frac{1}{2}-\gamma \cdot{ }^{7}$ Let $k_{0}, k_{1}$, and $k_{2}$ denote the number of clauses with zero, one, and two positive literals. Observe that $m=k_{0}+k_{1}+k_{2}$ and
$$
2 \operatorname{bias}(\Psi)=\sum_{i \in[n]}\left|\operatorname{pos}_{i}^{(2)}(\Psi)-\operatorname{neg}_{i}^{(2)}(\Psi)\right|=\left(2 k_{2}+k_{1}\right)-\left(k_{1}+2 k_{0}\right)=2\left(k_{2}-k_{0}\right)
$$

Let us now compute the expected number of satisfied AND clauses under the biased distribution described above. Note that a clause with two (not necessarily distinct) positive literals is satisfied with probability at least $\min \left\{\left(\frac{1}{2}+\gamma\right)^{2}, \frac{1}{2}+\gamma\right\}=\left(\frac{1}{2}+\gamma\right)^{2}$. Similarly, a clause with two negative literals is satisfied with probability at least $\left(\frac{1}{2}-\gamma\right)^{2}$, and a clause with a positive and negative literals (corresponding to different variables) is satisfied with probability $\left(\frac{1}{2}-\gamma\right)\left(\frac{1}{2}+\gamma\right)$.

$$
\begin{aligned}
\underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right] & =\sum_{i=0}^{2} k_{i} \cdot \underset{\sigma}{\operatorname{Pr}}[\text { a clause with } i \text { positive literals is satisfied by } \sigma] \\
& =k_{0} \cdot\left(\frac{1}{2}-\gamma\right)^{2}+k_{1} \cdot\left(\frac{1}{2}-\gamma\right)\left(\frac{1}{2}+\gamma\right)+k_{2} \cdot\left(\frac{1}{2}+\gamma\right)^{2} \\
& =\frac{k_{0}+k_{1}+k_{2}}{4}+\left(k_{2}-k_{0}\right) \cdot \gamma+\left(k_{2}-k_{1}+k_{0}\right) \cdot \gamma^{2} \\
& =\frac{m}{4}+\operatorname{bias}(\Psi) \cdot \gamma+\left(2\left(k_{2}+k_{0}\right)-m\right) \cdot \gamma^{2} \\
& \geq \frac{m}{4}+\operatorname{bias}(\Psi) \cdot \gamma+(2 \operatorname{bias}(\Psi)-m) \cdot \gamma^{2}
\end{aligned}
$$

where we used that $m=k_{0}+k_{1}+k_{2}$ and $2 \operatorname{bias}(\Psi)=2\left(k_{2}-k_{0}\right) \leq 2\left(k_{2}+k_{0}\right)$.
Since $\operatorname{bias}(\Psi) \in[0, m / 3]$, we can set $\gamma=\frac{\operatorname{bias}(\Psi)}{2(m-2 \operatorname{bias}(\Psi))} \in[0,0.5]$ and have that

$$
\underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right] \geq \frac{m}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))}
$$

Finally, it remains to show that $\frac{m}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))} \geq \frac{2(m+\operatorname{bias}(\Psi))}{9}$ :

$$
\begin{aligned}
\frac{m}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))} & =\frac{2(m+\operatorname{bias}(\Psi))}{9}+\frac{m-8 \operatorname{bias}(\Psi)}{36}+\frac{\operatorname{bias}(\Psi)^{2}}{4(m-2 \operatorname{bias}(\Psi))} \\
& =\frac{2(m+\operatorname{bias}(\Psi))}{9}+\frac{(m-8 \operatorname{bias}(\Psi))(m-2 \operatorname{bias}(\Psi))+9 \operatorname{bias}(\Psi)^{2}}{36(m-2 \operatorname{bias}(\Psi))} \\
& =\frac{2(m+\operatorname{bias}(\Psi))}{9}+\frac{(5 \operatorname{bias}(\Psi)-m)^{2}}{36(m-2 \operatorname{bias}(\Psi))} \\
& \geq \frac{2(m+\operatorname{bias}(\Psi))}{9}
\end{aligned}
$$

which holds for every $\operatorname{bias}(\Psi) \in[0, m / 3]$.

### 3.3 Algorithm for Max-2OR

For the case of Max-2OR, it is crucial to distinguish 1- and 2-clauses. Therefore, we treat clauses containing two identical literals as 1-clauses. We denote the number of 1 -clauses of $\Psi$ by $m_{1}$, and the number of 2-clauses by $m_{2}$. In particular, the total number of clauses is $m=m_{1}+m_{2}$.

[^6]In Lemmas 3.6 and 3.7 we give upper and lower bounds on val $_{\Psi}$ in terms of $m_{1}, m_{2}$, and bias $(\Psi)$, we postpone their proofs to Section 3.3.1. In this section we prove that the ratio between the presented lower and upper bounds is bounded by $\frac{\sqrt{2}}{2}$, and that there is a $O(\log n)$-space algorithm that sketches the lower bounds of Lemma 3.7 on val $_{\Psi}$.

When the bias of $\Psi$ is large (say, bias $(\Psi)=m$ ), it might be possible to satisfy all $m$ clauses of $\Psi$, so no non-trivial upper bounds on val $_{\Psi}$ can be proven in terms of bias in this case. Even if the bias is low (say, $\operatorname{bias}(\Psi)=0$ ), but the formula does not contain 1-clauses, it might still be possible to satisfy all clauses of $\Psi$. (E.g., if all clauses of $\Psi$ contain one positive and one negative literal.) It turns out that for the optimal approximation ratio, we need to bound from above val $\Psi_{\Psi}$ in the case of low bias and large number of 1-clauses.

Lemma 3.6. Let $\Psi$ be a Max-2OR instance with $m_{1} 1$-clauses, and $m_{2}$ 2-clauses. Then

$$
\operatorname{val}(\Psi) \leq \min \left\{m_{1}+m_{2}, \frac{m_{1}+2 m_{2}+\operatorname{bias}(\Psi)}{2}\right\}
$$

The trivial algorithm guarantees that for every Max-2OR instance $\Psi, \operatorname{val}_{\Psi} \geq m_{1} / 2+3 m_{2} / 4$. While this bound is tight in terms of $m_{1}$ and $m_{2}$, for instances with high bias $>m_{2} / 2$, we prove a better lower bound of val $\geq\left(m_{1}+m_{2}+\operatorname{bias}(\Psi)\right) / 2$. Clearly, this bound is not sufficient for a better than $1 / 2$-approximation in the case of low $\operatorname{bias}(\Psi)=0$. In order to handle this case, we design a distribution of assignments which in expectation satisfy a large number of clauses in formulas with low bias.

Lemma 3.7. Let $\Psi$ be a Max-2OR instance with $m_{1} 1$-clauses, and $m_{2}$ 2-clauses. Then

1. $\operatorname{val}_{\Psi} \geq \frac{m_{1}+m_{2}+\operatorname{bias}(\Psi)}{2}$;
2. if $\operatorname{bias}(\Psi) \leq m_{2}$, then

$$
\mathrm{val}_{\Psi} \geq \frac{m_{1}}{2}+\frac{3 m_{2}}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4 m_{2}}
$$

We will also use the following simple claim.
Claim 3.8. For every $x \geq 0, y>0$ :

$$
\frac{2 x+3 y+x^{2} / y}{4(x+y)} \geq \frac{\sqrt{2}}{2} .
$$

Proof. Let $z=\frac{x}{y}+1$, then

$$
\frac{2 x+3 y+x^{2} / y}{4(x+y)}=\frac{2 \frac{x}{y}+3+\frac{x^{2}}{y^{2}}}{4\left(\frac{x}{y}+1\right)}=\frac{z^{2}+2}{4 z}=\frac{z}{4}+\frac{1}{2 z} \geq \frac{\sqrt{2}}{2}
$$

by the inequality of arithmetic and geometric means.
Now we are ready to present an approximation algorithm for the Max-2OR problem.
Theorem 3.9 ( $\frac{\sqrt{2}}{2}$-approximation for Max-2OR). For any $\varepsilon \in(0,0.01)$, there exists a streaming algorithm that uses space $O\left(\varepsilon^{-2} \log n\right)$ and computes $\left(\frac{\sqrt{2}}{2}-\varepsilon\right)$-approximation for Max-2OR with success probability at least $3 / 4$.
Proof. We prove that Algorithm 3 computes a $\left(\frac{\sqrt{2}}{2}-\varepsilon\right)$-approximation by showing that (i) $v \leq \mathrm{val}_{\Psi}$, and (ii) $v \geq\left(\frac{\sqrt{2}}{2}-\varepsilon\right) \cdot \mathrm{val}_{\Psi}$, where $v$ is the output of the algorithm. Recall that by the guarantee of Theorem 2.2, with probability at least $3 / 4$ :

$$
(1-\delta) \operatorname{bias}(\Psi) \leq B \leq(1+\delta) \operatorname{bias}(\Psi)
$$

```
Algorithm \(3\left(\frac{\sqrt{2}}{2}-\varepsilon\right)\)-approximation streaming algorithm for Max-2OR
Input: \(\Psi\)-an instance of Max-2OR. Error parameter \(\varepsilon \in(0,0.01)\).
    1: Approximate the \(\ell_{1}\)-norm of the bias vector with error \(\delta=\varepsilon / 4\) (Theorem 2.2):
    Compute \(B \in(1 \pm \delta) \operatorname{bias}(\Psi)\).
    2: Count the number of 1- and 2- clauses \(m_{1}\) and \(m_{2}\).
    3: if \(B \in\left[0,(1-\delta) m_{2}\right]\) then
    Output: \(v=\frac{(1-\delta)^{2}\left(2 m_{1}+3 m_{2}+B^{2} / m_{2}\right)}{4}\).
    else
    Output: \(v=\frac{(1-\delta)\left(m_{1}+m_{2}+B\right)}{2}\).
```

(i) $\boldsymbol{v} \leq \boldsymbol{v a l}_{\Psi}$. First, note that $(1-\delta) B \leq(1-\delta)(1+\delta) \operatorname{bias}(\Psi) \leq \operatorname{bias}(\Psi)$. Next, if $B \leq(1-\delta) m_{2}$, then $\operatorname{bias}(\Psi) \leq B /(1-\delta) \leq m_{2}$, and, thus, $\operatorname{val}_{\Psi} \geq \frac{m_{1}}{2}+\frac{3 m_{2}}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4 m_{2}}$ by the second bound in Lemma 3.7. Then

$$
v=\frac{(1-\delta)^{2}\left(2 m_{1}+3 m_{2}+B^{2} / m_{2}\right)}{4} \leq \frac{\left(2 m_{1}+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}\right)}{4} \leq \operatorname{val}_{\Psi}
$$

If $B>(1-\delta) m_{2}$, then

$$
v=\frac{(1-\delta)\left(m_{1}+m_{2}+B\right)}{2} \leq \frac{m_{1}+m_{2}+\operatorname{bias}(\Psi)}{2} \leq \operatorname{val}_{\Psi}
$$

by the first bound in Lemma 3.7.
(ii) $v \geq\left(\frac{\sqrt{2}}{2}-\varepsilon\right) \cdot$ val $_{\Psi}$. Let us consider three cases.

1. $B \leq(1-\delta) m_{2}$ and $m_{1} \leq \operatorname{bias}(\Psi)$.

In this case the output of the algorithm is

$$
\begin{aligned}
v & =\frac{(1-\delta)^{2}\left(2 m_{1}+3 m_{2}+B^{2} / m_{2}\right)}{4} \\
& \geq \frac{(1-\delta)^{4}\left(2 m_{1}+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}\right)}{4} \\
& \geq \frac{(1-4 \delta)\left(2 m_{1}+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}\right)}{4}
\end{aligned}
$$

From the upper bound $\mathrm{val}_{\Psi} \leq m_{1}+m_{2}$ of Lemma 3.6, we have that

$$
\begin{aligned}
\frac{v}{\operatorname{val}_{\Psi}} & \geq(1-4 \delta) \cdot \frac{2 m_{1}+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}}{4\left(m_{1}+m_{2}\right)} \\
& \geq(1-4 \delta) \cdot \frac{2 \operatorname{bias}(\Psi)+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}}{4\left(\operatorname{bias}(\Psi)+m_{2}\right)} \\
& \geq(1-4 \delta) \cdot \frac{\sqrt{2}}{2}=(1-\varepsilon) \cdot \frac{\sqrt{2}}{2}
\end{aligned}
$$

where the second inequality follows from $m_{1} \leq \operatorname{bias}(\Psi)$, and the last inequality follows from Claim 3.8.
2. $B \leq(1-\delta) m_{2}$ and $m_{1}>\operatorname{bias}(\Psi)$.

From the upper bound $\operatorname{val}_{\Psi} \leq \frac{m_{1}+2 m_{2}+\operatorname{bias}(\Psi)}{2}$ of Lemma 3.6:

$$
\begin{aligned}
\frac{v}{\operatorname{val}_{\Psi}} & \geq(1-4 \delta) \cdot \frac{2 m_{1}+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}}{2\left(m_{1}+2 m_{2}+\operatorname{bias}(\Psi)\right)} \\
& \geq(1-4 \delta) \cdot \frac{2 \operatorname{bias}(\Psi)+3 m_{2}+\operatorname{bias}(\Psi)^{2} / m_{2}}{4\left(\operatorname{bias}(\Psi)+m_{2}\right)} \\
& \geq(1-4 \delta) \cdot \frac{\sqrt{2}}{2}=(1-\varepsilon) \cdot \frac{\sqrt{2}}{2},
\end{aligned}
$$

where the second inequality is due to $m_{1}>\operatorname{bias}(\Psi)$, and the last inequality is due to Claim 3.8.
3. $B>(1-\delta) m_{2}$.

From the bound in Lemma 3.6:

$$
\begin{aligned}
\operatorname{val}_{\Psi} & \leq \min \left\{m_{1}+m_{2}, \frac{m_{1}+2 m_{2}+\operatorname{bias}(\Psi)}{2}\right\} \\
& \leq \frac{1}{3} \cdot\left(m_{1}+m_{2}\right)+\frac{2}{3} \cdot \frac{m_{1}+2 m_{2}+\operatorname{bias}(\Psi)}{2} \\
& =\frac{2 m_{1}+3 m_{2}+\operatorname{bias}(\Psi)}{3} \\
& \leq \frac{2 m_{1}+3 m_{2}+B}{3(1-\delta)}
\end{aligned}
$$

In this case, the output of the algorithm is $v=\frac{(1-\delta)\left(m_{1}+m_{2}+B\right)}{2}$. Then

$$
\frac{v}{\mathrm{val}_{\Psi}} \geq \frac{3(1-\delta)^{2}}{2} \cdot \frac{m_{1}+m_{2}+B}{2 m_{1}+3 m_{2}+B} \geq \frac{3(1-\delta)^{2}}{2} \cdot \frac{m_{1}+m_{2}(2-\delta)}{2 m_{1}+4 m_{2}} \geq \frac{3(1-\delta)^{2}(2-\delta)}{8} \geq \frac{\sqrt{2}}{2}
$$

where the second inequality is due to $B>(1-\delta) m_{2}$, and the last one holds for every $\delta<0.01$.

