New Techniques for Proving Fine-Grained Average-Case Hardness

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Abstract

The recent emergence of fine-grained cryptography strongly motivates developing an average-case analogue of Fine-Grained Complexity (FGC).

Prior work [Goldreich-Rothblum 2018, Boix-Adserà et al. 2019, Ball et al. 2017] developed worst-case to average-case fine-grained reductions (WCtoACFG) for certain algebraic and counting problems over natural distributions and used them to obtain a limited set of cryptographic primitives. To obtain stronger cryptographic primitives based on standard FGC assumptions, ideally, one would like to develop WCtoACFG reductions from the core hard problems of FGC, Orthogonal Vectors (OV), CNF-SAT, 3SUM, All-Pairs Shortest Paths (APSP) and zero-k-clique. Unfortunately, it is unclear whether these problems actually are hard for any natural distribution. It is known, that e.g. OV can be solved quickly for very natural distributions [Kane-Williams 2019], and in this paper we show that even counting the number of OV pairs on average has a fast algorithm.

This paper defines new versions of OV, kSUM and zero-k-clique that are both worst-case and average-case fine-grained hard assuming the core hypotheses of FGC. We then use these as a basis for fine-grained hardness and average-case hardness of other problems. The new problems represent their inputs in a certain "factored" form. We call them "factored"-OV, "factored"-zero-k-clique and "factored"-3SUM. We show that factored-k-OV and factored kSUM are equivalent and are complete for a class of problems defined over Boolean functions. Factored zero-k-clique is also complete, for a different class of problems.

Our hard factored problems are also simple enough that we can reduce them to many other problems, e.g. to edit distance, *k*-LCS and versions of Max-Flow. We further consider counting variants of the factored problems and give WCtoACFG reductions for them for a natural distribution. Through FGC reductions we then get *average-case* hardness for well-studied problems like regular expression matching from standard *worst-case* FGC assumptions.

To obtain our WCtoACFG reductions, we formalize the framework of [Boix-Adserà et al. 2019] that was used to give a WCtoACFG reduction for counting *k*-cliques. We define an explicit property of problems such that if a problem has that property one can use the framework on the problem to get a WCtoACFG self reduction. We then use the framework to slightly extend Boix-Adserà et al.'s average-case counting *k*-cliques result to average-case hardness for counting arbitrary subgraph patterns of constant size in *k*-partite graphs.

The fine-grained public-key encryption scheme of [LaVigne et al.'20] is based on an average-case hardness hypothesis for the *decision* problem, zero-*k*-clique, and the known techniques for building such schemes break down for algebraic/counting problems. Meanwhile, the WCtoACFG reductions so far have only been for counting problems. To bridge this gap, we show that for a natural distribution, an algorithm that detects a zero-*k*-clique with high enough probability also implies an algorithm that can count zero-*k*-cliques with high probability. This gives hope that the FGC cryptoscheme of [LaVigne et al.'20] can be based on standard FGC assumptions.

1 Introduction

Fine-grained complexity (FGC) is an active research area that seeks to understand why many problems of interest have particular running time bounds t(n) that are easy to achieve with known techniques, but have not been improved upon significantly in decades, except by $t(n)^{o(1)}$ factors. FGC has produced a versatile set of tools that have resulted in surprising *fine-grained* reductions that together with popular hardness hypotheses explain the running time bottlenecks for a large variety of problems [Vas18]. The reductions of FGC have, for example, explained the difficulty of improving over the $n^{2-o(1)}$ time algorithms for Longest Common Subsequence (LCS) by giving a tight reduction from k-SAT, and thus showing that an improved LCS algorithm would violate the Strong Exponential Time Hypothesis (SETH) [ABV15].

There are three main problems, with associated hardness hypotheses about their running times, that FGC primarily uses as sources of hardness reductions (see [Vas18]). The three core hard problems are All Pairs Shortest Paths (APSP), hypothesized to require $n^{3-o(1)}$ time in n-node graphs¹, the 3SUM problem, hypothesized to require $n^{2-o(1)}$ time on n integer inputs, and the Orthogonal Vectors (OV) problem, hypothesized to require $n^{2-o(1)}$ time for n vector inputs of dimension $\omega(\log n)$ (the OV hypothesis is implied by SETH [Wil07]).

While it is unknown whether these three hypotheses are equivalent, some work suggests they might not be $[CGI^+16]$. There is a problem, Zero Triangle, on n node graphs that requires $n^{3-o(1)}$ time under both the 3SUM and the APSP hypothesis [VW18, VW13]. Zero Triangle asks if an n node graph with integer edge weights contains a triangle whose three edge weights sum to 0. A natural extension of Zero Triangle, zero-k-clique (where one wants to detect a k-clique with edge weight sum 0), is conjectured to require $n^{k-o(1)}$ time. There are also some simple to define problems on n node graphs that require $n^{3-o(1)}$ time under three core hardness hypotheses (SETH, APSP and 3SUM): Matching Triangles and Triangle Collection [AVY18].

Recently there has been increased interest in developing average-case fine-grained complexity (ACFGC), with a new type of *fine-grained cryptography* as a main motivation [BRSV17, BRSV18, GR18, LLV19, BBB19]. The main goal is to identify a problem P that requires some $t(n)^{1-o(1)}$ time on average for an easily sampled distribution, and then to build interesting cryptographic primitives from this problem, where any honest party only needs to run a very fast algorithm, in some $t'(n) \le O(t(n)^c)$ time for c much smaller than 1, while an adversary would need to run at least in $t(n)^{1-o(1)}$ time, unless problem P can be solved fast on average.

To obtain average-case fine-grained hard problems, one would like to be able to obtain worst-case to average-case fine-grained reductions for natural problems that are hypothesized to be fine-grained hard in the worst-case². This is what prior work does.

The problems for which fine-grained worst-case to average-case hardness reductions are known are mostly algebraic or counting problems, such as counting *k*-cliques [GR20, GR18, BRSV18, BBB19], or some problems involving polynomials. Some limited cryptographic primitives have been obtained from such problems, e.g. fine-grained proofs-of-work [BRSV18, BRSV17]. Building fine-grained one-way functions or fine-grained public key cryptography based on any worst-case FGC hardness assumption is still an open problem. Such primitives have been developed, based on plausible assumptions about the average-case complexity of zero-*k*-clique [LLV19]. This motivates the following question: *Is there a fine-grained worst-case to average-case reduction for zero-k-clique?*

¹All hypotheses are for the word-RAM model of computation with $O(\log n)$ bit words.

²Well, even more ideally, one would like to use problems that are provably unconditionally average-case hard, such as the problems from the known time-hierarchy theorems, but these problems are difficult to work with and there are no known techniques to build cryptography from them.

As prior work showed worst-case to average-case case reductions for counting cliques, a natural approach to obtaining worst-case to average-case reductions for the detection variant of zero-*k*-clique is to give a fine-grained reduction from counting to decision. A tight reduction is not known for the worst-case version of the problem. It turns out that a fine-grained reduction from counting to decision for zero-*k*-clique is possible in the average-case for a natural distribution with certain parameters, if the detection probability is high enough. We prove this in Section 7. While the parameters are currently not good enough to imply a worst-case to average-case reduction for (the decision version of) zero-*k*-clique, the reduction gives hope that the fine-grained public-key scheme of [LLV19] can eventually be based on a standard FGC (worst-case) hardness assumption.

The next natural question is whether worst-case to average-case reductions are possible for the other core problems of FGC, and in particular for OV (as it is as far as we know unrelated to zero-k-clique). Consider the most natural distribution for OV: given a fixed probability $p \in (0,1)$, one generates n vectors of dimension $d = \omega(\log n)$ by selecting for each vector v and $i \in [d]$ independently, v_i to be 1 with probability p and 0 otherwise. Kane and Williams [KW19] showed that for every p, there is an $\varepsilon_p > 0$ and an $O(n^{2-\varepsilon_p})$ time algorithm that solves OV on instances generated from the above distribution with high probability. Thus, for this distribution (if the OV conjecture is true), there can't be a fine-grained (n^2, n^2) -worst-case to average-case reduction for OV. In Section 6 we also show that even the counting version of OV, in which one wants to determine the number of pairs of orthogonal vectors, has a truly-subquadratic time algorithm that works with high probability over the same distribution. Thus, even counting OV cannot be average-case $n^{2-o(1)}$ -hard. (Though, it could be fine-grained average-case hard for a different time function. We leave this to future work.)

The first key contribution of this paper is in defining a new type of problem, a "factored problem" that is fine-grained hard from a core FGC assumption, whose counting version is average-case hard for a natural distribution again under a core FGC assumption, and that is also simple enough so that one can reduce it to well-studied problems and develop average-case hardness for them.

While developing worst-case to average-case reductions for our factored problems, we formalize the worst-case to average-case fine-grained reductions framework of Boix et al. [BBB19]. We identify a property of problems (the existence of a "good polynomial") that makes it possible for these problems to have such a worst-case to average-case reduction. Originally, [BBB19] gave average-case hardness for counting k-Cliques in Erdös-Renyi graphs using their framework. Along the way of generalizing their framework, we also obtain a worst-case to average-case reduction for counting copies of H for any k-node H, where the distribution for the average-case instance is again for Erdö-Renyi graphs. We achieve this using a new technique we call Inclusion-Edgesclusion.

In the rest of the introduction we will present our results mentioned in the above two paragraphs.

1.1 The factored problems

We call the problems we introduce "factored problems" (a full formal definition is in Section 2). To define them, let us first define a *factored vector*. Let b and g be positive integers. A (g,b)-factored vector, v, is made up of g sets $v[1], \ldots, v[g]$. Each set is a subset $v[i] \subseteq \{0,1\}^b$. Roughly speaking, a factored vector v represents many $b \cdot g$ binary vectors, namely a concatenation x_1, x_2, \ldots, x_g for each choice of a g-tuple of vectors $x_i \in v[i]$ for all i. For example, for g = 2 and b = 3, let v be a factored vector where $v[0] = \{001, 010\}$ and $v[1] = \{010, 110\}$. A natural interpretation of v is that it is a set of the following 4 binary vectors, by concatenating each member of v[0] with each member of v[1], that is $\{001010, 001110, 010010, 010110\}$.

Now, consider a function f that takes a 2b-bit input $x_1, \ldots, x_b, y_1, \ldots, y_b$ and returns a value in $\{0, 1\}$; we can consider f as a Boolean function. Then, for two factored vectors v and v' and a coordinate $i \in [g]$,

we can consider the number of pairs of *b*-bit vectors $x \in v[i], y \in v'[i]$ that f accepts. This is $accept_f(v, v', i) := \sum_{x \in v[i], y \in v'[i]} f(x_1, \dots, x_b, y_1, \dots, y_b)$, where $x = x_1 \dots x_b$ and $y = y_1 \dots y_b$. If we take the product $\prod_{i=1}^g accept_f(v, v', i)$, we would obtain the number of pairs of $b \cdot g$ -length vectors represented by v and v' that are accepted by f, where f is said to accept a pair of $b \cdot g$ -length vectors if it accepts each of the g pairs of chunks of g-length subvectors between positions f(i-1)b+1 to f(i-1)b+1 to

Then we can define the factored problem for f, F2-f that given two sets S and T of n(g,b)-factored vectors, computes the sum $\sum_{v \in S, v' \in T} \prod_{i=1}^g accept_f(v, v', i)$, i.e. the total number of pairs of vectors represented by vectors in S and T that are accepted by f. For technical reasons, we restrict the values $g = o(\lg(n)/\lg\lg(n))$ and $b = o(\lg(n))$, so that each factored vector can be represented with at most $gb2^b$ bits (g sets of at most 2^b vectors of length b).

Depending on the function f, we get different versions of a factored problem. If f on b-length vectors x and y, returns 1 iff $x \cdot y = 0$, then we get the factored OV problem F2-OV. If f returns 1 if the XOR of x and y is 0, we get the F2-XOR problem, and if f returns 1 iff x + y = 0 when viewed as integers, we get the F2-SUM problem.

More generally, f can be defined over $k \cdot b$ -length vectors, for integer $k \geq 2$, taking k-tuples of b-length binary vectors to $\{0,1\}$. Then analogously we can define Fk-f to compute the number of k-tuples of vectors represented by some k-tuple of factored vectors, one from each n-sized input set S_i , $i \in [k]$, so that f accepts the k-tuple. This way we can define Fk-OV, Fk-XOR, Fk-SUM etc, the factored versions of k-OV, k-XOR and k-SUM.

Similarly to these problems defined on k-tuples of sets of factored vectors, we define problems reminiscent to k-clique. Here f is a function that takes $\binom{k}{2}$ -tuples of b-length vectors to $\{0,1\}$, one is given a graph whose edges are labeled by factored vectors and the factored f k-clique problem, ffkC, asks to compute the number of $\binom{k}{2}$ -tuples of vectors that are accepted by f and are represented by the factored vectors labeling the edges of a k-clique in the graph. We focus in particular on the factored zero-k-clique problem, FZkC, in which f corresponds to returning whether the sum of $\binom{k}{2}$ k-bit numbers is k-clique problems.

1.2 Results for factored problems

We will summarize the results around our factored problems below. They appear in sections 3 and 4. We give a visual summary of our results in Figure 1. We use the shortened names for many of the problems in the figure. The results will concern both counting and decision versions of our factored problems. The decision versions ask whether the count is nonzero, whereas the counting versions ask for the exact count. When we want the counting version, we will place # in front of the name of the problem. See the Preliminaries (Section 2) for more details.

Summary. We first provide an overview summary of our results.

First we show that the factored versions of k-OV, k-SUM and k-XOR are all $n^{k-o(1)}$ -fine-grained hard under SETH. We also show that the factored version of zero-3-clique (FZ3C) is $n^{3-o(1)}$ -fine-grained hard based on any of the three core hypotheses of FGC (SETH, or the APSP or 3-SUM hypothesis). Additionally, we show that the counting versions of these factored problems are as hard in their natural uniform average-case as they are in the worst case. Moreover, we show that many natural problems, like counting regular expression matchings, reduce from our factored problems. This even implies fine-grained average-case hardness for these problems over some explicit distributions.

Thus our factored problems do three things simultaneously:

• Instead of trying to use the uniform average-case of the core problems of FGC as central problems

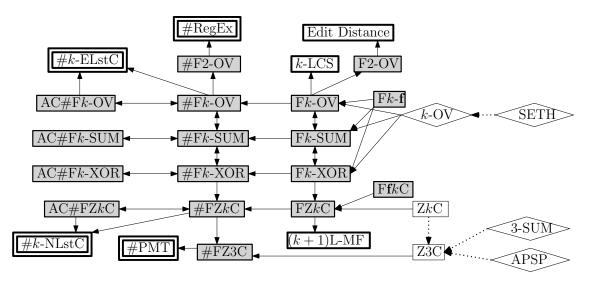


Figure 1: A summary of the reductions to and from factored problems in the paper. The problems in diamonds are the core problems of FGC. The full lines are reductions from this paper, while doted lines are pre-existing reductions. The problems in gray boxes are our factored problems. The problems in thick lined boxes are the problems we reduce from factored problems. For the problems surrounded by a thick-lined double box we have generated an explicit average case distributions on which they are hard (but it is not the uniform distribution). These results appear in Sections 3 and 4.

in a network of average-case reductions, we can use the factored versions of the core problems in FGC. For example, the counting variant of factored OV (#F2-OV) is hard in its uniform average case from the worst-case OV hypothesis. Generically, our factored problems serve as an alternative central problem for average-case hardness. To demonstrate this, in Section 4, we give reductions from counting factored problems to four problems in graph algorithms and sequence alignment (including counting regular expression matchings).

- The factored versions of the core problems are sufficiently expressive that they are complete for the large class of factored problems. In particular, Fk-OV, Fk-XOR, and Fk-SUM are complete for the class of problems of the form Fk-f over all f, while FZkC is complete for the class of problems FfkC over all f. Despite this expressiveness we are still able to reduce our factored problems to many natural problems. In section 4 we give fine-grained reductions from our factored problems to k-LCS, Edit Distance and a labeled version of Max Flow.
- Abboud et al. [AVY18] gave two problems, Triangle Collection and Matching Triangles that are hard from all three core assumptions in FGC. They also showed that one can reduce Triangle Collection 3 to several natural problems in graph algorithms. Unfortunately, however, neither Triangle Collection, nor Matching Triangles are known to be hard on average. One of our factored problems, FZ3C is also hard from all three core assumptions. Moreover, the counting version of FZ3C is additionally $n^{3-o(1)}$ hard in the average-case from all three core assumptions of FGC. Thus, problems that reduce from counting FZ3C get average-case hardness for some explicit average-case distribution. We give two examples of problems that reduce from counting FZ3C in Section 4. Hence if you are interested in

³Actually a version of the problem that is still hard under all three assumptions.

average-case hardness then counting FZ3C might be a better source for reductions than, say Matching Triangles or Triangle Collection.

Fine-grained hardness for factored problems. Here we show that our factored problems are fine-grained hard under standard FGC hypotheses.

We first show that a single call to a factored problem solves its non-factored counterpoint.

Theorem 1.1. In O(n) time, one can reduce an instance of size n of k-OV, k-XOR, k-SUM and ZkC to a single call to an instance of size $\tilde{O}(n)$ of Fk-OV, Fk-XOR, Fk-SUM and FZkC, respectively.

The above theorem holds both in the decision and counting context. It gives fine-grained hardness for the factored variants of all our problems, under the hypothesis that the original variants are hard. Note that k-XOR, k-SUM have $\tilde{O}(n^{\lceil k/2 \rceil})$ time algorithms. However, we have $n^{k-o(1)}$ conditional lower bounds for all of Fk-OV, Fk-XOR, Fk-SUM and FZkC. So, while we do get fine-grained hardness from the k-XOR and k-SUM hypotheses, this hardness is not tight. The hardness is tight from the k-OV and ZkC hypotheses however.

Now we give fine-grained hardness for FZ3C under all three core hypotheses from FGC.

Theorem 1.2. If FZ3C (even for $b = o(\log n)$ and $g = o(\log(n)/\log\log(n))$) can be solved in $O(n^{3-\varepsilon})$ time for some constant $\varepsilon > 0$, then SETH is false, and there exists a constant $\varepsilon' > 0$ such that 3-SUM can be solved in $O(n^{2-\varepsilon'})$ time and APSP can be solved in $O(n^{3-\varepsilon'})$ time.

Worst-case to average-case reductions for factored problems. We show that our factored problems admit fine-grained worst-case to average-case reductions. Our first theorem about this is a worst-case to average-case fine-grained reduction for the counting version of Fk- \mathfrak{f} for a natural distribution (defined in Definition 18). The proof appears in Section 3.

Theorem 1.3. Let μ be a constant such that $0 < \mu < 1$. Suppose that average-case #Fk- \mathfrak{f}^{μ} (see definition 18, this is an iid distribution which has ones with probability μ) can be solved in time T(n) with probability at least $1 - 1/(lg(n)^{kg} \lg \lg(n)^{kg})$. Then worst-case #Fk- \mathfrak{f} can be solved in time $\tilde{O}(T(n))^4$.

When $\mu = 1/2$ average-case #Fk-f^{μ} is average-case #Fk-f.

Thus, if we have worst-case fine-grained hardness for #Fk- \mathfrak{f} for some f, then we get average-case hardness for the same problem over a natural distribution. In particular, in the corollary below we obtain average-case hardness for #Fk-OV, #Fk-SUM, #Fk-XOR, based on the standard FGC hardness of k-OV, k-SUM, k-XOR (as implied by Theorem 1.1).

Corollary 1.4. If average-case #Fk-OV can be solved in time T(n) with probability $1-1/(lg(n)^{gk} \lg \lg(n)^{gk})$ then worst-case #Fk-OV can be solved in time $\tilde{O}(T(n))^4$.

If average-case # Fk-SUM can be solved in time T(n) with probability $1 - 1/(lg(n)^{gk} \lg \lg(n)^{gk})$ then worst-case # Fk-SUM can be solved in time $\tilde{O}(T(n))^4$.

If average-case #Fk-XOR can be solved in time T(n) with probability $1 - 1/(lg(n)^{gk} \lg \lg(n)^{gk})$ then worst-case #Fk-XOR can be solved in time $\tilde{O}(T(n))^4$.

Similarly, we obtain fine-grained average-case hardness for #FfkC, based on the fine-grained worst-case hardness of #FfkC.

⁴Note that given that $g = o(\lg(n)/\lg\lg(n))$ then a probability of $1 - 1/n^{\varepsilon}$ will be high enough for any $\varepsilon > 0$.

Theorem 1.5. Let μ be a constant and $0 < \mu < 1$. If average-case #FfkC $^{\mu}$ (see Definition 18, this is an iid distribution which has ones with probability μ) can be solved in time T(n) with probability $1 - 1/(\lg(n)^{k^2g} \lg \lg(n)^{k^2g})$ then worst-case #FfkC can be solved in time $\tilde{O}(T(n))^4$.

When $\mu = 1/2$ average-case #FfkC $^{\mu}$ is average-case #FfkC.

By Theorem 1.5, we have the following result for #FZkC in particular.

Corollary 1.6. If average-case #FZkC can be solved in time T(n) with probability $1 - 1/(lg(n)^{k^2g} \lg \lg(n)^{k^2g})$ then worst-case #FZkC can be solved in time $\tilde{O}(T(n))^4$.

Thus in particular we obtain fine-grained average-case hardness for counting factored zero-3-cliques, based on the hardness of zero-3-clique, and thus based on the APSP and 3-SUM hypotheses.

Completeness for Fk-OV, Fk-SUM, Fk-XOR and FZkC. Let $k \ge 2$ be a fixed integer. Consider the class of problems Fk- \mathfrak{f} defined over all boolean functions f on kb-length inputs. Our first sequence of results show that Fk-OV, Fk-SUM and Fk-XOR are complete for the class, so that a T(n) time algorithm for any of these problems would imply an $\tilde{O}(T(n))$ time algorithm for Fk- \mathfrak{f} for any f.

To prove this, we first show that Fk-XOR is complete for the class:

Theorem 1.7. If we can solve #Fk-XOR with g sets of k^3b length vectors in time T(n) then, for any f, we can solve a #Fk- \mathfrak{f} instance with g sets of b length vectors in time $T(n) + \tilde{O}(n)$ time.

We then show that Fk-OV, Fk-SUM and Fk-XOR are equivalent.

Theorem 1.8. If any of #Fk-OV, # Fk-SUM, or #Fk-XOR can be solved in T(n) time then all of #Fk-OV, # Fk-SUM, and #Fk-XOR can be solved in $\tilde{O}(T(n))$ time.

The above two theorems imply the final completeness theorem:

Theorem 1.9. If any of #Fk-OV, # Fk-SUM, or #Fk-XOR can be solved in T(n) time then #Fk- \mathfrak{f} can be solved in $\tilde{O}(T(n))$ time.

We also consider the class of problems (#)FfkC defined by Boolean functions f on $\binom{k}{2}b$ -length inputs. We show that (#)FZkC is complete for this class.

Theorem 1.10. If (#)FZkC can be solved in T(n) time then (#)F\(\frac{1}{2}kC\) for any f, can be solved in $\tilde{O}(T(n)+n^2)$ time.

Thus our factored problems corresponding to core problems in FGC, are the hard problems for natural classes of factored problems.

Fine-grained hardness for well-studied problems, based on the hardness of factored problems. The results we mention here appear in Section 4. The main upshot is that the factored problems are both hard and also simple enough to imply hardness for basic problems in graph and string algorithms. Some of the results are based on the hardness of FZ3C which implies hardness from all of SETH, 3-SUM and APSP. Some come from Fk-f which implies hardness from SETH.

Partitioned Matching Triangles. First we define the *Partitioned Matching Triangles* problem (PMT) as follows: Given g disjoint n-node graphs with node colors, is there a triple of colors a,b,c so that every one of the g graphs contains a triangle whose nodes are colored by a,b,c? The counting variant of PMT is to count the total number of such g-tuples of colored triangles.

Abboud et al. [AVY18] consider the related Matching Triangles problem mentioned earlier in the introduction, and show that it is hard from all three core FGC hypotheses. In the Matching Triangles problem one is given an integer T and a node-colored graph G and one wants to know if there is a triple of colors a, b, c so that there are at least T triangles in G colored by a, b, c.

We observe first that for the particular parameters for which Matching Triangles is shown to be hard in [AVY18], one can actually reduce Matching Triangles in a fine-grained way to Partitioned Matching Triangles (PMT), so that the latter problem is also hard from all three hypothesis. Furthermore, we give a powerful reduction to PMT from FZ3C. Moreover, our reduction also holds between the counting versions of the problems, so that we get fine-grained *average-case* hardness for counting PMT under all three hypotheses as well.

Theorem 1.11. If (#)Partitioned Matching Triangles can be solved in T(n) time, then we can solve (#)FZ3C in time $\tilde{O}(T(n) + n^2)$.

k-color Node Labeled st Connectivity. In the *k*-color Node Labeled *st* Connectivity Problem (*k*-NLstC) one is given an acyclic graph G = (V, E) with two designated nodes $s, t \in V$, and colors on all nodes in $V \setminus \{s, t\}$ from a set of colors C. One is then asked whether there is a path from S to S to S to S node colors.

We give a fine-grained reduction from FZkC to k-NLstC that also holds between the counting versions. Here in the counting version of k-NLstC we want to output the number of s-t paths through at most k colors, mod $\lceil 2^{2k \lg^2(n)} \rceil$.

Theorem 1.12. If a $O(|C|^{k-2}|E|^{1-\varepsilon/2})$ or $O(|C|^{k-2-\varepsilon}|E|)$ time algorithm exists for (counting mod $2^{2k\lg^2(n)}$) k-NLstC then a $O(n^{k-\varepsilon})$ algorithm exists for (#)FZkC.

The conditional lower bound of $(|C|^{k-2}|E|)^{1-o(1)}$ resulting from the above theorem is tight. In Appendix A we give the corresponding algorithm.

k-color Edge Labeled st Connectivity. The k-color Edge Labeled st Connectivity problem (k-ELstC) asks for a given acyclic graph with colored edges and given source s and target t, if there is a path from s to t that uses only k colors of edges.

We give conditional hardness for both the decision and counting version of the problem (where the counts are mod a small R). This also implies average-case hardness for the counting mod R problem under all three hardness hypotheses of FGC.

Theorem 1.13. If a $\tilde{O}(|E||C|^{k-1-\varepsilon})$ or $\tilde{O}(|E|^{1-\varepsilon}|C|^{k-1})$ time algorithm exists for (counting mod $2^{2k\lg^2(n)}$) k-ELstC, then a $\tilde{O}(n^{k-\varepsilon})$ algorithm exists for (#)Fk- \mathfrak{f} .

This is tight. Note this algorithm is slower (by a factor of |C|) than the node-labeled version, however it is optimal. The corresponding algorithm is in a theorem from Appendix A.

(k+1) Labeled Max Flow. The (k+1) Labeled Max Flow problem studied in [GCSR13] asks, given a capacitated graph G = (V, E) where the edges have colors, and $s, t \in V$, if there is a maximum flow from the source s to the sink t where number of distinct colors of the edges with non-zero flow is at most k+1.

Theorem 1.14. If (k+1)L-MF can be solved in T(n) time, then we can solve FZkC in time $\tilde{O}(T(n)+n^2)$.

