# Scheduling with Communication Delays via LP Hierarchies and Clustering 

Sami Davies* Janardhan Kulkarni ${ }^{\dagger}$ Thomas Rothvoss*<br>Jakub Tarnawski ${ }^{\dagger}$ Yihao Zhang*

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#### Abstract

We consider the classic problem of scheduling jobs with precedence constraints on identical machines to minimize makespan, in the presence of communication delays. In this setting, denoted by $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$, if two dependent jobs are scheduled on different machines, then at least $c$ units of time must pass between their executions. Despite its relevance to many applications, this model remains one of the most poorly understood in scheduling theory. Even for a special case where an unlimited number of machines is available, the best known approximation ratio is $2 / 3 \cdot(c+1)$, whereas Graham's greedy list scheduling algorithm already gives a $(c+1)$ approximation in that setting. An outstanding open problem in the top-10 list by Schuurman and Woeginger and its recent update by Bansal asks whether there exists a constant-factor approximation algorithm.

In this work we give a polynomial-time $O(\log c \cdot \log m)$-approximation algorithm for this problem, where $m$ is the number of machines and $c$ is the communication delay. Our approach is based on a Sherali-Adams lift of a linear programming relaxation and a randomized clustering of the semimetric space induced by this lift.


## 1 Introduction

Scheduling jobs with precedence constraints is a fundamental problem in approximation algorithms and combinatorial optimization. In this problem we are given $m$ identical machines and a set $J$ of $n$ jobs, where each job $j$ has a processing length $p_{j} \in \mathbb{Z}_{+}$. The jobs have precedence constraints, which are given by a partial order $\prec$. A constraint $j \prec j^{\prime}$ encodes that job $j^{\prime}$ can only start after job $j$ is completed. The goal is to find a schedule of jobs that minimizes makespan, which is the completion time of the last job. This problem is denoted by $\mathrm{P} \mid$ prec $\mid C_{\text {max }}$. In a seminal result from 1966, Graham Gra66] showed that the greedy list scheduling algorithm achieves a $\left(2-\frac{1}{m}\right)$-approximation. By now, our understanding of the approximability of this basic problem is almost complete: it had been known since the late ' 70 s, due to a result by Lenstra and Rinnooy Kan [LRK78, that it is NPhard to obtain better than 4/3-approximation, and in 2010 Svensson [Sve10] showed that, assuming

[^0]a variant of the Unique Games Conjecture [BK10], it is NP-hard to get a $(2-\varepsilon)$-approximation for any $\varepsilon>0$.

The above precedence-constrained scheduling problem models the task of distributing workloads onto multiple processors or servers, which is ubiquitous in computing. This basic setting takes the dependencies between work units into account, but not the data transfer costs between machines, which is critical in applications. A precedence constraint $j \prec j^{\prime}$ typically implies that the input to $j^{\prime}$ depends on the output of $j$. In many real-world scenarios, especially in the context of scheduling in data centers, if $j$ and $j^{\prime}$ are executed on different machines, then the communication delay due to transferring this output to the other machine cannot be ignored. This is an active area of research in applied data center scheduling literature, where several new abstractions have been proposed to deal with communication delays [CZM $\left.{ }^{+} 11, \widehat{\mathrm{GFC}^{+} 12}, \mathrm{HCG12}, \mathrm{SZA}^{+} 18, \mathrm{ZZC}^{+} 12, \mathrm{ZCB}^{+} 15, \mathrm{LYZ}^{+} 16\right]$. Another timely example is found in the parallelization of Deep Neural Network training (the machines being accelerator devices such as GPUs, TPUs, or FPGAs). There, when training the network on one sample/minibatch per device in parallel, the communication costs incurred by synchronizing the weight updates in fact dominate the overall running time [NHP ${ }^{+}$19]. Taking these costs into account, it turns out that it is better to split the network onto multiple devices, forming a "modelparallel" computation pipeline $\left[\mathrm{HCB}^{+} 19\right]$. In the resulting device placement problem, the optimal split crucially depends on the communication costs between dependent layers/operators.