### 3.3.1 Proofs of Lemma 3.6 and Lemma 3.7

Lemma 3.6. Let $\Psi$ be a Max-2OR instance with $m_{1} 1$-clauses, and $m_{2}$ 2-clauses. Then

$$
\operatorname{val}(\Psi) \leq \min \left\{m_{1}+m_{2}, \frac{m_{1}+2 m_{2}+\operatorname{bias}(\Psi)}{2}\right\}
$$

Proof. Since $m_{1}+m_{2}$ is the number of clauses in $\Psi$, the first bound val $(\Psi) \leq m_{1}+m_{2}$ holds trivially.
First we negate all variables of $\Psi$ with $\operatorname{bias}_{i}(\Psi)<0$. This transformation does not change $\operatorname{bias}(\Psi), m_{1}, m_{2}, \operatorname{val}(\Psi)$, and every assignment of the variables of the original instance can be uniquely mapped to a corresponding assignment for the new instance satisfying the same number of clauses. Therefore, without loss of generality, for every $i \in[n]$,

$$
\operatorname{pos}_{i}^{(1)}(\Psi)+\frac{\operatorname{pos}_{i}^{(2)}(\Psi)}{2}-\operatorname{neg}_{i}^{(1)}(\Psi)-\frac{\operatorname{neg}_{i}^{(2)}(\Psi)}{2} \geq 0
$$

Consider an assignment $\sigma$ to the variables of $\Psi$. We need to show that val ${ }_{\Psi}(\sigma) \leq \frac{m_{1}+2 m_{2}+\text { bias }(\Psi)}{2}$. Let $T$ be the set of (indices of) variables of $\sigma$ assigned the value 1 . Then the number of 1 clauses satisfied by $\sigma$ is

$$
S_{1}=\sum_{i \in T} \operatorname{pos}_{i}^{(1)}(\Psi)+\sum_{i \notin T} \operatorname{neg}_{i}^{(1)}(\Psi)
$$

Let $S_{2}$ denote the number of 2 -clauses satisfied by $\sigma$. We will show that

$$
\begin{equation*}
S_{2} \leq \min \left\{m_{2}, \operatorname{bias}(\Psi)+m_{1}+m_{2}-2 S_{1}\right\} \tag{3.10}
\end{equation*}
$$

First we show how (3.10) finishes the proof of the lemma, and then prove (3.10).
Indeed, then the number of clauses satisfied by $\sigma$ is bounded from above by

$$
\begin{aligned}
\operatorname{val}_{\Psi}(\sigma) & \leq S_{1}+S_{2} \\
& \leq S_{1}+\min \left\{m_{2}, \operatorname{bias}(\Psi)+m_{1}+m_{2}-2 S_{1}\right\} \\
& \leq S_{1}+\frac{m_{2}}{2}+\frac{\operatorname{bias}(\Psi)+m_{1}+m_{2}-2 S_{1}}{2} \\
& =\frac{m_{1}+2 m_{2}+\operatorname{bias}(\Psi)}{2} .
\end{aligned}
$$

Now we will prove the bound (3.10). The bound $S_{2} \leq m_{2}$ is trivial, since $m_{2}$ is the total number of 2 -clauses in the instance. The number of 2 -clauses satisfied by variables set to 1 is bounded from above by $\sum_{i \in T} \operatorname{pos}_{i}^{(2)}(\Psi)$, and the number of 2 -clauses satisfied by variables set to 0 is bounded by $\sum_{i \notin T} \operatorname{neg}_{i}^{(2)}(\Psi)$. Therefore,

$$
\begin{equation*}
S_{2} \leq \sum_{i \in T} \operatorname{pos}_{i}^{(2)}(\Psi)+\sum_{i \notin T} \operatorname{neg}_{i}^{(2)}(\Psi) \tag{3.11}
\end{equation*}
$$

Recall that

$$
\begin{align*}
\operatorname{bias}(\Psi) & =\sum_{i \in[n]} \operatorname{pos}_{i}^{(1)}(\Psi)+\frac{\operatorname{pos}_{i}^{(2)}(\Psi)}{2}-\operatorname{neg}_{i}^{(1)}(\Psi)-\frac{\operatorname{neg}_{i}^{(2)}(\Psi)}{2}  \tag{3.12}\\
m_{1} & =\sum_{i \in[n]} \operatorname{pos}_{i}^{(1)}(\Psi)+\operatorname{neg}_{i}^{(1)}(\Psi)  \tag{3.13}\\
m_{2} & =\sum_{i \in[n]} \frac{\operatorname{pos}_{i}^{(2)}(\Psi)}{2}+\frac{\operatorname{neg}_{i}^{(2)}(\Psi)}{2}  \tag{3.14}\\
-2 S_{1} & =-2 \sum_{i \in T} \operatorname{pos}_{i}^{(1)}(\Psi)-2 \sum_{i \notin T} \operatorname{neg}_{i}^{(1)}(\Psi) \tag{3.15}
\end{align*}
$$

and since $\operatorname{bias}_{i}(\Psi) \geq 0$ for every $i$ :

$$
\begin{equation*}
0 \geq-2 \sum_{i \notin T} \operatorname{bias}_{i}(\Psi)=-\sum_{i \notin T} 2 \operatorname{pos}_{i}^{(1)}(\Psi)+\operatorname{pos}_{i}^{(2)}(\Psi)-2 \operatorname{neg}_{i}^{(1)}(\Psi)-\operatorname{neg}_{i}^{(2)}(\Psi) \tag{3.16}
\end{equation*}
$$

Summing (3.12), (3.13), (3.14), (3.15), and (3.16) gives

$$
\operatorname{bias}(\Psi)+m_{1}+m_{2}-2 S_{1} \geq \sum_{i \in T} \operatorname{pos}_{i}^{(2)}(\Psi)+\sum_{i \notin T} \operatorname{neg}_{i}^{(2)}(\Psi) \geq S_{2}
$$

where the last inequality uses (3.11). This finishes the proof of (3.10) and the proof of the lemma.
Lemma 3.7. Let $\Psi$ be a Max-2OR instance with $m_{1} 1$-clauses, and $m_{2}$ 2-clauses. Then

1. $\mathrm{val}_{\Psi} \geq \frac{m_{1}+m_{2}+\operatorname{bias}(\Psi)}{2}$;
2. if $\operatorname{bias}(\Psi) \leq m_{2}$, then

$$
v a l_{\Psi} \geq \frac{m_{1}}{2}+\frac{3 m_{2}}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4 m_{2}}
$$

Proof. Without loss of generality, we assume that for every $i \in[n], \operatorname{bias}_{i}(\Psi) \geq 0$. (Again, we can negate all variables with $\operatorname{bias}_{i}(\Psi)<0$, and define a bijection between the assignments for the two formulas.) Therefore, for every $i \in[n]$,

$$
\operatorname{pos}_{i}^{(1)}(\Psi)+\frac{\operatorname{pos}_{i}^{(2)}(\Psi)}{2}-\operatorname{neg}_{i}^{(1)}(\Psi)-\frac{\operatorname{neg}_{i}^{(2)}(\Psi)}{2} \geq 0
$$

Let $p_{1}$ and $n_{1}$ be the numbers of 1-clauses with positive and negative literals in $\Psi$. Let $k_{0}, k_{1}$, and $k_{2}$ denote the numbers of 2 -clauses with 0,1 , and 2 positive literals. Then $m_{1}=p_{1}+n_{1}$, and $m_{2}=k_{0}+k_{1}+k_{2}$.

Note that

$$
\operatorname{bias}(\Psi)=\sum_{i} \operatorname{pos}_{i}^{(1)}(\Psi)+\frac{\operatorname{pos}_{i}^{(2)}(\Psi)}{2}-\operatorname{neg}_{i}^{(1)}(\Psi)-\frac{\operatorname{neg}_{i}^{(2)}(\Psi)}{2}=p_{1}-n_{1}+k_{2}-k_{0}
$$

Consider the distribution of assignments to the variables of $\Psi$, where every variable $x_{i}$ is assigned the value 1 independently with probability $\left(\frac{1}{2}+\gamma\right)$, for a parameter $\gamma \in[0,0.5]$ to be assigned later. The expected number of satisfied 1-clauses under this distribution is

$$
S_{1}=\sum_{i}\left(\frac{1}{2}+\gamma\right) \cdot \operatorname{pos}_{i}^{(1)}(\Psi)+\left(\frac{1}{2}-\gamma\right) \cdot \operatorname{neg}_{i}^{(1)}(\Psi)=\left(\frac{1}{2}+\gamma\right) p_{1}+\left(\frac{1}{2}-\gamma\right) n_{1}=\frac{m_{1}}{2}+\gamma\left(p_{1}-n_{1}\right)
$$

Since every 2-clause contains distinct variables, the expected number of satisfied 2-clauses is

$$
\begin{aligned}
S_{2} & =k_{0} \cdot\left(1-\left(\frac{1}{2}+\gamma\right)^{2}\right)+k_{1} \cdot\left(1-\left(\frac{1}{2}+\gamma\right)\left(\frac{1}{2}-\gamma\right)\right)+k_{2} \cdot\left(1-\left(\frac{1}{2}-\gamma\right)^{2}\right) \\
& =\frac{3 m_{2}}{4}+\gamma \cdot\left(k_{2}-k_{0}\right)-\gamma^{2} \cdot\left(k_{0}+k_{2}-k_{1}\right) \\
& \geq \frac{3 m_{2}}{4}+\gamma \cdot\left(k_{2}-k_{0}\right)-m_{2} \gamma^{2}
\end{aligned}
$$

Let us now compute the expected number of clauses satisfied by an assignment $\sigma$ from the distribution defined above.

$$
\begin{aligned}
\underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right]=S_{1}+S_{2} & \geq \frac{m_{1}}{2}+\gamma\left(p_{1}-n_{1}\right)+\frac{3 m_{2}}{4}+\gamma \cdot\left(k_{2}-k_{0}\right)-2 m_{2} \gamma^{2} \\
& =\frac{m_{1}}{2}+\frac{3 m_{2}}{4}+\gamma \operatorname{bias}(\Psi)-m_{2} \gamma^{2}
\end{aligned}
$$

First, we set $\gamma=\frac{1}{2}$ and derive the first bound:

$$
\operatorname{val}_{\Psi} \geq \underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right] \geq \frac{m_{1}+m_{2}+\operatorname{bias}(\Psi)}{2}
$$

Now, for the case where $\operatorname{bias}(\Psi) \leq m_{2}$, we set $\gamma=\frac{\operatorname{bias}(\Psi)}{2 m_{2}} \in[0,0.5]$, and derive the second bound:

$$
\operatorname{val}_{\Psi} \geq \underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right] \geq \frac{m_{1}}{2}+\frac{3 m_{2}}{4}+\frac{\operatorname{bias}(\Psi)^{2}}{4 m_{2}}
$$

## 4 Space Lower Bounds for Approximating Boolean Max-2CSP

In this section, we establish space lower bounds for streaming approximations for all Max-2CSPs. In Theorem 1.1 in Section 6 we will show that it suffices to prove lower bounds for Max- $\operatorname{CSP}(\mathcal{G})$ for the following
four cases $\mathcal{G} \in\{\mathrm{OR},\{\mathrm{TR}, \mathrm{OR}\}, \mathrm{XOR}, \mathrm{AND}\}$. A linear space lower bound for the case $\mathcal{G}=\mathrm{XOR}$ is proven by Kapralov and Krachun [KK19]. We use this result to prove a linear lower bound for the case $\mathcal{F}=\mathrm{OR}$ in Section 4.1. We prove the two remaining lower bounds by reductions from the communication complexity problem DBHP [KKS15]. In Section 4.2, we present a general framework for proving such lower bounds, while in Sections 4.3 and 4.4 we give specific reductions for the Max-2AND and Max-2OR problems. Finally, Sections 4.5 and 4.6 contain the proofs of some technical results used in the framework in Section 4.2.

### 4.1 From Max-2EXOR to Max-2EOR

In this section, we give a simple streaming reduction from Max-CUT to Max-2EOR, which asserts that a better than trivial 3/4-approximation for Max-2EOR would lead to a better then trivial 1/2-approximation for Max-CUT. Since the latter is known to require linear space [KK19], we get a linear lower bound on the space complexity of $(3 / 4+\varepsilon)$-approximations of Max-2EOR.

Lemma 4.1 (Folklore). Let $\Psi_{X O R}$ be a Max-2EXOR instance with $m$ clauses. Consider the following reduction from $\Psi_{\text {XOR }}$ to $\Psi_{O R}$, a Max-2EOR instance: For every clause $(x \oplus y)$ in $\Psi_{\text {XoR }}$, we add clauses $(x \vee y)$ and $(\neg x \vee \neg y)$ to $\Psi_{O R}$. Then

$$
v a l_{\Psi_{O R}}=m+v a l_{\Psi \times O R}
$$

Proof. It suffices to show that for every assignment $\sigma$, val $\Psi_{\Psi_{\text {OR }}}(\sigma)=m+\mathrm{val}_{\Psi_{\mathrm{xOR}}}(\sigma)$. Suppose $\sigma$ satisfies the clause $x \oplus y$ in $\Psi_{\text {XOR }}$, then $\sigma(x) \neq \sigma(y)$. In this case, $\sigma$ satisfies both the corresponding clauses, $(x \vee y)$ and $(\neg x \vee \neg y)$ in $\Psi_{\mathrm{OR}}$. On the other hand, if $\sigma$ does not satisfy $x \oplus y$ in $\Psi_{\mathrm{XOR}}$, then $\sigma(x)=\sigma(y)$. In this case, $\sigma$ satisfies exactly one of the corresponding clauses in $\Psi_{\text {OR }}$.

Corollary 4.2. For any constant $\varepsilon>0$, any streaming algorithm that $(3 / 4+\varepsilon)$-approximates Max-2EOR with success probability at least $3 / 4$ requires $\Omega(n)$ space.

Proof. Let ALG be a $(3 / 4+\varepsilon)$-approximate algorithm for Max-2EOR. We will show that there exists a streaming algorithm of the same space complexity as ALG which $(1 / 2+4 \varepsilon / 3)$-approximates Max-2EXOR. This, together with the $\Omega(n)$ space lower bound for $(1 / 2+\varepsilon)$-approximations for Max-2EXOR (Theorem 2.4), will finish the proof.

Given a Max-2EXOR instance $\Psi_{\text {XOR }}$ with $m$ clauses, we use Lemma 4.1 to convert it into a Max-2EOR instance $\Psi_{\text {OR }}$. Let $v$ be the output of the algorithm ALG on $\Psi_{\text {OR }}$, then we output $\max \{m / 2, v-m\}$ as an approximation to val $_{\Psi_{\text {xOR }}}$. It remains to show that $\left(\frac{1}{2}+\frac{4 \varepsilon}{3}\right) \cdot \operatorname{val}_{\Psi_{\mathrm{XOR}}} \leq \max \{m / 2, v-m\} \leq \operatorname{val}_{\Psi_{\mathrm{XOR}}}$.