This implies an $n^{k-1-o(1)}$ lower bound for kL-MF under all three FGC hypotheses. We also show that for the particular structured version of the problem given in our reduction, this lower bound is tight.

Regular Expression Matching. The Regular Expression Matching problem (studied e.g. in [BI16]) takes as input a regular expression (pattern) p of size m and a sequence of symbols (text) t of length n, and asks

if there is a substring of t that can be derived from p. The counting version of the problem, #Regular Expression Matching asks for the number of subset alignments of the pattern in the text mod an integer R, where $R = n^{o(1)}$. A classic algorithm constructs and simulates a non-deterministic finite automaton corresponding to the expression, resulting in the rectangular O(mn) running time for the detection version of the problem.

We give hardness from $\#F2\text{-OV} \pmod{R}$ which in turn implies average-case fine-grained hardness for counting regular expression matchings mod R, from SETH.

Theorem 1.15. Let R be an integer where $\lg(R)$ is subpolynomial. If you can solve $(\# \bmod R)$ regular expression matching in T(n) time, then you can solve $(\# \bmod R)$ F2-OV in $\tilde{O}(T(n)+n)$ time

Again, we show in Appendix A that for the particular "type" of pattern used in our reduction, this lower bound is tight.

LCS and Edit Distance. The k-LCS problem is a basic problem in sequence alignment. Given k sequences s_1, \ldots, s_k of length n, one is asked to find the longest sequence that appears in every s_i as a subsequence. k-LCS can be solved in $O(n^k)$ time with dynamic programming and requires $n^{k-o(1)}$ time under SETH, via a reduction from k-OV [ABV15]. Here we show that k-LCS is also fine-grained hard via a reduction from Fk-OV.

Theorem 1.16. A T(n) time algorithm for k-LCS with alphabet size O(k) implies a $\tilde{O}(T(n))$ algorithm for Fk-OV.

The Edit Distance problem is another famous sequence alignment problem. Here one is given two n length sequences a and b and one needs to compute the minimum number of symbol insertions, deletions and substitutions needed to transform a into b. Edit Distance can be solved in $O(n^2)$ time via dynamic programming, and requires $n^{2-o(1)}$ time under SETH, via a reduction from OV [BI15, BI18].

In section 4 we show that edit distance is also fine-grained hard from F2-OV.

Theorem 1.17. A T(n) time algorithm for Edit Distance implies a $\tilde{O}(T(n))$ algorithm for F2-OV.

1.2.1 Counting OV is Easy on Average

As mentioned earlier in the introduction we show that counting orthogonal vectors over the uniform distribution is easy in the average-case. Let $\#OV^{\mu,d}$ be the problem of solving orthogonal vectors on instances generated by sampling n vectors iid from the distribution over d bit vectors where every bit in the vector is sampled iid from the distribution that returns 1 with probability μ and returns 0 with probability $1 - \mu$.

Theorem 1.18. For all constant values of μ and all values of d there exists constants $\varepsilon > 0$ and $\delta > 0$ such that there is an algorithm for $\#OV^{\mu,d}$ that runs in time $\tilde{O}(n^{2-\delta})$ with probability at least $1 - n^{-\varepsilon}$.

1.2.2 Counting to Detection for $\mathbf{Z}k\mathbf{C}$

Our worst-case to average-case reductions show hardness for counting problems. We mentioned earlier in the introduction that stronger cryptographic primitives have been built from detection problems than from counting problems. In this paper we show that in the sufficiently low error regime there is a counting to detection reduction for the zero-k-clique problem. Unfortunately, this does *not* give a fine-grained one-way function from worst-case assumptions. However, it makes progress towards bridging the gap between the

problems we can show hard from the worst-case and those we can build powerful cryptographic primitives from.

Definition 1. An average case instance of ZkC (ACZkC) with range R takes as input a complete k-partite graph with n nodes in each partition. Every edge has a weight chosen iid from [0, R-1]. A clique is considered a zero k clique if the sum of the edges is zero mod R.

Theorem 1.19. Given a decision algorithm for ACZkC that runs in time $O(n^{k-\varepsilon})$ for some $\varepsilon > 0$ and succeeds with probability at least $1 - n^{-\omega(1)}$, there is a counting algorithm that runs in $O(n^{k-\varepsilon'})$ time for some $\varepsilon' > 0$ and succeeds with probability at least $1 - n^{-\omega(1)}$, where $\omega(1)$ here means any function that is asymptotically larger than constant.

1.2.3 Worst-Case to Average-Case Reductions

We define the notion of a good low-degree polynomial for the problem P (a GLDP(P)). We define the properties of a good low-degree polynomial in Definition 8. Intuitively these properties are that the function must be low degree, count the output of the problem, and have well structured monomials. We show that any problem P that has a GLDP(P) is hard in its uniform average case in appendix B. We do this using techniques from Boix-Adserà et al [BBB19]. We use the GLDP(\cdot) framework to show uniform average-case hardness for our counting factored problems (in section 3). We give the framework theorem statement below.

Theorem 1.20. Let μ be a constant such that $0 < \mu < 1$. Let P be a problem such that a function f exists that is a GLDP(P), and let d be the degree of f. Let A be an algorithm that runs in time T(n) such that when \vec{I} is formed by n bits each chosen iid from $Ber[\mu]$:

$$Pr[A(\vec{I}) = P(\vec{I})] \ge 1 - 1/\omega \left(\lg^d(n) \lg \lg^d(n) \right).$$

Then there is a randomized algorithm B that runs in time $\tilde{O}(n+T(n))$ such that for any for $\vec{I} \in \{0,1\}^n$:

$$Pr[B(\vec{I}) = P(\vec{I})] \ge 1 - O(2^{-\lg^2(n)}).$$

Boix-Adserà et al show that counting k cliques is as hard in Erdős-Rényi graphs as it is in the worst case. We use the GLDP(·) framework a second time to slightly generalize their result to show that counting any subgraph H in an Erdős-Rényi graph is at least as hard as counting subgraphs H in worst case k-partite graphs (in section 5).

Theorem 1.21. Let H have e edges and k vertices where $k = o(\sqrt{\lg(n)})$. Let A be an average-case algorithm for counting subgraphs H in Erdős-Rényi graphs with edge probability 1/b which takes T(n) time with probability $1 - 2^{-2k} \cdot b^{-k^2} \cdot (\lg(e) \lg \lg(e))^{-\omega(1)}$.

Then an algorithm exists to count subgraphs H in k-partite graphs in time $\tilde{O}(T(n))$ with probability at least $1 - \tilde{O}(2^{-\lg^2(n)})$.

1.3 Organization of the Paper

In the preliminaries section 2 we give a formal definition of our factored problems. We also define the problems that we use throughout the paper, and we give an introduction of the average-case framework which is defined formally in Appendix B. We show that the factored problems are hard, and give the worst-case to average-case reductions for the factored problems in section 3. In section 4, we show that our

factored problems can show hardness for many natural non-factored problems. We use the same framework that gives average-case hardness for the factored problems to show that counting arbitrary subgraphs in random graphs is hard in section 5. We give a fast algorithm for counting OV over the uniform average-case in section 6. We give counting to detection reduction for average-case zero-*k*-clique with high probability in section 7. Finally, we list problems that seem like promising future work in section 8.

We give the efficient algorithms for our factored problems and the problems that reduce from our factored problems in appendix A. We give the framework that generalizes the techniques of Boix-Adserà et al. in appendix B.

2 Preliminaries

We cover useful preliminaries for sections 3 and 4 in this section. We include preliminaries for Section 5, Appendix B, and proofs of algorithm running times in Appendix A.

2.1 Hypotheses about Core Problems of Fine-Grained Complexity

Definition 1. The 3-SUM Hypothesis [GO95] In the *k-SUM problem*, we are given an unsorted list *L* of *n* values (over \mathbb{Z} or \mathbb{R}) and want to determine if there are $a_1, \ldots, a_k \in L$ such that $\sum_{i=1}^k a_i = 0$. The counting version of *k*-SUM asks how many sets of *k* numbers $a_1, \ldots, a_k \in L$ sum to zero.

The *k*-SUM hypothesis states that that the *k*-SUM problem requires $n^{\lceil k/2 \rceil - o(1)}$ time [GO95].

This is equivalent to saying no $n^{\lceil k/2 \rceil - \varepsilon}$ time algorithm exists for k-SUM for constant $\varepsilon > 0$.

Definition 2. APSP Hypothesis [VW10a] APSP takes as input a graph G with n nodes (vertices), V and m edges, E. These edges are given weights in [-R,R] where $R = O(n^c)$ for some constant c. We must return the shortest path length for every pair of vertices $u, v \in V$. The length of a path is the sum of the edge weights for all edges on that path.

The APSP Hypothesis states that the APSP problem requires $n^{3-o(1)}$ time when $m = \Omega(n^2)$.

Definition 3. Strong Exponential Time Hypothesis (SETH) [IP01] Let c_k be the smallest constant such that there is an algorithm for k-CNF SAT that runs in $O(2^{c_k n + o(n)})$ time.

SETH states that there is no constant $\varepsilon > 0$ such that $c_k \le 1 - \varepsilon$ for all constant k.

Intuitively SETH states that there is no constant $\varepsilon > 0$ such that there is a $O(2^{n(1-\varepsilon)})$ time algorithm for k-CNF SAT for all constant values of k.

Definition 4. The k**-OV Hypothesis [Wil07]** In the k-OV problem, we are given k unsorted lists L_1, \ldots, L_k of n zero-one vectors of length d as input. If there are k vectors $v_1 \in L_1, \ldots, v_k \in L_k$ such that for $\forall i \in [1,d] \exists j \in [1,k]$ such that $v_i[j] = 0$ we call these k vectors an orthogonal k-tuple. One should return true if there is an orthogonal k-tuple in the input. The counting version of k-OV (#k-OV) asks for the number of orthogonal k-tuples.

The k-OV hypothesis states that that the k-OV problem requires $n^{k-o(1)}$ time [Wil07].

This is equivalent to saying no $O(n^{k-\varepsilon})$ time algorithm exists for k-OV for constant $\varepsilon > 0$.

2.2 Graphs

Definition 5. Let $H = (V_H, E_H)$ be a k-node graph with $V_H = \{x_1, \dots, x_k\}$.

An *H*-partite graph is a graph with *k* partitions $V_1, ..., V_k$. This graph must only have edges between nodes $v_i \in V_i$ and $v_j \in V_j$ if $e(x_i, x_j) \in E_H$. (See Figure 2)

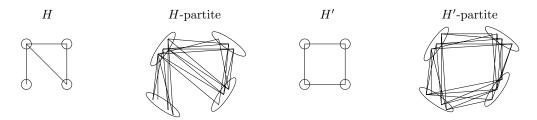


Figure 2: An example of the corresponding *H*-partite graphs.

2.3 Good Low-Degree Polynomials

We define the good low-degree polynomial for a problem P (GLDP(P)). In Appendix B we provide a framework which shows that if a problem P has a GLDP(P) then P is hard over the uniform average case. The proof of this framework is a generalization of the proof in Boix et al. [BBB19]. We use this to show average-case hardness for counting versions of factored problems and counting subgraphs in sections 3 and 5 respectively.

Definition 6. Let the polynomial f have n inputs x_1, \ldots, x_n . We say f is *strongly d-partite* if one can partition the inputs into d sets S_1, \ldots, S_d such that f can be written as a sum of monomials $\sum_i x_{1,i} \cdots x_{d,i}$, where every variable $x_{j,i}$ is from the partition S_j . That is, if there is a monomial $x_{i_1}^{c_1} \cdots x_{i_k}^{c_k}$ in f then it must be that $c_j = 1$ and for all $j \neq \ell$ if $x_{i_j} \in S_m$ then $x_{i_\ell} \notin S_m$.

Definition 7. Let $P(\vec{I})$ be the correct output for problem P given input \vec{I} .

Definition 8. Let n be the input size of the problem P, let P return an integer in the range [0, p-1] where p is a prime and $p < n^c$ for some constant c. A good low-degree polynomial for problem P (GLDP(P)) is a polynomial f over a prime finite field F_p where:

- If $\vec{I} = b_1, \dots, b_n$, then $f(b_1, \dots, b_n) = f(\vec{I}) = P(\vec{I})$ where b_i maps to either a zero or a one in the prime finite field.
- The function f has degree $d = o(\lg(n)/\lg\lg(n))$.
- The function f is strongly d-partite.

2.4 Factored Problems

We introduce a more expressive extension of k-SUM, k-OV, k-XOR, and ZkC. At a high level this extension takes every number or vector from the original problems and splits them up into $g = o(\lg(n)/\lg\lg(n))$ groups

of numbers or vectors with bit representations of size $b = o(\lg(n))$. If the original numbers had length ℓ , then $\ell \approx b \cdot g$. Then, we allow each group to contain multiple numbers or vectors.

We start by giving a definition of Fk-OV, then we give a small example of F2-OV. Next, we follow up with the analogously defined Fk-SUM ,Fk-XOR, and FZT. Finally, we give algorithms for these problems in the Appendix A.

2.4.1 Fk-OV, Intuition and Examples

Definition 9. A (g,b)-factored vector v is defined by g sets $(v[1],\ldots,v[g])$ where each $v[i]\subseteq\{0,1\}^b$ is a set of b-dimensional binary vectors.

For a set of vectors $\vec{w}_1, \dots, \vec{w}_k$ of the same dimension d, let is Orthogonal Tuple $(\vec{w}_1, \dots, \vec{w}_k)$ return 1 iff $\vec{w_1}, \dots, \vec{w_k}$ are orthogonal, i.e. iff $\sum_{a=1}^d \prod_{j=1}^k w_j[a] = 0$, where $w_j[a]$ is the a^{th} bit of the vector w_j . Now we define a useful operator, \circ for a set $\{Z_1, \dots, Z_k\}$ where each Z_i is a set of d-dimensional binary

vectors as follows.

$$\circ (Z_1, \dots, Z_k) := \sum_{\vec{w_1} \in Z_1, \dots, \vec{w_k} \in Z_k} isOrthogonalTuple(\vec{w_1}, \dots, \vec{w_k}).$$

Now, given k(g,b)-factored vectors v_1,\ldots,v_k the number of orthogonal vectors within those factored vectors is $\odot(v_1,...,v_k) := \prod_{i=0}^{g-1} \circ (v_1[i],...,v_k[i]).$

The input to Fk-OV is V_1, \ldots, V_k , where each V_i is a set of n(g, b)-factored vectors, where $g = o(\lg(n) / \lg \lg(n))$ and $b = o(\lg(n))$. The total number of orthogonal vectors in a given Fk-OV instance is

$$\sum_{v_1,\ldots,v_k\in V_1,\ldots,V_k} \circledcirc(v_1,\ldots,v_k).$$

The Fk-OV problem asks to determine whether $\sum_{v_1,\dots,v_k\in V_1,\dots,V_k} \otimes (v_1,\dots,v_k) > 0$.

An Example: We give a small example bellow. Consider F2-OV where g = 2 and b = 3. We give an example of factored vectors u, v and w:

$$u[0] = \{001,010\}$$
 $u[1] = \{001,010\}$
 $v[0] = \{000,010,110\}$ $v[1] = \{110,101\}$
 $w[0] = \{\}$ $w[1] = \{000,011,100,111\}$

First, note that $\otimes(w,u) = \otimes(w,v) = 0$ trivially because w[0] is the empty set. Empty sets are valid in this factored representation, but, rather degenerate. Next, note that $\odot(v,u)$ is $4\cdot 2=8$. For $\circ(u[0],v[0])$ all of (001,000), (001,010), (001,110), and (010,000) are orthogonal. For $\circ(u[1],v[1])$ both (001,110), and (010, 101) are orthogonal.

A Natural Interpretation: We can generate a k-OV instance by interpreting a factored vector as representing $|v_1| \cdot \dots \cdot |v_k|$ vectors. For example u in the above example would represent the following list of vectors:

As another example v would represent the following list of vectors:

000110,000101,010110,010101,110110,110101.

Finally, W represents no vectors, because w[0] is the empty set.

However, the number of vectors that can be represented by a single factored vector that has a $g2^b$ sized representation is 2^{bg} . While $g2^b$ is sub-polynomial, 2^{bg} can be super polynomial (e.g. if $b = g = \lg(n)^{3/2}$)!

2.4.2 Definitions for Fk-f, Fk-SUM, Fk-XOR, and FZT

Definition 10. Let $f: (\{0,1\}^b)^{\times k} \to \{0,1\}$ be a function taking k b-dimensional binary vectors to $\{0,1\}$. We can view f as a Boolean function.

Let us define an operator for f, \circ_f , that takes k factored vectors a_1, \ldots, a_k and computes the number of k-tuples of vectors, one in each a_i , that f accepts:

$$\circ_f(a_1, \dots, a_k) = \sum_{\vec{w_1} \in a_1 \dots \vec{w_k} \in a_k} f(\vec{w_1}, \dots, \vec{w_k}).$$

If v is a (g,b)-factored vector let, for $i \in [g]$, v[i] be the i^{th} set of vectors in v.

Given (g,b)-factored vectors v_1,\ldots,v_k the number of k-tuples of vectors accepted by f within those factored vectors is $\odot_f(v_1,\ldots,v_k)=\Pi_{i=0}^{g-1}\circ_f(v_1[i],\ldots,v_k[i])$. For each f, we define a problem Fk- \mathfrak{f} . The input to Fk- \mathfrak{f} is k sets, V_1,\ldots,V_k , of n (g,b)-factored vectors

For each f, we define a problem Fk- \mathfrak{f} . The input to Fk- \mathfrak{f} is k sets, V_1, \ldots, V_k , of n (g,b)-factored vectors each, where $g = o(\lg(n)/\lg\lg(n))$ and $b = o(\lg(n))$.

The total number k-tuples of vectors accepted by f in a given Fk-f instance is

$$\mathsf{F}k\text{-}f(V_1,\ldots,V_k) := \sum_{v_1,\ldots,v_k \in V_1,\ldots,V_k} \otimes_f (v_1,\ldots,v_k).$$

The Fk- \mathfrak{f} problem returns true iff Fk- $f(V_1,\ldots,V_k)>0$. More generally, the counting version #Fk- \mathfrak{f} of Fk- \mathfrak{f} asks to compute the quantity Fk- $f(V_1,\ldots,V_k)$.

Definition 11. Fk-XOR is the problem Fk- \mathfrak{f} where f is 1 if the componentwise XOR of the k given vectors is the 0 vector:

$$f(v_1,\ldots,v_k) = \begin{cases} 1, & \text{if } v_1 \oplus \ldots \oplus v_k = \vec{0} \\ 0, & \text{else} \end{cases}$$
.

Definition 12. Fk-SUM is the problem Fk-f where f that checks if the sum of the k vectors is the 0 vector:

$$f(v_1,\ldots,v_k) = \begin{cases} 1, & \text{if } v_1 + \ldots + v_k = 0 \\ 0, & \text{else} \end{cases}.$$

Definition 13. For an integer k, $\ell = \binom{k}{2}$ and a given function $f : \{0,1\}^{b\ell} \to \{0,1\}$, construed as taking ℓ -tuples of b-length binary vectors to $\{0,1\}$, let #FfkC be the problem of counting cliques in a graph whose edges are labeled with factored vectors, where a clique is counted with multiplicity the number of ℓ -tuples of vectors that f accepts and that appear in the ℓ factored vectors labeling the edges.

More formally, we change the definition of the operation $\odot_f(\cdot)$ to take as input k vertices v_1, \dots, v_k of a given graph G = (V, E) whose edges $(x, y) \in E$ are labeled by (g, b)-factored vectors $e_{x, y}$:

$$\otimes_f'(v_1,\ldots,v_k) = isClique(v_1,\ldots,v_k) \cdot \prod_{i=0}^{g-1} \circ_f (e_{v_1,v_2}[i],e_{v_1,v_3}[i],\ldots,e_{v_{k-1},v_k}[i]).$$

Above $isClique(v_1, ..., v_k)$ outputs 1 if $v_1, ..., v_k$ form a k-clique in G, and otherwise outputs 0.

We keep the definition of $\circ_f(\cdot)$ the same as before, but now its input is a list of ℓ sets of vectors that are the *i*th group of vectors of the factored vectors labeling the clique edges:

$$\circ_f(e_1[i],\ldots,e_\ell[i]) = \sum_{\vec{w_1} \in e_1[i] \ldots \vec{w_\ell} \in e_\ell[i]} f(\vec{w_1},\ldots,\vec{w_\ell}).$$

Finally, we let #FfkCbe the problem of computing

$$\mathrm{F} f k \mathrm{C}(G) := \sum_{\nu_1, \dots, \nu_k \in V} \otimes_f' (\nu_1, \dots, \nu_k).$$

Here, unlike for #F ℓ -f, we are only counting the sums of factored vectors when those factored vectors are on a set of $\ell = \binom{k}{2}$ edges that form a k clique. Let FfkCbe the detection version of the problem that returns 1 if FfkCG> 0 and 0 otherwise.

Definition 14. Factored Zero k-Clique, FZkC is the FfkC problem where f is the sum function for $\binom{k}{2}$ variables defined in the definition of Fk-SUM .

Definition 15. Factored Zero Triangle, FZT is FZ3C.

2.4.3 Hypotheses for Factored Problems

First we will define the hypotheses for our factored list problems.

In many lemma, theorem and definition statements we will use a structure where we put (#) before several problem or hypothesis names. This structure means that the statement is true for all non counting versions, or for all counting versions. For example, in the first line below the two implies statements are: "The Fk-OV hypothesis (i.e.Fk-OVH) states that Fk-OV requires $n^{k-o(1)}$ time." and "The Fk-OV hypothesis (i.e.Fk-OVH) states that Fk-OV requires Fk-OV requires Fk-OV requires Fk-OVH) states that Fk-OV requires Fk-OVH.

Definition 16. The (#)Fk-OV hypothesis (i.e.(#)Fk-OVH) states that (#) Fk-OV requires $n^{k-o(1)}$ time.

The (#) Fk-SUM hypothesis (i.e.(#)Fk-SUMH) states that (#) Fk-SUM requires $n^{k-o(1)}$ time.

The (#)Fk-XOR hypothesis (i.e.(#)Fk-XORH) states that (#)Fk-XOR requires $n^{k-o(1)}$ time.

The (#)Fk-f hypothesis (i.e.(#)Fk-fH) states that (#)Fk-f requires $n^{k-o(1)}$ time.

Now we will define the hypotheses for our factored clique problems.

Definition 17. The (#)FZkC hypothesis (i.e.(#)FZkCH) states that (#) FZkC requires $n^{k-o(1)}$ time. The (#)FfkC hypothesis (i.e.(#)FfkCH) states that (#) FfkC requires $n^{k-o(1)}$ time.

2.4.4 Average-Case for Factored Problems

We will separate the average-case distribution of factored problems into the normal case and a more-general parameterized case.

Definition 18. More General Average-Case Let $S_{b,\mu}$ be a distribution over sets of vectors from $\{0,1\}^b$. A set drawn from $S_{b,\mu}$ includes every vector $w \in \{0,1\}^b$ with probability μ .

Let $D_{g,b,\mu}$ be a distribution over factored vectors v where all g sets of v[i] are sampled iid from $S_{b,\mu}$. The average-case distribution for #Fk- \mathfrak{f}^{μ} samples every factored vector in its input iid from $D_{g,b,\mu}$.

The average-case distribution for $\#FfkC^{\mu}$ samples every factored vector in its input iid from $D_{e,b,u}$.

For the average-case we use in this paper we use $\mu = 1/2$. We feel this is the most natural distribution for our problem. We will occasionally call this the "uniform average-case" to emphasize that every set v[i] in every factored vector is chosen uniformly at random from all possible subsets of $\{0,1\}^b$.

Definition 19. The average-case distribution for #Fk-f samples every factored vector in its input iid from $D_{g,b,1/2}$.

The average-case distribution for #FfkC samples every factored vector in its input iid from $D_{g,b,1/2}$.

2.5 Problems harder than factored problems

Here we define problems that later are shown to be hard via reductions from the factored problems. We state the known results for each, and a simple algorithm for each is given in Appendix A that matches the lower bound we prove later.

Definition 20. The **Partitioned Matching Triangles (PMT)** problem takes as input $g = O(\log n / \log \log n)$ disjoint *n*-node graphs with node colors, and asks if there is a triple of colors with a triangle of that color triple in each of the g graphs. The counting version of the problem, #PMT, asks for the number of such g-tuples of colored triangle.

This problem is very similar to the Δ *Matching Triangles* problem defined in [AVY18], where given an n-node graph G with node colors, the problems asks if there is a triple color with Δ triangles of that color triple in G.

In [AVY18], 3SUM, APSP and SETH are reduced to Δ Matching Triangles where the instances produced can be represented as instances of Partitioned Matching Triangles instance for $g = \Delta$. So Partitioned Matching Triangles is hard from 3SUM, APSP and SETH. A related problem to PMT is the *node disjoint triangle packing* problem which asks to find a maximum size node-disjoint triangle packing in a given graph (see for example [CR02]). PMT is a natural mix of the Δ -matching-triangle and the node disjoint triangle packing problems.

Definition 21. Node Labeled k-Color st Connectivity (k-NLstC) takes as input a directed graph G with edge set E and vertex set V, two special nodes s and t, and a proper coloring of the vertices $c: V \setminus \{s,t\} \to C$, where C is a set of colors, so that the endpoints of every edge have different colors and s and t have all their neighbors colored distinctly. The input graph G is a layered graph, the vertex set V is partitioned into V_0, V_1, \ldots, V_ℓ , such that every directed edge goes from a node in set V_i to a node in set V_{i+1} for some $i \in \{0, \ldots, \ell-1\}$. The k-NLstC problem asks if there is a path from s to t that uses only k colors of nodes (where s and t are not counted for colors).

We also consider the problem of Counting k-NLstC mod R, in which we ask for the total number of paths from s to t that use at most k colors of nodes. We will generally use values of R such that $\lg(R)$ is subpolynomial, as this allows us to represent the count with a subpolynomial number of bits.

Definition 22. The k **Edge Labeled (directed/undirected)** st **Connectivity** (k-**ELstC**) problem takes as input a directed acyclic graph G = (V, E), two special vertices s and t and a coloring of the edges $c : E \to C$, where C is a set of colors.

k-ELstC asks, given this input can you pick k colors such that there is a path from s to t using only edges that are colored by one of those k colors? The counting version of k-ELstC, #k-ELstC asks for the number of paths from s to t mod R that use only k colors, where $\lg(R) = n^{o(1)}$.

Definition 23. The **Bounded Labeled Maximum Flow (BL-MF)** problem [GCSR13] takes as input a directed, capacitated, and edge-labeled graph G = (V, E) with a source node $s \in V$, a sink node $t \in V$, and a positive integer k, and asks if there is a maximum flow k from k to k in k such that the total number of different labels corresponding to arcs k we refer to the problem as k-MF.

BL-MF is the decision version of the maximum flow with the *minimum number of labels (MF-ML)* problem where we seek a maximum flow from s to t that uses the minimum number of labels. [GCSR13] uses this problem to model the purification of water during the distribution process. They show that BL-MF is NP-complete. Let $BL-MF^*$ be a slightly more restricted version of BL-MF where the number of edges of each label is o(n) and the edges attached to the sink and source have a special label l^* . We show a lower bound of $O(n^{k-1})$ for $kL-MF^*$ (and thus kL-MF) for fixed k, and show that it has a matching algorithm as well.