A classic model that captures the effect of data transfer latency on scheduling decisions is the problem of scheduling jobs with precedence and communication delay constraints, introduced by Rayward-Smith RS87 and Papadimitriou and Yannakakis PY90. The setting, denoted by $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$, is similar to the makespan minimization problem described earlier, except for one crucial difference. Here we are given a communication delay parameter $c \in \mathbb{Z}_{\geq 0}$, and the output schedule must satisfy the property that if $j \prec j^{\prime}$ and $j, j^{\prime}$ are scheduled on different machines, then $j^{\prime}$ can only start executing at least $c$ time units after $j$ had finished. On the other hand, if $j$ and $j^{\prime}$ are scheduled on the same machine, then $j^{\prime}$ can start executing immediately after $j$ finishes. In a closely related problem, denoted by $\mathrm{P} \infty \mid$ prec, $c \mid C_{\text {max }}$, a schedule can use as many machines as desired. The goal is to schedule jobs non-preemptively so as to minimize the makespan. In a non-preemptive schedule, each job $j$ needs to be assigned to a single machine and executed during $p_{j}$ consecutive timeslots. The problems $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$ and $\mathrm{P} \infty \mid$ prec, $c \mid C_{\max }$ are the focus of this paper.

Despite its theoretical significance and practical relevance, very little is known about the communication delay setting. A direct application of Graham's [Gra66] list scheduling algorithm yields a $(c+2)$-approximation, and no better algorithm is known for the problem. Over the years, the problem has attracted significant attention, but all known results, which we discuss below in Section [1.3, concern special settings, small communication delays, or hardness of approximation. To put this in perspective, we note that the current best algorithm for general $c$ GKMP08, which achieves an approximation factor of $2 / 3 \cdot(c+1)$, only marginally improves on Graham's algorithm while requiring the additional assumptions that the number of machines is unbounded and $p_{j}=1$. This is in sharp contrast to the basic problem $\mathrm{P} \mid$ prec $\mid C_{\max }$ (which would correspond to the case $c=0$ ), where the approximability of the problem is completely settled under a variant of the Unique Games Conjecture. This situation hints that incorporating communication delays in scheduling decisions requires fundamentally new algorithmic ideas compared to the no-delay setting. Schuurman and Woeginger SW99 placed the quest for getting better algorithms to the problem in their influential list of top-10 open problems in scheduling theory. In a recent MAPSP 2017 survey talk, Bansal Ban17 highlighted the lack of progress on this model, describing it as "not understood at all; almost completely open", and suggested that this is due to the lack of promising LP/SDP
relaxations.

### 1.1 Our Contributions

The main result of this paper is the following:
Theorem 1. There is a randomized $O(\log c \cdot \log m)$-approximation algorithm for $\mathrm{P}|\mathrm{prec}, c| C_{\max }$ with expected polynomial running time, where $c, p_{j} \in \mathbb{N}$.

In any non-preemptive schedule the number $m$ of machines is at most the number $n$ of jobs, so for the easier $\mathrm{P} \infty$ version of the problem, the above theorem implies the following:

Corollary 2. There is a randomized $O(\log c \cdot \log n)$-approximation algorithm for $\mathrm{P} \infty|\mathrm{prec}, c| C_{\max }$ with expected polynomial running time, where $c, p_{j} \in \mathbb{N}$.

For both problems one can replace either $c$ or $m$ by $n$, yielding a $O\left(\log ^{2} n\right)$-approximation algorithm. Our results make substantial progress towards resolving one of the questions in "Open Problem 3" in the survey of Schuurman and Woeginger [SW99], which asks whether a constant-factor approximation algorithm exists for $\mathrm{P} \infty \mid$ prec, $c \mid C_{\text {max }}$.

Our approach is based on a Sherali-Adams lift of a time-indexed linear programming relaxation for the problem, followed by a randomized clustering of the semimetric space induced by this lift. To our knowledge, this is the first instance of a multiple-machine scheduling problem being viewed via the lens of metric space clustering. We believe that our framework is fairly general and should extend to other problems involving scheduling with communication delays. To demonstrate the broader applicability of our approach, we also consider the objective of minimizing the weighted sum of completion times. Here each job $j$ has a weight $w_{j}$, and the goal is to minimize $\sum_{j} w_{j} C_{j}$, where $C_{j}$ is the completion time of $j$.

Theorem 3. There is a randomized $O(\log c \cdot \log n)$-approximation algorithm for $\mathrm{P} \infty \mid \mathrm{prec}, p_{j}=$ $1, c \mid \sum_{j} w_{j} C_{j}$ with expected polynomial running time, where $c \in \mathbb{N}$.