First, by Lemma 4.1

$$
v-m \leq \operatorname{val}_{\Psi_{\mathrm{OR}}}-m=\operatorname{val}_{\Psi_{\mathrm{XOR}}}
$$

Together with the trivial bound $\operatorname{val}_{\Psi_{\mathrm{xoR}}} \geq m / 2$, this establishes that $\max \{m / 2, v-m\} \leq \operatorname{val}_{\Psi_{\mathrm{xoR}}}$. Second,

$$
\begin{aligned}
\max \left\{\frac{m}{2}, v-m\right\} & \geq \frac{1}{3} \cdot \frac{m}{2}+\frac{2}{3} \cdot(v-m)=\frac{m}{6}+\frac{2}{3} \cdot\left(\frac{3}{4}+\varepsilon\right) \cdot \mathrm{val}_{\Psi_{\mathrm{OR}}}-\frac{2 m}{3} \\
& =\left(\frac{1}{2}+\frac{2 \varepsilon}{3}\right) \cdot\left(\mathrm{val}_{\Psi_{\mathrm{XOR}}}+m\right)-\frac{m}{2} \\
& =\left(\frac{1}{2}+\frac{2 \varepsilon}{3}\right) \cdot \mathrm{val}_{\Psi_{\mathrm{XOR}}}+\frac{2 \varepsilon m}{3} \\
& \geq\left(\frac{1}{2}+\frac{4 \varepsilon}{3}\right) \cdot \mathrm{val}_{\Psi_{\mathrm{XOR}}}
\end{aligned}
$$

### 4.2 Distributional Boolean Hidden Partition (DBHP) Problem

We prove lower bounds for Max-2EAND and Max-2OR in two steps. Recall that the goal of the players in DBHP is to distinguish between two distributions YES and NO. First, we show a reduction from DBHP to Max- $\operatorname{CSP}(\mathcal{G})$. This induces a YES and a NO distributions of instances of Max-CSP $(\mathcal{G})$, corresponding to the YES and NO cases of DBHP. Next, we show that with high probability there is a gap between the optimal value of instances from the YES and NO distributions. The ratio $\alpha$ between these optimal values will be the upper bound on the approximation ratio of space-efficient streaming algorithms. Informally, any $(\alpha+\varepsilon)$-approximate streaming algorithm with space $s$ distinguishes the distributions YES and NO, and, therefore, can be converted into a communication protocol for DBHP that uses $s$ bits of communication. Since Kapralov, Khanna, and Sudan [KKS15] proved that any communication protocol for DBHP requires at least $\Omega(\sqrt{n})$ bits of communication, the corresponding space lower bound for streaming algorithms follows.

Before presenting the framework for streaming lower bounds, we will need to define DBHP and slightly adjust it to our setting.

For $n \in \mathbb{N}$ and $p \in[0,1]$, by $G(n, p)$ we denote the Erdös-Rényi distribution of undirected graphs with $n$ vertices, where each edge is chosen independently with probability $p$.
Definition 4.3 (DBHP). Let $n \in \mathbb{N}, \beta \in(0,1 / 16)$ be parameters. Let $X^{*} \in\{0,1\}^{n}$ be a uniformly random vector, and $G$ be a random graph sampled from $G(n, 2 \beta / n)$. Let $r$ be the number of edges in $G$, and $M \in\{0,1\}^{r \times n}$ be the edge-vertex incidence matrix of $G$. We will consider the following three distributions of a vector $w \in\{0,1\}^{r}$.

- (YES distribution) $w=M X^{*} \in\{0,1\}^{r}$, where the arithmetic is over $\mathbb{F}_{2}$;
- (NO distribution) $w=\mathbf{1}+M X^{*} \in\{0,1\}^{r}$, where $\mathbf{1} \in \mathbb{F}_{2}^{r}$ is the all 1 s vector, and the arithmetic is over $\mathbb{F}_{2}^{r}$;
- $\overline{\mathbf{N O}}$ distribution) $w$ be uniformly sampled from $\{0,1\}^{r}$.

For a pair of distinct distributions $\mathcal{D} \neq \mathcal{D}^{\prime} \in\{\boldsymbol{Y} \boldsymbol{E S}, \mathbf{N O}, \overline{\mathbf{N O}}\}$, we consider the following decisional 2-player one-way communication problem $\operatorname{DBHP}_{\mathcal{D}, \mathcal{D}^{\prime}}(n, \beta)$. Alice receives $X^{*} \in\{0,1\}^{n}$, and Bob receives $(M, w)$ as their private inputs, where $w$ is sampled from $\mathcal{D}$ or $\mathcal{D}^{\prime}$ with probability $1 / 2$. A communication protocol $\Pi$ for $\operatorname{DBHP}_{\mathcal{D}, \mathcal{D}^{\prime}}(n, \beta)$ consists of a message $m$ sent from Alice to Bob. The complexity of the protocol $\Pi$ is the length of the message $m:|\Pi|:=|m|$. The goal of the players is to distinguish between the distributions $\mathcal{D}$ and $\mathcal{D}^{\prime}$, and the success probability of $\Pi$ is defined as $\operatorname{Pr}_{(M, w) \sim \mathcal{D}}[$ Bob outputs $\mathcal{D}] / 2+$ $\operatorname{Pr}_{(M, w) \sim \mathcal{D}^{\prime}}\left[\right.$ Bob outputs $\left.\mathcal{D}^{\prime}\right] / 2$.
[KKS15] showed that for any constant $\delta>0$, any protocol that solves $\operatorname{DBHP}_{\mathbf{Y E S}, \overline{\mathbf{N O}}}(n, \beta)$ with success probability $(1 / 2+\delta)$ requires $\Omega\left(\beta^{3 / 2} \sqrt{n}\right)$ bits of communication. The next lemma shows that the same lower bound extends to the DBHP ${ }_{\text {YeS, NO }}$ problem by an application of the triangle inequality.

Lemma 4.4 (A modification of [KKS15, Lemma 5.1]). Let $\beta \in\left(n^{-1 / 10}, 1 / 16\right)$ and $s \in\left(n^{-1 / 10}, 1\right)$ be parameters. Any protocol $\Pi$ for $\operatorname{DBHP}_{Y E S, N O}(n, \beta)$ that uses $s \sqrt{n}$ bits of communication cannot distinguish between the $\boldsymbol{Y E S}$ and $\boldsymbol{N O}$ distributions with success probability more than $1 / 2+c \cdot\left(\beta^{3 / 2}+s\right)$ for some constant $c>0$ and all large enough $n$.

For completeness, we present a proof of Lemma 4.4 in Section 4.5. For ease of exposition, now we will use $\operatorname{DBHP}(n, \beta)$ to denote $\operatorname{DBHP} \mathbf{Y E S}, \mathrm{NO}(n, \beta)$.

Finally, note that the graph $G$ in the definition of DBHP is extremely sparse (in expectation it has $r \approx \beta n<0.1 n$ edges), and, thus, it is not immediately useful for designing hard instances of Max-2CSP problems. In order to overcome this issue, [KKS15] used DBHP where Bob receives a collection of $T$ messages all sampled either from the YES or NO distribution. Now the union of the $T$ sparse graphs received by Bob can be used in reductions to Max-2CSPs.
Definition 4.5 (DBHP with $T$ messages). For any $\beta \in(0,1 / 16)$ and $n, T \in \mathbb{N}$, we define $\operatorname{DBHP}(n, \beta, T)$ as follows. Let $X^{*} \in\{0,1\}^{n}$ be a uniformly random vector, and for $1 \leq t \leq T$, let $G_{i}$ be a random graph
sampled from $G(n, 2 \beta / n)$, and $M_{i}$ be the edge-vertex incidence matrix of $G_{i}$. Alice receives $X^{*}$, and Bob receives a list $\left(M_{1}, w_{1}\right), \ldots,\left(M_{T}, w_{T}\right)$, where with probability $1 / 2$ all $w_{t}=M_{t} X^{*}$ ( $\boldsymbol{Y E S}$ case), and with probability $1 / 2$ all $w_{t}=\mathbf{1}+M_{t} X^{*}$ ( $\mathbf{N O}$ case). The goal of the players is to have a non-trivial advantage over a random guess in distinguishing between the two distributions, while only communication from Alice to Bob is allowed.

Reduction from DBHP. A reduction from $\operatorname{DBHP}(n, \beta, T)$ to $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ is defined by a pair of algorithms, $\mathcal{A}$ and $\mathcal{B}$. Alice receives her input vector $X^{*} \in\{0,1\}^{n}$, runs $\mathcal{A}$ on the input $X^{*}$, and outputs a set of $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$-clauses. Bob receives a collection of $T$ pairs $\left(M_{t}, w_{t}\right)$, applies $\mathcal{B}$ to each of them, and outputs $T$ sets of Max- $\operatorname{CSP}(\mathcal{G})$-clauses. Finally, the resulting instance of the $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ problem is the union of clauses from $\mathcal{A}\left(X^{*}\right), \mathcal{B}\left(M_{1}, w_{1}\right), \ldots, \mathcal{B}\left(M_{T}, w_{T}\right)$.

The reduction above naturally induces two distributions $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ of $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ instances, corresponding to the YES and NO distributions of $\left(M_{t}, w_{t}\right)$. Let us pick some $v^{Y}$ and $v^{N}$, such that $\operatorname{Pr}_{\Psi \sim \mathcal{D}^{Y}}\left[\operatorname{val}_{\Psi} \geq v^{Y}\right]>1-o(1)$ and $\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}}\left[\operatorname{val}_{\Psi} \leq v^{N}\right]>1-o(1)$. Note that for any $\alpha>v^{N} / v^{Y}$, an $\alpha$-approximate streaming algorithm for $\operatorname{Max} \operatorname{CSP}(\mathcal{G})$ distinguishes the two distributions $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ with high probability. The following theorem states that any streaming algorithm that distinguishes these two distributions, requires space $\Omega(\sqrt{n})$. In particular, any streaming $\alpha$-approximation for $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ requires space at least $\Omega(\sqrt{n})$.

Theorem 4.6 (Reduction from DBHP with $T$ messages). Let $c>0$ be the constant from Lemma 4.4. For every $T \in \mathbb{N}, 0<\beta \leq 1 /(10 c T)^{2 / 3}$, and reduction $(\mathcal{A}, \mathcal{B})$ from $\operatorname{DBHP}$ to $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$, any streaming algorithm that distinguishes $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ with success probability at least $3 / 4$ requires space at least $\frac{1}{40 c T} \cdot \sqrt{n}$.

The proof of Theorem 4.6 follows the proofs in [KKS15] by using the standard hybrid argument as well as the data processing inequality for total variation. We postpone the details of the proof of Theorem 4.6 to Section 4.6, and first describe reductions from DBHP to Max-2EAND and Max-2OR in Sections 4.3 and 4.4, respectively.

### 4.3 From DBHP to Max-2EAND

Now, we describe the reduction from DBHP to Max-2EAND. In order to describe the reduction, it suffices to specify the parameters $\beta$ and $T$, and the algorithms $\mathcal{A}^{\text {EAND }}$ and $\mathcal{B}^{\text {EAND }}$. Recall that we associate a vector $X^{*} \in\{0,1\}^{n}$ with the set of its ones: $X \subseteq[n], X=\left\{i: X_{i}=1\right\}$. Also, recall that the input of Bob, $(M, w)$, consists of an edge-vertex incidence matrix $M \in\{0,1\}^{r \times n}$ and a vector $w \in\{0,1\}^{r}$. In particular, every row of $M$ has exactly two ones.

## Reduction from DBHP to Max-2EAND

- Let $c>0$ be the constant from Lemma 4.4. For a given error parameter $\varepsilon \in(0,1)$, let $T=\left(10000 / \varepsilon^{2}\right)^{3} \cdot(10 c)^{2}$ and $\beta=\frac{1}{(10 c T)^{2 / 3}}$ such that $\beta T=10000 / \varepsilon^{2}$.
- $\mathcal{A}^{\text {EAND }}\left(X^{*}\right)$ : Sample $\beta n T / 4$ independent pairs $(i, j) \in X^{*} \times \overline{X^{*}}$, and for each of them output the clause $\left(x_{i} \wedge \neg x_{j}\right)$.
- $\mathcal{B}^{\text {EAND }}(M, w)$ : Let $r$ be the number of rows in $M$. For each $1 \leq k \leq r$ with $w_{k}=1$, let the 1 s in the $k^{\text {th }}$ row of $M$ be at the $i^{\text {th }}$ and $j^{\text {th }}$ positions, then output two clauses: $\left(x_{i} \wedge \neg x_{j}\right)$ and $\left(\neg x_{i} \wedge x_{j}\right)$.

Lemma 4.7. For any $\varepsilon \in(0,1)$, let $\left(\beta, T, \mathcal{A}^{E A N D}, \mathcal{B}^{E A N D}\right)$ be the parameters described in the above reduction. For a Max-2EAND instance $\Psi$, let $m_{\Psi}$ denote the number of clauses in $\Psi$. Then

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{Y}}^{\left(\beta, T, \mathcal{A}^{E A N D}, \mathcal{B}^{E A N D}\right)} \boldsymbol{}\left[v a l_{\Psi}<\left(\frac{3}{5}-\varepsilon\right) \cdot m_{\Psi}\right]=o(1)
$$

and

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{E A N D}, \mathcal{B}^{E A N D}\right)}\left[v a l_{\Psi}>\left(\frac{4}{15}+\varepsilon\right) \cdot m_{\Psi}\right]=o(1) .
$$

We prove Lemma 4.7 in Section 5.2. An immediate corollary of Theorem 4.6 and Lemma 4.7 is the desired lower bound for streaming approximation of Max-2EAND.

Corollary 4.8. For any constant $\varepsilon \in(0,1)$, any streaming algorithm that $(4 / 9+\varepsilon)$-approximates Max-2EAND with success probability at least $3 / 4$ requires $\Omega(\sqrt{n})$ space.

### 4.4 From DBHP to Max-2OR

Now, we describe the reduction from DBHP to Max-2OR. Again, it suffices to specify the parameters $\beta$ and $T$, and the algorithms $\mathcal{A}^{\mathrm{OR}}$ and $\mathcal{B}^{\mathrm{OR}}$.

## Reduction from DBHP to OR

- Let $c>0$ be the constant from Lemma 4.4. For a given error parameter $\varepsilon \in(0,1)$, let $T=\left(10000 / \varepsilon^{2}\right)^{3} \cdot(10 c)^{2}$ and $\beta=\frac{1}{(10 c T)^{2 / 3}}$ such that $\beta T=10000 / \varepsilon^{2}$.
- $\mathcal{A}^{\mathrm{OR}}\left(X^{*}\right)$ : Sample $\frac{\sqrt{2}-1}{2} \cdot \beta n T$ independent copies of $i \in X^{*}$, and for each of them output the 1-clause $\left(x_{i}\right)$. Sample another $\frac{\sqrt{2}-1}{2} \cdot \beta n T$ independent copies of $j \in \overline{X^{*}}$, and for each of them output the 1-clause ( $\left.\neg x_{j}\right)$.
- $\mathcal{B}^{\mathrm{OR}}(M, w)$ : Let $r$ be the number of rows in $M$. For each $1 \leq k \leq r$ with $w_{k}=1$, let the the 1 s in the $k^{\text {th }}$ row of $M$ be at the $i^{\text {th }}$ and $j^{\text {th }}$ positions, then output two clauses: $\left(x_{i} \vee x_{j}\right)$ and $\left(\neg x_{i} \vee \neg x_{j}\right)$.

Lemma 4.9. For any $\varepsilon \in(0,1)$, let $\left(\beta, T, \mathcal{A}^{O R}, \mathcal{B}^{O R}\right)$ be the parameters described in the above reduction. For a Max-2OR instance $\Psi$, let $m_{\Psi}$ denote the number of clauses in $\Psi$. Then

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{Y}}{\left.\operatorname{Pr}, T, \mathcal{A}^{\circ R}, \mathcal{B}^{\circ R}\right)}\left[v a l_{\Psi}=m_{\Psi}\right]=1
$$

and

$$
\underset{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{\circ R, \mathcal{B} O R}\right)}{\operatorname{Pr}}\left[v a l_{\Psi}>\left(\frac{\sqrt{2}}{2}+\varepsilon\right) \cdot m_{\Psi}\right]=o(1) .
$$

The proof of Lemma 4.9 is presented in Section 5.3. Now, the desired lower bound for any streaming approximations for Max-2OR immediately follows from Theorem 4.6 and Lemma 4.9.
Corollary 4.10. For any constant $\varepsilon \in(0,1)$, any streaming algorithm that $(\sqrt{2} / 2+\varepsilon)$-approximates Max-2OR with success probability at least $3 / 4$ requires $\Omega(\sqrt{n})$ space.