Definition 24. The **Regular Expression Matching** problem [BI16] takes as input a regular expression (pattern) p of size m and a sequence of symbols (text) t of length n, and asks if there is a substring of t that can be derived from p. The counting version of the problem, #Regular Expression Matching asks for the number of subset alignments of the pattern of the pattern in the text mod an integer R, where $R = n^{o(1)}$.

A classic algorithm constructs and simulates a non-deterministic finite automaton corresponding to the expression, resulting in the rectangular O(mn) running time.

Definition 25. The (counting) k-Longest Common Subsequence ((#)k-LCS) problem (see for example [IF92]) takes as input k sequences P_1, \ldots, P_k of length n over an alphabet Σ . Let ℓ be the length of the longest sequence X such that X appears in all of P_1, \ldots, P_k (in the same order).

k-LCS asks for the value of ℓ , while #*k*-LCS asks to compute ℓ and also the total *number* of common subsequences of length ℓ .

More formally, define $C_\#(X_i)$ to be the total number k-tuples of ℓ sequence locations in each of our k strings such that those locations map onto the sequence X_i for all k strings when X_i is of length ℓ . Let $X_1, X_2, \ldots X_j$ be all possible sequences of length ℓ that appear in all of P_1, \ldots, P_k . For the #k-LCS problem we ask for the value of ℓ and the value of $\Pi_{i-1}^j C_\#(X_i)$.

Definition 26. The **Edit Distance** problem (see for example [BI15]) takes as input two sequences x and y over an alphabet Σ , and asks to output the edit distance EDIT(x,y) which is equal to the minimum number of symbol insertions, symbol deletions or symbol substitutions needed to transform x into y.

3 Factored Problems are Hard

In this section we will first show the simple result that Fk-OV, Fk-SUM, Fk-XOR, and FZT are all at least as hard as their non-factored variants. Second, we will show a worst-case to average-case reduction from Fk-OV to itself. We will also show the corresponding worst-case to average-case reductions for Fk-SUM, Fk-XOR, and FZT. Third, we will show many worst-case reductions between these factored problems. Notably, Fk-OV, Fk-SUM, and Fk-XOR are all equivalent up to sub-polynomial factors. Additionally, FZT is $n^{3-o(1)}$ hard from F3-OV (and thus equivalently hard from F3-SUM, and F3-XOR). Notably this means that the Ff3CH is implied by SETH, the 3-SUM hypothesis, and the APSP hypothesis. Figure 3 summarized the reductions of this section.

Remember that algorithms for these problems are given in Appendix A. We give $O(n^{k+o(1)})$ algorithms for Fk-OV, Fk-SUM, FZkC and Fk-XOR.

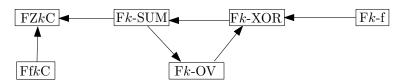


Figure 3: Map of the reductions

3.1 Factored Versions are Harder

Consider any problem where we have k sets of vectors V_1, \ldots, V_k and we want to compute the number of k-tuples of vectors $v_1 \in V_1, \ldots, v_k \in V_k$ of length l = bg, such that $\hat{f}(v_1, \ldots, v_k) = 1$, for some function $\hat{f}: \{0,1\}^{l \times k} \to \{0,1\}$. Call this problem k- \hat{f} . Note that k-SUM, kXOR, and kOV are examples of such problems. We show that these problems can be solved using their factored version.

For any vector v of length bg and for any $j=1,\ldots,g$, let v^j be the subvector of v that starts at the (j-1)b+1th bit and ends at the jbth bit. Suppose that there is a function $f:\{0,1\}^{b\times k}\to\{0,1\}$ such that $\hat{f}(v_1,\ldots,v_k)=1$ if and only if $\Pi_{j=1}^g f(v_1^j,\ldots,v_k^j)=1$. We call f the factored version of \hat{f} . In other words, the function \hat{f} can be applied more locally, on subvectors of length g. Note that for most problems including the problems we work with, this property holds.

Now we can easily reduce $k - \hat{f}$ to $Fk - \hat{f}$. Let the resulting $Fk - \hat{f}$ instance be the following: For any bg length vector v in the $k - \hat{f}$ instance, let the factored version of v have sets $v[j] = \{v^j\}$. By the property mentioned, it is straightforward to see that this instance of $Fk - \hat{f}$ is equivalent to $k - \hat{f}$.

For the k-SUM problem it is less obvious how to solve it with Fk-SUM. For the k-SUM problem we can use the nearly linear hash functions to reduce all numbers to the range $[-n^k, n^k]$ [Pat10]. Additionally, we can reduce k-SUM in the range $[-n^k, n^k]$ to a version where every number is instead a vector with g numbers with g bits each [ALW14], where $g \cdot b = k \lg(n)$. We consider a sum of g vectors to be a zero sum if the vectors sum to the zero vector. To ask if g numbers of length g and the numbers are each g bits. But, we need to guess the g-1 carries, this is a total of g0 guesses. If g1 guesses. If g3 guesses are each g4 bits is sub-polynomial, and so we can go through all these guesses. This vectorized version of the g5 guesses are each g6 guesses are each g7 guesses. This vectorized version of the g8 guesses are each g9 guesses.

Similar to the approach for solving $k-\hat{f}$ problems using $Fk-\hat{f}$, we can reduce ZkC to FZkC. Here the function \hat{f} (which is the sum function) gets $\binom{k}{2}$ vectors as input instead of k vectors, and these vectors should have the property that they form the edges of a k-clique in the graph. If f is the factored version of \hat{f} , then we have that $\hat{f}(e_1,\ldots,e_{\binom{k}{2}})=1$ and $e_1,\ldots,e_{\binom{k}{2}}$ are the edges of a k-clique if and only if $\Pi_{j=1}^g f(e_1^j,\ldots,e_{\binom{k}{2}}^j)=1$ and $e_1,\ldots e_{\binom{k}{2}}$ are edges of a k-clique. So again, the FZkC instance that is equivalent to the ZkC instance is that for each vector e of an edge, we let $e[j]=\{e^j\}$. So we have the following theorem. Note that our reductions form a one-to-one correspondence between each solution in a $k-\hat{f}$ instance and the corresponding $Fk-\hat{f}$ instance, and hence they work for the counging version of our problems as well.

REMINDER OF THEOREM 1.1 In O(n) time, one can reduce an instance of size n of k-OV, k-XOR, k-SUM and ZkC to a single call to an instance of size $\tilde{O}(n)$ of Fk-OV, Fk-XOR, Fk-SUM and FZkC respectively.

Proof. We can split a number or vector in the original problem into a vector with g numbers of length b. This reduction step is trivial for k-OV, ZkC, and k-XOR. To be explicit:

• k-OV: Let $d = o(\lg^2(n))$ be the dimension. Given k lists of n vectors L_1, \ldots, L_k we will produce k lists of factored vectors L'_1, \ldots, L'_k . Let v[x : y] be a vector formed by taking all the bits from x^{th} bit to the

 y^{th} bit. For every vector $v_i \in L_i$ take the vector $v_i[bj+1,b(j+1)] = v_i^j$ where $b = \sqrt{d}$ and $j \in [1,g]$ where $g = \sqrt{d}$. We create a factored vector v_i' from v_i by creating a vector where the j^{th} subset of $\{0,1\}^b$ is just a set with the single vector v_i^j .

- k-XOR: There is a random reduction for k-XOR which shrinks the vectors to length $d = k \lg(n)$ bits. Given two lists of n vectors L_1 and L_2 we will produce two lists of factored vectors L'_1 and L'_2 . Let v[x:y] be a vector formed by taking all the bits from x^{th} bit to the y^{th} bit. For every vector $v_i \in L_i$ take the vector $v_i[bj+1,b(j+1)] = v_i^j$ where $b = \sqrt{d}$ and $j \in [1,g]$ where $g = \sqrt{d}$. We create a factored vector v'_i from v_i by creating a vector where the j^{th} subset of $\{0,1\}^b$ is just a set with the single vector v_i^j .
- ZkC and k-SUM: We reduce the range of numbers with linear hash functions to the range $[-n^k, n^k]$. We want to split each $k\lg(n)$ bit number into $g = \sqrt{k\lg(n)}$ numbers length $b = \sqrt{k\lg(n)}$ bits. If we guess all g caries then we can replace the question of if $k(\binom{k}{2})$ numbers sum to zero to if g sets of $k(\binom{k}{2})$ numbers each sum to zero (see [ALW14]). So, for all $O(k^g)$ possible guesses of carries we form a factored vector for an edge by having g subsets of $\{0,1\}^b$ that each have one number. The j^{th} set has the j^{th} number created by splitting the original number (possibly updated by our guess of the carry).

3.2 Worst-Case to Average-Case Reductions for Factored Problems

We will use our framework from Section B to show that these factored versions are as hard on average as they are in the worst case.

#Fk-f We give a polynomial for #Fk-f. We represent every factored vector v with $g2^b$ variables. The variable $x_{v[i]}(\vec{s})$ is a 1 if $s \in v[i]$ and 0 otherwise. We create such a variable for all $i \in [0, g-1]$ and all $s \in \{0, 1\}^b$. Let S_f be the subset of k tuples of vectors in $\{0, 1\}^b$ such that $f(s_1, \ldots, s_k) = 1$.

$$f_{ckfunc}(ec{X}) = \sum_{v_1 \in V_1, ..., v_k \in V_k} \left(\prod_{i \in [0,g-1]} \left(\sum_{(s_1, ..., s_k) \in S_f} x_{v_1[i]}(ec{s_1}) \cdots x_{v_k[i]}(ec{s_k})
ight)
ight).$$

Lemma 3.1. $f_{ckfunc}(\vec{X})$ is a GLDP(#Fk- \mathfrak{f}) (see Definition 8)

Proof. We will show that each property of a good polynomial is met by f_{ckfunc} .

- If $\vec{I} = b_1, \dots, b_n$, then $f_{ckfunc}(b_1, \dots, b_n) = f_{ckfunc}(\vec{I}) = P(\vec{I})$ where b_i maps to either a zero or a one in the prime finite field: f_{ckfunc} and #Fk-f count the same thing. Note that the inner summation is computing \circ_f , the product is computing \circ_f . Thus the overall sum is computing #Fk-f.
- The function f_{ckfunc} has degree $d = o(\lg(n)/\lg\lg(n))$: f_{ckfunc} has degree kg which, when k is constant is $o(\lg(n)/\lg\lg(n))$ by the definition of g.
- The function f_{ckfunc} is strongly d-partite: Every monomial is formed by exactly one copy of a $x_{v_j[i]}(\vec{s})$ variable for every $j \in [0, g-1]$ and $i \in [1, k]$. These form our partitions and make the function strongly kg partite.

Now we can say that the average case version of #Fk-f is as hard as the worst case version.

REMINDER OF THEOREM 1.3 Let μ be a constant such that $0 < \mu < 1$. If average-case #Fk- \mathfrak{f}^{μ} (see Definition 18) can be solved in time T(n) with probability $1 - 1/(lg(n)^{kg} \lg \lg(n)^{kg})$ then worst-case #Fk- \mathfrak{f} can be solved in time $\tilde{O}(T(n))^5$.

When $\mu = 1/2$ average-case #Fk- \mathfrak{f}^{μ} is average-case #Fk- \mathfrak{f} .

Proof. This follows from Theorem 1.20 and Lemma 3.1. The dimension of the GLDP(#Fk-f) is kg. By our construction of f_{ckfunc} every set has every possible string as a variable. By the construction of the framework from theorem 1.20, every bit will be selected as a 1 uniformly at random with probability μ . So, given the construction of f_{ckfunc} every set will have every possible string included with probability μ . So the distribution induced by our framework matches our defined average-case distribution.

Finally by definition 19 when $\mu = 1/2$ average-case #Fk-f^{μ} is average-case #Fk-f.

REMINDER OF COROLLARY 1.4 By Theorem 1.3, we have the following result:

If average-case #Fk-OV can be solved in time T(n) with probability $1-1/(lg(n)^{gk} \lg \lg(n)^{gk})$ then worst-case #Fk-OV can be solved in time $\tilde{O}(T(n))^5$.

If average-case # Fk-SUM can be solved in time T(n) with probability $1 - 1/(lg(n)^{gk} \lg \lg(n)^{gk})$ then worst-case # Fk-SUM can be solved in time $\tilde{O}(T(n))^5$.

If average-case # Fk-XOR can be solved in time T(n) with probability $1 - 1/(lg(n)^{gk} \lg \lg(n)^{gk})$ then worst-case # Fk-XOR can be solved in time $\tilde{O}(T(n))^5$.

#FfkC Now we will give the #FfkC polynomial. Once again we will represent every factored vector v with $g2^b$ variables. The variable $x_{v[i]}(\vec{s})$ is a 1 if $s \in v[i]$ and 0 otherwise. We create such a variable for all $i \in [0,g-1]$ and all $s \in \{0,1\}^b$. Once again, let S_f be the subset of $\binom{k}{2}$ tuples of vectors in $\{0,1\}^b$ such that $f(s_1,\ldots,s_{\binom{k}{2}})=1$. Finally for convenience let $E_1,\ldots,E_{\binom{k}{2}}$ be the $\binom{k}{2}$ partitions of edges in the input of FfkC and let $\ell=\binom{k}{2}$ to make notation easier to read. Let S_E be the set of all ℓ tuples of edges e_1,\ldots,e_ℓ that form a clique. In an abuse of notation we will also use e_i to represent the factored vector associated with the edge e_i .

$$f_{ffkc}(ec{X}) = \sum_{e_1, \dots, e_\ell \in S_E} \left(\prod_{i \in [0,g-1]} \left(\sum_{(s_1, \dots, s_\ell) \in S_f} x_{e_1[i]}(ec{s_1}) \cdots x_{e_\ell[i]}(ec{s_\ell}) \right) \right).$$

Lemma 3.2. $f_{ffkc}(\vec{X})$ is a GLDP(#F $\hat{y}kC$) (see Definition 8).

Proof. We will show that each property of a good polynomial is met by f_{ffkc} .

- If $\vec{I} = b_1, \dots, b_n$, then $f_{ffk}(b_1, \dots, b_n) = f_{ffkc}(\vec{I}) = P(\vec{I})$ where b_i maps to either a zero or a one in the prime finite field: f_{ffkc} and #Fk-f count the same thing. Note that the inner summation is computing \circ_f , the product is computing \circ_f . Thus the overall sum is computing #Fk-f.
- The function f_{ffkc} has degree $d = o(\lg(n)/\lg\lg(n))$: f_{ffkc} has degree $\ell g < k^2 g$ which, when k is constant is $o(\lg(n)/\lg\lg(n))$ by the definition of g.

⁵Note that given that $g = o(\lg(n)/\lg\lg(n))$ then a probability of $1 - 1/n^{\varepsilon}$ will be high enough for any $\varepsilon > 0$.

• The function f_{ffkc} is strongly d-partite: Every monomial is formed by exactly one copy of a $x_{e_j[i]}(\vec{s})$ variable for every $j \in [0, g-1]$ and $i \in [1, \ell]$. These form our partitions and make the function strongly ℓg partite.

REMINDER OF THEOREM 1.5 Let μ be a constant and $0 < \mu < 1$. If average-case #F\(\frac{t}{k}C^{\mu}\) (see Definition 18) can be solved in time T(n) with probability $1 - 1/(lg(n)^{k^2g} \lg\lg(n)^{k^2g})$ then worst-case #F\(\frac{t}{k}C\) can be solved in time $\tilde{O}(T(n))^5$.

When $\mu = 1/2$ average-case #FfkC^{μ} is average-case #FfkC.

Proof. This follows from Theorem 1.20 and Lemma 3.2. The dimension of the GLDP(#FfkC) is $\binom{k}{2}g < k^2g$. By our construction of f_{ffkc} every set has every possible string as a variable. By the construction of the framework from theorem 1.20, every bit will be selected as a 1 uniformly at random with probability μ . So, given the construction of f_{ffkc} every set will have every possible string included with probability μ . So the distribution induced by our framework matches our defined average-case distribution.

Finally by definition 19 when $\mu = 1/2$ average-case #FfkC^{mu} is average-case #FfkC.

REMINDER OF COROLLARY 1.6 By Theorem 1.5, we have the following result:

If average-case # FZkC can be solved in time T(n) with probability $1 - 1/(lg(n)^{k^2g} \lg \lg(n)^{k^2g})$ then worst-case #FZkC can be solved in time $\tilde{O}(T(n))^5$.

Reductions to Counting Factored Problems Imply Average Case Hardness Over Some Distribution Assume a problem #P exists such that an algorithm for it running in $T(n)^{1-\varepsilon}$ implies a violation of #Fk-#F or #F or #F or #F or #F or #F or #F into instances of #P. In that case we can describe a distribution D over which problem #P is $T(n)^{1-o(1)}$ hard on average from #Fk-#F or #F o

Thus, reductions from problems #P to #Fk-f or #FfkC give explicit hard average-case distributions for problems #P.

3.3 Factoring is Expressive: Worst-Case Reductions

Our factored versions of these problems are very expressive. This allows us to show hardness from these factored problems.

3.3.1 Completeness

We will now show that Fk-OV, Fk-XOR, and Fk-SUM are all complete for Fk- \mathfrak{f} for all functions \mathfrak{f} . We do this by showing Fk-XOR solves Fk- \mathfrak{f} . Then, the equivalence between Fk-OV, Fk-XOR, and Fk-SUM implies they are all complete for Fk- \mathfrak{f} . We will also show, using similar techniques, that FZkC is complete for $F\mathfrak{f}k$ C for all functions \mathfrak{f} .

This is a reminder of Theorem 1.7, however, we add an additional statement to the theorem. We give an explicit function that we use to build this reduction.

REMINDER OF THEOREM 1.7 If we can solve #Fk-XOR with g sets of k^3b length vectors in time T(n) then we can solve #Fk- \mathfrak{f} instance with g sets of b length vectors in time $T(n) + \tilde{O}(n)$.

Additionally, Let v_1, \ldots, v_k be k factored vectors each with g subsets of $\{0,1\}^b$. Let f_{XOR} be the function that returns 1 if the k input vectors xor to zero and otherwise returns 0. There is a function $\gamma_{f \to XOR,k}(\cdot,\cdot)$ that takes as input a factored vector with g subsets of $\{0,1\}^b$ and an index and returns a new factored vector with g subsets of $\{0,1\}^{O(b)}$. This function $\gamma_{f \to XOR,k}$ runs in $\tilde{O}(2^b \cdot g)$ time for each vector and:

$$\odot_{f_{XOR}}(\gamma_{\mathfrak{f}\to XOR,k}(\nu_1,1),\ldots,\gamma_{\mathfrak{f}\to XOR,k}(\nu_k,k)) = \odot_{\mathfrak{f}}(\nu_1,\ldots,\nu_k).$$

Proof. Consider a Fk-f instance and let v_i be a factored vector from the i^{th} list of it. Given the factored vector v_i from Fk-f we will describe how to make the factored vector v_i' for our Fk-XOR instance. This transformation will be $\gamma_{j\to XOR,k}(\cdot,\cdot)$. We will describe the transformation for $\gamma_{j\to XOR,k}(v_i,i)$. We do this by doing the same transformation on each set $v_i[j]$ where $j \in [1,g]$. We transform each set by performing the same transformation on every vector $u_i \in v_i[j]$. We describe this transformation in the next paragraph.

Given a vector u_i of length b we produce at most $2^{b(k-1)}$ new vectors of length k^3b . These vectors represent all possible k tuples which include u_i as as the i^{th} vector. We want to include a k tuple vector only if f of that k tuple evaluates to 1. And we want our new long vectors to to return true if we are comparing vectors in Fk-f instance that do indeed have exactly that k tuple of vectors.

More formally, let one possible k tuple that includes u_i as the i^{th} vector be $(w_1, \ldots, w_{i-1}, u_i, w_{i+1}, \ldots, w_k)$. If $f(w_1, \ldots, w_{i-1}, u_i, w_{i+1}, \ldots, w_k) = 1$, then we create a k^3b -length vector for u_i with this kb length vector by considering every possible tuple (x, y, z) where $x, y, z \in [1, k]$: We set aside b bits for every possible tuple (in sorted order by the tuple). We want to use these to check if the x^{th} vector and y^{th} vector agree about what tuple they are considering as follows:

- If the tuple is (x, x, z) we write the all zeros string, for the rest of the cases assume the first two indices are non-equal.
- If the tuple is (x, i, z) we write w_z (or u_i if z = i) in the b bits.
- If the tuple is (i, y, z) we write w_z (or u_i if z = i) in the b bits.
- If the tuple is (x, y, z) and $x, y \neq i$ then we write the all zeros vector of length b.

If we are comparing k of these new vectors each of which representing the same tuple (w_1, \ldots, w_k) then the new vector xors to zero. Consider a given group of b bits that corresponds to (x, y, z). Only two of our vectors have non-zero entries here, the x^{th} and y^{th} vectors. Both wrote down w_z if they were representing the same k-tuple. A vector xored to itself produces the zero vector, so we get the zero vector.

If we are comparing k of these new vectors and not all of the vectors agree about what tuples they are comparing then we will not xor to the zero vector. Say the x^{th} and y^{th} vectors disagree about what the z^{th} element of the tuple is. Then the b bits corresponding to (x,y,z) will still have only two vectors with non-zero contributions. We will be xoring two vectors which are not equal, this will xor to some non-zero string.

Thus there is a one-to-one correspondence between k-tuples of vectors that evaluate to one in the Fk- \mathfrak{f} version and k-tuples of vectors that xor to the zero vector in the new Fk-XOR version. Thus, the counts both give as output are equal.

As a result, we can transform an instance of Fk- \mathfrak{f} with g groups of b length vectors into an instance of Fk-XOR with g groups of k^3b length vectors in time $O(n \cdot g \cdot 2^b \cdot 2^{(k-1)b} \cdot k^3b)$. We restrict k to be constant and $b = o(\lg(n))$, thus the time for the conversion is $\tilde{O}(n)$. In the new version the count of the number of

Fk-XOR vectors that xor to zero is the same as the count of the number of Fk- \mathfrak{f} vectors that evaluate to 1 on the function. So, a T(n) algorithm for Fk-XOR with g groups and k^3b bits implies a $T(n) + \tilde{O}(n)$ algorithm for Fk- \mathfrak{f} .

3.3.2 fk-OV, fk-SUM, and Fk-XOR are Equivalent and Complete

Intuitively, we can use our subsets of $\{0,1\}^b$ to do guesses that reduce from one problem to another.

Lemma 3.3. If (#)Fk-OV can be solved in time T(n) then (#)Fk-XOR can be solved in $\tilde{O}(T(n))$ time.

Proof. Say we are given an Fk-XOR instance, with k lists of factored vectors, each with g subsets of g-bit vectors. We will follow the structure of Theorem 1.7. Say we are given a factored vector from list g, v_i . Consider the g-bit subset g-bi

- If the tuple is (x,x,z) we write the 2b bit all zeros string. For the rest of these assume the first two indices are not equal.
- If the tuple is (i, x, z) then write $w_z \parallel \bar{w_z}$ (for convenience let $w_i = u_i$).
- If the tuple is (x, i, z) then write $\overline{w}_z \parallel w_z$ (for convenience let $w_i = u_i$).
- If the tuple is (x, y, z) and $x \neq i$ and $y \neq i$ then we put the all ones string.

Now k of these constructed vectors will be orthogonal only if $w_1 \oplus ... \oplus w_k = \vec{0}$, all the vectors w_i existed in the original lists, and the constructed vectors all agree on the tuple $(w_1, ..., w_k)$.

So, with our constructed vectors the count of the number of vectors that are orthogonal will remain the same. The new instance will have the same number of factored vectors, n, but the vectors will have g subsets of $\{0,1\}^{2k^3b}$. An algorithm which runs in T(n) on this Fk-OV instance will run in T(n) on the Fk-XOR instance.

Next we reduce (#)Fk-XOR to (#) Fk-SUM . In Fk-XOR we want to know if k vectors xor to zero, which is very similar to asking if k numbers sum to zero. The difference is entirely carries. So, we can pad the instance, and then guess carries.

We will use the Fk-SUM variant where we ask if k-1 numbers sum to equal exactly the last number.

Lemma 3.4. If (#) Fk-SUM can be solved in time T(n) then (#)Fk-XOR can be solved in $\tilde{O}(T(n))$ time. Additionally, Let v_1, \ldots, v_k be k factored vectors each with g subsets of $\{0,1\}^b$. Let f_{XOR} be the function that returns 1 if the k input vectors xor to zero and otherwise returns 0. Let f_{SUM} be the function that returns 1 if the k input vectors sum to zero and otherwise returns 0. There is a function $\gamma_{XOR \to SUM,k}(\cdot,\cdot)$ that takes

as input a factored vector with g subsets of $\{0,1\}^b$ and an index and returns a new factored vector with g subsets of $\{0,1\}^{O(b)}$. This function $\gamma_{XOR \to SUM,k}$ runs in $\tilde{O}(2^b \cdot g)$ time for each vector and:

$$\odot_{f_{SUM}}(\gamma_{XOR \to SUM,k}(\nu_1,1),\ldots,\gamma_{XOR \to SUM,k}(\nu_k,k)) = \odot_{\mathfrak{f}}(\nu_1,\ldots,\nu_k).$$

Proof. We will describe the transformation $\gamma_{XOR \to SUM,k}$ below. Let v_1, \ldots, v_k be k factored vectors from a Fk-XOR instance. Let $v_1[j], \ldots, v_k[j]$ be the j^{th} subset of b-bit vectors from each of the factored vectors. Let $v_i[j][h][\ell]$ be the ℓ^{th} bit of the h^{th} vector in the set $v_i[j]$.

We will turn every bit from $v_i[j][h][\ell]$ into $\lceil \lg(k) \rceil + 1$ bits in a new number. If i < k then this new longer string is $\lceil \lg(k) \rceil$ zeros followed by the bit $v_i[j][h][\ell]$. If i = k, then every vector $v_k[j][h]$ turns into many vectors in the k^{th} set of the Fk-SUM instance: If $v_k[j][h][\ell] = 0$ then we use our $\lceil \lg(k) \rceil + 1$ bits to represent all of the **even** numbers in [0, k-1]. If $v_k[j][h][\ell] = 1$ then we use our $\lceil \lg(k) \rceil + 1$ bits to represent all of the **odd** numbers in [0, k-1]. So we produce $O(k^b)$ vectors for $v_k[j][h]$.

If a k-tuple of vectors forms a zero vector in k-xor then we get exactly one k-sum. The number of sets stays the same but the length of vectors in those sets grows from b to $(\lceil \lg(k) \rceil + 1)b$ length vectors. This is a constant and so if Fk-SUM can be solved in time T(n) then Fk-XOR can be solved in $\tilde{O}(T(n))$ time. \square

REMINDER OF THEOREM 1.8 If any of #Fk-OV, # Fk-SUM, or #Fk-XOR can be solved in T(n) time then all of #Fk-OV, # Fk-SUM, and #Fk-XOR can be solved in $\tilde{O}(T(n))$ time.

Proof. This follows from Lemmas 3.3, 3.4, and Theorem 1.7.