No non-trivial approximation ratio was known for this problem prior to our work.

### 1.2 Our Techniques

As we alluded earlier, there is a lack of combinatorial lower bounds for scheduling with communication delays. For example, consider Graham's list scheduling algorithm, which greedily processes jobs on $m$ machines as soon as they become available. One can revisit the analysis of Graham Gra66] and show that there exists a chain $Q$ of dependent jobs such that the makespan achieved by list scheduling is bounded by

$$
\frac{1}{m} \sum_{j \in J} p_{j}+\sum_{j \in Q} p_{j}+c \cdot(|Q|-1)
$$

The first two terms are each lower bounds on the optimum - the 3rd term is not. In particular, it is unclear how to certify that the optimal makespan is high because of the communication delays. However, this argument suffices for a $(c+2)$-approximation, since $p_{j} \geq 1$ for all $j \in J$.

As pointed out by Bansal Ban17], there is no known promising LP relaxation. To understand the issue let us consider the special case $\mathrm{P} \infty\left|\mathrm{prec}, p_{j}=1, c\right| C_{\text {max }}$. Extending, for example, the LP of Munier and König [MK97], one might choose variables $C_{j}$ as completion times, as well as decision variables $x_{j_{1}, j_{2}}$ denoting whether $j_{2}$ is executed in the time window [ $C_{j_{1}}, C_{j_{1}}+c$ ) on
the same machine as $j_{1}$. Then we can try to enforce communication delays by requiring that $C_{j_{2}} \geq C_{j_{1}}+1+(c-1) \cdot\left(1-x_{j_{1}, j_{2}}\right)$ for $j_{1} \prec j_{2}$. Further, we enforce load constraints $\sum_{j_{1} \in J} x_{j_{1}, j_{2}} \leq c$ for $j_{2} \in J$ and $\sum_{j_{2} \in J} x_{j_{1}, j_{2}} \leq c$ for $j_{1} \in J$. To see why this LP fails, note that in any instance where the maximum dependence degree is bounded by $c$, one could simply set $x_{j_{1}, j_{2}}=1$ and completely avoid paying any communication delay. Moreover, this problem seems to persist when moving to more complicated LPs that incorporate indices for time and machines.

A convenient observation is that, in exchange for a constant-factor loss in the approximation guarantee, it suffices to find an assignment of jobs to length- $c$ intervals such that dependent jobs scheduled in the same length- $c$ interval must be assigned to the same machine. (The latter condition will be enough to satisfy the communication delay constraints as, intuitively, between every two length-c intervals we will insert an empty one.) In order to obtain a stronger LP relaxation, we consider an $O(1)$-round Sherali-Adams lift of an inital LP with indices for time and machines. From the lifted LP, we extract a distance function $d: J \times J \rightarrow[0,1]$ which satisfies the following properties:
(i) The function $d$ is a semimetric.
(ii) $C_{j_{1}}+d\left(j_{1}, j_{2}\right) \leq C_{j_{2}}$ for $j_{1} \prec j_{2}$.
(iii) Any set $U \subseteq J$ with a diameter of at most $\frac{1}{2}$ w.r.t. $d$, satisfies $|U| \leq 2 c$.

Here we have changed the interpretation of $C_{j}$ to the index of the length-c interval in which $j$ will be processed. Intuitively, $d\left(j_{1}, j_{2}\right)$ can be understood as the probability that jobs $j_{1}, j_{2}$ are not being scheduled within the same length- $c$ interval on the same machine. To see why a constant number of Sherali-Adams rounds are helpful, observe that the triangle inequality behind $(i)$ is really a property depending only on triples $\left\{j_{1}, j_{2}, j_{3}\right\}$ of jobs and an $O(1)$-round Sherali-Adams lift would be locally consistent for every triple of variables.