### 4.5 Proof of Lemma 4.4

In this section, we show that the hardness of $\operatorname{DBHP}_{\mathbf{Y E S}, \overline{\mathbf{N O}}}(n, \beta)$ proved in [KKS15] can be easily extended to the hardness of $\operatorname{DBHP}_{\mathrm{YES}, \mathrm{NO}}(n, \beta)$.

Lemma 4.4 (A modification of [KKS15, Lemma 5.1]). Let $\beta \in\left(n^{-1 / 10}, 1 / 16\right)$ and $s \in\left(n^{-1 / 10}, 1\right)$ be parameters. Any protocol $\Pi$ for $\operatorname{DBHP}_{\boldsymbol{Y E S}, \mathrm{NO}}(n, \beta)$ that uses $s \sqrt{n}$ bits of communication cannot distinguish between the YES and NO distributions with success probability more than $1 / 2+c \cdot\left(\beta^{3 / 2}+s\right)$ for some constant $c>0$ and all large enough $n$.

Proof. Let us consider a protocol $\Pi$ that uses $s \sqrt{n}$ bits of communication to distinguish between the YES and NO distributions. For an Alice's input $X^{*}$, we denote the message that Alice sends to Bob by $\Pi\left(X^{*}\right)$. For each $\mathcal{D} \in\{\mathbf{Y E S}, \mathbf{N O}, \overline{\mathbf{N O}}\}$, let $\mathcal{P}_{\mathcal{D}}$ be the distribution of $\left(M, \Pi\left(X^{*}\right), w\right)$ where $\left(X^{*}, M, w\right) \sim \mathcal{D}$.

The equation (12) in $[\text { KKS15 }]^{8}$ shows that in this case

$$
\left\|\mathcal{P}_{\mathbf{Y E S}}-\mathcal{P}_{\overline{\mathrm{NO}}}\right\|_{t v d}=O\left(\beta^{3 / 2}+s\right) .
$$

Observe that when $\left(X^{*}, M, w\right) \sim \overline{\mathbf{N O}}$, both $\left(M, \Pi\left(X^{*}\right), w\right)$ and $\left(M, \Pi\left(X^{*}\right), \mathbf{1}+w\right)$ are distributed according to $\mathcal{P}_{\overline{\mathrm{NO}}}$. (Indeed, in this case $w \in\{0,1\}^{r}$ is uniformly random, and independent of the choices of $X^{*}$ and $M$. ) Also, from the definitions of the distributions YES and NO, when ( $\left.X^{*}, M, w\right) \sim$ YES (and, thus, $\left.\left(M, \Pi\left(X^{*}\right), w\right) \sim \mathcal{P}_{\mathrm{YES}}\right)$, we have that $\left(M, \Pi\left(X^{*}\right), \mathbf{1}+w\right)$ is distributed according to $\mathcal{P}_{\mathrm{NO}}$.

Further, by the data processing inequality (see Proposition 2.6), adding the constant vector $\mathbf{1}$ to the variable $w$ in $\left(M, \Pi\left(X^{*}\right), w\right)$ does not increase the total variation distance. Thus, we have

$$
\left\|\mathcal{P}_{\mathrm{NO}}-\mathcal{P}_{\overline{\mathrm{NO}}}\right\|_{\text {tvd }} \leq\left\|\mathcal{P}_{\mathrm{YES}}-\mathcal{P}_{\overline{\mathrm{NO}}}\right\|_{\text {tvd }}=O\left(\beta^{3 / 2}+s\right) .
$$

Finally, by the triangle inequality (see Proposition 2.6),

$$
\left\|\mathcal{P}_{\mathbf{Y E S}}-\mathcal{P}_{\mathbf{N O}}\right\|_{t v d} \leq\left\|\mathcal{P}_{\mathbf{Y E S}}-\mathcal{P}_{\overline{\mathbf{N O}}}\right\|_{t v d}+\left\|\mathcal{P}_{\mathbf{N O}}-\mathcal{P}_{\overline{\mathbf{N O}}}\right\|_{t v d}=O\left(\beta^{3 / 2}+s\right) .
$$

From the definition of the total variation distance, we have that the success probability of Bob in distinguishing YES from NO is at most $1 / 2+O\left(\beta^{3 / 2}+s\right)$, which completes the proof.

### 4.6 Proof of Theorem 4.6

Before presenting the proof of Theorem 4.6, we will show that a streaming algorithm ALG for distinguishing the distributions $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ can be turned into a protocol for $\operatorname{DBHP}(n, \beta)$.
Lemma 4.11. Let $n, T, s \in \mathbb{N}, \beta \in\left(n^{-1 / 10}, 1 / 16\right)$, and let $(\mathcal{A}, \mathcal{B})$ be a reduction from DBHP to Max-CSP(G). Suppose that a streaming algorithm $\boldsymbol{A L G}$ distinguishes $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ using space $s$ with probability at least $1 / 2+\Delta$, then there is a one-way protocol for $\operatorname{DBHP}(n, \beta)$ using at most $s$ bits of communication that succeeds with probability at least $1 / 2+\Delta /(2 T)$.

Proof. First we fix the randomness of the algorithm ALG so that the resulting deterministic algorithm succeeds with probability at least $1 / 2+\Delta$ (by the averaging argument). Next, by triangle inequality, ALG can distinguish either (i) $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{\bar{N}}(\beta, T, \mathcal{A}, \mathcal{B})$ or (ii) $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{\bar{N}}(\beta, T, \mathcal{A}, \mathcal{B})$ with probability at least $1 / 2+\Delta / 2$. Without loss of generality, let us assume it is case (i) while the case (ii) can be analyzed similarly.

Now, for each $i=0,1, \ldots, T$, let $S_{i}^{Y}$ (resp., $S_{i}^{\bar{N}}$ ) be the state of ALG after receiving $\mathcal{A}\left(X^{*}\right), \mathcal{B}\left(M_{1}, w_{1}\right), \ldots, \mathcal{B}\left(M_{i}, w_{i}\right)$, where the inputs are sampled from the YES (resp., $\left.\overline{\mathbf{N O}}\right)$ distribution. Note that $\left\{S_{i}^{Y}\right\}$ and $\left\{S_{i}^{\bar{N}}\right\}$ are random variables, and $\left\|S_{0}^{Y}-S_{0}^{\bar{N}}\right\|_{\text {tvd }}=0$ while the success probability of

[^7]ALG guarantees that $\left\|S_{T}^{Y}-S_{T}^{\bar{N}}\right\|_{t v d} \geq \Delta / 2$. By the hybrid argument and the triangle inequality for the total variation distance (Proposition 2.6), there exists $i^{*} \in[T-1]$ such that

$$
\begin{equation*}
\left\|S_{i^{*}+1}^{Y}-S_{i^{*}+1}^{\bar{N}}\right\|_{t v d}-\left\|S_{i^{*}}^{Y}-S_{i^{*}}^{\bar{N}}\right\|_{t v d} \geq \frac{\Delta}{2 T} \tag{4.12}
\end{equation*}
$$

This indicates that the $\left(i^{*}+1\right)^{\text {th }}$ inputs (i.e., $\left.\mathcal{B}\left(M_{i}, w_{i}\right)\right)$ are sufficient for distinguishing between the YES and the $\overline{\mathbf{N O}}$ cases with non-trivial probability. Specifically, let $\tilde{S}^{Y}$ (resp. $\tilde{S}^{\bar{N}}$ ) be the distribution of the states of ALG, when it starts with a state from $S_{i^{*}}^{Y}$ and receives one input $\mathcal{B}(M, w)$ where $(M, w)$ is sampled from the YES (resp. $\overline{\mathbf{N O}}$ ) distributions.

Claim 4.13. Let $\tilde{S}^{Y}$ and $\tilde{S}^{\bar{N}}$ be the random variables defined above, then $\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{\text {tvd }} \geq \frac{\Delta}{2 T}$.
Proof. First, by the triangle inequality for the total variation distance (see Proposition 2.6), we have

$$
\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{t v d} \geq\left\|\tilde{S}^{Y}-S_{i^{*}+1}^{\bar{N}}\right\|_{t v d}-\left\|\tilde{S}^{\bar{N}}-S_{i^{*}+1}^{\bar{N}}\right\|_{t v d}
$$

Note that $\tilde{S}^{Y}=S_{i^{*}+1}^{Y}$ by definition, and $\left\|\tilde{S}^{\bar{N}}-S_{i^{*}+1}^{\bar{N}}\right\|_{t v d} \leq\left\|S_{i^{*}}^{Y}-S_{i^{*}}^{\bar{N}}\right\|_{t v d}$ by the data processing inequality. Concretely, we apply item 2 of Proposition 2.6 with $X=S_{i^{*}}^{Y}, Y=S_{i^{*}}^{\bar{N}}, W=\mathcal{B}\left(M_{i^{*}+1}, w_{i^{*}+1}\right)$ where $\left(M_{i^{*}+1}, w_{i^{*}+1}\right) \sim \overline{\mathbf{N O}}$, and $\left.f=\mathbf{A L G}\right)$. Note that since $\left(M_{i^{*}+1}, w_{i^{*}+1}\right) \sim \overline{\mathbf{N O}}$ we have $W$ being independent to both $X$ and $Y$. This is the reason why we need to work on YES versus $\overline{\mathbf{N O}}$ rather than directly using YES versus NO. Finally, this together with (4.12), gives the desired bound

$$
\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{t v d} \geq\left\|S_{i^{*}+1}^{Y}-S_{i^{*}+1}^{\bar{N}}\right\|_{t v d}-\left\|S_{i^{*}}^{Y}-S_{i^{*}}^{\bar{N}}\right\|_{t v d} \geq \frac{\Delta}{2 T}
$$

Finally, we use ALG to design a protocol for $\operatorname{DBHP}(n, \beta)$. Note that Alice and Bob have the description of the algorithm ALG, and, therefore, know the distributions $\tilde{S}^{Y}$ and $\tilde{S}^{\bar{N}}$. In particular, they both know the value of $i^{*}$. Moreover, since Alice and Bob know $i^{*}$, they know the distributions $\tilde{S}^{Y}$ and $\tilde{S}^{\bar{N}}$.

```
Algorithm 4 A protocol for \(\operatorname{DBHP}(n, \beta)\) using ALG
Input: Alice receives input \(X^{*}\), and Bob receives inputs \((M, w)\).
Goal: Distinguish between \(M X^{*}=w\) (YES case) and \(M X^{*}=\mathbf{1}-w(\overline{\mathbf{N O}}\) case).
    1: Alice samples a state \(S_{A}\) of ALG from the distribution \(S_{i^{*}}^{Y}\), conditioned on the first input being \(\mathcal{A}\left(X^{*}\right)\).
    Alice sends \(S_{A}\) to Bob. Since ALG uses \(s\) bits of memory, \(\left|S_{A}\right| \leq s\).
    Bob executes ALG with the initial state \(S_{A}\) on the input \(\mathcal{B}(M, w)\). Let \(S\) be the resulting state of ALG.
    Bob outputs YES if \(\operatorname{Pr}\left[\tilde{S}^{Y}=S\right] \geq \operatorname{Pr}\left[\tilde{S}^{\bar{N}}=S\right]\); otherwise Bob outputs \(\overline{\text { NO }}\).
```

Let $\Omega_{Y}=\left\{S: \operatorname{Pr}\left[\tilde{S}^{Y}=S\right] \geq \operatorname{Pr}\left[\tilde{S}^{\bar{N}}=S\right]\right\}$ and $\Omega_{\bar{N}}=\left\{S: \operatorname{Pr}\left[\tilde{S}^{Y}=S\right]<\operatorname{Pr}\left[\tilde{S}^{\bar{N}}=S\right]\right\}$, then

$$
\begin{aligned}
\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{t v d} & =\frac{1}{2} \sum_{S \in \Omega_{Y}} \operatorname{Pr}\left[\tilde{S}^{Y}=S\right]-\operatorname{Pr}\left[\tilde{S}^{\bar{N}}=S\right]+\frac{1}{2} \sum_{S \in \Omega_{\bar{N}}} \operatorname{Pr}\left[\tilde{S}^{\bar{N}}=S\right]-\operatorname{Pr}\left[\tilde{S}^{Y}=S\right] \\
& =\left(\sum_{S \in \Omega_{Y}} \operatorname{Pr}\left[\tilde{S}^{Y}=S\right]-\operatorname{Pr}\left[\tilde{S}^{\bar{N}}=S\right]\right)-\frac{1}{2}
\end{aligned}
$$

This implies that Bob correctly identifies the YES distribution with probability

$$
\underset{(M, w) \sim \mathbf{Y E S}}{\operatorname{Pr}}[\text { Bob outputs YES }]=\sum_{S \in \Omega_{Y}} \operatorname{Pr}\left[\tilde{S}^{Y}=S\right] \geq \frac{1}{2}+\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{t v d}
$$

Similarly,

$$
\underset{(M, w) \sim \overline{\mathbf{N O}}}{\operatorname{Pr}}[\text { Bob outputs } \overline{\mathbf{N O}}] \geq \frac{1}{2}+\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{t v d}
$$

Thus, the above protocol solves $\operatorname{DBHP}(n, \beta)$ with success probability at least

$$
\frac{1}{2} \underset{(M, w) \sim \text { YES }}{\operatorname{Pr}}[\text { Bob outputs YES }]+\frac{1}{2} \operatorname{Pr}_{(M, w) \sim \overline{\mathbf{N O}}}\left[\text { Bob outputs } \overline{\mathbf{N O}]} \geq \frac{1}{2}+\left\|\tilde{S}^{Y}-\tilde{S}^{\bar{N}}\right\|_{t v d} \geq \frac{1}{2}+\frac{\Delta}{2 T}\right.
$$

where the last inequality is due to Claim 4.13 .
Now we are ready to finish the proof of Theorem 4.6.
Theorem 4.6 (Reduction from DBHP with $T$ messages). Let $c>0$ be the constant from Lemma 4.4. For every $T \in \mathbb{N}, 0<\beta \leq 1 /(10 c T)^{2 / 3}$, and reduction $(\mathcal{A}, \mathcal{B})$ from $\operatorname{DBHP}$ to $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$, any streaming algorithm that distinguishes $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ with success probability at least $3 / 4$ requires space at least $\frac{1}{40 c T} \cdot \sqrt{n}$.
Proof. Consider a streaming algorithm that distinguishes between the distributions $\mathcal{D}^{Y}(\beta, T, \mathcal{A}, \mathcal{B})$ and $\mathcal{D}^{N}(\beta, T, \mathcal{A}, \mathcal{B})$ with probability at least $3 / 4$ using space $S$. Then, by Lemma 4.11 , there exists a protocol for $\operatorname{DBHP}(n, \beta)$ with at most $S$ bits of communication, and success probability $1 / 2+1 /(8 T)$.

On the other hand, by Lemma 4.4, any protocol for $\operatorname{DBHP}(n, \beta)$ with success probability

$$
\frac{1}{2}+c \cdot\left(\beta^{3 / 2}+\frac{1}{40 c T}\right) \stackrel{\beta \leq 1 /(10 c T)^{2 / 3}}{\leq} \frac{1}{2}+\frac{1}{8 T}
$$

must use $S \geq \frac{1}{40 c T} \cdot \sqrt{n}$ bits of communication.