REMINDER OF THEOREM 1.9 If any of #Fk-OV, # Fk-SUM, or #Fk-XOR can be solved in T(n) time then #Fk-f can be solved in $\tilde{O}(T(n))$ time.

Proof. Use Theorem 1.7 and Theorem 1.8.

By Theorem 1.8 and Theorem 1.9 we get the following corollary.

Corollary 3.5. #Fk-OVH, #Fk-SUMH, and #Fk-XORH are all equivalent. Moreover, #Fk-OV is implied by #Fk-f for any function f.

3.3.3 Factored zero-k-clique is hard from fk-OV, fk-SUM, fk-XOR, and FfkC

Lemma 3.6. If (#)FZkC is solved in time T(n) then (#) Fk-SUM is solved in time $\tilde{O}(T(n))$.

Proof. Consider a Fk-SUM instance with lists L_1, \ldots, L_k . We will build the k-partite graph of our FZkC instance to have vertex sets V_1, \ldots, V_k . For every factored number $x_i \in L_i$ from the Fk-SUM instance we create a node $v_i \in V_i$ where all edges going from v_i to any vertex in V_{i+1} have the value x_i on them. All edges going from v_i to nodes in V_j where $j \neq i-1$ and $j \neq i+1$ are given the special factored vector where every set contains only the all zeros string.

Now, when three nodes are selected v_1, v_2, \dots, v_k the corresponding edges have a zero sum iff the corresponding x_1, x_2, \dots, x_k are a zero sum.

Theorem 3.7. If (#)FZkC can be solved in T(n) time then all of (#)Fk-OV, (#) Fk-SUM, and (#)Fk-XOR can be solved in $\tilde{O}(T(n))$ time.

Proof. This follows from Lemma 3.6 and Theorem 1.8.

Now we will show that FZkC is complete for the set of all problems FfkC for all functions f.

REMINDER OF THEOREM 1.10 If (#)FZkC can be solved in T(n) time then (#)F\(\frac{1}{3}kC\) for any f, can be solved in $\tilde{O}(T(n) + n^2)$ time.

Proof. We will use the transforms $\gamma_{\hat{\mathfrak{f}}\to XOR,\binom{k}{2}}(\cdot,\cdot)$ and $\gamma_{XOR\to SUM,\binom{k}{2}}(\cdot,\cdot)$ from Theorem 1.7 and Lemma 3.4 respectively.

Let G be the k-partite graph we take as input from (#)FfkC, now label the $\ell = \binom{k}{2}$ edge sets as E_1, \ldots, E_ℓ . Now, for every factored vector $e_i \in E_i$ run the following transform: $\hat{\gamma}_\ell(e_i, i) = \gamma_{XOR \to SUM, \binom{k}{2}}(\gamma_{f \to XOR, \ell}(e_i, i), i)$. This causes the output factored vector to have g subsets of $\{0,1\}^{O(b)}$. The transformation takes $\tilde{O}(2^{O(b)}g)$ time per vector, which is $\tilde{O}(1)$ time per vector. The output vectors have the property that

$$igotimes_{f_{SUM}}(\hat{\gamma}_{\ell}(e_1,1),\ldots,\hat{\gamma}_{\ell}(e_{\ell},\ell)) = igotimes_{\mathfrak{f}}(e_1,\ldots,e_{\ell}).$$

Because \odot_f' (the function used in our factored clique problem definition) is equal to $\odot_f \cdot isClique(e_1, \dots, e_\ell)$, by running this transformation we will have that:

$$\odot'_{f_{SUM}}(\widehat{\gamma}_{\ell}(e_1,1),\ldots,\widehat{\gamma}_{\ell}(e_{\ell},\ell))=\odot'_{\mathfrak{f}}(e_1,\ldots,e_{\ell}).$$

Thus, we can run the transformation $\hat{\gamma}_{\ell}$ in time $\tilde{O}(n^2)$ (because n^2 is the input size). Additionally, the output of the counting or detection variants of the FZkC on the transformed input will be exactly equal to the output of FfkC on the original input. Thus, if we can solve (#)FZkC in time T(n) we can solve (#) FfkC in time $\tilde{O}(T(n)+n^2)$.

3.3.4 Factored Zero Triangle is hard from SETH, 3-SUM and APSP

Lemma 3.8. The FZ3CH is implied by any one of SETH, the 3-SUM hypothesis, or the APSP hypothesis.

Proof. A violation of FZ3CH implies a violation of F3-OVH by Lemma 3.7. A violation of F3-OVH implies a violation of SETH [Wil07].

A violation of FZ3CH implies that a $O(n^{1-\varepsilon})$ time algorithm exists for the zero triangle problem for some $\varepsilon > 0$. A $O(n^{1-\varepsilon})$ time algorithm for zero triangle implies a violation of the 3-SUM hypothesis and the APSP hypothesis [VW10b].

So if any one of the three core hypotheses of fine-grained complexity (SETH, 3-SUM, and APSP) are true then FZ3CH is true. \Box

REMINDER OF THEOREM 1.2 If FZ3C (even for $b = o(\log n)$ and $g = o(\log(n)/\log\log(n))$) can be solved in $O(n^{3-\varepsilon})$ time for some constant $\varepsilon > 0$, then SETH is false, and there exists a constant $\varepsilon' > 0$ such that 3-SUM can be solved in $O(n^{2-\varepsilon'})$ time and APSP can be solved in $O(n^{3-\varepsilon'})$ time.

Proof. This follows from Lemma 3.8. A $O(n^{3-\varepsilon})$ time algorithm implies a violation of FZ3CH. If FZ3CH is false then all of SETH, the APSP hypothesis, the 3-SUM hypothesis are false.

4 Implications from Factored Variants

In this section we will show that a series of problems reduce from both counting and decision versions of FZ3C, FZkC, and Fk-f.

The reductions from the counting variant of FZ3C generate counting problems that are hard in the average-case from SETH, 3-SUM, and APSP. The reductions from the counting variant of FZkC or Fk-f generate counting problems that are hard in the average-case from SETH. As a result, in this section we produce a suite of problems that are fine-grained hard from the most popular hypotheses of fine-grained complexity.

In this section we give explicit tight fine-grained reductions from factored problems to many other problems. We will quickly summarize the results of this section.

We give four tight fine-grained reductions from counting versions of our factored problems. We reduce #PMT from #FZ3C, #k-NLstC from #FZkC and #k-ELstC from #Fk-f. Finally, we reduce counting regular expression matching to #F2-OV.

We also give three tight fine-grained reductions that only work from the detection versions of our factored problems. We reduce (k + 1)L-MF to FZkC. We reduce Edit Distance to F2-OV and k-LCS from Fk-OV.

4.1 The Partitioned Matching Triangles Problem solves the Factored Zero Triangle Problem

REMINDER OF THEOREM 1.11 If (#)Partitioned Matching Triangles (PMT) can be solved in T(n) time, then we can solve (#) FZ3C in time $\tilde{O}(T(n) + n^2)$.

Proof. Let G = (U, V, W) be an instance of FZ3C, where each edge e is a factored vector. For notation convenience let uv[j] refer to the j^{th} set of the factored vector on the edge from u to v.

We define an instance of PMT as a set of g graphs G_j for j = 1, ..., g. We define G_j as follows. For every $u \in U$ add a copy of u in G_j with color u. For every $v \in V$ and $w \in W$, add vertices v_x and w_x for all $x = -2^{b+1}, ..., 2^{b+1}$, with color v and w respectively. Note that since $b = o(\log n)$, G_j has $\tilde{O}(n)$ nodes.

Now we attach $u \in G_j$ to $v_x \in G_j$ if $x \in uv[j]$. We connect v_x to w_y if $y - x \in vw[j]$ and we connect w_y to u if $-y \in wu[j]$.

We prove that the FZ3C instance and the PMT instance are equivalent. For this, consider a zero triangle uvw, where the vectors $x_j \in uv[j]$, $y_j \in vw[j]$ and $z_j \in wu[j]$ are picked to have sum zero for each j. This corresponds to the triangles $uv_{x_j}w_{y_j+x_j}$ in G_j for each j, where all these triangles are of color (u,v,w). Conversely, any set of g triangles of color (u,v,w) in G_j s should be of the form $uv_{x_j}w_{y_j}$, and hence from the definition of the PMT instance we have that for each j, $x_j \in uv[j]$, $y_j - x_j \in vw[j]$ and $-y_j \in wu[j]$ and so they correspond to a zero uvw triangle.

4.2 k-Node Labeled st Connectivity is hard from Factored Zero-k-Clique

We will show that counting k-NLstC mod $2^{2k\lg^2(n)}$ is hard from #FZkCH. The generated graph will be a dense DAG. Recall that this implies an explicit average-case distribution over which counting k-NLstC mod $2^{2k\lg^2(n)}$ is hard from worst case FZkCH, SETH, the 3-SUM hypothesis, and the APSP hypothesis.

REMINDER OF THEOREM 1.12 If a $O(|C|^{k-2}|E|^{1-\varepsilon/2})$ or $O(|C|^{k-2-\varepsilon}|E|)$ time algorithm exists for (counting mod $2^{2k\lg^2(n)}$) k-NLstC then a $O(n^{k-\varepsilon})$ algorithm exists for (#)FZkC.

Proof. Let $G = (V_1, ..., V_k)$ be an instance of FZkC. We reduce this instance to an instance of k-NLstC as follows. We begin by adding the special node s and the special node t. We will build g gadgets and put them after each other serially. The nodes in the gadgets will be assigned colors associated to the nodes of G. Each gadget will be designed to check if given k colors (and thus k nodes in G) whether the ith subset of the $\binom{k}{2}$ factored vectors represented do have a zero sum. See Figure 4 for a representation of our construction.

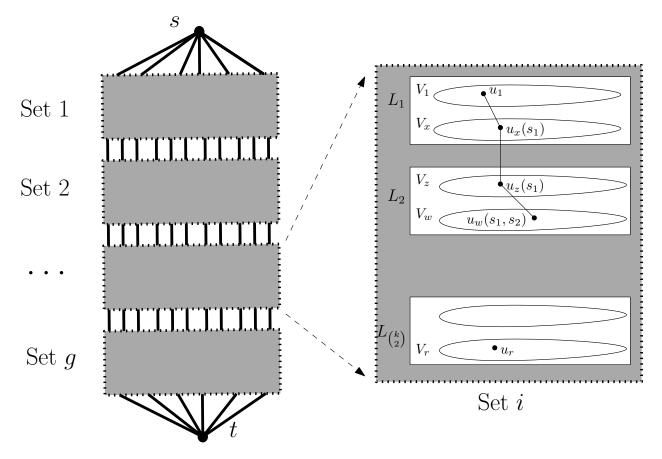


Figure 4: Left: The gadget structure for *k*-NLstC. Right: Inside of a set gadget.

The gadget for set i consists of $\binom{k}{2}$ layers $L_1,\ldots,L_{\binom{k}{2}}$. Each layer L_j represents the edges from V_x to V_y for some $x,y\in\{1,\ldots,k\}$ as follows: L_j consists of two layers itself, one for V_x and one for V_y . For each vertex $u_x\in V_x$, we add $2^{b(j-1)}$ nodes $u_x(s_1,\ldots,s_{j-1})$ where $s_1,\ldots,s_{j-1}\in\{0,1\}^b$, and we color these nodes with the color u_x . So we have a total of $n2^{b(j-1)}$ vertices. These vertices represent that we have chosen a particular node $u_x\in V_x$ and that in the first j edges we have chosen the j-1 vectors s_1,\ldots,s_{j-1} from the sets of the previous j-1 edges. For the second layer of L_j , for each $u_y\in V_y$ we add 2^{bj} nodes $u_y(s_1,\ldots,s_j)$ where $s_1,\ldots,s_j\in\{0,1\}^b$, and we color these nodes with the color u_y . We add edges between nodes $u_x(s_1,\ldots,s_{j-1})$ and $u_y(s_1,\ldots,s_{j-1},s_j)$ iff the i^{th} set of the factored vector of edge (u_x,u_y) contains the string s_j , i.e. $s_j\in u_xu_y[i]$.

Now to specify the edges between layers, suppose that layer L_{j+1} deals with the edges between V_z and V_w . For every j-tuple (s_1, \ldots, s_j) , add an edge from $u_y(s_1, \ldots, s_j)$ in L_j to $u_z(s_1, \ldots, s_j)$ in L_{j+1} for every $u_y \in V_y$ and $u_z \in V_z$. If z = y we can skip this step and just use the same set nodes.

Finally, we do something special for layer $L_{\binom{k}{2}}$. Lets say that layer $L_{\binom{k}{2}}$ summarizes the edges between

 V_x and V_y . For the nodes associated to V_y , instead of having vertices $u_y(s_1, \ldots, s_{\binom{k}{2}})$, we put only one vertex u_y . We connect $u_x(s_1, \ldots, s_{\binom{k}{2}-1})$ to u_y if and only if the set i of the edge (u_x, u_y) has a vector $s_{\binom{k}{2}}$ such that the vectors $s_1, \ldots, s_{\binom{k}{2}-1}, s_{\binom{k}{2}}$ sum to zero. Note that given a fixed choice of $s_1, \ldots, s_{\binom{k}{2}-1}$ there is a single vector $s_{\binom{k}{2}}$ such that they all sum to zero together.

This forms a layered directed graph, where edges go from layer L_i to layer L_{i+1} . We also assume that L_1 represents the edges from V_1 to V_x for some x. We add an edge from s to all vertices of the first layer of L_1 and an edge from all vertices in the last layer of $L_{\binom{k}{2}}$ to t. A representation of the layers is represented in Figure 4.

By this construction, a path with colors u_1, u_2, \dots, u_k that goes through the j^{th} gadget represents a zero sum within the j^{th} sets on the $\binom{k}{2}$ edges between u_1, \dots, u_k .

In our graph the number of colors |C| is O(n) and $|E| = O(n^2)$ so a $O(|C|^{k-2}|E|^{1-\varepsilon/2})$ algorithm and a $O(|C|^{k-2-\varepsilon}|E|)$ algorithm both run in $O(n^{k-\varepsilon})$ time. The number of solutions to both problems is the same, thus the counts are the same. The maximum count of FZkC is $2^{gb}n^k = O(2^{\lg^2(n) + k\lg(n)})$. Notably, this is less than $2^{2k\lg^2(n)}$, so the count from the k-NLstC instance will be less than the count for the FZkC instance. \square

Note that we can count k-NLstC mod $2^{2k\lg(n)^2}$ with |C|=n and $|E|=n^2$ in time $\tilde{O}(n^k)$.

Corollary 4.1. *If* #FZkCH *is true then* #k-NLstC (mod R) *takes* $|C|^{k-2\pm o(1)}|E|^{1\pm o(1)}$ (where $\lg(R) = n^{o(1)}$).

Proof. By Theorem 1.12 if #FZkCH is true then # k-NLstC (mod R) takes at least $|C|^{k-2-o(1)}|E|^{1-o(1)}$ time. By Theorem A.4 there is a $|C|^{k-2+o(1)}|E|^{1+o(1)}$ time algorithm for counting k-NLstC mod R.

4.3 *k*-Edge Labeled st Connectivity is hard from Factored *k* Function Problems (F*k*-f)

In this subsection we will show hardness from The edge labeled version of st connectivity. This reduction will get hardness from #Fk-f. Note that while k-NLstC has a $\tilde{O}(C^{k-2}E)$ algorithm, k-ELstC has a more expensive $\tilde{O}(C^{k-1}E)$ algorithm. In this section we will show that the k-ELstC algorithm is optimal up to sublinear factors if Fk-fH is true (note that this algorithm is thus also implied to be tight by SETH).

While our reduction to k-NLstC generated a dense graph, our reduction to k-ELstC generates a *sparse* graph. The sparsity allows for a tight reduction to the Fk-fH problem. However, because that our reduction requires sparsity to be tight, we have not been able to reduce FZkC to k-ELstC.

REMINDER OF THEOREM 1.13 If a $\tilde{O}(|E||C|^{k-1-\varepsilon})$ or $\tilde{O}(|E|^{1-\varepsilon}|C|^{k-1})$ time algorithm exists for (counting mod $2^{2k\lg^2(n)}$) k-ELstC, then a $\tilde{O}(n^{k-\varepsilon})$ algorithm exists for (#)Fk- \mathfrak{f} .

Proof. Given an instance of Fk-f which takes k lists V_1, \ldots, V_k of factored vectors, we produce an instance of k-ELstC with $\tilde{O}(n)$ colors and $\tilde{O}(n)$ edges. In the Fk-f instance, let $u_j[i]$ be the i^{th} subset of the vector $u_j \in V_j$. We use the vectors in the Fk-f instance as colors in the k-ELstC instance.

We start by adding two nodes s and t. We will make g gadgets, G_1, \ldots, G_g , where G_i handles the i^{th} set of the factored vectors, i.e. $u_j[i]$ for all $u_j \in V_j$ for all j. In each gadget G_i we have k layers of vertices L_1^i, \ldots, L_k^i , where the vertex set L_j^i represents the factored vectors in V_j . Finally we attach these gadgets one after the other serially.

For $j \le k$ the layer L_j^i has two layers itself, one with $n2^{bj}$ nodes and one with 2^{bj} nodes. For each node $u_j \in V_j$, we add the nodes $u_j(s_1, \ldots, s_j)$ to the first layer of L_j^i for all $s_1, \ldots, s_j \in \{0, 1\}^b$, so adding $n2^{bj}$ nodes in total. For the second layer of L_j^i , we add nodes $l_i(s_1, \ldots, s_j)$ for all $s_1, \ldots, s_j \in \{0, 1\}^b$. For each vertex u_j , we add a matching from the 2^{bj} nodes associated to vector u_j to the nodes in the second layer,

connecting $u_j(s_1,...,s_j)$ to $l_i(s_1,...,s_j)$. We color these edges with u_j . Note that this is how we achieve sparsity. Every other layer has $\tilde{O}(1)$ nodes in it. So every node (other than s and t) has an out-degree of $\tilde{O}(1)$.

We add edges from the second layer of L^i_j to the first layer of L^i_{j+1} . For $u_{j+1} \in V_{j+1}$, we connect $l_i(s_1, \ldots, s_j) \in L^i_j$ to $u_{j+1}(s_1, \ldots, s_j, s_{j+1}) \in L^i_{j+1}$ if, and only if, $s_{j+1} \in u_{j+1}[i]$. We color this edge with u_{j+1} .

The full effect of this means that by layer L_k^i a path from the beginning to the end of the gadget with the colors of a given set of k vectors implies those vectors have the corresponding set of vectors in their i^{th} sets. We will only add outgoing edges from nodes in the second layer of L_k^i only if $f(s_1, \ldots, s_k) = 1$.

We add edges between gadgets G_i and G_{i+1} by adding edges between the second layer of L_k^i and the first layer of L_1^{i+1} as follows. We connect the node $l_i(s_1,\ldots,s_k)\in G_i$ to $u_1(s_1')\in G_{i+1}$ for some $u_1\in V_1$ if and only if $f(s_1,\ldots,s_k)=1$ and $s_1'\in u_1[i+1]$. We color this edge with u_1 .

Now we deal with s and t. We add edges from s to all nodes $u_1 \in L_1^1$ if $s_1 \in u_1[1]$. These edges are colored with u_1 . Further, we add edges from the first layer of L_k^g to t directly, removing the second layer of L_k^g . We only add edges from $u_k(s_1, \ldots, s_k)$ for $u_k \in V_k$ to t iff $f(s_1, \ldots, s_k) = 1$. We color this edge with u_k .

First, note that we always add edges between two layers of size o(n) and $\tilde{O}(n)$, so adding at most $\tilde{O}(n)$ edges between them. Since we have O(1) layers, our graph has $\tilde{O}(n)$ edges in total.

Given this graph setup, if we pick k colors for example associated with $u_1(1), u_2(2), \dots, u_k(k)$ then the number of paths from s to t using only those colors of edges will correspond to the outcome of

$$\odot(u_1(1), u_2(2), \ldots, u_k(k))$$

as defined in the preliminaries. As a result, the sum over all k tuples of colors will be the count of the output of the Fk- \mathfrak{f} instance. The count of a Fk- \mathfrak{f} instance is at most $n^k 2^{bg} = o(2^{2k \lg^2(n)})$. So if $R = \Omega(2^{2k \lg^2(n)})$ the count mod R and the count are the same.

Corollary 4.2. If Fk- $\mathfrak{f}H$ (#Fk- \mathfrak{f}) is true then k-ELstC(#k-ELstC mod R) takes $|C|^{k-1\pm o(1)}|E|^{1\pm o(1)}$ (when $\lg(R) = n^{o(1)}$).

Proof. By Theorem 1.13 if Fk-fH (#Fk-fH) is true then k-ELstC (#k-ELstC mod R) takes at least $|C|^{k-1-o(1)}|E|^{1-o(1)}$ time

By Theorem A.5 there is a $|C|^{k-1+o(1)}|E|^{1+o(1)}$ time algorithm for counting mod *R k*-ELstC.

4.4 (k+1) Labeled Max Flow solves Factored Zero-k-Clique

REMINDER OF THEOREM 1.14 If (k+1)L-MF can be solved in T(n) time, then we can solve FZkC in time $\tilde{O}(T(n)+n^2)$.

Proof. We use the set gadgets from Theorem 1.12 and instead of placing them serially, we make a parallel network G' as shown in Figure 5. More particularly, let SG_1, \ldots, SG_g be the set gadgets from Theorem 1.12. Add s_1, \ldots, s_g with a source node s to the graph. Add t_1, \ldots, t_g with a sink node t to the graph. This completes the definition of the vertices of G'.

We attach s to all s_j and all t_j to t with label l^* for $j=1,\ldots,g$. For each j, we attach s_j to all the nodes in the first layer of SG_j , which is a copy of V_1 . Let the label of any s_ju edge be u where $u \in V_1$. Connect all the nodes in the last layer of SG_j to t_j for all j. Suppose that the last layer of SG_j corresponds to V_y . Let the label of any edge u_yt_j be u_y . Let the label of any edge (r,z) in any set gadget SG_j be the same as the color of r, since G' is supposed to be an edge-labeled graph. All the edges are unit capacitated. This completes the definition of G'.

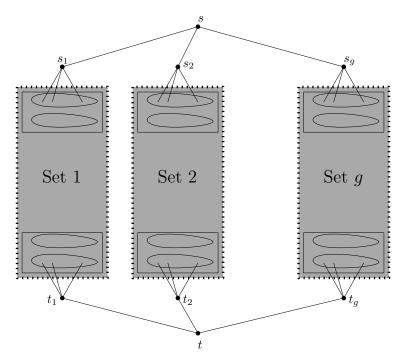


Figure 5: (k+1) labeled max flow instance structure.

First note that the maximum flow is at most g since the outdegree of s is g and the graph is unit-capacitated. So the flow going through each set gadget is at most 1, which means that there is at least one path from s_j to t_j through SG_j . From Theorem 1.12 any zero weight k-clique corresponds to g paths, one in each set gadget, using g labels corresponding to the g vertices of the clique. So any zero weight g-clique corresponds to a g from g to g from g to g from g to g for each g with all the g paths having the same g labels, which corresponds to a zero g-clique by Theorem 1.12.

4.5 Regular Expression Matching is hard from Factored OV

We are going to reduce F2-OV to regular expression matching. First, we define type and depth of a regular expression. Intuitively, the structure of the operations in a regular expression is called its *type*, which is represented by a tree with nodes labeled with operations. Let \bullet be an arbitrary operator. A tree T with root node \bullet means that all the first level operations of a regular expression E of are \bullet , i.e. $E = A_1 \bullet A_2 \bullet \dots \bullet A_\ell$, where A_i s are regular expressions. The type of each A_i can be the subtree with any of the children of the root node as its root. The *depth* of a regular expression is the longest root-leaf path in the type tree of the regular expression.

We reduce (#)F2-OV to (#)regular expression matching where the pattern is a depth 5 pattern of type T_0 shown in Figure 6, and we give an O(mn) algorithm for counting such patterns in O(mn) time (Theorem A.9 in the Appendix).

REMINDER OF THEOREM 1.15 Let R be an integer where $\lg(R)$ is subpolynomial. If you can solve (# mod R) regular expression matching in T(n) time, then you can solve (# mod R) F2-OV in $\tilde{O}(T(n)+n)$ time.



Figure 6: Type T_0 of the regular expression of Theorem 1.15. Where "|" is the OR operator, "·" is the concatenation operator, and * is the Kleen star operator.

Proof. We use the proof of Theorem 1 in [BI16] that shows hardness for patterns of type " $|\cdot|$ ", where " $|\cdot|$ " is the OR operator, " \cdot " is the concatenation operator and the type tree is a path of length two with node $|\cdot|$, and $|\cdot|$ respectively. In [BI16] authors start with any 2OV instances (A,C) where $A = \{a^1, \ldots, a^n\}$ and $C = \{c^1, \ldots, c^n\}$ are sets of n vectors of dimension d and reduce it to an instance of regular expression matching with pattern p constructed from A (and independent from C) and text t constructed from C (and independent from A) both of O(dn) size, where any orthogonal pair (a,c) with $a \in A$ and $c \in C$ corresponds to an alignment of p on a substring of t, and conversely any alignment of p on t corresponds to an orthogonal pair (a,c). More particularly, pattern $p = VG(a^1)|\ldots|VG(a^n)$ consists of the OR of vector gadgets $VG(a) = CG(a_1) \cdot \ldots \cdot CG(a_d)$ where CG is a coordinate gadget and a_i is the ith bit of vector a. Each coordinate gadget is aligned on a single bit. The text $t = VG'(c^1)2 \dots 2VG'(c^n)$ consists of vector gadgets $VG'(c) = c_1c_2 \dots c_d$ which is the bit representation of the vector c. We have that $a^i \cdot c^j = 0$ iff there is an alignment of $VG(a^i)$ on $VG'(c^j)$. As a result, the number of orthogonal pairs in (A,C) is the number of subset alignments of p on t.

We use the above construction for our factored vectors. Note that if w is a factored vector, for any $j \in \{1, ..., g\}$ we can construct a pattern (or a text) of length O(|w[j]|b) using the vectors in w[j] which have length b.

Consider an instance (U,V) of F2-OV where $U=\{u^1,\ldots,u^n\}$ and $V=\{v^1,\ldots,v^n\}$ are sets of n factored vectors. We construct the pattern P using U and the text T using V. We first construct the pattern. Let $p_i[j]$ be the pattern corresponding to $u^i[j]$ using the construction of [BI16] for $i=1,\ldots,n$ and $j=1,\ldots,g$. Note that the symbols used in $p_i[j]$ are 0,1. Let $p_i'[j]=[0|1|2]^*\cdot p_i[j]\cdot [0|1|2]^*$, where "*" is the Kleen star operator. Define the "pattern factored vector gadget" PVG(i) for u^i as follows:

$$PVG(i) = p'_i[1] \cdot 3 \cdot p'_i[2] \cdot 3 \cdot \dots \cdot 3 \cdot p'_i[g]$$

Let the pattern *P* be the following:

$$P = PVG(1)|PVG(2)|\dots|PVG(n)$$

Note that the length of P is $\tilde{O}(n)$, since we have that $|p_i[j]| = O(|u^i[j]|) = O(2^b.b) = o(n)$, $|p_i'[j]| = O(p_i[j])$ and the number of occurrence of the symbol 3 is (g-1)n. The number of symbols in a * expression is also O(gn) since the number of * expressions is 2ng. So the total number of symbols in P is $\tilde{O}(n)$, and as a result the length of P is $\tilde{O}(n)$.