We will now outline how to round such an LP solution. For jobs whose LP completion times are sufficiently different, say $C_{j_{1}}+\Theta\left(\frac{1}{\log (n)}\right) \leq C_{j_{2}}$, we can afford to deterministically schedule $j_{1}$ and $j_{2}$ at least $c$ time units apart while only paying a $O(\log n)$-factor more than the LP. Hence the critical case is to sequence a set of jobs $J^{*}=\left\{j \in J \left\lvert\, C^{*} \leq C_{j} \leq C^{*}+\Theta\left(\frac{1}{\log (n)}\right)\right.\right\}$ whose LP completion times are very close to each other. Note that by property (ii), we know that any dependent jobs $j_{1}, j_{2} \in J^{*}$ must have $d\left(j_{1}, j_{2}\right) \leq \Theta\left(\frac{1}{\log (n)}\right)$. As $d$ is a semimetric, we can make use of the rich toolset from the theory of metric spaces. In particular, we use an algorithm by Calinescu, Karloff and Rabani [KR04]: For a parameter $\Delta>0$, one can partition a semimetric space into random clusters so that the diameter of every cluster is bounded by $\Delta$ and each $\delta$-neighborhood around a node is separated with probability at most $O(\log (n)) \cdot \frac{\delta}{\Delta}$. Setting $\delta:=\Theta\left(\frac{1}{\log (n)}\right)$ and $\Delta:=\Theta(1)$ one can then show that a fixed job $j \in J^{*}$ will be in the same cluster as all its ancestors in $J^{*}$ with probability at least $\frac{1}{2}$, while all clusters have diameter at most $\frac{1}{2}$. By (iii), each cluster will contain at most $2 c$ many (unit-length) jobs, and consequently we can schedule all the clusters in parallel, where we drop any job that got separated from any ancestor. Repeating the sampling $O(\log n)$ times then schedules all jobs in $J^{*}$. This reasoning results in a $O\left(\log ^{2} n\right)$-approximation for this problem, which we call $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\text {max }}$. With a bit of care the approximation factor can be improved to $O(\log c \cdot \log m)$.

Finally, the promised $O(\log c \cdot \log m)$-approximation for the more general problem $\mathrm{P} \mid$ prec, $c \mid$ $C_{\max }$ follows from a reduction to the described special case $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\text {max }}$.

### 1.3 History of the Problem

Precedence-constrained scheduling problems of minimizing the makespan and sum of completion times objectives have been extensively studied for many decades in various settings. We refer the
reader to [Mic18, LLKS93, PST04, AMMS08, Sve09] for more details. Below, we only discuss results directly related to the communication delay problem in the offline setting.

Approximation algorithms. As mentioned earlier, Graham's Gra66 list scheduling algorithm yields a $(c+2)$-approximation for $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$, and a $(c+1)$-approximation for the $\mathrm{P} \infty$ variant. For unit-size jobs and $c \geq 2$, Giroudeau, König, Moulai and Palaysi GKMP08 improved the latter $\left(\mathrm{P} \infty\left|\mathrm{prec}, p_{j}=1, c \geq 2\right| C_{\max }\right)$ to a $\frac{2}{3}(c+1)$-approximation. For unit-size jobs and $c=1$, Munier and König MK97] obtained a 4/3-approximation via LP rounding ( $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c=1 \mid C_{\text {max }}$ ); for the $P$ variant, Hanen and Munier [HM01] gave an easy reduction from the $\mathrm{P} \infty$ variant that loses an additive term of 1 in the approximation ratio, thus yielding a $7 / 3$-approximation. Thurimella and Yesha [?] gave a reduction that, given an $\alpha$-approximation algorithm for $\mathrm{P} \infty \mid$ prec, $c, p_{j}=1 \mid C_{\max }$, would yield a $(1+2 \alpha)$-approximation algorithm for $\mathrm{P} \mid$ prec, $c, p_{j}=1 \mid C_{\text {max }}$.

For a constant number of machines, a hierarchy-based approach of Levey and Rothvoss [R16] for the no-delay setting ( $\mathrm{P} m \mid$ prec, $p_{j}=1 \mid C_{\text {max }}$ ) was generalized by Kulkarni, Li, Tarnawski and Ye KLTY20 to allow for communication delays that are also bounded by a constant. For any $\varepsilon>0$ and $\hat{c} \in \mathbb{Z}_{\geq 0}$, they give a nearly quasi-polynomial-time $(1+\varepsilon)$-approximation algorithm for $\mathrm{P} m \mid$ prec, $p_{j}=1, c_{j k} \leq \hat{c} \mid C_{\text {max }}$. The result also applies to arbitrary job sizes, under the assumption that preemption of jobs is allowed, but migration is not.