## 5 Analysis for the gap of Max-2EAND and Max-2OR instances

The goal of this section is to prove Lemma 4.7 and Lemma 4.9. We analyse the structure of DBHP in Section 5.1 and present an intuitive and graphical view of the reductions. After that, we give the proofs for Lemma 4.7 and Lemma 4.9 in Section 5.2 and Section 5.3 respectively.

Notation for an assignment. In this section, we interchangeably work with one of the following representations for $\sigma$ in order to simplify the presentation. Previously, $\sigma$ was defined as a function that maps $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ to $\{0,1\}$. It can be represented by a vector in $\{0,1\}^{n}$ which has $\sigma\left(x_{i}\right)$ as its $i^{\text {th }}$ coordinate. It can also be represented by the set $\left\{i \in[n]: \sigma\left(x_{i}\right)=1\right\}$.

### 5.1 A graphical view of DBHP

Here we introduce a graphical view of DBHP which will provide a more intuitive lens to understand the reductions. Recall that in DBHP, Bob has private inputs $M \in\{0,1\}^{r}$ and $w \in\{0,1\}^{r}$, where $M$ is the edge-incidence matrix of an $n$-vertex graph $G$ and $w$ is an indicator vector. Specifically, $M$ corresponds to a graph sampled from $G(n, 2 \beta / n)$ and $r$ denotes the number of edges in this graph. We focus on the subgraph $H \subseteq G$ that contains only those edges from $M$ whose corresponding entry in $w$ is 1 . We examine the distributions of this subgraph $H$ under different input distributions to DBHP. Recall that we are interested in two input distributions to DBHP: YES and NO. In both of these distributions, we first sample a hidden partition $X^{*} \in\{0,1\}^{*}$ and then sample $T$ independent graphs from $G(n, 2 \beta / n)$ where the edge-vertex incidence matrices of these graphs are denoted as $\left\{M_{t}\right\}_{t \in[T]}$. In the YES distribution, $w_{t}=M_{t} X^{*}$ and in the NO distribution, $w_{t}=\mathbf{1}-M_{t} X^{*}$. We will abuse notation and call the corresponding distributions of the subgraph $H$ as YES and NO respectively. We summarize the properties of these distributions in the following lemma.


Figure 2: For a random graph on vertex set [n], we partition the edges into two sets: (i) edges that lie across $X^{*}$ and $\overline{X^{*}}$ and (ii) edges that lie in $X^{*}$ or $\overline{X^{*}}$. In the YES distribution, only the (i) type edges are present in $H$. In the NO distribution, only the (ii) type edges are present in $H$.

Lemma 5.1 (Graphical view of DBHP). For any $n \in \mathbb{N}$ large enough and $\varepsilon \in(0,0.25)$, let $T=\left(10000 / \varepsilon^{2}\right)^{3} \cdot(10 c)^{2}$ and $\beta=\frac{1}{(10 c T)^{2 / 3}}$ such that $\beta T=10000 / \varepsilon^{2}$. Let $\boldsymbol{Y E S}$ and $\boldsymbol{N O}$ be the distributions of the subgraph $H$ induced from $\operatorname{DBHP}(n, \beta, T)$ as described above, and let $m_{D B H P}$ denote the total number of edges in $H$. For every $X^{*}, \sigma \in\{0,1\}^{n}$, define $m_{\text {cross }}(\sigma)$ to be the number of edges $(i, j)$ such that (i) $\sigma\left(x_{i}\right) \neq \sigma\left(x_{j}\right)$ and (ii) $X_{i}^{*}=X_{j}^{*}$. We have the following.

- (Size of $\left.X^{*}\right)$ For each distribution YES, $\boldsymbol{N O}$ and for any constant $\varepsilon^{\prime} \in(0,1)$ such that $\varepsilon^{\prime} \geq \varepsilon / 10$, we have

$$
\operatorname{Pr}\left[\left|\left|X^{*}\right|-\frac{n}{2}\right|>\varepsilon^{\prime} \cdot n\right]=o(1) .
$$

- (Number of edges) For each distribution YES, NO and for any constant $\varepsilon^{\prime} \in(0,1)$ such that $\varepsilon^{\prime} \geq \varepsilon / 10$, we have

$$
\operatorname{Pr}\left[\left|m_{D B H P}-\frac{\beta n T}{2}\right|>\varepsilon^{\prime} \cdot \beta n T\right]=o(1) .
$$

- ( $\mathbf{N O}$ distribution) For any constant $\varepsilon^{\prime} \in(0,1)$ such that $\varepsilon^{\prime} \geq \varepsilon / 10$, we have

$$
\underset{N O}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}, m_{\text {cross }}(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{n^{2}}\right) \cdot 2 \beta n T+\varepsilon^{\prime} \cdot \beta n T\right]=o(1) .
$$

Proof.

- (Size of $\left.X^{*}\right)$ For any $\varepsilon^{\prime} \in(0,1)$, we have

$$
\operatorname{Pr}\left[\left|\left|X^{*}\right|-\frac{n}{2}\right|>\varepsilon^{\prime} \cdot n\right] \leq \frac{2 \cdot \sum_{i=0}^{\left\lceil n / 2-\varepsilon^{\prime} \cdot n\right\rceil}\binom{n}{i}}{2^{n}}=2^{-\Omega_{\varepsilon^{\prime}}(n)}=o(1)
$$

- (Number of edges) Here we only prove the case for YES distribution while the other case can be proved similarly. Also, we only show the upper bound for $m_{\text {DBHP }}-\beta n T / 2$ while the lower bound can be proved symmetrically.

Let $\varepsilon^{\prime \prime} \in(0,1)$ be error parameters that will be chosen in the end according to $\varepsilon^{\prime}$. First, by Lemma 2.3, we have for any constant $\varepsilon^{\prime \prime} \in(0,1)$,

$$
\begin{equation*}
\operatorname{Pr}_{\mathbf{Y E S}}\left[| | X^{*}\left|-\frac{n}{2}\right|>\varepsilon^{\prime \prime} \cdot n\right]=o(1) . \tag{5.2}
\end{equation*}
$$

Denote the event where $-\varepsilon^{\prime \prime} n \leq\left|X^{*}\right|-n / 2 \leq \varepsilon^{\prime \prime} n$ as GOOD. Now, for each $t \in[T]$, let $m_{t}=\left\|M_{t} X^{*}\right\|_{1}$, i.e., the number of edges in $M_{t}$ that crosses $X^{*}$ and $\overline{X^{*}}$, we have

$$
\underset{\mathrm{YES}}{\mathbb{E}}\left[m_{t} \mid \mathrm{GOOD}\right] \leq\left(\frac{1}{2}+\varepsilon^{\prime \prime}\right) \cdot \beta n
$$

Further, note that when conditioning on $X^{*}, m_{t}$ are independent, thus by Chernoff bound, when $n$ is large enough, we have

$$
\operatorname{Pr}_{\mathrm{YES}}\left[\left.\sum_{t \in[T]} m_{t}-\left(\frac{1}{2}+\varepsilon^{\prime \prime}\right) \cdot \beta n T>\varepsilon^{\prime \prime} \cdot \frac{\beta n T}{2} \right\rvert\, \mathrm{GOOD}\right]=o(1) .
$$

As $m_{\text {DBHP }}=\sum_{t \in[T]} m_{t}$, by choosing $\varepsilon^{\prime \prime}=\varepsilon^{\prime} / 3$, we conclude that

$$
\operatorname{Pr}\left[\left|m_{\mathrm{DBHP}}-\frac{\beta n T}{2}\right|>\varepsilon^{\prime} \cdot \frac{\beta n T}{2}\right]=o(1)
$$

- (NO distribution) First, for each $t \in[T]$, let $m_{\text {cross }}^{(t)}(\sigma)$ denote the number of cross edges (i.e., $\sigma\left(x_{i}\right) \neq$ $\sigma\left(x_{j}\right)$ and $\left.X_{i}^{*}=X_{j}^{*}\right)$ from $M_{t}$. Note that $m_{\text {cross }}=\sum_{t \in[T]} m_{\text {cross }}^{(t)}(\sigma)$. Next, observe that for each $t \in[T], m_{\text {cross }}^{(t)}(\sigma)$ is a sum of $\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|$ independent Bernoulli random variables with expectation $2 \beta / n$. Also, random variables $m_{\text {cross }}^{(1)}(\sigma), \ldots, m_{\text {cross }}^{(T)}(\sigma)$ are independent to each other because we have fixed $X^{*}$ and $\sigma$. Thus, by Chernoff bound (i.e., Lemma 2.3), we have

$$
\begin{aligned}
& \operatorname{Pr}\left[m_{\text {cross }}(\sigma)>\left(\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|\right) \cdot \frac{2 \beta}{n} \cdot T+\varepsilon^{\prime} \beta n T\right] \\
< & \exp \left(-\frac{\varepsilon^{\prime 2} \beta^{2} n^{2} T^{2}}{2 \beta n T}\right)=\exp \left(-\frac{\varepsilon^{\prime 2} \beta n T}{2}\right)<2^{-10 n}
\end{aligned}
$$

where the last inequality is due to the choice of $T=10000 / \varepsilon^{2}$ and $\varepsilon^{\prime} \geq \varepsilon / 10$. Finally, by union bound, we have

$$
\operatorname{Pr}\left[\exists \sigma \in\{0,1\}^{n}, m_{\mathrm{cross}}(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{n^{2}}\right) \cdot 2 \beta n T+\varepsilon^{\prime} \beta n T\right]=o(1)
$$

### 5.2 The gap of Max-2EAND instances

In this subsection, we complete the proof of the following lemma using the graphical view of DBHP.
Lemma 4.7. For any $\varepsilon \in(0,1)$, let $\left(\beta, T, \mathcal{A}^{E A N D}, \mathcal{B}^{E A N D}\right)$ be the parameters described in the above reduction. For a Max-2EAND instance $\Psi$, let $m_{\Psi}$ denote the number of clauses in $\Psi$. Then

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{Y}}^{\left(\beta, T, \mathcal{A}^{E A N D}, \mathcal{B}^{E A N D}\right)} \boldsymbol{}\left[v a l_{\Psi}<\left(\frac{3}{5}-\varepsilon\right) \cdot m_{\Psi}\right]=o(1)
$$

and

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{E A N D}, \mathcal{B}^{E A N D}\right)}\left[v a l_{\Psi}>\left(\frac{4}{15}+\varepsilon\right) \cdot m_{\Psi}\right]=o(1) .
$$

Proof. Recall that $\mathcal{A}^{\text {EAND }}\left(X^{*}\right)$ uses $X^{*}$ to sample $\beta n T / 4$ independent copies of $(i, j) \in X^{*} \times \overline{X^{*}}$ and outputs $\left(x_{i} \wedge \neg x_{j}\right)$. On the other hand, for each row of $M$ with the $i^{\text {th }}$ and $j^{\text {th }}$ entry being 1 , if the corresponding entry of $w$ is $1, \mathcal{B}^{\text {EAND }}(M, w)$ outputs $\left(x_{i} \wedge \neg x_{j}\right)$ and $\left(\neg x_{i} \wedge x_{j}\right)$.

- (YES distribution) Consider the following assignment $\sigma$ :

$$
\sigma\left(x_{i}\right)=\left\{\begin{array}{l}
1, i \in X^{*} \\
0, \text { otherwise }
\end{array}\right.
$$

Under this assignment, each clause $\left(x_{i} \wedge \neg x_{j}\right)$ generated by $\mathcal{A}^{\text {EAND }}$ is satisfied since $(i, j) \in X^{*} \times \overline{X^{*}}$. For every pair of clauses $\left(x_{i} \wedge \neg x_{j}\right)$ and $\left(\neg x_{i} \wedge x_{j}\right)$ generated by $\mathcal{B}^{\text {EAND }}$, exactly one of them is satisfied by $\sigma$. Therefore, $\sigma$ satisfies $\beta n T / 4+m_{\text {DBHP }}$ clauses while the total number of clauses is $m_{\Psi}=$ $\beta n T / 4+2 m_{\text {DBHP }}$.
From Lemma 5.1, we know that $m_{\text {DBHP }} \in(1 \pm \varepsilon / 10) \cdot \beta n T / 2$ with probability at least $1-o(1)$. Thus, we have $m_{\Psi} \in(5 / 4 \pm \varepsilon / 10) \cdot \beta n T$ and $\sigma$ satisfies at least $(1-\varepsilon / 10) \cdot \beta n T$ clauses with probability $1-o(1)$. Therefore, $\operatorname{val}(\sigma) \geq(3 / 5-\varepsilon) \cdot m_{\Psi}$ with probability at least $1-o(1)$.

- (NO distribution) Consider any fixed assignment $\sigma \in\{0,1\}^{n}: \sigma$ satisfies a clauses $\left(x_{i} \wedge \neg x_{j}\right)$ generated by $\mathcal{A}^{\text {EAND }}$ if and only if $\sigma\left(x_{i}\right)=1$ and $\sigma\left(x_{j}\right)=0$. Let $a(\sigma)$ be the random variable that denotes the number of clauses generated by $\mathcal{A}^{\text {EAND }}$ which are satisfied by $\sigma$. Observe that $a(\sigma)$ is the sum of $\beta n T / 4$ independent Bernoulli random variables with mean $\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right| /\left(\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right|\right)$. By Chernoff bound (i.e., Lemma 2.3), we have

$$
\underset{\text { NO }}{\operatorname{Pr}}\left[a(\sigma)>\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right|} \cdot \frac{\beta n T}{4}+\frac{\varepsilon \beta n T}{10}\right]<2^{-10 n} .
$$

Applying the union bound, we have

$$
\begin{equation*}
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}^{n}, a(\sigma)>\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right|} \cdot \frac{\beta n T}{4}+\frac{\varepsilon \beta n T}{10}\right]=o(1) . \tag{5.3}
\end{equation*}
$$

Now, let us consider the clauses generated by $\mathcal{B}^{\text {EAND }}$ : an edge $(i, j)$ in $M_{t}$ is selected by $w_{t}$ if and only if $X^{*}$ contains both $i$ and $j$, or contains neither (i.e., $X_{i}^{*}=X_{j}^{*}$ ). Observe that exactly one of ( $x_{i} \wedge \neg x_{j}$ ) and $\left(\neg x_{i} \wedge x_{j}\right)$ is satisfied by $\sigma$ if and only if $\sigma\left(x_{i}\right) \neq \sigma\left(x_{j}\right)$; otherwise, both are unsatisfied. Therefore, the number of clauses satisfied by $\sigma$ is exactly $m_{\text {cross }}(\sigma)$, i.e., the number of edges $(i, j)$ such that (i) $X_{i}^{*}=X_{j}^{*}$ and (ii) $\sigma\left(x_{i}\right) \neq \sigma\left(x_{j}\right)$. Therefore, the total number of satisfied clauses is given by

$$
\begin{equation*}
\operatorname{val}_{\Psi}(\sigma)=a(\sigma)+m_{\operatorname{cross}}(\sigma) \tag{5.4}
\end{equation*}
$$

By Lemma 5.1, we have

$$
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}^{n}, m_{\text {cross }}(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{n^{2}}\right) \cdot 2 \beta n T+\frac{\varepsilon \beta n T}{10}\right]=o(1) .
$$