Now we construct the text. Let $t_i[j]$ be the text corresponding to $v^i[j]$ using the construction in [BI16] for i = 1, ..., n and j = 1, ..., g. Note that the symbols used in $t_i[j]$ are 0, 1, 2. Define the "text factored vector gadget" TVG(i) for v_i as follows:

$$TVG(i) = t_i[1]3t_i[2]3...3t_i[g]$$

Let the pattern *T* be the following:

$$T = TVG(1)4TVG(2)4\dots4TVG(n)$$

Similar to P, the length of T is $\tilde{O}(n)$.

Now we have to show that there is a one to one correspondence between orthogonal vectors in the F2-OV instance and the number of subset alignments of P on T. First consider an orthogonal pair (u_i, v_j) , where for each $k = 1, \ldots, g$ the vector chosen from $u^i[k]$ is the r_k th vector $u^i[k][r_k]$, and the vector chosen from $v^j[k]$ is the s_k th vector $v^j[k][s_k]$. So we have that for each k, $u^i[k][r_k]$, $v^j[k][s_k] = 0$. This means that there is a subset alignment of $p_i[k]$ on $t_j[k]$ corresponding to $u^i[k][r_k]$ and $v_j[k][s_k]$ for all k. We use the $[0|1|2]^*$ parts of $p_i'[k]$ to cover the rest of $t_j[k]$ and thus we get a full alignment of $p_i'[k]$ on $t_j[k]$. Having these alignments for each k, they extend uniquely to a full alignment of PVG(i) on TVG(i).

Conversely, suppose that there is a subset alignment of P on T. Note that the first level of the pattern consists of ORs, so any alignment should choose some $i \in \{1, ..., n\}$ and align PSG(i) on T. On the other hand, symbol 4 is not used in the pattern P. So P should be aligned on TSG(j) for some j. So assume that in this subset alignment, PSG(i) is aligned on TSG(j). Since there are exactly g-1 symbols "3" that are concatenated in PSG(i) and there are exactly g-1 symbols "3" in TSG(j), the 3s should be aligned to each other. So $p_i'[k]$ is fully aligned to $t_j[k]$. Recall that $p_i'[k] = [0|1|2]^* \cdot p_i[k] \cdot [0|1|2]^*$. So $p_i[k]$ should be aligned to $t_j[k]$. So by the construction of [BI16] there are unique r_k and s_k where the vector gadget for $u^i[k][r_k]$ is fully aligned to the vector gadget for $v^j[k][s_k]$, which means that these two vectors are orthogonal. Since this is true for every k, u^i and v^j are orthogonal, and hence the number of factored orthogonal vectors in the F2-OV instance equals to the number of subset alignments of P on T.

4.6 Longest Common Subsequence and Edit Distance

We are going to look at the k-LCS problem in this subsection. We show that k-LCS is hard from Fk-OV. We note that this sort of proof should also work for other string similarity measures. In a work of Bringmann and Künnemann they show a general framework for proving hardness for string comparisons on two strings from 2-OV [BK15]. Presumably this framework can be expanded to work for Fk-OV, however, generating this framework is out of the scope of this paper. We will note however that the only additional gadget you seem to need to solve Fk-OV is a selector gadget for at most $\tilde{O}(1)$ strings each of length $\tilde{O}(1)$. This means even expensive gadgets are acceptable.

We will first show that weighted k-LCS is hard from Fk-OV.

Definition 1. Weighted Longest Common Subsequence (WLCS) [ABV15] For k sequences P_1, \ldots, P_k of length n over an alphabet Σ and a weight function $w : \Sigma \to [K]$, let X be the sequence that appears in all of P_1, \ldots, P_k as a subsequence and maximizes the expression $W(X) = \sum_{i=1}^{|X|} w(X[i])$. We say that X is the WLCS of P_1, \ldots, P_k and write WLCS $(P_1, \ldots, P_k) = W(X)$. The Weighted Longest Common Subsequence problem asks to output WLCS (P_1, \ldots, P_k) .

We will then use this lemma from a previous work to show that k-LCS is hard from k-WLCS if the weights are small enough [ABV15].

Lemma 4.3. If the k-LCS of k sequences of length O(Kn) over Σ can be computed in time T(n) then the k-WLCS of k sequences of length n over Σ with weights $w : \Sigma \to [K]$ can be computed in $\tilde{O}(T(n)K)$ time [ABV15].

We want to use the ideas and gadgets of Abboud, Backurs and Vassilevska Williams [ABV15]. We basically want to ask, given sets V_1, \ldots, V_k of factored vectors, are there k factored vectors $v_1 \in V_1, \ldots, v_k \in V_k$ such that for all i there exist vectors $u_1 \in v_1[i], \ldots, u_k \in v_k[i]$ such that those vectors are orthogonal. Notably, once you have specified the factored vectors and the index i what remains is a (small) orthogonal vectors instance. The construction from [ABV15] produces a fixed longest common sub-sequence value if there

is an orthogonal k tuple. So, if we can construct a setup where the output of the WLCS is basically a concatenation of gadgets for each index i then we will get the value we want for any given pair of vectors. We will need to add some gadgets to force the WLCS to "pick" a set of vectors.

Notably, "selector gadgets" from [ABV15] serve the purpose of forcing the WLCS to choose which k factored vectors to compare. And, if we have a gadget for every subset (so every $i \in [1,g]$) and put a high value set of symbols between them it forces the gadgets to not interact or loose that value. For this we want a "parallel gadget". Thus, we get a WLCS that is roughly the concatenation of the WLCS of each of the g gadgets. This gives us the desired result.

Gadgets of General Use First we will describe the selector gadget.

Lemma 4.4. As input we are given k lists L_1, \ldots, L_k each of which contain n strings of length at most ℓ (e.g. $s_{i,1}, \ldots, s_{i,n} \in L_i$ and $|s_{i,j}| \leq \ell$) with an alphabet Σ and weights in the range [K]. Let M be the maximum value of $WLCS(s_{1,j_1}, \ldots, s_{k,j_k})$ over all choices of $j_1, \ldots, j_k \in [1,n]$.

We can generate a k-WLCS instance P_1, \ldots, P_k with k symbols added to Σ , the new range of weights being $[2\ell Kn]$ and length $|P_i| = O(n^2 + n\ell)$, such that WLCS $(P_1, \ldots, P_k) = C_{sel} + M$ for $C_{sel} = (2kn)(2\ell Kn)$.

Proof. We introduce k symbols $@_1, @_2, \ldots, @_k$ for this selector gadget. We assign a weight of $2\ell Kn$ to all the symbols $@_j$ (note this is larger than the total weight of any given string $s_{i,j}$). For convenience by $@_j^{(x)}$ we mean x copies of the symbol $@_1$.

We first define a helper gadget for separating our strings

$$STG_i(s) = @_i @_{i-1}^{(2n)} \dots @_1^{(2n)} s @_1^{(2n)} \dots @_{i-1}^{(2n)} @_i.$$

We can now define our output strings:

$$P_i = \bigotimes_{k=1}^{(2n)} \dots \bigotimes_{i=1}^{(2n)} STG_i(s_{i,1}) \dots STG_i(s_{i,n}) \bigotimes_{i=1}^{(2n)} \dots \bigotimes_{k=1}^{(2n)} ...$$

Note that every $STG_i()$ gadget is of length $O(n^2 + \ell)$ and O(n) $STG_i()$ gadgets are used. The additional @ symbols make up at most kn symbols on every string. So the total length of each string P_i is at most $O(n^2 + n\ell)$. The largest weight we use is for the @ symbols, they have a weight of $2\ell Kn$. There are k @ symbols, so we increase the alphabet by k.

Let ? be some string made of symbols from the original alphabet (so no @ symbols). We will use this to make arguing easier. We claim that the weighted longest common subsequence will look like this:

Let us argue for this claim. First, every symbol $@_i$ appears only 2n times in string P_i so it can not appear more often. Second, in the string P_1 the only place that symbols $@_k, \ldots, @_2$ appear is at the start and end of the string in the order presented above. For $@_k$ it appears only 2n times in the string P_k . In every other string $@_k$ only appear at the start and end of strings. To match all 2n copies of the $@_k$ symbol we must align a single $STG_k(s)$ gadget with the other strings P_i for i < k. Given that we are matching a single $STG_k(s)$ string note that the only locations that $@_{k-1}, \ldots, @_1$ symbols appear are around the string s in decreasing and then increasing order. So, if we do try to match all 2n copies of every symbol $@_i$ we must get k-WLCS that looks like the above.

Now we will argue that you can match 2n copies of every symbol $@_i$. Consider the string P_i , if you pick any single $STG_i(s_{i,j})$, all the symbols $@_i$ and the "intro" and "outro" strings of $@_k^{(2n)} \dots @_{i+1}^{(2n)}$ and $@_{i+1}^{(2n)} \dots @_k^{(2n)}$ together make a string of the form:

$$(a_k^{(2n)} \dots (a_i^{(2j-1)} \dots (a_1^{(2n)})) = (a_1^{(2n)} \dots (a_i^{(2n-2j+1)} \dots (a_k^{(2n)}))$$

If we match k of these we get our claimed string where $y_i = 2j - 1$.

Now we must argue that one wants to match all $@_i$ symbols possible. Note that every $@_i$ symbol is worth more than all non @ symbols in the entire string. Given this, we must prefer matching all @ symbols to any other goal.

So, given that the k-WLCS will have the described form we can now note the following. The ? that appears must be the k-WLCS of k strings $s_{i,j}$. In every P_i in order to match all 2n symbols @ $_i$ and only have non-@ symbols in the middle of the string one must select a single STG() gadget to be included in the k-WLCS.

So the *k*-WLCS will include 2n copies of $@_i$ symbols and the *k*-WLCS of the *k* strings that have the largest *k*-WLCS.

Now we will describe the parallel gadget.

Lemma 4.5. As input we are given k lists L_1, \ldots, L_k each containing g strings of length at most ℓ (e.g. $s_{i,1}, \ldots, s_{i,g} \in L_i$ and $|s_{i,j}| \leq \ell$) with an alphabet Σ and weights in the range [K]. Let

$$M = \sum_{j=1}^{g} WLCS(s_{1,j},\ldots,s_{k,j}).$$

We can generate a k-WLCS instance P_1, \ldots, P_k with 1 symbol added to the alphabet Σ , the new range of weights being $[2\ell Kg]$ and length $|P_i| = O(n\ell)$, such that $WLCS(P_1, \ldots, P_k) = C_{par} + M$ for $C_{par} = 2\ell Kg(g-1)$.

The count of # $WLCS(P_1, ..., P_k)$ will be

$$\Pi_{j=1}^g \#WLCS(s_{1,j},\ldots,s_{k,j}).$$

So the multiplication of all the matched k tuple counts.

Proof. We create a new character \$ with weight $2\ell kn$ which is larger than M. Create each string P_i as follows

$$P_i = s_{i,1} \$ s_{i,2} \$ \dots \$ s_{i,g}.$$

Now aligning the g-1 symbols s has such impact it swamps everything else. So the WLCS will force comparisons of the first s tuple $(s_{1,1}, \ldots, s_{(k,1)})$, then the next s tuple and so on. Given this, the count of the number of longest weighted subsequences is simply the multiplication of how many ways to achieve the longest subsequence for each of our s s-tuples.

Building Factored Vector Gadgets

Lemma 4.6. Let w_1, \ldots, w_k be vectors of length $b = o(\lg(n))$. There are k vector gadgets $VG_1(\cdot), \ldots, VG_k(\cdot)$ such that $WLCS(VG_1(w_1), \ldots, VG_k(w_k))$ is some constant C_{VG} if the k vectors w_1, \ldots, w_k are orthogonal and is $C_{VG} - 1$ otherwise.

This uses an alphabet of size 2k + 2, weights of size $\tilde{O}(1)$ and the length of each w_i is $\tilde{O}(1)$.

Proof. We introduce two symbols 0 and 1 where w(0) = w(1) = 1. For each vector w_i we construct all possible kb length zero-one strings of the following form: $\{0,1\}^{(i-1)b}w_i\{1,0\}^{(k-i)b}$. That is, we generate all possible bk length zero-one strings where the bits from position b(i-1)+1 to position bi form w_i . Call this set of strings $S_i'(w_i)$. Now we generate the set $S_i(w_i) \subseteq S_i'(w_i)$ by including only strings where the vectors formed by the first b bits, the second b bits, the third b bits, etc form a k tuple of vectors that are k-orthogonal. So $S_i(w_i)$ is a representation of all tuples of k vectors of length b where w_i is the ith vector and the k vectors are k-orthogonal.

Now note that the only way that there is one string from each set $S_1(w_1), \ldots, S_k(w_k)$ such that the weighted longest common subsequence of those strings is kb is that if those strings match perfectly. The only way for there to be k perfectly matching strings is if vectors w_1, \ldots, w_k are orthogonal (the string would otherwise be excluded).

So we have generated k lists of at most $2^{b(k-1)}$ strings of length bk. To ensure all the lists are the same length we will pad all the lists to length $2^{b(k-1)}$ with empty strings. We can now use the selector gadget (Lemma 4.4) to wrap around these lists. This will add k new symbols and make the gadgets have length $O(2^{2b(k-1)} + 2^{b(k-1)}bk)$ with weights in range $[2bk2^{b(k-1)}]$. We note that this length is $\tilde{O}(1)$ and this weight is also $\tilde{O}(1)$ because b is constrained to be $b = o(\lg(n))$. Call this construction $VG'_i(\cdot)$ for the i^{th} vector.

So right now if we have an orthogonal vector k tuple we get a weighted longest common subsequence of weight $y = 2k2^{b(k-1)}(2bk2^{b(k-1)}) + bk$. But, for some inputs the optimal weight could be much lower (like $2k2^{b(k-1)}(2bk2^{b(k-1)})$).

So, we will add another layer of a selector (from Lemma 4.4) around $VG'_i(w_i)$ as follows: Our lists will be of length two. The i^{th} list will be $VG'_i(w_i)$ and 0^{y-1} . So if the vectors aren't orthogonal the second option will lower bound the weight of the longest subsequence. This layer of selector adds another k symbols to our alphabet. It multiplies our weight by $\tilde{O}(1)$. Our weight remains $\tilde{O}(1)$.

Now if the vectors are orthogonal we get weight $C_{sel} + y$ and $C_{sel} + y - 1$ otherwise where C_{sel} is set by our selector gadget.

Lemma 4.7. Let Z_1, \ldots, Z_k each be a subset of $\{0,1\}^b$. There are k set vector gadgets $SVG_1(\cdot), \ldots, SVG_k(\cdot)$ such that $WLCS(SvG_1(Z_1), \ldots, SVG_k(Z_k))$ is some constant C_{SVG} if the k sets of vectors have $\circ(Z_1, \ldots, Z_k) > 0$ and is $C_{SVG} - 1$ otherwise.

This uses an alphabet of size 3k+2, weights of size $\tilde{O}(1)$ and has a length of $\tilde{O}(1)$.

Proof. We first construct vector gadgets $VG_i(w_i)$ of Lemma 4.6 for all $w_i \in Z_i$. Let the list L_i consist of all the vector gadgets $VG_i(w_i)$ for all $w_i \in Z_i$. We use these lists to make the selector gadget of Lemma 4.4.

Note that Z_i is a set of at most 2^b zero-one vectors of length b. So the expense of the selector gadget is polynomial in the length, weight, and number of input strings. All these numbers are $\tilde{O}(1)$ so the cost of this selector gadget is $\tilde{O}(1)$. If there is an orthogonal k tuple within these sets then the optimal weight will be $C_{sel} + C_{VG}$, if there are not then the optimal weight will be $C_{sel} + C_{VG} - 1$.

This adds another set of k symbols for a total of 3k + 2.

Lemma 4.8. Let v_1, \ldots, v_k each be a factored vector with g sets containing g-bit vectors. There are g set vector gadgets $FVG_1(\cdot), \ldots, FVG_k(\cdot)$ such that $WLCS(FVG_1(v_1), \ldots, FVG_k(v_k))$ is some constant C_{FVG} if the g sets of vectors have g (g) and is g and is g otherwise.

This gadget uses an alphabet of size 3k + 3 has weights of $\tilde{O}(1)$ and has a length of $\tilde{O}(1)$.

Proof. Let v be a factored vector with g sets called $v[1], \ldots, v[g]$.

To build $FVG_i(v)$ we want to concatenate the gadgets $SVG_i(v[1]), \ldots, SVG_i(v[g])$. We will use the parallel gadget for this (Lemma 4.5).

We have that $\odot(v_1,\ldots,v_k)>0$ only if for all $j\in[1,g]$ we have $\circ(v_1[j],\ldots,v_k[j])>0$. So, we want to know if the sum of all the k-WLCS of all k tuples of string $SVG_1(v_1[j]),\ldots,SVG_k(v_k[j])$ are C_{SVG} . If they are all C_{SVG} , then $\odot(v_1,\ldots,v_k)>0$.

There are g set vector gadgets each of length $\tilde{O}(1)$ and with symbols of weight $\tilde{O}(1)$. The parallel gadgets weights and length depend polynomialy on g and the weights and length of the input strings. Notably, all these values are $\tilde{O}(1)$ so the length and weight of the $FVG_i(\cdot)$ will both be $\tilde{O}(1)$.

The number of symbols increases by 1 over the symbols in SVG_i . So we have 3k + 3 symbols.

WLCS and LCS We now give a reduction from Fk-OV to k-WLCS in the worst case.

Theorem 4.9. A T(n) time algorithm for k-WLCS with alphabet size O(k) and weights in the range $[\tilde{O}(1)]$ implies a $\tilde{O}(T(n))$ algorithm for Fk-OV.

Proof. Let the Fk-OV instance be given as k lists V_1, \ldots, V_k each containing n factored vectors $v \in V_i$. Every factored vector has g subsets of $\{0,1\}^b$. Recall that $g = o(\lg(n)/\lg\lg(n))$ and $b = o(\lg(n))$.

We will be reducing this to an instance of k-WLCS where we have k strings P_1, \ldots, P_k . These strings will have length $\tilde{O}(n)$ and weights that range from 1 to a number that is $\tilde{O}(1)$.

We will produce our k-WLCS instance by wrapping an alignment gadget around our factored vector gadgets from Lemma 4.8. We are going to use the alignment gadget from [ABV15] (see the proof of Lemma 14 in that paper) as follows.

We introduce k+1 new symbols: $8,9,3_2,3_3,\ldots,3_k$. Let $Q=|P_k|$. For the weights of these symbols, we set $w(3_i)=B_i$ and we set $B=B_k>D$ where D is the largest possible weight of a factored vector gadget FVG which we defined in Lemma 4.8. The length of a FVG(v) is $\tilde{O}(1)$ and the weight of every symbol is $\tilde{O}(1)$ so D is $\tilde{O}(1)$. We set $B=B_k=(10kD)^2$. We set $B_i=(2k)^{k-i}B$. We set $B_i=(2k)^{k-i}B$.

$$w(8) = w(9) \gg w(3_2) \gg w(3_3) \gg \ldots \gg w(3_k) \gg D.$$

We will use parentheses bellow. They do not represent symbols, they are there to assist in readability and to help convey repetitions for example (\$#)³ means \$#\$#\$#.

Now we produce gadgets to wrap our factored vector gadgets. Let

$$FVG_1'(s) = 8FVG_1(s)9$$

and

$$FVG_i'(s) = 8FVG_i(s)9(3_2...3_i)^Q.$$

Define the factored vector \vec{e} to be the vector formed by g empty sets (so a vector that gets the worst match possible). We define the concatenation operator. Let $|_{v \in V_i}FVG'_i(v)|$ be the concatenation of the $FVG'(\cdot)$ applied to every factored vector v in the input list V_i .

Now we will define the strings P_i :

$$P_i = (3_{i+1} \dots 3_k)^Q (3_2 \dots 3_i) (FVG'_i(\vec{e}))^{(i-1)n} \left(|_{v \in V_i} FVG'_i(v) \right) (FVG'_i(\vec{e}))^{(i-1)n} (3_{i+1} \dots 3_k)^Q$$

Given the choices of weights for the symbols $8,9,3_2,\ldots,3_k$, a weighted longest common subsequence must contain the maximum possible number of each symbol. Given the construction of the alignment gadgets, there are weighted longest common subsequences that contain the maximum possible number of each symbol individually, simultaneously. For a more formal treatment see Lemma 14 in [ABV15]. The length of these strings is $\tilde{O}(n)$ and the weights are of size $\tilde{O}(1)$. Recall the $FVG(\cdot)$ constructions have length $\tilde{O}(1)$ each.

Our alphabet use for FVG is O(k) symbols and we have added k+1 symbols so the total number of symbols is O(k).

Further note that the optimal k-WLCS will align exactly n k-tuples of $FVG(\cdot)$ s. This means the length of the optimal k-WLCS will be some constant C_{tot} , plus n(X-1) if there are no k-tuples (v_1, \ldots, v_k) such that $\odot(v_1, \ldots, v_k) > 0$. Otherwise the optimal k-WLCS will be at least $C_{tot} + n(X-1) + 1$. This allows us to solve the detection problem for Fk-OV with one call to k-WLCS on strings of length $\tilde{O}(n)$ and weights in the range $\tilde{O}(1)$.

REMINDER OF THEOREM 1.16 A T(n) time algorithm for k-LCS with alphabet size O(k) implies a $\tilde{O}(T(n))$ algorithm for Fk-OV.

Proof. We use Theorem 4.9 and Lemma 4.3. The weights of the instance produced in Theorem 4.9 are $\tilde{O}(1)$ and the length of strings is $\tilde{O}(n)$. So we can reduce Fk-OV to k-WLCS and then reduce that instance of k-WLCS to k-LCS.

Edit Distance We will use the following Lemma to obtain hardness for Edit Distance.

Lemma 4.10 (Restated from Theorem C.2 from [Kus19]). An algorithm for WLCS (k-WLCS where k=2) that runs in $O(n^{2-\varepsilon})$ time for some constant $\varepsilon > 0$ implies a $O(n^{2-\delta})$ time algorithm for Edit Distance for some constant $\delta > 0$. [Kus19]

Thus Theorem 4.9 and Lemma 4.10 give us the following theorem.

REMINDER OF THEOREM 1.17 A T(n) time algorithm for Edit Distance implies a $\tilde{O}(T(n))$ algorithm for F2-OV.

Fk-OVH and LCS and Edit Distance Theorem 1.16 and Theorem 1.17 give us the following corollary.

Corollary 4.11. If Fk-OVH is true then k-LCS requires $n^{k-o(1)}$ time. If F2-OVH is true then Edit Distance requires $n^{2-o(1)}$ time.

5 Average Case Hardness for Subgraph Counting

Here we demonstrate the power of the framework in Section B to show average case hardness for counting subgraphs H with k vertices, where $k = o(\sqrt{\lg(n)/\lg\lg(n)})$. If the sub-graph H is sufficiently sparse then some larger k can be tolerated. Notably, for this section, as long as the number of edges is $e = o(\lg(n)/\lg\lg(n))$ then our worst case to average case reduction has sub-polynomial overhead.

Using the framework we can immediately show that counting subgraphs H in what are roughly H-partite Erdős-Rényi graphs (see Definition 1) is hard. We use our Inclusion/Edgesculsion Lemma from Section 5.2 to extend this result to counting subgraphs H in Erdős-Rényi graphs, and show that this problem is average case hard as well. We start by a few definitions.

Definition 1. The **counting** H **sub-graphs in a** H**-partite fashion (CHGHP)** problem takes as input a k-node graph H and a H-partite n-node graph G with vertex set partition V_1, \ldots, V_k , and asks for the count of the number of sub-graphs of G that have exactly one node from each of the k partitions and contain the graph H.

Definition 2. The uniform counting H sub-graphs in a H-partite fashion (UCHGHP) problem takes as input a k-node graph H and an H-partite n-node graph G with vertex set partition V_1, \ldots, V_k , where every edge between partitions that have edges in H is chosen to exist iid with probability μ . The problem asks for the count of the number of sub-graphs of G that have exactly one node from each of the k partitions and contain the graph H.

Note that CHGHP is a worst-case problem whereas UCHGHP is an average-case problem. Notably, UCHGHP is the uniform distribution over inputs to CHGHP.

5.1 Reducing counting H subgraphs in H-partite fashion to uniform counting

We start by reducing CHGHP to UCHGHP. Our ultimate goal is to reduce CHGHP to counting *H* subgraphs in an Erdős-Rényi graph.

Lemma 5.1. Let H be a k-node graph with vertices $V_H = \{x_1, \ldots, x_k\}$ and G a H-partite n-node graph with vertex set partition V_1, \ldots, V_k . Let \vec{E} be the set of variables $\{e(v_i, v_j) | i \neq j, v_i \in V_i, v_j \in V_j\}$ when an edge variable is a 1 if that edge exists and 0 if the edge is absent in G. Let $h(v_1, \ldots, v_k)$ be a function that multiples $e(v_i, v_j)$ if $x_i x_j$ is an edge in H for all $i, j \in [1, k]$ where $i \neq j$. If p is a prime in $[2n^k, n^{2k}]$, the following function returns the output of CHGHP on G:

$$f(\vec{E}) = \sum_{\nu_1 \in V_1, \dots, \nu_k \in V_k} h(\nu_1, \dots, \nu_k) \pmod{p}.$$

Proof. Consider the function h: If v_1, \ldots, v_k in that particular order contain the graph H it returns 1, otherwise it returns 0. Specifically, we are checking if our particular permutation of these variables completely covers the (arbitrary) permutation of variables associated with the input sub-graph H.

Now f sums over all choices of k nodes from each partition and counts how many instances of the sub graph appear in each. There is no double counting because every set of k nodes differs by at least one node.

Lemma 5.2. The function f defined in Lemma 5.1 is a good low-degree polynomial for CHGHP if the number of edges in H is $o(\lg(n)/\lg\lg(n))$.

Proof. To prove the lemma, first note that f is a polynomial over a prime finite field F_p for some prime $p \in [2n^k, n^{2k}]$, and the number of monomials in f is $O(n^k \cdot k!)$, which is polynomial. By Lemma 5.1 the function f returns the same value as CHGHP when it is given zero-one inputs.

Let $|E_H|$ be the number of edges in H. The function f has degree $|E_H| = o(\lg(n)/\lg\lg(n))$. In fact given constant k, f has constant degree. This is because f is formed with a sum over monomials $h(v_1, \ldots, v_k)$, which have degree $|E_H| \le {k \choose 2}$.

Finally, the function f is strongly $|E_H|$ -partite. There are $|E_H|$ partitions of edges. The function f is a sum over calls to h where h takes as input one variable from each of those edge partitions and multiplies all of them.

Corollary 5.3. Let $d = \binom{k}{2}$ and $k = o(\sqrt{\lg(n)/\lg\lg(n)})$. If an algorithm exists to solve UCHGHP in time T(n) with probability $1 - 1/\omega \left(\lg^d(n)\lg\lg^d(n)\right)$, then an algorithm exists to solve CHGHP in time $\tilde{O}(T(n) + n^2)$ with probability at least $1 - O\left(2^{-\lg^2(n)}\right)$.