Hardness. Hoogeveen, Lenstra and Veltman HLV94 showed that even the special case $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c=1 \mid C_{\max }$ is NP-hard to approximate to a factor better than $7 / 6$. For the case with bounded number of machines (the P variant) they show $5 / 4$-hardness. These two results can be generalized for $c \geq 2$ to $(1+1 /(c+4))$-hardness GKMP08] and $(1+1 /(c+3))$-hardness [BGK96], respectively ${ }^{2}$

Duplication model. The communication delay problem has also been studied (to a lesser extent) in a setting where jobs can be duplicated (replicated), i.e., executed on more than one machine, in order to avoid communication delays. This assumption seems to significantly simplify the problem, especially when we are also given an unbounded number of machines: already in 1990, Papadimitriou and Yannakakis PY90 gave a rather simple 2-approximation algorithm for $\mathrm{P} \infty \mid$ prec, $p_{j}, c_{j k}$, dup $\mid C_{\text {max }}$. Observe that this result holds even when communication delays are unrelated (they depend on the pair of jobs). The only non-trivial approximation algorithm for arbitrary $c$ and a bounded number of machines is due to Lepere and Rapine [LR02], who gave an asymptotic $O(\log c / \log \log c)$-approximation for $\mathrm{P} \mid \operatorname{prec}, p_{j}=1, c$, dup $\mid C_{\text {max }}$. On the hardness side, Papadimitriou and Yannakakis PY90] showed NP-hardness of $\mathrm{P} \infty \mid \mathrm{prec}, p_{j}=1, c$, dup $\mid C_{\text {max }}$ (using a large delay $c=\Theta\left(n^{2 / 3}\right)$ ).

Besides being seemingly easier to approximate, we also believe that the replication model is less applicable in most real-world scenarios due to the computation and energy cost of replication, as well as because replication is more difficult to achieve if the computations are nondeterministic in some sense (e.g. randomized).

Many further references can be found in (VLL90, GKMP08, Dro09, GK07, CC91, JKS93, MH97].

[^1]
## 2 Preliminaries

### 2.1 The Sherali-Adams Hierarchy for LPs with Assignment Constraints

In this section, we review the Sherali-Adams hierarchy which provides an automatic strengthening of linear relaxations for $0 / 1$ optimization problems. The authorative reference is certainly Laurent Lau03, and we adapt the notation from Friggstad et al. [FKK ${ }^{+} 14$. Consider a set of variable indices $[n]=\{1, \ldots, n\}$ and let $U_{1}, \ldots, U_{N} \subseteq[n]$ be subsets of variable indices. We consider a polytope

$$
K=\left\{x \in \mathbb{R}^{n} \mid \tilde{A} x \geq \tilde{b}, \quad \sum_{i \in U_{k}} x_{i}=1 \quad \forall k \in[N], \quad 0 \leq x_{i} \leq 1 \quad \forall i \in[n]\right\},
$$

which we also write in a more compact form as $K=\left\{x \in \mathbb{R}^{n} \mid A x \geq b\right\}$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$. We note that we included explicitly the "box constraints" $0 \leq x_{i} \leq 1$ for all variables $i$. Moreover, the constraint matrix contains assignment constraints of the form $\sum_{i \in U_{k}} x_{i}=1$. This is the aspect that is non-standard in our presentation.

The general goal is to obtain a strong relaxation for the integer hull $\operatorname{conv}\left(K \cap\{0,1\}^{n}\right)$. Observe that any point $x \in \operatorname{conv}\left(K \cap\{0,1\}^{n}\right)$ can be interpreted as a probability distribution $X$ over points $K \cap\{0,1\}^{n}$. We know that any distribution can be described by the $2^{n}$ many values $y_{I}=\operatorname{Pr}\left[\bigwedge_{i \in I}\left(X_{i}=1\right)\right]$ for $I \subseteq[n]$ - in fact, the probability of any other event can be reconstructed using the inclusion-exclusion formula, for example $\operatorname{Pr}\left[X_{1}=1\right.$ and $\left.X_{2}=0\right]=y_{\{1\}}-y_{\{1,2\}}$. While this is an exact approach, it is also an inefficient one. In order to obtain a polynomial-size LP, we only work with variables $y_{I}$ where $|I| \leq O(1)$. Hence, for $r \geq 0$, we denote $\mathcal{P}_{r}([n]):=\{S \subseteq[n]| | S \mid \leq r\}$ as all the index sets of size at most $r$.