Since $\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right| \leq \frac{n^{2}}{4}$,

$$
\begin{equation*}
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}^{n}, m_{\mathrm{cross}}(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right|}\right) \cdot \frac{\beta n T}{2}+\frac{\varepsilon \beta n T}{10}\right]=o(1) \tag{5.5}
\end{equation*}
$$

Let $p=\left|\sigma \cap X^{*}\right| /\left|X^{*}\right| \in[0,1]$ and $q=\left|\bar{\sigma} \cap \overline{X^{*}}\right| /\left|\overline{X^{*}}\right| \in[0,1]$.
Combining (5.3), (5.4), and (5.5), we get

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{\text {EAND }}\right), \mathcal{B} \text { EAND }}\left[\exists \sigma\{0,1\}^{n}, \operatorname{val}_{\Psi}(\sigma)>\left(\frac{p q+2 p(1-p)+2 q(1-q)}{4}\right) \cdot \beta n T+\frac{\varepsilon \beta n T}{5}\right]=o(1) .
$$

We have $\frac{p q+2 p(1-p)+2 q(1-q)}{4}=\frac{8-(3 p-2)^{2}-(3 q-2)^{2}-3(p-q)^{2}}{24} \leq 1 / 3$. Since $m_{\Psi} \in(5 / 4 \pm \varepsilon / 10) \cdot \beta n T$ with probability $1-o(1)$, we conclude that

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{\text {EAND }}\right), \mathcal{B}^{\text {EAND }}}\left[\exists \sigma\{0,1\}^{n}, \operatorname{val}_{\Psi}(\sigma)>\left(\frac{4}{15}+\varepsilon\right) \cdot m_{\Psi}\right]=o(1) .
$$

### 5.3 The gap of Max-2OR instances

In this subsection, we complete the proof of the following lemma using the graphical view of DBHP.
Lemma 4.9. For any $\varepsilon \in(0,1)$, let $\left(\beta, T, \mathcal{A}^{O R}, \mathcal{B}^{O R}\right)$ be the parameters described in the above reduction. For a Max-2OR instance $\Psi$, let $m_{\Psi}$ denote the number of clauses in $\Psi$. Then

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{Y}}{\left.\operatorname{Pr}, T, \mathcal{A}^{O R}, \mathcal{B}^{\circ R}\right)}\left[v a l_{\Psi}=m_{\Psi}\right]=1
$$

and

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{O R}, \mathcal{B}^{\circ R}\right)}\left[v a l_{\Psi}>\left(\frac{\sqrt{2}}{2}+\varepsilon\right) \cdot m_{\Psi}\right]=o(1) .
$$

Proof. Recall that $\mathcal{A}^{\mathrm{OR}}\left(X^{*}\right)$ uses $X^{*}$ to sample $\frac{\sqrt{2}-1}{2} \cdot \beta n T$ independent copies of $i \in X^{*}$ and another $\frac{\sqrt{2}-1}{2} \cdot \beta n T$ independent copies of $j \in \overline{X^{*}}$ and output $\left(x_{i}\right)$ as well as $\left(\neg x_{j}\right)$. On the other hand, for each row of $M$ with the $i^{\text {th }}$ and $j^{\text {th }}$ entry being 1 , if the corresponding entry of $w$ is $1, \mathcal{B}^{\text {OR }}$ outputs $\left(x_{i} \vee x_{j}\right)$ and $\left(\neg x_{i} \vee \neg x_{j}\right)$.

- (YES distribution) Consider the following assignment $\sigma$ :

$$
\sigma\left(x_{i}\right)=\left\{\begin{array}{l}
1, i \in X^{*} \\
0, \text { otherwise }
\end{array}\right.
$$

. Under this assignment, every clause of the form $\left(x_{i}\right)$ or of the form $\left(\neg x_{j}\right)$ generated by $\mathcal{A}^{\text {OR }}$ is satisfied because $i \in X^{*}$ and $j \in \overline{X^{*}}$. Similarly, every pair of clauses $\left(x_{i} \vee x_{j}\right)$ and $\left(\neg x_{i} \vee \neg x_{j}\right)$ generated by $\mathcal{B}^{\mathrm{OR}}$ are also satisfied since in the YES distribution, $(i, j) \in X^{*} \times \overline{X^{*}}$. Thus, val ${ }_{\Psi}=m_{\Psi}$ as desired.

- (NO distribution) Consider any fixed assignment $\sigma \in\{0,1\}^{n}$ : Let $a(\sigma)$ be the random variable that denotes the number of clauses generated by $\mathcal{A}^{\text {EAND }}$ which are satisfied by $\sigma$. Observe that $a(\sigma)$ is the sum of $\frac{\sqrt{2}-1}{2} \beta n T \geq 100 n$ independent Bernoulli random variables with mean $\left|\sigma \cap X^{*}\right| /\left|X^{*}\right|$ and $\frac{\sqrt{2}-1}{2} \beta n T \geq 100 n$ independent Bernoulli random variables with mean $\left|\bar{\sigma} \cap \overline{X^{*}}\right| /\left|\overline{X^{*}}\right|$. By Chernoff bound (i.e., Lemma 2.3), we have

$$
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[a(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right|}{\left|X^{*}\right|}+\frac{\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{\left|\overline{X^{*}}\right|}\right) \cdot \frac{\sqrt{2}-1}{2} \beta n T+\frac{\varepsilon \beta n T}{15}\right]<2^{-10 n} .
$$

Applying the union bound, we have

$$
\begin{equation*}
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}^{n}, a(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right|}{\left|X^{*}\right|}+\frac{\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{\left|\overline{X^{*}}\right|}\right) \cdot \frac{\sqrt{2}-1}{2} \beta n T+\frac{\varepsilon \beta n T}{15}\right]=o(1) . \tag{5.6}
\end{equation*}
$$

Now, consider the clauses generated by $\mathcal{B}^{\text {OR }}$ : an edge $(i, j)$ in $M_{t}$ is selected by $w_{t}$ if and only if $X_{i}^{*}=X_{j}^{*}$. Observe that both the clauses $\left(x_{i} \vee x_{j}\right)$ and $\left(\neg x_{i} \vee \neg x_{j}\right)$ are satisfied if and only if $\sigma\left(x_{i}\right) \neq \sigma\left(x_{j}\right)$; otherwise, exactly one of them is satisfied. Therefore, the number of satisfied clauses among the clauses generated by $\mathcal{B}^{\text {OR }}$ is $m_{\text {DBHP }}+m_{\text {cross }}(\sigma)$. Therefore, the total number of satisfied clauses is given by

$$
\begin{equation*}
\operatorname{val}_{\Psi}(\sigma)=a(\sigma)+m_{\mathrm{DBHP}}+m_{\mathrm{cross}}(\sigma) \tag{5.7}
\end{equation*}
$$

By Lemma 5.1, we have

$$
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}^{n}, m_{\mathrm{cross}}(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{n^{2}}\right) \cdot 2 \beta n T+\frac{\varepsilon \beta n T}{15}\right]=o(1) .
$$

and

$$
\begin{equation*}
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[m_{\mathrm{DBHP}}>\frac{\beta n T}{2}+\varepsilon \cdot \frac{\beta n T}{15}\right]=o(1) . \tag{5.8}
\end{equation*}
$$

Since $\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right| \leq \frac{n^{2}}{4}$,

$$
\begin{equation*}
\underset{\mathrm{NO}}{\operatorname{Pr}}\left[\exists \sigma \in\{0,1\}^{n}, m_{\text {cross }}(\sigma)>\left(\frac{\left|\sigma \cap X^{*}\right| \cdot\left|\bar{\sigma} \cap X^{*}\right|+\left|\sigma \cap \overline{X^{*}}\right| \cdot\left|\bar{\sigma} \cap \overline{X^{*}}\right|}{\left|X^{*}\right| \cdot\left|\overline{X^{*}}\right|}\right) \cdot \frac{\beta n T}{2}+\frac{\varepsilon \beta n T}{15}\right]=o(1) \tag{5.9}
\end{equation*}
$$

Let $p=\left|\sigma \cap X^{*}\right| /\left|X^{*}\right| \in[0,1]$ and $q=\left|\bar{\sigma} \cap \overline{X^{*}}\right| /\left|\overline{X^{*}}\right| \in[0,1]$. Combining (5.6), (5.7), (5.9) and (5.8), we get

$$
\operatorname{Pr}_{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{\circ \mathrm{R}}\right), \mathcal{B}^{\circ \mathrm{R}}}\left[\exists \sigma\{0,1\}^{n}, \operatorname{val}_{\Psi}(\sigma)>\left(\frac{(p+q)(\sqrt{2}-1)}{2}+\frac{1}{2}+\frac{p(1-p)+q(1-q)}{2}\right) \cdot \beta n T+\frac{\varepsilon \beta n T}{5}\right]=o(1) .
$$

We have $\frac{p+q}{2} \cdot(\sqrt{2}-1)+\frac{1}{2}+\frac{p(1-p)+q(1-q)}{2}=1-\frac{(p-\sqrt{2} / 2)^{2}+(q-\sqrt{2} / 2)^{2}}{2} \leq 1$. Since $m_{\Psi} \in(\sqrt{2} \pm \varepsilon / 10) \cdot \beta n T$ with probability $1-o(1)$, we conclude that

$$
\underset{\Psi \sim \mathcal{D}^{N}\left(\beta, T, \mathcal{A}^{\circ \mathrm{R}}\right), \mathcal{B}^{\circ \mathrm{R}}}{\operatorname{Pr}}\left[\exists \sigma\{0,1\}^{n}, \operatorname{val}_{\Psi}(\sigma)>\left(\frac{\sqrt{2}}{2}+\varepsilon\right) \cdot m_{\Psi}\right]=o(1) .
$$

## 6 Proof of Theorem 1.1

Theorem 1.1. Let $\mathcal{F} \subseteq\{T R, O R, X O R, A N D\}$ be a set of allowed binary predicates. Let $\alpha_{\mathcal{F}}=\min _{\mathcal{G} \subseteq \mathcal{F}} \alpha_{\mathcal{G}}$, where $\alpha_{\mathcal{G}}$ is given in Table 1.

For every $\varepsilon>0$, there exists an $\left(\alpha_{\mathcal{F}}-\varepsilon\right)$-approximate streaming algorithm for $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ that uses space $O\left(\varepsilon^{-2} \log n\right)$. On the other hand, any $\left(\alpha_{\mathcal{F}}+\varepsilon\right)$-approximate streaming algorithm for Max-CSP $(\mathcal{F})$ requires space $\Omega(\sqrt{n})$.

Proof. Note that for $\mathcal{G}$ listed in Table 1, the space lower bounds for Max-CSPG are proven in Corollary 4.2, Corollary 4.10, Theorem 2.4, and Corollary 4.8, respectively. Then the space lower bound for any $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ directly follows from the fact that for $\mathcal{G} \subseteq \mathcal{F}$, any hard instance for $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$ is also a hard instance of $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$.

We provide a case-by-case analysis to prove the upper bounds.

Case I $-\arg \min _{\mathcal{G} \subseteq \mathcal{F}} \alpha_{\mathcal{G}} \in\{\mathbf{T R}, \mathbf{O R},\{\mathbf{T R}, \mathbf{O R}\}\}:$ In this case, $\mathcal{F}=\{\mathrm{OR}\}, \mathcal{F}=\{\mathrm{TR}\}$, or $\mathcal{F}=\{\mathrm{OR}, \mathrm{TR}\}$, and each of these cases is covered in Table 1. The corresponding upper bounds for these cases are proven in Proposition 3.1 and Theorem 3.9.

Case II $-\arg \min _{\mathcal{G} \subset \mathcal{F}} \alpha_{\mathcal{G}}=$ XOR: In this case, $\mathcal{F} \subseteq\{O R, X O R, T R\}$. Consider any instance $\Psi$ of $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$. A random assignment satisfies every constraint in $\Psi$ with probability at least $1 / 2$. Therefore, the trivial streaming algorithm that counts the number of clauses, $m$ and outputs $m / 2$ achieves $1 / 2$ approximation (see Proposition 3.1).

| Type $\mathcal{G}$ | Tight <br> bound | Previous bound |  |
| :---: | :---: | :---: | :---: |
|  | $\alpha_{\mathcal{G}}$ | $\alpha_{\mathcal{G}}^{\mathrm{pr}}$ | Reference |
| TR | 1 | 1 | Folklore |
| OR | $\frac{3}{4}$ | $\left[\frac{3}{4}, 1\right]$ | Folklore |
| $\{$ TR, OR $\}$ | $\frac{\sqrt{2}}{2}$ | $\left[\frac{\sqrt{5}-1}{2}, 1\right]$ | $[\mathrm{LS} 79]$ |
| XOR | $\frac{1}{2}$ | $\frac{1}{2}$ | $[\mathrm{KK} 19]$ |
| AND | $\frac{4}{9}$ | $\left[\frac{2}{5}, \frac{1}{2}\right]$ | $[\mathrm{GVV} 17]$ |

Table 1: Summary of known and new approximation factors $\alpha_{\mathcal{G}}$ for $\operatorname{Max}-\operatorname{CSP}(\mathcal{G})$. We have suppressed ( $1 \pm \varepsilon$ ) multiplicative factors.

Case III - $\arg \min _{\mathcal{G} \subseteq \mathcal{F}} \alpha_{\mathcal{G}}=$ AND: In this case, $\mathcal{F} \subseteq\{$ AND, OR, XOR,TR $\}$. Any Boolean constraint $f(x)$ of length at most 2 can be expressed as the disjunction of (at most 4) AND constraints $\left\{f_{i}(x)\right\}$ such that for any assignment $\sigma$, the number of satisfied constraints among $\left\{f_{i}(\sigma)\right\}$ is exactly 1 if $f(\sigma)=1$; otherwise, it is 0 . Therefore, any instance $\Psi$ of $\operatorname{Max}-\operatorname{CSP}(\mathcal{F})$ can be reduced to $\Psi^{\prime}$, an instance of Max-2AND with the same optimal value. Now the approximation algorithm for Max-2AND from Theorem 3.4 finishes the proof.

## 7 A tight approximation algorithm for Max- $k$ SAT

It follows from Theorem 1.1 that for every $\varepsilon>0$, any $(\sqrt{2} / 2+\varepsilon)$-approximate streaming algorithm for Max- $k$ SAT requires space $\Omega(\sqrt{n})$ since any Max- $k$ SAT instance is also an instance of Max-2SAT. In this section, we prove that this lower bound is indeed tight, i.e., we show that for any $\varepsilon>0$, there exists a streaming algorithm that uses space $\mathcal{O}\left(\varepsilon^{-2} \log n\right)$ and computes $(\sqrt{2} / 2-\varepsilon)$-approximation for Max- $k$ SAT.