Proof. If $k = o(\sqrt{\lg(n)/\lg\lg(n)})$ then the size of the edge set in H, E_H is at most $\binom{k}{2} = d = o(\lg(n)/\lg\lg(n))$. Using Theorem 1.20 we simply need that a good low-degree polynomial for CHGHP exists. By Lemma 5.2, the function f from Lemma 5.1 is a GLDP(CHGHP).

Corollary 5.4. Let H be a sub-graph with an edge set E_H where $|E_H| = o(\lg(n)/\lg\lg(n))$. Let $d = |E_H|$. If an algorithm exists to solve UCHGHP in time T(n) with probability $1 - 1/\omega(\lg^d(n)\lg\lg^d(n))$, then an algorithm exists to solve CHGHP in time $\tilde{O}(T(n) + n^2)$ with probability at least $1 - O\left(2^{-\lg^2(n)}\right)$.

Proof. We have that E_H is at most $\binom{k}{2} = d = o(\lg(n)/\lg\lg(n))$. Using Theorem 1.20 we simply need that a good low-degree polynomial for CHGHP exists. By Lemma 5.2, the function f from Lemma 5.1 is a GLDP(CHGHP).

5.2 Inclusion-Edgesculsion

In Corollary 5.4 we show that counting subgraphs H in Erdős-Rényi H-partite graphs quickly with a high enough probability implies fast algorithms for counting H-subgraphs in the worst case. We now want to extend this to fully Erdős-Rényi graphs. Specifically, we want to show that counting H-subgraphs in Erdős-Rényi quickly with a high enough probability implies a fast algorithm for counting H-subgraphs in the worst case. To acheive this goal we introduce our Inclusion-Edgesclusion technique. We begin with a few definitions.

Definition 3. Let G be a k-partite Erdős-Rényi graph with every edge included with probability 1/b where b is a constant integer. Let the vertex partitions of G be V_1, \ldots, V_k and the edge partitions be $E_{i,j} \ \forall i, j \in [1,k]$ where i < j.

Label all $|V_i| \cdot |V_j|$ edges with numbers in [1,b] as follows. Edges that exist in G are labeled 1. The rest of the edges are uniformly at random assigned labels from [2,b]. For $\ell \in [1,b]$, let $E_{i,j}^{\ell}$ be the set of all edges of label ℓ .

Let $G^{(\ell_1)(\ell_2)\dots(\ell_{\binom{k}{2}})}$ be the graph formed by choosing edge sets $E_{1,2}^{\ell_1}, E_{1,3}^{\ell_2}, \dots, E_{k-1,k}^{\ell_{\binom{k}{2}}}$. Let S_G be the set of all possible $b^{\binom{k}{2}}$ graphs $G^{(\ell_1)(\ell_2)\dots(\ell_{\binom{k}{2}})}$. Note when b=2 these sets of edges are $E_{i,j}^{(1)}=E_{i,j}$ and $E_{i,j}^{(2)}=\bar{E}_{i,j}$.

Definition 4. Let G be a k-partite Erdős-Rényi graph with every edge included with probability 1/b where b is a constant integer. Let the vertex partitions be V_1, \ldots, V_k . Let the edge partitions be $E_{i,j} \ \forall i, j \in [1,k]$ where i < j.

Let a labeled subgraph L of H in G be a subgraph of H where every vertex is assigned a unique label from [1,k].

Define the count of the number of labeled subgraphs L in G to be the number of not-necessarily induced subgraphs L where every vertex in L with label ℓ comes from V_{ℓ} in the original graph.

We want to reduce UCHGHP to counting subgraphs H in Erdős-Rényi graphs. A uniformly random H-partite graph only has edges between partitions corresponding to edges in H. However, an Erdős-Rényi graph would have edges within partitions and between partitions that don't correspond to edges in H. So, if we add these random edges we will over count subgraphs H, including subgraphs H that appear outside of the original H-partite graph.

We solve this problem by creating multiple graphs. Each graph individually looks like it is sampled from the Erdős-Rényi distribution. However, these graphs are correlated. We use a variant of an inclusion-exclusion argument (hence the name "inclusion-edgesclusion") to count the subgraphs H that appear in the original H-partite graph.

We will start with a warm up lemma.

Lemma 5.5 (Warm Up Lemma). Let C_G be the count of the number of k-node subgraphs H in a complete k-partite graph with the same edge partitioning as G where exactly one node of the subgraph is in each partition in G.

Let C_{S_G} be the sum of the subgraphs H in all graphs $G^{(\ell_1)(\ell_2)...(\ell_{\binom{k}{2}})}$ in S_G where each of the partitions of $G^{(\ell_1)(\ell_2)...(\ell_{\binom{k}{2}})}$ has exactly one vertex of the subgraph.

Then, $C_G = C_{S_G}$.

Proof. If a subgraph H_0 exists and has one vertex in each partition, then there is exactly one choice of $G^{(\ell_1)(\ell_2)...(\ell_{\binom{k}{2}})} \in S_G$ that will contain it. The choice of $G^{(\ell_1)(\ell_2)...(\ell_{\binom{k}{2}})}$ that picks the edge sets that H_0 's edges lay in. Every H that exists in the complete graph will appear in exactly one of these $G^{(\ell_1)(\ell_2)...(\ell_{\binom{k}{2}})}$, so the counts of both are the same.

What should you get out of this lemma intuitively? Consider what happens if we sum all H that involve exactly one edge from $E_{i,j}^1, E_{i,j}^2, \ldots$ or, $E_{i,j}^b$ (as defined in Definition 3). Then, we are getting the sum of all H that would exist if $E_{i,j}$ were complete. We can use this idea to count the subgraphs that use *particular* edge partitions, while every $E_{i,j}^{(\ell)}$ looks uniformly random. To do this count, we develop a few lemmas and then we proceed to our main counting result in Lemma 5.9.

Counting Small Subgraphs We will argue that we can count labeled subgraphs H recursively. We start by arguing the base cases. Below are give fast algorithms for counting small labled subgraphs. By counting labeled subgraphs H in a graph G with partitions V_1, \ldots, V_k , we mean that the vertex set of H is labeled with $1, \ldots, k$, and we want every copy of H to have a copy of x_i in Y_i where x_i is the vertex with label i in H.

Lemma 5.6. Let G be a graph with n nodes, m edges and k labeled partitions of the vertices $V_1, ..., V_k$ (G is not necessarily k partite).

Given a labeled k-node tree H with vertices, counting the number of such labeled trees in G takes O(m+n) time.

Proof. Pick a root of the tree H. Let (u_0, p_0) be the root and its label p_0 . Let U_i be the set of all tuples of vertices and their labels in the tree at level i. Let h be the height of the tree.

Thus, the set U_h only contains leaves, and every node in U_h has one sub-tree that includes it and no nodes below it.

For all $(u_{h-1}, p_{h-1}) \in U_{h-1}$, where p_{h-1} is the label of u_{h-1} , we look at the vertex set $V_{p_{h-1}}$. For all nodes in $V_{p_{h-1}}$ we are going to count the number of labeled sub-trees that include it and the nodes below it. We can do this in linear time over the edges between the relevant partitions. Save all the computed values.

Now, we can do this for level h-2, using our pre-existing counts. We can propagate these up the tree until we reach our root and count the total number of labeled trees H in the graph.

Lemma 5.7. Let G be a graph with n nodes, m edges and k labeled partitions of the vertices V_1, \ldots, V_k (G is not necessarily k partite).

If we have the counts of all labeled subgraphs of H in G of size less than s vertices, we can compute the number of labeled subgraphs in G that are the union of two disconnected labeled subgraphs of H of size s or less.

Proof. Let one be labeled subgraph L and the other be labeled subgraph L'. Given that they share no vertices, we can simply multiply the number of subgraphs L and L'.

Lemma 5.8. Let G be a graph with n nodes, m edges and k labeled partitions of the vertices V_1, \ldots, V_k (G is not necessarily k partite).

We can compute all counts of subgraphs in G with 2 vertices or fewer in $\tilde{O}(m+n)$ time.

Proof. All subgraphs with 1 edge are trees. So by Lemma 5.6 we can compute all subgraphs with 2 edges or fewer in $\tilde{O}(m)$ time.

The Recursive Step of Inclusion-Edgesclusion This next lemma is the core step. We will use all counts of subgraphs with a small number of edges to count those with more edges. At its core this relies on the fact that if we sum together the counts of the number of subgraphs H with all possible combinations of complimentary edge sets this roughly gives us a count of the number of subgraphs when that edge partition is a complete bipartite edge set.

Lemma 5.9. Let G be a labeled k-partite graph with n nodes per partition.

Say we are given the counts of the number of subgraphs H in all graphs S_G (see Definition 3).

Additionally, say we are given the counts of all less than or equal to v vertex labeled subgraphs of H with [0,e] edges.

Let L *be a labeled subgraph of* H *with* v *vertices and* e+1 *edges.*

Using both of these counts we can count the number of not-necessarily induced subgraphs L in G in time $O(k! \cdot 2^{k^2} + b^{k^2})$.

Proof. Let H have $v_H = k$ vertices and e_H edges. Let the subgraph L be given as a list of v vertices labeled as being in partitions i_1, \ldots, i_v and e + 1 edges between partitions i_x and i_y where $x, y \in [1, v]$. Let S_E be the set of all such pairs (x, y).

Consider \bar{S}_E , the set of all pairs of partitions not in S_E . Then consider the subset of instances in S_G where the edges between partitions in S_E (for example E_{i_1,i_2}) are all set to be the version labeled (1) $(E_{i_1,i_2}^{(1)})$. Call this subset $S_G[L]$.

Take the counts of the number of subgraphs H that appear in all graphs in $S_G[L]$ and sum them together, call this count $c_{S_G[L]}$. What will this count contain? It will count the number of subgraphs H that appear if the graph G were to have complete bipartite graphs between all pairs of partitions in \bar{S}_E , weighted by how many edges in S_E that subgraph uses. If a specific subgraph H appears in the graph G where ℓ of its edges are in the \bar{S}_E partitions then it is counted $b^{\binom{k}{2}-e-1-\ell}$ times. We include that many copies of graphs in $S_G[L]$ that include this particular H.

Given that L is a labeled subgraph of H, at least one labeling of H will share all e+1 edges and v vertices of L. There may be many valid labelings for the e_H-e-1 unaccounted for edges and k-v unaccounted for vertices.

We want to count all H that happen to have a labeling that matches the e+1 edges of L, and not count those that share only some of these edges. Luckily, given the counts of all small subgraphs we can count how many subgraphs H exist that match up only partially with L and remove these from the count $c_{S_G[L]}$.

For a subgraph to match up only partially with L, it must match up with some labeled subgraph of L, L' must have v vertices and at most e edges. We have the counts of all labeled subgraphs with v vertices and at most e edges. We want to remove from $c_{S_G[L]}$ the count of all subgraphs H that overlap with L' and share no edges with L-L'.

Let $G_{L,L'}$ be a graph on k vertices where all edges in L' are included, all edges in L-L' are excluded and all other edges are included. Let $c_{G_{L,L'}}$ be the count of the number of subgraphs H that exist in this graph. Note we can compute this in O(k!) and we do this computation on at most $O(2^{k^2})$ graphs.

Let L' have $e_{L'}$ edges and $v_{L'}$ vertices. Let $c_{L'}$ be the count of all labeled subgraphs L' that exist in G. The count of all subgraphs H which overlap exactly with L' (sharing no edges with L-L') that are counted in $c_{S_G[L]}$ is

$$c_{L'} \cdot c_{G_{L,L'}} \cdot n^{k-v_{L'}} \cdot b^{\binom{k}{2}-e-e_H+e_{L'}}.$$

Lets break down this value. First, of course the number of labeled subgraphs L' that appear in the original graph each contribute proportionally. A choice of a particular labeled subgraph L' fixes $v_{L'}$ of the k vertices, but the rest of the vertices could be any of the available n vertices per partition. Now, given a fixed choice of k vertices and e_H edges this subgraph may still appear in multiple graphs in $S_G[L]$. Specifically, it will appear in all graphs where we haven't "fixed" the edge set. This is a total of $b^{\binom{k}{2}-e-e_H+e_{L'}}$ graphs.

So, for all $O(2^{k^2})$ labeled subgraphs of L we can compute their contribution to $c_{S_G[L]}$ and subtract out this contribution. This leaves only a count of subgraphs H that overlap with L exactly. To compute the number of subgraphs L we simply divide this number by $c_{G_{L,L}} \cdot n^{k-\nu} \cdot b^{\binom{k}{2}-e_H}$.

The total time for this computation is, at most $O(2^{k^2} \cdot k! + b^{k^2})$. If $k = o(\sqrt{\lg(n)})$ and b is a constant, then this term is sub-polynomial.

Lemma 5.10. Let G be a graph with n nodes, m edges and k labeled partitions of vertices V_1, \ldots, V_k . Given the count of all labeled subgraphs of H in G with less than v vertices, we can count all labeled sub-graphs with v vertices and at most v-1 edges in $\tilde{O}(m)$ time.

Proof. There are two cases. The subgraph is connected (only possible when we have exactly v-1 edges), or it is disconnected.

If the subgraph is connected then it is a tree, by Lemma 5.6 we have can count this labeled tree in $\tilde{O}(m)$ time.

If the subgraph is disconnected then it is made up of disconnected labeled subgraphs with less than v vertices. We have the count of each of these on their own, thus by repeated applications of Lemma 5.7 we can count these with overhead the number of subgraphs which is at most v, and thus also $\tilde{O}(m)$.

Reducing to UCHGHP First we reduce counting labeled copies of H in a k-partite Erdős-Rényi graph to counting H in Erdős-Rényi graphs. We then note that by picking a particular labeling this solves the problem of UCHGHP. Finally, we use our previous reduction from CHGHP to UCHGHP to get our desired result: a reduction from CHGHP to counting H subgraphs in Erdős-Rényi graphs.

Lemma 5.11. Let H have e edges and k vertices. Let A be an average-case algorithm for counting "unlabeled" subgraphs H in k-partite Erdős-Rényi graphs with edge probability 1/b which takes T(n) time with probability $1 - \varepsilon / \left(2^k \cdot b^{k^2}\right)$.

The number of "labeled" copies of subgraph H in k-partite $Erd \tilde{o}s$ -Rényi graphs with edge probability 1/b can be computed in time $\tilde{O}(2^{k^2} \cdot m + 2^k \cdot b^{k^2} \cdot T(n))$ with probability at least $1 - \varepsilon$.

Proof. We want to count only subgraphs that use exactly one vertex from each partition. We can make 2^k calls to A using standard inclusion/exclusion to count only subgraphs with exactly one edge in each partition. Call this algorithm A'.

Let $C(v, \ell)$ be a list of tuples of all labeled subgraphs J with v vertices and ℓ edges with the associated count of the number of labeled subgraphs J in G.

By Lemma 5.10 we can compute $C(v, \ell)$ in time $|C(v, \ell)| \cdot \tilde{O}(m)$ if $\ell \le v - 1$.

By Lemma 5.9 if we can compute $C(v,\ell)$ for all $\ell \leq \ell^*$ then we can compute $C(v,\ell^*+1)$ given calls to A' on all graphs in S_G . Note each of these steps uses the *same* set of calls to A' on all graphs in S_G .

We can bound $|S_G| \le b^{k^2}$. With this we can say that we make at most b^{k^2} calls to A', meaning we make at most $2^k \cdot b^{k^2}$ calls to A.

We can bound the total sum of all $|C(v,\ell)|$ by 2^{k^2} (every possible choice of a subset of edges in the complete graph on k vertices). This gives a time bound of $\tilde{O}(2^{k^2} \cdot m + 2^k \cdot b^{k^2} \cdot T(n))$.

We make $2^k \cdot b^{k^2}$ calls to A, if they are all correct then we give the correct answer to the labeled H question. If A succeeds with probability at least $1 - \varepsilon / \left(2^k \cdot b^{k^2}\right)$, then, by the union bound $2^k \cdot b^{k^2}$ calls to A will all succeed with probability at least $1 - \varepsilon$.

Lemma 5.12. Let H have e edges and k vertices where $k = o(\lg(n)/\lg\lg(n))$. Let A be an average-case algorithm for counting subgraphs H in Erdős-Rényi graphs with edge probability 1/b which takes T(n) time with probability $1-2^{-2k} \cdot b^{-k^2} \cdot (\lg(e)\lg\lg(e))^{-\omega(1)}$

Then an algorithm exists to count subgraphs H in H-partite graphs (CHGHP) in time $\tilde{O}(T(n))$ with probability at least $1 - O(2^{-\lg^2(n)})$.

Proof. By Lemma 5.11, A implies a $\tilde{O}(T(n))$ algorithm for counting the number of labeled copies of subgraph H in k-partite Erdős-Rényi graphs with edge probability 1/b with probability $1-2^{-k}(\lg(e)\lg\lg(e))^{-\omega(1)}$.

We need to add random edges within each partition to get a truly Erdős-Rényi graph. Luckily, we can use traditional inclusion-exclusion to count how many subgraphs don't include exactly one vertex in each partition. This introduces another 2^k calls. By the union bound this causes the probability of success to be at least $1 - (\lg(e) \lg \lg(e))^{-\omega(1)}$.

Now note that counting labeled copies of subgraph H in k-partite Erdős-Rényi graphs solves UCHGHP with edge probability 1/b with a single call. Given an instance of UCHGHP label the vertices of the subgraph H in the input instance, between all other partitions add random edges with probability 1/b.

Now apply Lemma 5.4. An algorithm for UCHGHP that succeeds with probability $1 - (\lg(e) \lg \lg(e))^{-\omega(1)}$ in time T(n) implies an algorithm for CHGHP that runs in time $\tilde{O}(T(n) + n^2)$ and succeeds with probability $1 - O(2^{-\lg^2(n)})$.

REMINDER OF THEOREM 1.21 Let H have e edges and k vertices where $k = o(\sqrt{\lg(n)})$. Let A be an average-case algorithm for counting subgraphs H in Erdős-Rényi graphs with edge probability 1/b which takes T(n) time with probability $1-2^{-2k} \cdot b^{-k^2} \cdot (\lg(e) \lg \lg(e))^{-\omega(1)}$.

Then an algorithm exists to count subgraphs H in k-partite graphs in time $\tilde{O}(T(n))$ with probability at least $1 - \tilde{O}(2^{-\lg^2(n)})$.

Proof. From Lemma 5.12 we know that A implies a $\tilde{O}(T(n))$ time algorithm for counting H in H-partite graphs.

Now, given an input of a graph G that is k-partite graph we can produce all e choose $\binom{k}{2}$ graphs that have only e sets of edges between the partitions. From these we can select only those that are H partite (the number of these will vary based on H's shape). The number of these graphs is at most 2^{k^2} , which by our restriction on k is $\tilde{O}(1)$. Call the set of these H partite graphs S_H .

We use the result from Lemma 5.12 to count the results on each of these graphs. By the union bound we will get the correct answer on every graph with probability at least $1 - \tilde{O}(2^{-\lg^2(n)})$. The sum these counts over all $G' \in S_H$ is equal to the number of H in the original graph.

6 Counting OV is Easy on Average

Previous work has shown that detecting if there is at least one orthogonal vector in a set of n vectors is possible in sub-quadratic time [KW19]. So, the next natural candidate problem that we might hope to show hard with our framework would be the *counting* version of average-case Orthogonal Vectors problem (OV). However, even the counting version of orthogonal vectors has truly subquadratic algorithm, as we will prove below.

Definition 1. The counting μ -uniform d-dimensional Orthogonal Vectors problem ($\#OV^{\mu,d}$) takes as input two lists of n zero-one vectors, where each vector is d-dimensional. All $2 \cdot n \cdot d$ bits are chosen iid where a one is selected with probability μ . The output is the *count* of the number of vectors that are orthogonal (whose dot product is zero).

We will consider constant μ for this section. We built up a few lemmas to prove the following theorem. **REMINDER OF THEOREM 1.18** For all constant values of μ and all values of d there exists constants $\varepsilon > 0$ and $\delta > 0$ such that there is an algorithm for $\#OV^{\mu,d}$ that runs in time $\tilde{O}(n^{2-\delta})$ with probability at least $1 - n^{-\varepsilon}$.

We start by showing that if vectors are very long we are unlikely to have an orthogonal vector pair.

Lemma 6.1. A # $OV^{\mu,d}$ instance has at most a $n^2 \cdot e^{-\mu^2 \cdot d}$ probability of having at least one pair of orthogonal vectors

Proof. Any given pair of vectors has a probability of $(1 - \mu^2)^d$ of being an orthogonal pair. The probability that some vector is an orthogonal pair is at most $n^2 \cdot (1 - \mu^2)^d$ which is at most $n^2 \cdot e^{-\mu^2 \cdot d}$.

Lemma 6.2. If $d > (1+\delta)2\lg(e)\lg(n)$ for some constant $\delta > 0$ then there is a constant $\mu = (1+\delta)^{-1/4}$ such that $\#OV^{\mu,d}$ instance has at least a $1-1/n^{\varepsilon}$ probability of having no orthogonal vectors for some constant ε .

Proof. Using Lemma 6.1 and plugging in our value of d we have that the probability of an $\#OV^{\mu,d}$ instance having an orthogonal vector is at most $n^2 \cdot (n^2)^{-(1+\delta)\mu^2}$. If $\mu = (1+\delta)^{-1/4}$ then we can bound the probability by $n^{2(1-(1+\delta)^{1/2})}$. For $\delta > 0$ we have that $(1+\delta)^{1/2} > 1$, and so $1-(1+\delta)^{1/2}$ is a negative constant. Thus there is some positive constant ε (for example $\varepsilon = -2(1-(1+\delta)^{1/2})$) such that the probability there are no orthogonal vectors in a $\#OV^{\mu,d}$ instance is at least $1-1/n^{\varepsilon}$.

A straightforward Corollary of Lemma 6.2 is the following.

Corollary 6.3. For all constant μ there is a constant $\delta = 1/\mu^4 - 1$ such that for $d > (1+\delta)2\lg(e)\lg(n)$ a $\#OV^{\mu,d}$ instance has at least a $1 - 1/n^{\varepsilon}$ probability of having no orthogonal vectors for some constant ε .

We use the following theorem appearing in [CW16] in the proof of Theorem 1.18.

Theorem 6.4. Given a vector of dimension $d = c\lg(n)$ there is a $\tilde{O}(n^{2-1/O(\lg(c))})$ time algorithm that succeeds with probability 1 on instances of $\#OV^{\mu,d}$ in returning the count of the number of orthogonal vector pairs for every vector if one exists, regardless of μ . [CW16]

Finally, we return to the proof of Theorem 1.18. We show that even the counting version of the uniform average-case OV has a subquadratic algorithm.

Proof of Theorem 1.18. Let the dimension be $d = c \lg(n)$. By Corollary 6.3 if $c > 2 \lg(e)/\mu^4$ then there is some $\varepsilon > 0$ such that there are no orthogonal vectors with probability at least $1 - n^{-\varepsilon}$. Notably, this gives us an $\tilde{O}(d)$ time algorithm where we return a count of zero if the dimension is larger than $2 \lg(e) \lg(n)/\mu^4$ that succeeds with probability at least $1 - n^{-\varepsilon}$.

When $c \leq 2\lg(e)/\mu^4$ we will run the algorithm from Theorem 6.4. This runs in $\tilde{O}(n^{2-1/\lg(c)})$ time and is correct with probability 1. This is at its worst a run time of $\tilde{O}(n^{2-1/\lg(2\lg(e)\mu^{-4})})$. So $\delta = \mu^4/\lg(2\lg(e)\mu^{-4})$, μ is a constant so δ is also a constant.

7 Counting to Detection Reduction for Average-Case ZkC

In fine-grained complexity the primary technique used for worst-case to average-case reductions has used the technique described by [BRSV17]. This technique produces average-case hardness for computing the output of functions over a finite field. These problems are fundamentally counting problems. The issue with counting problems is that they are much harder to build cryptographic objects out of.

Here we give a reduction from Counting to Detection for ZkC in the average case (ACZkC). Notably, such a reduction **does not exist** in the worst case in fine-grained complexity. This makes the assumption that average case ZkC detection with high probability requires $n^{k-o(1)}$ time more plausible. The assumption that ZkC detection is hard with probability 1/100 can be used to make fine-grained public-key cryptography [LLV19] (though the assumption that average-case ZkC is hard with probability $1-n^{-o(1)}$ should be sufficient). There is a gap here between the probabilities we are describing, $1-1/\Omega(n^k)$, and the probabilities used for fine-grained cryptography, $1-1/n^{o(1)}$. However, this makes a step forward in closing the gap between the problems we can show are average-hard from worst-case assumptions and those we can build cryptography from.

Let us define average-case ZkC.

Definition 1. An average case instance of ZkC (ACZkC) with range R takes as input a complete k-partite graph with n nodes in each partition. Every edge has a weight chosen iid from [0, R-1]. A clique is considered a zero k clique if the sum of the edges is zero mod R.

The idea of our reduction from counting to detection uses the fact that average-case ZkC is easy when R is small and there are very few solutions when R is large. In the worst-case we can reduce detecting ACZkC to counting $n^{k-\varepsilon}$ ACZkCs when $\varepsilon > 0$. So, intuitively we are using the fact that when R is small we can use a fast algorithm for counting. When R is larger there are $n^{k-\varepsilon}$ solutions, so we can use a reduction to show that faster detection solves counting those small number of solutions.

First we will prove that when the range is small there is a fast algorithm. Then, we will show that a search algorithm counts very well when the range is exactly $R = n^k$. We will then show that this gives a generic counting to search reduction. Next, we will provide a search to decision reduction. Finally, we will give the counting to detection statement.

Note that throughout this section we assume the function p(n) is a monotonically *non-increasing function*. Additionally, when we say an algorithm succeeds in the average case with probability p, this is randomness over *both* the input and the random coins flipped in the algorithm.

Small Range is Easy

Lemma 7.1. There is a $\tilde{O}(R^2n^{\omega\lceil k/3\rceil})$ time algorithm for ACZkC.

Proof. Take the graph as a k-partite graph. Group together k/3 partitions of nodes. If k is not a multiple of 3 then make groups of $\lceil k/3 \rceil$ partitions and $\lfloor k/3 \rfloor$ partitions. Then, in each group of partitions create a node for every possible set of $\lceil k/3 \rceil$ or $\lfloor k/3 \rfloor$ nodes one from each partition. The total number of nodes is $O(n^{\lceil k/3 \rceil})$.

Consider two nodes v and u where v represents x nodes and u represents y nodes. Add an edge between u and v only if all x+y represented nodes form a clique. The weight on the edge between u and v is the sum of half the weight of all edges within the clique of x nodes represented by v, half of the weight of all the edges within the clique of y nodes represented by u, and the weight of all edges going between the x nodes in v and the y nodes in u.

Now, the weights of the edges are still in the range [0, R]. We want to find a zero triangle in this new graph. We can guess the edge weights of two of the edges in the triangle, which forces the third value. Then, we produce a graph with only the edges of the guessed weights, then use matrix multiplication. All told this takes $O(R^2 \cdot (n^{\lceil k/3 \rceil})^{\omega})$ time. This can be simplified to $\tilde{O}(R^2 n^{\omega \lceil k/3 \rceil})$ time.

We can have a slight improvement in the running time of Lemma 7.2.

Lemma 7.2. There is a $\tilde{O}(R^2n^{(\omega(k-2)/3)+2})$ time algorithm for ACZkC.

Proof. Let g be the largest integer such that $3g \le k$. Note that $3g \ge k - 2$.

If 3g = k then by Lemma 7.1 an algorithm exists which runs in time $\tilde{O}(R^2 n^{\omega \lceil k/3 \rceil})$ time, which is $\tilde{O}(R^2 n^{(\omega(k-2)/3)+2})$.

If 3g = k - 1 then pick one partition, for every node in this partition we create a zero k - 1 clique instance and use Lemma 7.1 to get a $\tilde{O}(R^2n^{\omega\lceil(k-1)/3+1\rceil})$ time algorithm, which is $\tilde{O}(R^2n^{(\omega(k-2)/3)+2})$.

If 3g = k-2 then pick one partition, for every node in this partition we create a zero k-2 clique instance and use Lemma 7.1 to get a $\tilde{O}(R^2n^{(\omega(k-2)/3)+2})$ time algorithm.

Lemma 7.2 gives the following corollary.

Corollary 7.3. If $R = O(n^{(k-2-\omega(k-2)/3-\varepsilon)/2})$ then there is a $\tilde{O}(n^{k-\varepsilon})$ time algorithm for ACZkC with range R.

High Probability Counting for $R = n^k$ When the range is n^k we want to count efficiently with very high probability. We will do this by first proving two helper lemmas.

Lemma 7.4. The probability that an instance of ACZkC with range $R = n^k$ has at least $k^k \lg^{2k}(n)$ solutions is $2^{-\Omega(\lg^2(n))}$.

Proof. If there are at least $(k \lg^2(n))^k$ zero cliques then there is at least one set of $\lg^2(n)$ cliques such that each zero clique has at least one node not shared by any other zero clique. After all at least $k \lg^2(n)$ distinct nodes must be involved in these $(k \lg^2(n))^k$ zero cliques.

If a zero clique has a node not shared with the other cliques then whether or not it is a zero clique is uncorrelated with the other zero cliques. So, the probability that there are $(k \lg^2(n))^k$ zero cliques is at most the probability that out of n^k independent trials $\lg^2(n)$ return true when the probability of a trial returning true is $1/R = 1/n^k$. By the Chernoff bound we get the probability of this event is less than $2^{-\lg^2(n)/3}$ which is $2^{-\Omega(\lg^2(n))}$.

Lemma 7.5. Using a search algorithm, \mathscr{A} , that succeeds with probability 1-p on an instances of ACZkC with $n/(k\lg^2(n))$ nodes per partition and edge weights in the range $R=\theta(n^k)$ in time T(n) we can count the number of solutions (or list all those solutions) in a ACZkC instance in time $\tilde{O}(T(n)+n^{k-1})$ with probability at least $1-pk^k\lg^{2k+2}(n)-2^{-\Omega(\lg^2(n))}$.

Proof. Let the input ACZkC instance be the graph G with edge set E and vertex set V. First, note that with probability $1 - 1/2^{\Omega(\lg^2(n))}$ there are at most $s = k^k \lg^{2k}(n)$ zero k-cliques (ZKCs).

Now consider a given ZKC c in G. Imagine creating a new instance G' that is a subset of G by selecting a random subset of n/x nodes from each partition. The ZKC c is in G' with probability x^{-k} . Now consider a clique c' which is in G and shares no nodes with c. Given that c is in G' the probability that c' is also in G' is at most x^{-k} . If there are at most ℓ cliques in G then the probability that a given clique c is in G' and no disjoint cliques (cliques that share no vertices with c) are in G' is at least: $x^{-k}(1-x^{-k})^{\ell}$. Further note that the sub-graph G' has total variation distance 0 from ACZkC instances with n/x nodes per partition and range n^k .

Consider the algorithm \mathcal{B}_x . It creates an empty set S_B that it will fill with cliques it finds. It generates G' at random by selecting a random set of n/x nodes from each partition. Then it runs \mathscr{A} on G'. If \mathscr{A} returns a clique c, check that it is a ZKC. If it is, further exhaustively check that there is no clique that shares a node with c this takes $O(k(n/x)^{k-1})$ time (you can simply check all sets of k nodes involving one node in the clique). Any cliques it finds in this search are added to S_B and S_B is returned. \mathscr{B}_x takes $O(T(n) + n^{k-1})$ time. If:

- 1. a ZKC c is in G',
- 2. A returns correctly, and
- 3. there are no ZKCs in G' which share no vertices with c

then \mathscr{B}_x will include c in S_B . Because \mathscr{A} returned a ZKC and it was either c or a clique that shared a node with c. In the later case our exhaustive search would find it. Given a specific clique c and \mathscr{A} returning correctly $c \in S_B$ with probability $x^{-k}(1-x^{-k})^{\ell}$.

Consider the case where $\ell \leq s$ and $x = k \lg^2(n) = s^{1/k}$. Then $\mathcal{B}_{k\lg^2(n)}$ returns a given c with probability at least $s^{-1}(1-s^{-1})^s \geq \frac{1}{4s}$. If \mathscr{A} is returning correctly every trial is independent. Thus if we run $\mathscr{B}_{k\lg^2(n)} = 4s\lg^2(n)$ times we will find the clique c with probability at least $1 - (1 - 1/(4s))^{4s\lg^2(n)} \geq 1 - 2^{-\Omega(\lg^2(n))}$. The probability we find all the ZKCs (given that there are at most s ZKCs) is, by union bound at least $1 - s2^{-\Omega(\lg^2(n))} = 1 - 2^{-\Omega(\lg^2(n))}$.

After making $4s \lg^2(n)$ calls to $\mathscr{B}_{k\lg^2(n)}$ we will have made $4s \lg^2(n)$ calls to \mathscr{A} . Using union bound all of these will succeed with probability at least $1 - 4s \lg^2(n) p$.

So the time we take is $O(4s\lg^2(n)T(n) + 4s\lg^2(n)n^{k-1})$ which is $\tilde{O}(T(n) + n^{k-1})$. Our success probability requires the union of the number of cliques being less than s, \mathscr{A} returning correctly on all calls, and the randomness in $\mathscr{B}_{k\lg^2(n)}$ allowing us to return all cliques. Thus our probability of success is at least $1 - 4s\lg^2(n)p - 2^{-\Omega(\lg^2(n))} - 2^{-\Omega(\lg^2(n))}$. This can be simplified to a success probability of $1 - 4k^k\lg^{2k+2}(n)p - 2^{-\Omega(\lg^2(n))}$.

Counting to Search We will start by describing the self reduction for ACZkC. This is a folklore self-reduction in the worst case and was analyzed in the average case in [LLV19].

Lemma 7.6. Given an instance, I, of average case ZkC with range R with kn nodes it can be split into $(n/x)^k$ instances $I_1, \ldots, I_{(n/x)^k}$ each with kx nodes such that:

- 1. The distribution over each I_i is the average case distribution with kx nodes and range R. (Though two instances I_i and $I_{i'}$ may be correlated.)
- 2. The number of solutions in instance I (#Solutions(I)) is equal to the sum of solutions in all the instances $I_1, \ldots, I_{(n/x)^k}$ ($\sum_{i=1}^{(n/x)^k}$ #Solutions(I_i)).

Proof. Note the k-partite graph in the instance I and note each partition of vertices V_1, \ldots, V_k . We create a random partition of each vertex set into n/x sets of x vertices. Name the subsets of V_i , $V_i[j]$ where $j \in [1, n/x]$. The $(n/x)^k$ subinstances are formed by taking the intersection of k subsets one from each of the k partitions: $V_1[j_1] \cup \ldots \cup V_k[j_k]$ for all possible k tuples $(j_1, \ldots, j_k) \in [n/x]^k$.

For the first claim, note that for any given instance I_i we simply have a random selection of n/x nodes from an average case instance. So every edge is chosen iid from [R]. This is indeed the distribution of an average case ZkC instance. We will note that two separate instances may be correlated. For example the instance formed by $V_1[j_1] \cup V_2[j_2] \cup V_3[j_3] \ldots \cup V_k[j_k]$ and the instance formed by $V_1[j_1] \cup V_2[j_2] \cup V_3[j_3'] \ldots \cup V_k[j_k']$ will share all edges between sections $V_1[j_1]$ and $V_2[j_2]$. Of course union bounds can still be used to bound error between these $(n/x)^k$ instances.

For the second claim, any ACZkC witness has k nodes one from each partition: $v_1 \in V_1, \dots, v_k \in V_k$. Every witness appears in exactly one sub-instance. A given witness (v_1, \dots, v_k) will appear only in the instance formed by a union of the subsets $V_1[j_1] \cup \dots \cup V_k[j_k]$ where $v_i \in V_i[j_i]$ in every subset.

Lemma 7.7. Let p(n) be a monotonically non-increasing function.

Assume an algorithm exists for the search version of ACZkC with range $R \in [k^k \lg^{2k}(n)n^k, 2k^k \lg^{2k}(n)n^k]$ that succeeds with probability at least 1 - p(n) and runs in time $O(n^{k-\varepsilon})$ where $\varepsilon > 0$. Let $k' = 2 + \omega(k-2)/3$. Then there is an algorithm for counting the number of ZkC in an average case instance for any positive integer R with probability at least $1 - 2^{-\Omega(\lg^2(n))} - p\left(n^{(k-k'-\delta)/(2k)}/(k\lg^2(n))\right) \cdot n^{(k'+\delta)/2} \cdot k^k \lg^{k2}(n)$ that runs in time $\tilde{O}(n^{k-\delta})$ for some constant $\delta > 0$.

Proof. Let us call the search algorithm \mathscr{A} . There are two cases to consider. $R \leq n^k$ and $R > n^k$.

If $R > n^k$ then we can use nearly linear hashing (see [Pat10]) to reduce our range down to n^k . There may be false positives here, however, the instance will look uniformly random (we are hashing large uniformly random numbers). So we can use Lemma 7.4 to say that there will be at most $s = k^k \lg^{k2}(n)$ solutions (false positives or true positives) with probability at least $1 - 2^{-\Omega(\lg^2(n))}$. Now, we can use the algorithm from Lemma 7.5 to list all solutions with probability at least $1 - p(n/(k\lg^2(n)))s - 1/2^{\Omega(\lg^2(n))}$. For each listed solution we can check if it is a false positive and only count the actual cliques. This will return the true number of cliques with probability at least $1 - p(n/(k\lg^2(n)))s - 2^{-\Omega(\lg^2(n))}$. This requires s calls to s so it takes time s. This constrains s calls to s.

If $R \le n^{(k-k'-\varepsilon')/2}$ where $k' = 2 + \omega(k-2)/3$, then by Corollary 7.3 there is a $O(n^{k-\varepsilon'})$ time algorithm that succeeds with probability 1.

If $n^{(k-k'-\varepsilon')/2} < R < n^k$ then we will use the average case self reduction for ACZkC (see Lemma 7.6 or [LLV19]) to reduce the problem to problems of size $x = R^{1/k}$, so now we have that $R = x^k$ where x is our new smaller input size. We can now call $\mathscr A$ on all these instances. Note that $s \ge n^{(k-k'-\varepsilon')/(2k)}$. Further note that the total number of instances is $(n/x)^k \le n^{(k'+\varepsilon')/2}$. So, if $\mathscr A$ succeeds with probability 1 - p(n) then by union bound these independent instances will succeed with probability at least $1 - p(x/(k \lg(x))) \cdot (n/x)^k$.

Now note that this is at least $1 - p(n^{(k-k'-\varepsilon')/(2k)}) \cdot n^{(k'+\varepsilon'/(k\lg(n)))/2}$. If there is an algorithm running in time $O(x^{k-\varepsilon})$ for all $(n/x)^k$ problems then the running time is $\tilde{O}(n^{k-\varepsilon(k-k'-\varepsilon')/(2k)})$. Notably $\varepsilon(k-k'-\varepsilon')/(2k) > 0$

We want $\delta < \varepsilon'$ and $\delta < \varepsilon(k-k'-\varepsilon')/(2k) < \varepsilon/2$. If we choose $\delta < \varepsilon' = \varepsilon/2$ then this meets all of our constraints. In every case the algorithm succeeds with probability at least $1 - 2^{-\Omega(\lg^2(n))} - p(n^{(k-k'-\delta)/(2k)}) \cdot n^{(k'+\delta)/2} \cdot s$ and runs in time $\tilde{O}(n^{k-\delta})$ when $\delta < \varepsilon' = \varepsilon/2$.

Search to Decision

Lemma 7.8. Let p(n) be a monotonically non-increasing function.

Given a detection algorithm that runs in $O(n^{k-\delta})$ time for some $\delta > 0$ and has success probability at least 1 - p(n) we can produce a search algorithm that runs in time $\tilde{O}\left(n^{k-\epsilon k} + n^{k-\delta(1-\epsilon)}\right)$ for any constant $1/2 > \epsilon > 0$ and has success probability at least $1 - p(n^{1-\epsilon})n^{k\epsilon}$.

Specifically for $\varepsilon = 1/2$ this can be bounded as $\tilde{O}(n^{k-\delta/2})$ time and probability at least $1 - p(n^{1/2})n^{k/2}$.

Proof. We use the classic self reduction for ACZkC producing instances of size $n^{1-\varepsilon}$. For this we randomly split each partition of vertices into n^{ε} groups of $n^{1-\varepsilon}$ nodes. We form all $n^{k\varepsilon}$ possible sub-problems and run the detection algorithm on them. On any instance that returns true we brute force the problem in $O(n^{k-k\varepsilon})$ time. We of course can stop as soon as we find a clique.

The probability that none of our $n^{k\varepsilon}$ instances produces a false positive is at least $1 - p(n^{1-\varepsilon})n^{k\varepsilon}$. If we have no false positives then our running time is $O(n^{k-k\varepsilon} + n^{k\varepsilon}(n^{1-\varepsilon})^{k-\delta})$. This can be simplified to $O(n^{k-k\varepsilon} + n^{k-\delta(1-\varepsilon)})$

Counting to Decision

Lemma 7.9. Let p(n) be a monotonically non-increasing function.

Given a decision algorithm for ACZkC that runs in time $O(n^{k-\varepsilon})$ for some $\varepsilon > 0$ and succeeds with probability at least 1 - p(n) there is a counting algorithm that runs in $O(n^{k-\varepsilon'})$ for some $\varepsilon' > 0$ and succeeds with probability at least $1 - 2^{-\lg^2(n)} - p(n^{1/25})n^k$.

Proof. Use Lemma 7.8 when $\varepsilon = 1/2$ and Lemma 7.7. When combing our numbers we find that the probability is at least $1 - p(n^{1/25})n^k$.

Note this is not tight, by tuning ε and plugging in an improved value for the matrix multiplication constant you will get a tighter result. This bound is sufficient for our purposes so we leave it as is.

REMINDER OF THEOREM 1.19 Given a decision algorithm for ACZkC that runs in time $O(n^{k-\varepsilon})$ for some $\varepsilon > 0$ and succeeds with probability at least $1 - n^{-\omega(1)}$, there is a counting algorithm that runs in $O(n^{k-\varepsilon'})$ time for some $\varepsilon' > 0$ and succeeds with probability at least $1 - n^{-\omega(1)}$, where $\omega(1)$ here means any function that is asymptotically larger than constant.

Proof. We plug in $n^{-\omega(1)}$ for p(n) in Lemma 7.9. Note that the second error term, $2^{-\lg^2(n)}$ is $n^{-\omega(1)}$.

8 Future Work

Average-case fine-grained complexity still has a lot of unexplored areas. We suggest the following open problems that directly relate to results of this work.

General Questions What other natural non-factored problems are hard from factored problems (either Fk- \mathfrak{f} and $F\mathfrak{f}kC$)? We give three problems in section 4 where we only show their detection version is hard. Can one show that a counting version of (k+1)L-MF, k-LCS, or Edit Distance is hard from counting factored problems? Recall that such a reduction would imply average case hardness over some distribution for the problem reduced to. We show hardness for the uniform average case for #Fk- \mathfrak{f} and $\#F\mathfrak{f}kC$, can one show hardness for other natural worst case distributions of these problems?

Cryptography and Counting vs Detection In Section 7 we show that detecting ZkC with high probability in the average case implies fast algorithms for counting with high probability in the average case.

- Counting to detection in the high error regime: Can you show that a detection algorithm for average-case ZkC that succeeds with probability 1-1/(polylog(n)) implies an algorithm for counting ZkC with probability 2/3? If such a reduction exists in the high error regime you can build cryptography protocols from an assumption about the difficulty of counting ZkC on average [LLV19].
- Worst case ZkC to counting ZkC on average: Can we reduce the worst case hardness of ZkC to average case #ZkC? What about k-SUM? If you can prove this for ZkC and prove the previous higherror regime reduction, then you can build fine-grained cryptography from a worst-case assumption about the complexity of ZkC.
- Counting to detection for other problems: A similar proof technique that we use for ZkC should work for the 3-SUM problem. For this style of reduction we need: (1) an efficient average-case self-reduction for the problem, (2) the number of witnesses to be small on average when some parameter R is large, and (3) an efficient algorithm when R is small. All of these exist for 3-SUM, however, there isn't an efficient self reduction for k-SUM for k > 3. Can another approach work to show counting to detection results for problems like k-SUM, k-LCS, etc?

Using/Extending the Good Low-Degree Polynomial Framework A few directions that could be taken with respect to our framework are the following:

- Can the framework be extended to handle multiple outputs? For example, the problem of multiplying two zero-one matrices?
- Can we find new problems P that have GLDP(P)?
- Can the framework be improved? For example, could it be improved to handle polynomials of (slightly) greater degree? Can the strong *k*-partiteness condition be weakened?

LCS and Edit Distance In Section 4.6 we cover LCS and Edit Distance. We have two open problems from this section we want to highlight.

• Making a framework for string distance lower bounds from factored problems: Bringmann and Künnemann [BK15] create a framework for proving $n^{2-o(1)}$ lower bounds from SETH. We believe this same framework can be extended to work for F2-OV by adding a requirement of a selection gadget. It also seems that this framework could be extended to contain Fk-OV. Relatedly, can k-median distance and k-center-edit-distance be reduced to Fk-OV?

• Getting tight hardness for #k-LCS or #k-WLCS: We note that the counting versions of k-LCS and k-WLCS both have algorithms that run in time $n^{k+o(1)}$ (see Appendix A). Given our construction, the counting versions of k-LCS and k-WLCS count $\odot(v_1,\ldots,v_k)$ is given as input the k strings $FVG(v_1)$, ..., $FVG(v_k)$. However, unfortunately, the counting versions of k-LCS and k-WLCS do not return # Fk-OV given our construction of P_1,\ldots,P_k . This is due to using an alignment gadget instead of a selector gadget. If we used the selector gadget, the count of longest common subsequences would be the sum over the counts of all $FVG(v_1),\ldots,FVG(v_k)$ where $\odot(v_1,\ldots,v_k)>0$. This would result in the count being exactly the output of # Fk-OV. However, the strings produced by our reduction would have length n^2 and weights of size $\tilde{O}(n)$. So, we would get a lower bound of $n^{k/2-o(1)}$ for #k-WLCS, and a lower bound of $n^{k/3-o(1)}$ for #k-LCS. A more efficient selector gadget would yield tight lower bounds for # k-LCS and # k-WLCS, including in the average-case. We suggest this as a potential topic for future work.

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A Removed Algorithms

A.1 Algorithms for Factored Problems

These algorithms are straightforward be case they are simply brute force.

Lemma A.1. Fk-f can be solved in $\tilde{O}(n^k)$ time.

Proof. For Fk-f, we want to run $\odot(v_1,\ldots,v_k)$ on every set of k vectors. To do this we need to compute $\circ(v_1[i],\ldots,v_k[i])$ for all $i\in[0,g]$. Running $\circ(\cdot)$ takes $O(\Pi_{j\in[1,k]}|v_j[i]|)$ time. We can use the upper bound $|v_j[i]| \leq 2^b$. So computing $\circ(\cdot)$ takes at most $O(2^{bk})$ time. Thus, computing $\circ(\cdot)$ takes at most $O(g \cdot 2^{bk})$. Thus, computing Fk-f takes at most $O(n^k \cdot g \cdot 2^{bk})$ time. We bounded g and g to be g to g takes at most g to g takes at most g

Thus, computing Fk-f takes at most $O(n^k \cdot g \cdot 2^{bk})$ time. We bounded g and b to be $o(\lg(n))$ so $g \cdot 2^{bk}$ is subpolynomial. Thus, Fk-f takes at most $\tilde{O}(n^k)$ time.

Lemma A.2. FfkC (#FfkC) can be solved in time $\tilde{O}(n^k)$.

Proof. For every k tuple of nodes in the graph we want to evaluate $\odot(\cdot)$. If we can evaluate $\odot(\cdot)$ in $\tilde{O}(1)$ time then we can count or detect in $\tilde{O}(n^k)$ time.

Evaluating $isClique(\cdot)$ can be done in $\tilde{O}(1)$ time. Evaluating the multiplication, given the results of the function $\circ(\cdot)$ can be done in $\tilde{O}(g)$ time. Evaluating $\circ(\cdot)$ should require at most $\tilde{O}(2^{\binom{k}{2}b})$ time as the function f has a total truth table size of $2^{\binom{k}{2}b}$ and we simply need to evaluate how many entries of the truth table are 1 while we simultaneously have that vector.

Finally, we note that $g = \tilde{O}(1)$ and $2^{\binom{k}{2}b} = \tilde{O}(1)$. So evaluating $\odot(\cdot)$ can be done in $\tilde{O}(1)$ time.

A.2 Algorithms for Problems Harder than Factored Problems

Theorem A.3. Counting partitioned matching triangles (#PMT) can be solved in $\tilde{O}(n^3)$ time.

Proof. Let the g graphs be G_1, \ldots, G_g in our PMT instance, and let n_r^j be the number of nodes of color r in G_j . For all triple of colors (c_1, c_2, c_3) and all j, we count the number of triangles of these colors in G_j . We can do this by inspecting every triple of nodes of color (c_1, c_2, c_3) in time $n_{c_1}^j n_{c_2}^j n_{c_3}^j$. Since $\sum_r n_r^j = n$ for all j, we have that $\sum_{j=1}^g \sum_{(c_1, c_2, c_3)} n_{c_1}^j n_{c_2}^j n_{c_3}^j = O(gn^3) = \tilde{O}(n^3)$.

Theorem A.4. (Counting mod R) k-NLstC has a $\tilde{O}(|C|^k + |C|^{k-2}|E|)$ time algorithm for all $k \geq 2$ (when $\lg(R)$ is sub-polynomial).

Proof. We can guess the k colors and use BFS to discover if s is connected to t. This takes O(|E|) time.

If we are counting instead of detecting paths from s to t then we want to extend the BFS approach by associating an additional number to each node. Every node will keep a value of the number of paths from s to that node mod R. These numbers will require a sub-polynomial number of bits to represent as $\lg(R)$ is bounded to be subpolynomial. In a layered graph we can compute the number of paths from s to a node v in layer i by summing the number of paths from s to u for all u that are neighbors of v that are in layer i-1. We can go through the graph by computing these numbers layer by layer staring at layer L_0 . This also takes O(|E|) time.

Let E_{c_i,c_j} be the set of all edges between nodes of colors c_i and c_j . Let E_{s,c_i} be the set of all edges between s and nodes with color c_i . Let E_{t,c_i} be the set of all edges between t and nodes with color c_i .

Our running time is:

$$\sum_{c_1, c_2, \dots, c_k \in C} \left(\left(\sum_{i \in [1, k]} |E_{s, c_i}| + |E_{t, c_i}| \right) + \sum_{i, j \in [1, k]} |E_{c_i, c_j}| \right).$$

We know that $|E_{s,c_i}| + |E_{t,c_i}| \le 2$ from the problem definition. We also know that $|E_{c_i,c_i}| = 0$. So we can simplify to:

$$\tilde{O}\left(k|C|^k + k^2 \sum_{c_1, c_2, \dots, c_k \in C} |E_{c_1, c_2}|\right).$$

Then we can use the fact that $\sum_{c_1,c_2\in C} |E_{c_1,c_2}| = |E|$ and that k is a constant to get:

$$\tilde{O}\left(|C|^k+|C|^{k-2}|E|\right)$$
.

Theorem A.5. There is an algorithm for (counting mod R) k-ELstC that runs in time $\tilde{O}(|C|^{k-1}|E|)$ (when $\lg(R) = n^{o(1)}$).

Proof. We do an exhaustive search for all k colors. Once we guess k colors c_1, \ldots, c_k then we simply run a O(|E|) time algorithm for (directed/undirected) reachability on this input. If we are counting paths mod R then we use the fact that the graph is a directed acyclic graph to count the number of paths from s to every node, so we can go through a normal breadth first search but keeping the count mod R. Because $\lg(R) = n^{o(1)}$ we can track these sums in sub-polynomial time.

We start by sorting our edges by their color (so that given a guess of colors we can in $k \lg(n)$ time give pointers to the full set of all edges of that color). Let e(c) be the number of edges of color c. Then our running time can be given as:

$$\sum_{c_1,...,c_k \in [1,|C|]} e(c_1) + ... + e(c_k) + k \lg(n).$$

Consider a particular one of the additive parts of this sum: $\sum_{c_1,...,c_k \in [1,n]} e(c_i)$. This can be re-written as:

$$\sum_{c_1,...,c_{i-1},c_{i+1},...,c_k \in [1,|C|]} \left(\sum_{c_i \in [1,|C|]} e(c_i) \right).$$

Which is

$$\sum_{c_1,\dots,c_{i-1},c_{i+1},\dots,c_k\in[1,|C|]}|E|=|C|^{k-1}|E|.$$

So the total running time is $k|C|^k + k|C|^{k-1}|E|\lg(R)\lg(n) + |E|\lg(n)$. The $k|C|^k$ time comes from running all of our small instances. The $k\lg(n)$ coming from the need to give pointers into where our k colors of edges are stored. The factor of $\lg(R)$ comes from tracking the count mod R in the counting version. And finally, the $|E|\lg(n)$ comes from sorting our edges according to color.

Theorem A.6. There is an algorithm for detecting (k+1)L- MF^* on an n-node graph that runs in $\tilde{O}(n^k)$ time.

Proof. First we run a max flow algorithm on the graph to obtain the value |F| of the max flow. Since the graph is unit-capacitated, this can be done in $O(n\sqrt{m}) = O(n^2)$ time.

Recall that the edges connected to the source s and the sink t have a special label l^* . So this label must be among the k+1 labels. Now for any choice of k labels l_1, \ldots, l_k , we consider the subgraph induced on the edges with labels in l_1, \ldots, l_k, l^* , and we run a max flow algorithm on this graph. If the max flow value on this graph equals |F|, we are done. Otherwise if all these graphs have maximum flow less than |F|, there is no max flow with k+1 labels. Note that the max flow in each small graph takes o(n) time since for each label the number of edges with that label is o(n).

Now we turn to regular expressions matching problem, and state an efficient algorithm for counting the number of alignments of the pattern on sub-strings of the text. First we state two lemmas.

Lemma A.7. Let M be an NFA with no cycles of length more than 1. Let a computation of a string t in M be a sequence of states from the start state to the accept state of M that produces t. Then given a text T and a fixed integer R where $\log R$ is sub-polynomial, there is an algorithm that computes the number of computations of substrings of T in M mod R in O(m|T|) time, where m is the number of edges of M.