We first extend the notion of bias which we defined in Section 2 to Max- $k$ SAT instances. Let $\Psi$ be an instance of Max- $k$ SAT with $m=|\Psi|$ clauses. For $r \in \mathbb{N}$, an $r$-clause is a clause that depends on $r$ variables. We denote the total number of $r$-clauses by $m_{r}$, and the total number of clauses that depend on at least $r$ variables by $m_{\geq r}$. We use $\operatorname{pos}_{i}^{(r)}(\Psi)$ for the number of $r$-clauses where the variable $x_{i}$ appears positively. Similarly, $\operatorname{neg}_{i}^{(r)}(\Psi)$ denotes the number of $r$-clauses containing $\neg x_{i}$.
Definition 7.1. The bias of a variable $x_{i}$ of an instance $\Psi$ of Max- $k S A T$ is defined as

$$
\operatorname{bias}_{i}(\Psi)=\left|\sum_{r} \frac{1}{2^{r-1}}\left(\operatorname{pos}_{i}^{(r)}(\Psi)-\operatorname{neg}_{i}^{(r)}(\Psi)\right)\right|
$$

The bias vector of $\Psi$ is a vector $\boldsymbol{b} \in \mathbb{R}^{n}$, where $\boldsymbol{b}_{i}=$ bias $_{i}(\Psi)$. Finally, the bias of the formula $\Psi$ is defined as the sum of biases of its variables:

$$
\operatorname{bias}(\Psi)=\sum_{i=1}^{n} \operatorname{bias}_{i}(\Psi)=\sum_{i=1}^{n}\left|\sum_{r} \frac{1}{2^{r-1}}\left(\operatorname{pos}_{i}^{(r)}(\Psi)-\operatorname{neg}_{i}^{(r)}(\Psi)\right)\right|
$$

In Lemmas 7.2 and 7.3 we give upper and lower bounds on val ${ }_{\Psi}$ in terms of bias $(\Psi)$ and $m_{r}$. We postpone their proofs to Sections 7.1 and 7.2. In this section we prove that the ratio between the presented lower and upper bounds is bounded by $\frac{\sqrt{2}}{2}$, and that there is a $O(\log n)$-space algorithm that sketches the lower bounds of Lemma 7.3 on $\mathrm{val}_{\Psi}$. The following lemma gives an upper bound on $\mathrm{val}_{\Psi}$.

Lemma 7.2. Let $\Psi$ be a Max-kSAT instance. Then

$$
v a l_{\Psi} \leq \min \left\{\sum_{j=1}^{k} m_{j}, \frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k}\left(\frac{2^{j}+j-2}{2^{j}}\right) m_{j}\right\}
$$

For lower bounds, the trivial algorithm guarantees that for every Max-kSAT instance $\Psi$, val ${ }_{\Psi} \geq \sum_{j=1}^{k}(1-$ $\left.2^{-j}\right) m_{j}$. This bound is not sufficient for proving tight space lower bounds for streaming algorithms, so we improve it as follows. For instances with high bias, we prove a (stronger) lower bound of val $\geq \frac{\text { bias }(\Psi)}{2}+$ $\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}$. In order to handle the case of low bias, we design a distribution of assignments which in expectation satisfy a large number of clauses in formulas with low bias. To summarize, we have the following lemma on the lower bound for $\mathrm{val}_{\Psi}$.

Lemma 7.3. Let $\Psi$ be a Max-kSAT instance. Then

1. $\operatorname{val}_{\Psi} \geq \frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}$;
2. if $\operatorname{bias}(\Psi) \leq \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}$, then

$$
v a l_{\Psi} \geq \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{\operatorname{bias}(\Psi)^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2^{j}-j-1}{2^{j-2}}\right) m_{j}\right)} .
$$

Now we are ready to present an approximation algorithm for the Max-kSAT problem.

```
Algorithm \(5(\sqrt{2} / 2-\varepsilon)\)-approximation streaming algorithm for Max- \(k\) SAT \()\)
Input: \(\Psi\)-an instance of Max- \(k\) SAT. Error parameter \(\varepsilon \in(0,0.01)\).
    1: Approximate the \(\ell_{1}\)-norm of the bias vector with error \(\delta=\varepsilon / 8\) (Theorem 2.2):
    Compute \(B \in(1 \pm \delta) \operatorname{bias}(\Psi)\).
    2: Use \(k\) counters to count the number of \(j\)-clauses, \(m_{j}\), for all \(j \in[k]\).
    if \(B \in\left[0,(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)\right]\) then
    Output: \(v=\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{(1-\delta)^{2} B^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2 j-j-1}{2^{j-2}}\right) m_{j}\right)}\).
    else
    Output: \(v=\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}+\frac{(1-\delta) B}{2}\).
```

Theorem 1.2. For every $\varepsilon>0$, there exists an $(\sqrt{2} / 2-\varepsilon)$-approximate streaming algorithm for Max-SAT that uses space $O\left(\varepsilon^{-2} \log n\right)$. On the other hand, for any $k \geq 2, \varepsilon>0$ any $(\sqrt{2} / 2+\varepsilon)$-approximate streaming algorithm for Max- $k S A T$ requires space $\Omega(\sqrt{n})$.

Proof. We prove that Algorithm 5 computes a $\left(\frac{\sqrt{2}}{2}-\varepsilon\right)$-approximation by showing that (i) $v \leq \operatorname{val}_{\Psi}$, and (ii) $v \geq\left(\frac{\sqrt{2}}{2}-\varepsilon\right) \cdot$ val $_{\Psi}$, where $v$ is the output of the algorithm. Recall that by the guarantee of Theorem 2.2, with probability at least $3 / 4$ :

$$
(1-\delta) \operatorname{bias}(\Psi) \leq B \leq(1+\delta) \operatorname{bias}(\Psi)
$$

(i) $\boldsymbol{v} \leq \operatorname{val}_{\Psi}$. If $B>(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$, then

$$
v=\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}+\frac{(1-\delta) B}{2} \leq \sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}+\frac{\left(1-\delta^{2}\right) \operatorname{bias}(\Psi)}{2} \leq \operatorname{val}_{\Psi}
$$

by the first bound in Lemma 7.3.
If $B \leq(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$, then $\operatorname{bias}(\Psi) \leq B /(1-\delta) \leq\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$, and, thus, val $\Psi$ $\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{\operatorname{bias}(\Psi)^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2^{j}-j-1}{2^{j-2}}\right) m_{j}\right)}$ by the second bound in Lemma 7.3. Then

$$
\begin{aligned}
v & =\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{(1-\delta)^{2} B^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2^{j}-j-1}{2^{j-2}}\right) m_{j}\right)} \\
& \leq \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{\left(1-\delta^{2}\right) \operatorname{bias}(\Psi)^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2^{j}-j-1}{2^{j-2}}\right) m_{j}\right)} \leq \operatorname{val}_{\Psi}
\end{aligned}
$$

(ii) $\boldsymbol{v} \geq(1-\varepsilon) \cdot \frac{\sqrt{2}}{2} \mathbf{v a l}_{\Psi}$. We will in fact prove the stronger lower bound of $v \geq(1-\varepsilon) \cdot \frac{3}{4} \mathrm{val}_{\Psi}$ when the bias is high, i.e., $B>(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$. We will prove the desired bound by considering the following two cases: (1) $m_{1}>\sum_{j=2}^{k}\left(4-(j+4) 2^{1-j}\right) m_{j}$, and $(2) m_{1} \leq \sum_{j=2}^{k}\left(4-(j+4) 2^{1-j}\right) m_{j}$. If $m_{1}>\sum_{j=2}^{k}\left(4-(j+4) 2^{1-j}\right) m_{j}$, then

$$
B+m_{1} \geq(1-\delta) \sum_{j=2}^{k}\left(8-(6 j+12) 2^{-j}\right) m_{j} \geq(1-\delta) \sum_{j=2}^{k}\left(6-(2 j+12) 2^{-j}\right) m_{j}
$$

Rearranging the terms, we get

$$
\begin{aligned}
v & =\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}+\frac{(1-\delta) B}{2} \\
& \geq(1-\delta)\left(\frac{B+m_{1}}{2}+\sum_{j=2}^{k} \frac{j}{2^{j}} m_{j}\right) \\
& \geq(1-\delta)\left(\frac{3}{4} \cdot \frac{B+m_{1}}{2}+\frac{B+m_{1}}{8}+\sum_{j=2}^{k} \frac{j}{2^{j}} m_{j}\right) \\
& \geq(1-\delta)^{2} \cdot \frac{3}{4} \cdot\left(\frac{B}{2}+\sum_{j=1}^{k}\left(1+(j-2) 2^{-j}\right) m_{j}\right) \\
& \geq(1-\delta)\left(1-\delta^{2}\right) \cdot \frac{3}{4} \cdot\left(\frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k}\left(1+(j-2) 2^{-j}\right) m_{j}\right) \\
& \geq(1-\varepsilon) \cdot \frac{3}{4} \cdot \operatorname{val}_{\Psi},
\end{aligned}
$$

by the bound in Lemma 7.2.

If $m_{1} \leq \sum_{j=2}^{k}\left(4-(j+4) 2^{1-j}\right) m_{j}$, then

$$
\begin{aligned}
v & =\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}+\frac{(1-\delta) B}{2} \\
& \geq(1-\delta)\left(\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}+\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-1}} m_{j}\right) \\
& =(1-\delta)\left(\frac{m_{1}}{2}+\sum_{j=2}^{k} \frac{3 m_{j}}{4}+\sum_{j=2}^{k}\left(\frac{5}{4}-(j+2) 2^{-j}\right) m_{j}\right) \\
& =(1-\delta)(\frac{m_{1}}{2}+\sum_{j=2}^{k} \frac{3 m_{j}}{4}+\underbrace{\sum_{j=2}^{k}\left(1-(j / 2+2) 2^{-j}\right) m_{j}}_{\geq m_{1} / 4}+\underbrace{\sum_{j=2}^{k}\left(1 / 4-j \cdot 2^{-(j+1)}\right) m_{j}}_{\geq 0}) \\
& \geq(1-\delta) \cdot \frac{3}{4} \cdot \sum_{j=1}^{k} m_{j} \geq(1-\varepsilon) \cdot \frac{3}{4} \cdot \mathrm{val}_{\Psi} .
\end{aligned}
$$

We will now prove the lower bound of $v \geq(1-\varepsilon) \cdot \frac{\sqrt{2}}{2}$ val $_{\Psi}$ when the bias is low, i.e., $B \leq(1-$ $\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$. We have the following two claims.
Claim 7.4. If $\sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j} \leq B \leq(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$, then

$$
\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{B^{2}}{4 \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}} \geq \frac{\sqrt{2}}{2} \sum_{j=1}^{k} m_{j}
$$

Claim 7.5. If $B \leq \sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j}$, then

$$
\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{B^{2}}{4 \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}} \geq \frac{\sqrt{2}}{2}\left(\frac{B}{2}+\sum_{j=1}^{k} \frac{2^{j}+j-2}{2^{j}} m_{j}\right)
$$

Let us postpone the proof for the two claims to Section 7.3 and Section 7.4 and complete the proof for Theorem 1.2 assuming their correctness. If $\sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j} \leq B \leq(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$, then by Claim 7.4 we have

$$
\begin{aligned}
v & =\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{B^{2}}{4 \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}} \geq \frac{\sqrt{2}}{2} \sum_{j=1}^{k} m_{j} \\
& \geq \frac{\sqrt{2}}{2} \cdot \mathrm{val}_{\Psi}
\end{aligned}
$$

If $B \leq \sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j}$, then by Claim 7.5 , we have

$$
\begin{aligned}
v= & \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{B^{2}}{4 \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}} \geq \frac{\sqrt{2}}{2}\left(\frac{B}{2}+\sum_{j=1}^{k} \frac{2^{j}+j-2}{2^{j}} m_{j}\right) \\
& \geq(1-\delta) \cdot \frac{\sqrt{2}}{2}\left(\frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k} \frac{2^{j}+j-2}{2^{j}} m_{j}\right)
\end{aligned}
$$

$(\because$ Lemma 7.2$) \geq(1-\delta) \cdot \frac{\sqrt{2}}{2} \cdot \operatorname{val}_{\Psi} \geq(1-\varepsilon) \cdot \frac{\sqrt{2}}{2} \cdot \operatorname{val}_{\Psi}$.

We conclude that $v \geq(1-\varepsilon) \cdot \frac{\sqrt{2}}{2}$ val $_{\Psi}$. This completes the proof of Theorem 1.2.

### 7.1 Proof of Lemma 7.2

Lemma 7.2. Let $\Psi$ be a Max-kSAT instance. Then

$$
\mathrm{va}_{\Psi} \leq \min \left\{\sum_{j=1}^{k} m_{j}, \frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k}\left(\frac{2^{j}+j-2}{2^{j}}\right) m_{j}\right\}
$$

Proof. Let $k$ be the length of the largest clause in $\Psi$. Since $\sum_{j=1}^{k} m_{j}$ is the number of clauses in $\Psi$, the first bound $\mathrm{val}_{\Psi} \leq \sum_{j=1}^{k} m_{j}$ holds trivially. To show the second bound, we first negate all the variables of $\Psi$ with $\operatorname{bias}_{i}(\Psi)<0$. This transformation does not change $\operatorname{bias}(\Psi), \operatorname{val}_{\Psi}$, and $m_{j}$, for all $j$, and every assignment of the variables of the original instance can be uniquely mapped to a corresponding assignment for the new instance satisfying the same number of clauses. Therefore, without loss of generality, for every $i \in[n]$,

$$
\sum_{j=1}^{k} \frac{1}{2^{j-1}}\left(\operatorname{pos}_{i}^{(j)}(\Psi)-\operatorname{neg}_{i}^{(j)}(\Psi)\right)=\operatorname{bias}_{i}(\Psi) \geq 0
$$

Consider an assignment $\sigma$ to the variables of $\Psi$. We need to show that $\operatorname{val}_{\Psi}(\sigma) \leq \frac{m_{1}+2 m_{2}+\text { bias }(\Psi)}{2}+\frac{9 m_{\geq 3}}{8}$. Let $T$ be the set of (indices of) variables of $\sigma$ assigned the value 1 . We denote by $S_{j}$ the number of $j$-clauses satisfied by $\sigma$. We will prove that the number of $k$-clauses satisfied by $\sigma$ is bounded by

$$
\begin{equation*}
S_{k} \leq \min \left\{m_{k}, 2^{k-2} \operatorname{bias}(\Psi)+\sum_{j=1}^{k} j \cdot 2^{k-j-1} \cdot m_{j}-\sum_{j=1}^{k-1} 2^{k-j} \cdot S_{j}\right\} \tag{7.6}
\end{equation*}
$$

First we show how (7.6) finishes the proof of the lemma, and then prove (7.6).
Indeed, then the number of clauses satisfied by $\sigma$ is bounded from above by

$$
\begin{aligned}
\operatorname{val}_{\Psi}(\sigma) & \leq \sum_{j=1}^{k} S_{j} \\
& \leq \sum_{j=1}^{k-1} S_{j}+\min \left\{m_{k}, 2^{k-2} \operatorname{bias}(\Psi)+\sum_{j=1}^{k} j \cdot 2^{k-j-1} \cdot m_{j}-\sum_{j=1}^{k-1} 2^{k-j} \cdot S_{j}\right\} \\
& \leq \sum_{j=1}^{k-1} S_{j}+\frac{2^{k-1}-1}{2^{k-1}} \cdot m_{k}+\frac{1}{2^{k-1}}\left(2^{k-2} \operatorname{bias}(\Psi)+\sum_{j=1}^{k} j \cdot 2^{k-j-1} \cdot m_{j}-\sum_{j=1}^{k-1} 2^{k-j} \cdot S_{j}\right) \\
& =\frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k} j \cdot 2^{-j} \cdot m_{j}+\left(1-2^{-k+1}\right) m_{k}+\sum_{j=1}^{k-1}\left(1-2^{-j+1}\right) S_{j} \\
& \leq \frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k}\left(1+(j-2) 2^{-j}\right) m_{j},
\end{aligned}
$$

where the last inequality follows since $S_{j}$ is trivially upper bounded by $m_{j}$.

### 7.2 Proof of Lemma 7.3

Lemma 7.3. Let $\Psi$ be a Max-kSAT instance. Then

1. $v a l_{\Psi} \geq \frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}$;
2. if $\operatorname{bias}(\Psi) \leq \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}$, then

$$
\mathrm{val}_{\Psi} \geq \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{\operatorname{bias}(\Psi)^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2^{j}-j-1}{2^{j-2}}\right) m_{j}\right)}
$$

Proof. Let $k$ be the length of the largest clause in $\Psi$. Without loss of generality, we assume that for every $i \in[n]$, $\operatorname{bias}_{i}(\Psi) \geq 0$. (Again, we can negate all variables with $\operatorname{bias}_{i}(\Psi)<0$, and define a bijection between the assignments for the two formulas.) Therefore, for every $i \in[n]$,

$$
\sum_{j=1}^{k} \frac{1}{2^{j-1}}\left(\operatorname{pos}_{i}^{(j)}(\Psi)-\operatorname{neg}_{i}^{(j)}(\Psi)\right)=\operatorname{bias}_{i}(\Psi) \geq 0
$$

Consider the greedy assignment $\sigma$ which assigns 1 to every variable. Under this assignment,

$$
\operatorname{val}_{\Psi}(\sigma) \geq \sum_{i \in[n]} \sum_{j=1}^{k} \frac{\operatorname{pos}_{i}^{(j)}(\Psi)}{j}
$$

Recall that

$$
\begin{aligned}
\operatorname{bias}(\Psi) & =\sum_{i \in[n]} \sum_{j=1}^{k} \frac{1}{2^{j-1}}\left(\operatorname{pos}_{i}^{(j)}(\Psi)-\operatorname{neg}_{i}^{(j)}(\Psi)\right) \\
m_{j} & =\sum_{i \in[n]} \frac{\operatorname{pos}_{i}^{(j)}(\Psi)}{j}+\frac{\operatorname{neg}_{i}^{(j)}(\Psi)}{j}
\end{aligned}
$$

It follows that

$$
\operatorname{val}_{\Psi}(\sigma) \geq \frac{\operatorname{bias}(\Psi)}{2}+\sum_{j=1}^{k} \frac{j}{2^{j}} m_{j}
$$

This proves the first inequality in Lemma 7.3.
To prove the second inequality in Lemma 7.3, we consider a distribution of assignments to the variables of $\Psi$, where every variable $x_{i}$ is assigned the value 1 independently with probability $\left(\frac{1}{2}+\gamma\right)$, for a parameter $\gamma \in[0,1 / 2]$ to be assigned later. The expected number of satisfied $j$-clauses under this distribution is

$$
\begin{aligned}
S_{j} & \geq \sum_{r=0}^{j} m_{j}^{(r)} \cdot\left(1-\left(\frac{1}{2}-\gamma\right)^{r}\left(\frac{1}{2}+\gamma\right)^{j-r}\right) \\
& \geq\left(1-2^{-j}\right) \sum_{r=0}^{j} m_{j}^{(r)}+\left(\sum_{r=0}^{j} m_{j}^{(r)} \cdot \frac{2 r-j}{2^{j-1}}\right) \gamma-\sum_{d=2}^{j} \sum_{r=0}^{j} m_{j}^{(r)} \cdot \frac{1}{2^{j-d}} \cdot\binom{j}{d} \gamma^{d} \\
& =\left(1-2^{-j}\right) m_{j}+\left(\sum_{r=0}^{j} m_{j}^{(r)} \cdot \frac{2 r-j}{2^{j-1}}\right) \gamma-m_{j} \sum_{d=2}^{j} \frac{1}{2^{j-d}} \cdot\binom{j}{d} \gamma^{d} \\
& =m_{j}+\left(\sum_{r=0}^{j} m_{j}^{(r)} \cdot \frac{2 r-j}{2^{j-1}}\right) \gamma-m_{j}\left(\left(\frac{1}{2}+\gamma\right)^{j}-\frac{j}{2^{j-1}} \gamma\right) \\
& \geq m_{j}+\left(\sum_{r=0}^{j} m_{j}^{(r)} \cdot \frac{2 r-j}{2^{j-1}}\right) \gamma-m_{j}\left(2^{-j}+\frac{2^{j}-j-1}{2^{j-2}} \gamma^{2}\right)
\end{aligned}
$$

where the last inequality follows by applying the following inequality which holds for all $0 \leq x \leq 1$

$$
(1+x)^{n} \leq 1+n x+\left(2^{n}-n-1\right) x^{2}
$$

Let $m_{j}^{(r)}$ denote the number of $j$-clauses with $r$ positive literals. Note that a $j$-clause with $r$ positive literals contributes $\frac{2 r-j}{2^{j-1}}$ to the total bias. Therefore,

$$
\operatorname{bias}(\Psi)=\sum_{j=1}^{k} \sum_{r=0}^{j} \frac{2 r-j}{2^{j-1}} \cdot m_{j}^{(r)} .
$$

Let us now compute the expected number of clauses satisfied by an assignment $\sigma$ from the distribution defined above.

$$
\begin{aligned}
\underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right]=\sum_{j=1}^{k} S_{j} & \geq \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\gamma \cdot\left(\sum_{j=1}^{k} \sum_{r=0}^{j} m_{j}^{(r)} \cdot \frac{2 r-j}{2^{j-1}}\right)-\gamma^{2} \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j} \\
& =\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\gamma \cdot \operatorname{bias}(\Psi)-\gamma^{2} \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j} .
\end{aligned}
$$

For the case where $\operatorname{bias}(\Psi) \leq \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}$, we set $\gamma=\frac{\text { bias }(\Psi)}{2\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)} \in[0,1 / 2]$, and derive the second bound:

$$
\begin{aligned}
\operatorname{val}_{\Psi} \geq \underset{\sigma}{\mathbb{E}}\left[\operatorname{val}_{\Psi}(\sigma)\right] & \geq \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{\operatorname{bias}(\Psi)^{2}}{2\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)}-\frac{\operatorname{bias}(\Psi)^{2}}{4\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)^{2}}\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right) \\
& \geq \sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{\operatorname{bias}(\Psi)^{2}}{4\left(\sum_{j=2}^{k}\left(\frac{2^{j}-j-1}{2^{j-2}}\right) m_{j}\right)} .
\end{aligned}
$$

### 7.3 Proof of Claim 7.4

We will use the following claim.

Claim 7.7. For every $x \geq 0, y>0, y \geq a \geq 0$ :

$$
\frac{2 x+3 y+x^{2} / y-2 a}{4(x+y)-3 a} \geq \frac{\sqrt{2}}{2} .
$$

Proof. Let $L=2 x+3 y+x^{2} / y$ and $M=4(x+y)>3 a$. From Claim 3.8, $L / M \geq \frac{\sqrt{2}}{2}>\frac{2}{3}$. First we observe that

$$
\frac{L-2 a}{M-3 a}-\frac{L}{M}=\frac{a(3 L-2 M)}{M(M-3 a)} \geq 0
$$

for every $a \geq 0$ and $M>3 a$. Now

$$
\frac{2 x+3 y+x^{2} / y-2 a}{4(x+y)-3 a}=\frac{L-2 a}{M-3 a} \geq \frac{L}{M} \geq \frac{\sqrt{2}}{2},
$$

where the last inequality uses Claim 3.8 again.
We are now ready to prove Claim 7.4.
Claim 7.4. If $\sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j} \leq B \leq(1-\delta)\left(\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}\right)$, then

$$
\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{B^{2}}{4 \sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}} \geq \frac{\sqrt{2}}{2} \sum_{j=1}^{k} m_{j}
$$

Proof. Let

$$
V=\frac{\sum_{j=1}^{k} \frac{2^{j}-1}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}}}{4 \sum_{j=1}^{k} m_{j}} .
$$

Now it remains to show that $V \geq \sqrt{2} / 2$. First, observe that

$$
V \geq \frac{\sum_{j=1}^{k}\left(4-2^{2-j}\right) m_{j}}{4 \sum_{j=1}^{k} m_{j}} \geq \frac{1}{2}
$$

Also, if $x / y \geq 1 / 2$ for non-negative $x, y$, then we have $x / y \geq(x+a) /(y+2 a)$ for all $a \geq 0$. As $B \geq$ $\sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j}$, we can lower bound $V$ as follows.

$$
\begin{aligned}
V & =\frac{\sum_{j=1}^{k} \frac{2^{j}-1}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2^{j-j-1}}{2^{j-2}} m_{j}}}{4 \sum_{j=1}^{k} m_{j}} \\
& \geq \frac{\sum_{j=1}^{k} \frac{2^{j}-1}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}}+2\left(B-\sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j}\right)}{4 \sum_{j=1}^{k} m_{j}+4\left(B-\sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j}\right)} \\
& =\frac{2 B+\sum_{j=1}^{k} \frac{2^{j}+j-3}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2^{j-j-1}}{2^{j-2}} m_{j}}}{4 B+\sum_{j=1}^{k} \frac{2^{j}+2 j-4}{2^{j-2}} m_{j}} .
\end{aligned}
$$

Now, let $x=B, y=\sum_{j=2}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}$, and $a=\sum_{j=1}^{k} \frac{2^{j}-2 j}{2^{j-2}} m_{j}$, and rewrite the above quantity as follows

$$
\begin{aligned}
& =\frac{2 x+3 y+x^{2} / y-2 a}{4(x+y)-3 a} \\
& \geq \frac{\sqrt{2}}{2},
\end{aligned}
$$

where the last inequality follows from Claim 7.7 and $y \geq a$.

### 7.4 Proof of Claim 7.5

Claim 7.5. If $B \leq \sum_{j=1}^{k} \frac{2-j}{2^{j-1}} m_{j}$, then

$$
\sum_{j=1}^{k}\left(1-2^{-j}\right) m_{j}+\frac{B^{2}}{4 \sum_{j=2}^{k} \frac{2^{j-j-1}}{2^{j-2}} m_{j}} \geq \frac{\sqrt{2}}{2}\left(\frac{B}{2}+\sum_{j=1}^{k} \frac{2^{j}+j-2}{2^{j}} m_{j}\right) .
$$

Proof. Let

$$
V=\frac{\sum_{j=1}^{k} \frac{2^{j}-1}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=\frac{2 j^{j} j-1}{2 j}}^{2 j-2} m_{j}}}{2 B+\sum_{j=1}^{k} \frac{2^{j+j-j-2}}{2^{j j-2} m_{j}}} .
$$

Now it remains to show that $V \geq \sqrt{2} / 2$. Let $R=\sum_{j=1}^{k} \frac{2-j}{2^{j-2}} m_{j}-2 B \geq 0$. Since the denominator of $V$ is not less than both $R$ and the numerator of $V$, we have that

$$
\begin{aligned}
& V \geq \frac{\sum_{j=1}^{k} \frac{2^{j}-1}{2 j-2} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2 j-j-1}{2 j-2} m_{j}}-R}{2 B+\sum_{j=1}^{k} \frac{2 j+j-2}{2^{j-2}} m_{j}-R} \\
& =\frac{\sum_{j=1}^{k} \frac{2^{j}-1}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2 j-j-1}{2 j^{j-2}} m_{j}}-\left(\sum_{j=1}^{k} \frac{2-j}{2^{j-2}} m_{j}-2 B\right)}{2 B+\sum_{j=1}^{k} \frac{2^{j j+j-2}}{2^{j-2}} m_{j}-\left(\sum_{j=1}^{k} \frac{2-j}{2^{j-2}} m_{j}-2 B\right)} \\
& =\frac{2 B+\sum_{j=1}^{k} \frac{2^{j}+j-3}{2^{j-2}} m_{j}+\frac{B^{2}}{\sum_{j=2}^{k} \frac{2 j-j-1}{2^{j-2}} m_{j}}}{4 B+\sum_{j=1}^{k} \frac{2^{j}+2 j-4}{2^{j-2}} m_{j}} .
\end{aligned}
$$

Now, let $x=B, y=\sum_{j=1}^{k} \frac{2^{j}-j-1}{2^{j-2}} m_{j}$, and $a=\sum_{j=1}^{k} \frac{2^{j}-2 j}{2^{j-2}} m_{j}$, and rewrite the above quantity as follows.

$$
\begin{aligned}
& =\frac{2 x+3 y+x^{2} / y-2 a}{4(x+y)-3 a} \\
& \geq \frac{\sqrt{2}}{2},
\end{aligned}
$$

where the last inequality follows from Claim 7.7 and $y \geq a$.

## Open Questions

Our work gives optimal approximation ratios for all Boolean maximum constraint satisfaction problems with constraints of length at most two. It would be interesting to understand the complexity of constraint languages with arity greater than two, and larger alphabet sizes.

In terms of lower bounds, we show that better than $\frac{4}{9}$ - and $\frac{\sqrt{2}}{2}$-approximations for Max-2-AND and Max-2-OR require space $\Omega(\sqrt{n})$. Can we improve these space lower bounds to $\Omega(n)$, matching the space requirements of standard algorithms that give $(1-\varepsilon)$-approximation?

## Acknowledgement

We thank Madhu Sudan for many helpful discussions, and for pointing out a mistake in an old proof. We also thank Mitali Bafna for spotting an error in our previous algorithm for Max-kSAT. Finally, we thank the anonymous referees for their several useful suggestions which helped improve the presentation of the paper.

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[^1]:    ${ }^{1}$ In this work we focus on randomized streaming algorithms that make one pass over the input in a fixed (adversarial) order, and return the correct answer with probability $3 / 4$.
    ${ }^{2}$ Although formally Max-CUT is a special case of Max-2XOR where all constraints are of the form $x_{i} \oplus x_{j}=1$, it can be shown that these two problems are equivalent.

[^2]:    ${ }^{3}$ While Theorem 1.1 states the bound for the Max-2AND problem only, it is easy to see that the proof in Section 4.3 gives the same bound even for Max-DICUT.

[^3]:    ${ }^{4}$ For uniformity reasons, our definition of bias differs from the definition in [GVV17] by a multiplicative factor of 2 .

[^4]:    ${ }^{5}$ We only apply this transformation to Max-2AND instances, because here it plays in our favor. For example, an AND clause with repeated literals is satisfied by a uniform random assignment with probability $1 / 2$, while an AND clause with distinct variables is satisfied with probability only $1 / 4$. For the case of OR, a clause with repeated literals would be satisfied only with probability $1 / 2$, while an OR clause with distinct variables would be satisfied with probability $3 / 4$.

[^5]:    ${ }^{6}$ Indeed, given a instance $\Psi$ where $\operatorname{pos}_{i}^{(2)}(\Psi)<\operatorname{neg}_{i}^{(2)}(\Psi)$, we can consider the instance $\Psi^{\prime}$ where every $x_{i}$ is replaced with $\neg x_{i}$, and vice versa. We have that $\operatorname{pos}_{i}^{(2)}\left(\Psi^{\prime}\right) \geq \operatorname{neg}_{i}^{(2)}\left(\Psi^{\prime}\right)$, $\operatorname{bias}(\Psi)=\operatorname{bias}\left(\Psi^{\prime}\right)$, and every assignment for $\Psi^{\prime}$ is uniquely mapped to the corresponding assignment for $\Psi$ satisfying the same number of clauses.

[^6]:    ${ }^{7}$ Note that if we set $\gamma=0$, the algorithm becomes the trivial random sampling, and if we set $\gamma=0.5$, the algorithm becomes the greedy algorithm from [GVV17].

[^7]:    ${ }^{8}$ In [KKS15], they use $D^{1}, D^{2}$ to denote $\mathcal{P}_{\text {YES }}$ and $\mathcal{P}_{\overline{\text { NO }}}$. They also used $P$ instead of $\Pi$.