Proof. All numbers are taken mod R. Let Q be the set of states of M, and let $\Delta: Q \times \Sigma \to P(Q)$ be the transition function of M, where Σ is the alphabet, and P(Q) is the power set of Q. Recall that we have an edge from state s to state s' if $s' \in \Delta(s, \sigma)$ for $\sigma \in \Sigma \cup \{\varepsilon\}$, where ε is the empty string. Note that for any state $s \in Q$, $s \notin \Delta(s, \varepsilon)$. We can assume that there is only one accept state with no outgoing edge (and hence no self-loops).

Since M has no cycles other than self-loops, it has a topological ordering s_1, s_2, \ldots, s_r where s_1 is the start state, s_r is the accept state, r is the number of states of M and there is no edge from state s_i to s_j if i > j. We compute the number of computations of substrings of T in M by dynamic programming. Let |T| = n, and let T_i be the postfix of T starting at i for $i = 1, \ldots, n+1$, where T_{n+1} is the empty string. Let M_i be the NFA obtained from M by having s_i as the start state. For $i = 1, \ldots, r$ and $j = 1, \ldots, |T|$, let f(i, j) be the number of computations of prefixes of T_j by M_i . So $\sum_{i=1}^n f(1, j)$ is what we have to compute.

As the base case, we have that f(r, n+1) = 1. Let $N_{out}(s_i)$ be the set of outgoing neighbors of s_i , i.e. we have that $s_j \in N_{out}(s_i)$ if there is an edge from s_i to s_j . Similarly we define $N_{in}(s_i)$ to be the set of incoming neighbors of s_i .

Fix i, j. Suppose that we have computed f(i', j') for all $i' \ge i$ and $j' \ge j$ where i' + j' > i + j. We compute f(i, j) as follows.

$$f(i,j) = \sum_{s_{\ell} \in \Delta(s_i, T[j])} f(i+1, \ell) + \sum_{s_{\ell} \in \Delta(s_i, \varepsilon)} f(i, \ell)$$

Note that $s_i \notin \Delta(s_i, \varepsilon)$, so we can compute this sum, which takes $O(|N_{out}(s_i)|)$ to compute. Hence the computation of all f(i, j)s takes O(mn) time.

Lemma A.8. If E is a regular expression of the type T_0 (see Figure 6), there is an NFA equivalent to E that has no cycle of length more than 1. This MFA has O(|E|) edges.

Proof. Let a sub-type of a regular expression type T be a type shown by a sub-tree of the tree of T. We show that for any regular expression of type T_0 or any sub-type of T_0 , there is an NFA equivalent to E that has no cycle length more than 1. Recall that ε is the empty string.

So let E be a regular expression of any sub-type of T_0 . We construct the NFA of E in a recursive manner. As the base case, suppose that E has length 1. So it consists of only one symbol a, for which a two state

NFA suffices: Let s_1 be the starting state and s_2 be the accept state, and let e be an edge from s_1 to s_2 with value a (equivalently, the transition function Δ is $\Delta(s_1, a) = \{s_2\}$).

If E has length more than 1, it is of the form $A \bullet B$ or A*, where \bullet is one of the operators concatenation ("·") or OR ("|"), and A and B are two regular expressions of a sub-type of T_0 . Let M_A and M_B be the NFAs corresponding to A and B respectively, with s_A, s_B as the corresponding start states and t_A, t_B as the corresponding accept states.

So we have three cases:

- 1. Concatenation: suppose that $\bullet = \cdot$. Define M to be the MFA that consists of M_A and M_B , with an edge added from t_A to s_B with value ε . Let s_A be the start state of M and t_B be the accept state of M.
- 2. Or: suppose that $\bullet = |$. Let s be a new state, which has an edge of value ε to s_A and s_B . Mark s as the start state of M. Let t be a new state, where there is an edge from t_A and t_B to t with value ε . Let t be the accept state.
- 3. Star: Suppose that $E = A^*$. Since E is of a subtype of T_0 , A must be of type "|". So it is the OR of some symbols. Let the set of these symbols be Q_A . Then define M to have 3 states, s_E as the start state, t_E as the accept state, and s as a middle state where there is a self-loop from s to itself with all symbols in Q_A as its values, an edge from s_A to s and an edge form s to t_A with empty string s as their value.

It is straightforward to see that this NFA is equivalent to E, so that each alignment of E on a text is equivalent to a computation of the text by the NFA M. Note that in each case we add O(1) edges. So the total number of edges is O(|E|).

Combining Lemma A.7 and A.8 gives us the following Theorem.

Theorem A.9. Given a regular expression E, a text T and a fixed integer R where $\log R$ is sub-polynomial, there is an algorithm that counts the number of alignments of E on substrings of T mod R in O(|T||E|) time.

Theorem A.10. There is an algorithm for #k-WLCS mod R which runs in $\tilde{O}(n^k)$ time when $\lg(R) = o(\lg(n))$.

Proof. Take P_1, \ldots, P_k to be the input sequences. Recall that $w(P_{\ell}[i])$ is the weight of the symbol at position i in the ℓ^{th} string.

We will use dynamic programming. We will have a cell in our table for every k tuples of locations in the strings i_1, \ldots, i_k . Every cell will contain two pieces of information:

- $\ell(i_1,\ldots,i_k)$ the length of the longest common subsequence(s) of the substrings $P_1[:i_1],\ldots,P_k[:i_k]$.
- $C(i_1, ..., i_k)$ is the count of the number of longest common subsequences mod R. This will have a $n^{o(1)}$ bit representation due to our restriction on R.

We start by initializing all cells associated with locations i_1, \ldots, i_k where any $i_j = 0$. These cells are initialized to $\ell(i_1, \ldots, i_k) = 0$ and $C(i_1, \ldots, i_k) = 1$, as there is only one way to have a zero length string.

Let the total sum of a cell be $\sum_{j=1}^{k} i_j$, we will fill cells out in order by there total sum, starting with zero and moving to kn. Any cell that has a i_j value equal to zero will be left with its initialization.

When filling the cell there are two cases: when $P_1[i_1] = P_2[i_2] = \dots = P_k[i_k]$, and when that isn't true. We define some helpful notation. Let $\vec{v} = i_1, \dots, i_k$ and let $eq_\ell(\vec{v}, \vec{u})$ be a function that returns 1 if $\ell(\vec{v}) = \ell(\vec{u})$. Let $\vec{v}_{(-1)}$ be the vector $i_1 - 1, \dots, i_k - 1$. Let $S(\vec{v})$ be a set of all vectors \vec{u} such that for all indices j we have

that $\vec{u}[j] = \vec{v}[j] + \{0, -1\}$ excluding \vec{v} and $\vec{v}_{(-1)}$. So all the smaller neighboring vectors of \vec{v} , excluding the strictly smaller vector (note these may differ from \vec{v} in $1, 2, \dots, k-1$ locations). By our order of computation all cells associated with $S(\vec{v})$ and $\vec{v}_{(-1)}$ will have been computed by the time we are computing the cell \vec{v} .

We will start with the case where $P_1[i_j] \neq P_2[i_{j'}]$. Our length is the maximal length seen so far.

$$\ell(i_1,\ldots,i_k) = \max_{\vec{u}\in S(\vec{v})} (\ell(\vec{u})).$$

This is maximizing over all possible previous choices of longest common subsequence. We know our current last symbols can't all be included in the LCS.

For setting C: We want to look only at entries that are longest common subsequences, so naively you might think to just sum all the counts from the earlier cells that hit our max length of $\ell(\vec{v})$. But, we will have an inclusion exclusion issue. Consider the case of k=2, i.e. 2-LCS. If C(i-1,j)=x, C(i,j-1)=y, and C(i-1,j-1)=z then C(i,j)=x+y-z. This is because x captures both all the longest sequences between $P_1[:i-1]$ and $P_2[:j-1]$ as well as those that use the symbol in location $P_1[i]$. The parallel statement is true for y. So we are double counting those longest common subsequences that appear in both $P_1[:i-1]$ and $P_2[:j-1]$, so we subtract out that double counting. In order to handle this smoothly we will define a more involved version of $S(\vec{v})$. Let $S_r(\vec{v})$ contain the subset of vectors $\vec{u} \in S(\vec{v}) \cup \vec{v}_{(-1)}$ where $\left(\sum_{j=1}^k \vec{v}[j]\right) - \left(\sum_{j=1}^k \vec{u}[j]\right) = r$. So $S_r(\vec{v})$ is the set of vectors that have r indices that are smaller than \vec{v} . Now, after all this lead up, our value for C is the following:

$$C(\vec{v}) = \sum_{r=1}^{k} (-1)^{r} \sum_{\vec{u} \in S_{r}(\vec{v})} eq_{\ell}(\vec{v}, \vec{u}) C(\vec{u}).$$

We need to mod this by R so that the total bits in the representation is not too large.

So in $\tilde{O}(2^kR)$ time per cell we can compute #k-LCS. There are a total of n^k cells so the total time for this algorithm is $\tilde{O}(n^k)$.

Now we will deal with the case of $P_1[i_1] = P_2[i_2] = \ldots = P_k[i_k]$. First let us set $\ell(\cdot)$:

$$\ell(i_1,\ldots,i_k) = \ell(i_1-1,i_2-1,\ldots,i_k-1) + w(P_1[i_1]).$$

This works because we have a matching symbol. Our new longest common subsequence at this location will have a length one longer than the longest sequence that existed using none of the current symbols.

For setting C: We want to count two non-overlapping sets. One set is the weighted longest common subsequences at location $\vec{v}_{(-1)}$. The other set is all the strings that use some but not all of the symbols from our current location \vec{v} . For counting this we need inclusion exclusion like before.

$$C(\vec{v}) = C(\vec{v}_{(-1)}) + \sum_{r=1}^{k} (-1)^{r} \sum_{\vec{u} \in S_{r}(\vec{v})} eq_{\ell}(\vec{v}, \vec{u}) C(\vec{u}).$$

This counts all longest sequences that include the current symbols indicated by \vec{v} by including the count of $C(\vec{v}_{(-1)})$, it also counts all alternate ways to achieve a longest common subsequence of this length using at least one of these symbols by the summation. We need to mod this by R so that the total bits in the representation is not too large.

Corollary A.11. There is an algorithm for #k-LCS mod R which runs in $\tilde{O}(n^k)$ time when $\lg(R) = o(\lg(n))$.

Proof. The #k-LCS problem is a special case of #k-WLCS problem where $w(\cdot)$ is the constant function that returns 1.

B Framework for Generating Uniform Average Case Hardness

B.1 Preliminaries

B.1.1 Notation

Definition 1. We use $x \sim \mathbb{F}_p^n$ to mean that x is drawn uniformly at random from all p^n values in the support of \mathbb{F}_p^n .

B.1.2 Getting Nearly Uniform Bit Strings from Finite Field Elements

Adserà et. al show that counting cliques is hard on average over the uniform distribution where every edge exists iid [BBB19].

Theorem B.1. Let $Z_i = Ber[\mu]$ where $\mu \in (0,1)$. Then let $Y \equiv \sum_{i=0}^t Z_i \cdot 2^i \pmod{p}$. Let the total variation distance between Y and UNIF[0,p-1] be Δ . Then there exists a constant C such that if $t \geq C \cdot \mu^{-1} \cdot (1-\mu)^{-1} \cdot \lg(p/\epsilon^2) \cdot \lg(p)$, then $\Delta \leq \varepsilon$ [BBB19].

Theorem B.2. If you are given an input with n numbers x_1, \ldots, x_n each chosen from UNIF[1, p-1] there exists a sampling procedure which runs in time $O(n\lg^3(n)t(1/p-\varepsilon)^{-1})$ that, with probability at least $1-2^{-\lg^2(n)}$, produces a new set of numbers $I=x_1',\ldots,x_n'$ such that:

- 1. $x_i' \equiv x_i \mod p \text{ for all } i$.
- 2. Each x'_i is t bits long where $t \ge C \cdot \mu^{-1} \cdot (1-\mu)^{-1} \cdot \lg(p/\varepsilon^2) \cdot \lg(p)$.
- 3. I is total variation distance $n\varepsilon$ from the distribution where every bit of x_i' is iid sampled from $Ber[\mu]$. (inspired by [BBB19])

Proof. Let $Z_i = Ber[\mu]$ where $\mu \in (0,1)$. Then let Y be the distribution formed by $\sum_{i=0}^t Z_i \cdot 2^i \pmod{p}$.

Consider the procedure to generate x_i' where we sample a number y from Y, if $y \equiv x_i \pmod{p}$ then $x_i' = y$, else repeat. We take O(t) time to produce a sample. We succeed with the probability that $y \equiv x_i \pmod{p}$. This probability is at least $\frac{1}{p} - \varepsilon$, because ε is the total variation distance of Y and UNIF[1, p-1]. Thus, the time to produce a single sample in expectation is $O(t(1/p-\varepsilon)^{-1})$. To fail $\Theta(t(1/p-\varepsilon)^{-1}\lg^3(n))$ times in a row will happen with probability at most $1 - 2^{-2\lg^2(n)}$. If we fail $\Theta(t(1/p-\varepsilon)^{-1}\lg^3(n))$ times in a row simply halt the program and throw an error.

We run this procedure for all n numbers, thus taking at most $O(nt(1/p-\varepsilon)^{-1}\lg^3(n))$ time to succeed with probability at least $1-n2^{-2\lg^2(n)} \ge 1-2^{-\lg^2(n)}$.

The total variation distance from each individual x_i' to the uniform distribution is ε and there are n inputs in total. Thus, the total variation distance is at most $n\varepsilon$ by the union bound.

Corollary B.3. If you are given an input with n numbers x_1, \ldots, x_n each chosen from UNIF[1, p-1] there exists a sampling procedure which runs in time $O(n\lg^3(n)t(1/p-1/n^3)^{-1})$ that, with probability at least $1-2^{-\lg^2(n)}$, produces a new set of numbers $I=x_1',\ldots,x_n'$ such that:

- 1. $x_i' \equiv x_i \pmod{p}$ for all i.
- 2. Each x'_i is t bits long where $t \ge C \cdot \mu^{-1} \cdot (1-\mu)^{-1} \cdot (\lg(p) + 6\lg(n)) \cdot \lg(p)$.
- 3. I is total variation distance $1/n^2$ from the distribution where every bit of x_i' is iid sampled from $Ber[\mu]$.

Proof. Simply plug in $\varepsilon = 1/n^3$ to Theorem B.2.

B.2 The framework

In this section we are going to show that any problem P with a $GLDP(\cdot)$ is hard over the uniform average case. We define $GLDP(\cdot)$ in Definition 8.

First, we want to convert our problem over a polynomial large finite field to a problem over many $O(\lg(n))$ sized finite fields. We will use the Chinese Remainder Theorem (CRT) to do this.

Lemma B.4. Let P be some problem with output in range $[1, n^c]$. Let P_p be the same problem as P, but where $P_p(\vec{I}) \equiv P(\vec{I}) \pmod{p}$.

Let f be a GLDP(P). Let f_1, \ldots, f_s be a set of s polynomials where $s = O(\lg(n)/\lg\lg(n))$. We define f_i as the same polynomial as f, but over finite field F_{p_i} where $p_i = \Theta(\lg(n))$ and all p_i are distinct.

Then, for all i, f_i is a $GLDP(P_{p_i})$.

Finally, given $f_i(\vec{I})$ for all $i \in [1,s]$ we can return $P(\vec{I})$.

Proof. If $f(\vec{I}) = P(\vec{I})$ then trivially $f(\vec{I}) \equiv P(\vec{I}) \pmod{p}$. As a result $f_i(\vec{I}) \equiv P_{p_i}(\vec{I}) \equiv P(\vec{I}) \pmod{p_i}$. If f has degree d then f_i also has degree d (it certainly has at most d, because f is strongly d-partite they will in fact be equal).

If f is d-partite then so is f_i .

Thus, f_i is a GLDP(P_{p_i}).

Given $f_i(\vec{I})$ for all $i \in [1,s]$ we know $P_{p_i}(\vec{I})$ for all $i \in [1,s]$. We can use the Chinese Remainder Theorem to find the value of P as long as $\prod_{i=1}^s p_i \ge n^c$. By the prime number theorem there is a sufficiently large constant c' such that there are more than $2c \lg(n)/\lg \lg(n)$ primes between $\lg(n)$ and $c' \lg(n)$. If we choose these primes to be $p_1, \ldots, p_{s=2c \lg(n)/\lg \lg(n)}$ then $\prod_{i=1}^s p_i \ge n^{2c} \ge n^c$.

Now we want to apply a worst-case to average case reduction for each f_i separately. We can use Lemma 1 from [BRSV17] to achieve this.

Lemma B.5. Consider positive integers n, d, and p, and an $\varepsilon \in (0,1/3)$ such that d > 9, p is prime and p > 12d. Suppose that for some polynomial $f : \mathbb{F}_p^n \to \mathbb{F}_p$ of degree at most⁶ d, there is an algorithm A running in time T(n) such that when x is drawn uniformly at random from all inputs \mathbb{F}_p^n :

$$Pr[A(x) = f(x)] \ge 1 - \varepsilon.$$

Then there is a randomized algorithm B that runs in time $O(nd^2log^2(p) + d^3 + T(n)d)$ such that for any $x \in \mathbb{F}_p^n$:

$$Pr[B(x) = f(x)] \ge 2/3.$$

[BRSV17]

Notably, we demand that $d = o(\lg(n)/\lg\lg(n))$ and we use $p = \Theta(\lg(n))$, so p > 12d. The running time, given these choices, is $\tilde{O}(n + T(n))$ time.

Corollary B.6. Assume an f exists that is GLDP(P). Then, let f_1, \ldots, f_s be the polynomials described in B.4. Let A be an algorithm that runs in time T(n) such that when $x \sim \mathbb{F}_{p_i}^n$:

$$Pr[A(x) = f_i(x)] \ge 3/4,$$

⁶Ball et al. simply say a polynomial of degree d, however, unsurprisingly, their proof does not require the polynomial be of degree at least 9 to work.

for all i. Then there is a randomized algorithm B that runs in time $\tilde{O}(n+T(n))$ such that for any $\vec{I} \in \{0,1\}^n$:

$$Pr[B(\vec{I}) = P(\vec{I})] \ge 1 - O(2^{-\lg^2(n)}).$$

Proof. We use Lemma B.5 for each polynomial f_i . It follows that having an algorithm A for computing f_i over the uniform input $\mathbb{F}_{p_i}^n$ that succeeds with probability 3/4 implies that a randomized algorithm B_i' exists that succeeds with probability 2/3.

We can now create an algorithm B_i by running B'_i for $\Theta(\lg^3(n))$ times and pick the most common output, this will return the correct answer with probability at least $1 - 2^{-\lg^{2.5}(n)}$.

Now if all of B_1, \ldots, B_s return the correct answer then we can use the CRT trick of Lemma B.4 to compute the value of $P(\vec{I})$. All of B_1, \ldots, B_s return the correct answer with probability at least $1 - s2^{-\lg^{2.5}(n)} = 1 - O(\lg(n)/\lg\lg(n))2^{-\lg^{2.5}(n)} < 1 - O\left(2^{-\lg^2(n)}\right)$

So, we now want to show that solving random instances of P can solve random instances $f_i(x)$ where $x \sim \mathbb{F}_p^n$. To do this we will use the sampling procedure described in Corollary B.3. We will also use the fact that $P_{p_i}(x) = f_i(x)$ when x is a zero and one input.

Lemma B.7. Assume a d degree polynomial f exists that is GLDP(P). Then, let f_1, \ldots, f_s and p_1, \ldots, p_s be the polynomials and primes described in Lemma B.4.

Let A be an algorithm that runs in time T(n) such that when \vec{I} is formed by n bits each chosen iid from $Ber[\mu]$ where $\mu \in (0,1)$ is a constant, then:

$$Pr[A(\vec{I}) = P(\vec{I})] \ge 1 - 1/\omega \left(\lg^d(n)\lg\lg^d(n)\right).$$

Then there is a B that runs in time $\tilde{O}(n+T(n))$ such that when $x \sim \mathbb{F}_n^n$:

$$Pr[B(x) = f_i(x)] > 3/4,$$

for all f_i .

Proof. Let D_{μ} be the distribution over inputs where each of the n bits is chosen iid from $Ber[\mu]$, that is one is chosen with probability μ and zero is chosen with probability $1 - \mu$. Recall that when we say $\vec{Z} \sim D_{\mu}$ we mean that \vec{Z} is drawn from the distribution D_{μ} . We will use an abuse of notation where we run $f_i(\vec{Z})$, when we do this we mean that one should interpret the n length bit vector as n values from F_{p_i} where 0 maps to $0 \in F_{p_i}$ and 1 maps to $1 \in F_{p_i}$. Additionally when we have a vector v we will use v[j] to represent the j^{th} number in v.

In this proof we will show how to use $P(\vec{Z})$ to solve instances of $f_i(\vec{Z})$ for all i. Note that we can simply take the output of $P(\vec{Z})$ modulo p_i . So we want to use $f_i(\vec{Z})$ where $\vec{Z} \sim D_\mu$ to solve $f_i(z)$ where $z \sim \mathbb{F}_{p_i}^n$.

Let f' be the function f_i but taken over the integers instead of F_{p_i} . Note that this is the same f' regardless of f_i . We have that if $x \in \mathbb{F}_{p_i}^n$ then $f'(x) \equiv f_i(x) \pmod{p_i}$. Furthermore, if we make a new input x' where $x'[j] \equiv x[j] \pmod{p_i}$ for all $j \in [1,n]$ then $f'(x) \equiv f_i(x) \pmod{p_i}$. So, given an input $x \sim \mathbb{F}_{p_i}^n$ we will take the sampling procedure of Corollary B.3 and make a new input x', where x'[j] is a $t = O(\mu^{-1} \cdot (1 - \mu)^{-1} \cdot (\lg(p_i) + 6\lg(n)) \cdot \lg(p_i))$ bit number. Note that because μ is constant and neither zero nor one and $p_i = \Theta(\lg(n))$ then $t = O(\lg(n)\lg\lg(n))$. Furthermore, any given number x'[j] has the property that the distribution over its binary representation has total variation distance $\leq 1/n^3$ from the distribution where all

t bits are chosen iid from $Ber[\mu]$. Thus, all tn bits in our new input x' have total variation distance at most $1/n^2$ from the distribution where all tn bits are chosen iid from $Ber[\mu]$.

Now, we can compute the value of f'(x') with t^d calls to f' where every call has a zero one input. Every monomial is formed by one variable from each of the d partitions. Let m be the number of monomials. So we can write our polynomial f' as follows:

$$f'(x') = \sum_{j=1}^{m} y_{k_{j,1}} \cdot y_{k_{j,2}} \cdots y_{k_{j,d}},$$

where $y_{k_{j,\ell}}$ is a variable from the ℓ^{th} partition S_{ℓ} . The input x' is formed with n of these input variables $y_{k_{j,\ell}}$.

We can break down this multiplication for every bit. Let $y_{k_{j,\ell}}[r]$ be the r^{th} bit of $y_{k_{j,\ell}}$. Now we can rewrite our sum. Recall that $g_{f'}(v_1, \ldots, v_d)$ is the function such that f' can be written as a sum of calls to $g_{f'}$, where v_{ℓ} is a variable from partition S_{ℓ} :

$$f'(x') = \sum_{j=1}^{m} \left(\sum_{r_1, \dots, r_d \in [0, t-1]} 2^{r_1 + \dots + r_d} \cdot y_{k_{j,1}}[r_1] \cdot y_{k_{j,2}}[r_2] \cdots y_{k_{j,d}}[r_d] \right).$$

Put in words, we can multiply d numbers each of t bits by making a weighted sum over the t^d multiplications of the bits of the d numbers.

Now, we want to create t^d inputs $\hat{x}_1, \dots, \hat{x}_{t^d}$. They are formed by taking all possible choices of r_1, \dots, r_d where each r_ℓ is an integer in [0, t-1]. Given a choice of r_1, \dots, r_d we create a new input \hat{x}_j by taking all variables in S_ℓ and making their value in \hat{x}_j be the r_ℓ^{th} bit of that variable in x'.

Now, call $A(\hat{x}_j)$ for all $j \in [1, t^d]$. Note that $P(\hat{x}_j) \equiv f_i'(\hat{x}_j) \equiv f_i(\hat{x}_j) \pmod{p_i}$. So, if $A(\hat{x}_j) = P(\hat{x}_j)$ for all $j \in [1, t^d]$ then we can return the value of $f'(x') \equiv f_i(x) \pmod{p}$.

By the definition of A in this Lemma, A must succeed on any individual random input $x \sim D_{\mu}$ with probability $1 - 1/\omega(\lg^d(n) \lg \lg^d(n))$. The total variation distance of any \hat{x}_j from D_{μ} is at most $1/n^2$. So A must succeed on any one given random input \hat{x}_j with probability $1 - 1/\omega(\lg^d(n) \lg \lg^d(n)) - 1/n^2$ which is $1 - 1/\omega(\lg^d(n) \lg \lg^d(n))$.

Our inputs \hat{x}_j are not iid from each other, however, if A is correct with probability 1-q on a given input from \hat{x}_j then A must be correct with probability at least $1-qt^d$ on t^d inputs \hat{x}_j at once.

So, A will return correct answers for all t^d inputs \hat{x}_j at once with probability at least $1 - 1/\omega(1)$. Given these correct answers we can compute $f_i(x)$, for all f_i . So, an algorithm B exists that makes t^d calls to A and takes $n \lg^4(n)t$ time to produce our new sampled input x' from x.

B returns f_i correctly with probability at least $1 - 1/\omega(1) > 3/4$.

B takes a total time of $O(t^dT(n)+n)$. We have that $t=O(\lg(n)\lg\lg(n))$ and $d=o(\lg(n)/\lg\lg(n))$ (by our definition of GLDP(P)). Thus, $t^d=n^{o(1)}$. So we have that *B* runs in time $\tilde{O}(T(n)+n)$.

This next theorem gives a worst case to average case reduction for P.

REMINDER OF THEOREM 1.20 Let μ be a constant such that $0 < \mu < 1$. Let P be a problem such that a function f exists that is a GLDP(P), and let d be the degree of f. Let A be an algorithm that runs in time T(n) such that when \vec{l} is formed by n bits each chosen iid from $Ber[\mu]$:

$$Pr[A(\vec{I}) = P(\vec{I})] \ge 1 - 1/\omega \left(\lg^d(n)\lg\lg^d(n)\right).$$

Then there is a randomized algorithm B that runs in time $\tilde{O}(n+T(n))$ such that for any for $\vec{I} \in \{0,1\}^n$:

$$Pr[B(\vec{I}) = P(\vec{I})] \ge 1 - O(2^{-\lg^2(n)}).$$

Proof. We will use Lemma B.7 and Corollary B.6 to get this result.

Note that the algorithm *A* here can be used as the algorithm *A* in Lemma B.7.

Furthermore, note that the algorithm B of Lemma B.7 has the same requirements as the algorithm A of Corollary B.6.

So, given the algorithm A of this theorem we can produce the algorithm B from Corollary B.6.

The algorithm *B* of Corollary B.6 has the same properties of the algorithm *B* described in this theorem.

Thus, algorithm *A* implies that an algorithm *B* exists.