Definition 4. Let $S A_{r}(K)$ be the set of vectors $y \in \mathbb{R}^{\mathcal{P}_{r+1}([n])}$ satisfying $y_{\emptyset}=1$ and

$$
\sum_{H \subseteq J}(-1)^{|H|} \cdot\left(\sum_{i=1}^{n} A_{\ell, i} y_{I \cup H \cup\{i\}}-b_{\ell} y_{I \cup H}\right) \geq 0 \quad \forall \ell \in[m]
$$

for all $I, J \subseteq[n]$ with $|I|+|J| \leq r$.
The parameter $r$ in the definition is usually called the rank or number of rounds of the SheraliAdams lift. It might be helpful for the reader to verify that for $I=J=\emptyset$, the constraint simplifies to $\sum_{i=1}^{n} A_{\ell, i} y_{\{i\}} \geq b_{\ell} y_{\emptyset}=b_{\ell}$, which implies that $\left(y_{\{1\}}, \ldots, y_{\{n\}}\right) \in K$. Moreover it is instructive to verify that for any feasible integral solution $x \in K \cap\{0,1\}^{n}$ one can set $y_{I}:=\prod_{i \in I} x_{i}$ to obtain a vector $y \in S A_{r}(K)$.

Theorem 5 (Properties of Sherali-Adams). Let $y \in S A_{r}(K)$ for some $r \geq 0$. Then the following holds:
(a) For $J \in \mathcal{P}_{r}([n])$ with $y_{J}>0$, the vector $\tilde{y} \in \mathbb{R}^{\mathcal{P}_{r+1-|J|}([n])}$ defined by $\tilde{y}_{I}:=\frac{y_{I U J}}{y_{J}}$ satisfies $\tilde{y} \in S A_{r-|J|}(K)$.
(b) One has $0 \leq y_{I} \leq y_{J} \leq 1$ for $J \subseteq I$ and $|I| \leq r+1$.
(c) If $|J| \leq r+1$ and $y_{i} \in\{0,1\} \forall i \in J$, then $y_{I}=y_{I \backslash J} \cdot \prod_{i \in I \cap J} y_{i}$ for all $|I| \leq r+1$.
(d) For $J \subseteq[n]$ with $|J| \leq r$ there exists a distribution over vectors $\tilde{y}$ such that $(i) \tilde{y} \in S A_{r-|J|}(K)$, (ii) $\tilde{y}_{i} \in\{0,1\}$ for $i \in J$, (iii) $y_{I}=\mathbb{E}\left[\tilde{y}_{I}\right]$ for all $I \subseteq[n]$ with $|I \cup J| \leq r+1$ (this includes in particular all $\left.I \in \mathcal{P}_{r+1-|J|}([n])\right)$.
(e) For $I \subseteq[n]$ with $|I| \leq r$ and $k \in[N]$ one has $y_{I}=\sum_{i \in U_{k}} y_{I \cup\{i\}}$.
(f) Take $H \subseteq[N]$ with $|H| \leq r$ and set $J:=\bigcup_{k \in H} U_{k}$. Then there exists a distribution over vectors $\tilde{y}$ such that (i) $\tilde{y} \in S A_{r-|H|}(K)$, (ii) $\tilde{y}_{i} \in\{0,1\}$ for $i \in J$, (iii) $y_{I}=\mathbb{E}\left[\tilde{y}_{I}\right]$ for all $I \in \mathcal{P}_{r+1-|H|}([n])$.
Proof. For (a)-(d), we refer to the extensive coverage in Laurent Lau03. We prove (e) and (f) which are non-standard and custom-tailored to LPs with assignment constraints:
(e) Fix $I \subseteq[n]$ with $|I| \leq r$. We apply (d) to obtain a distribution over $\tilde{y}$ with $\tilde{y} \in S A_{r-|I|}(K)$ so that $\tilde{y}_{i} \in\{0,1\}$ for $i \in I$. Then

$$
\sum_{i \in U_{k}} y_{I \cup\{i\}} \stackrel{\text { linearity }}{=} \mathbb{E}\left[\sum_{i \in U_{k}} \tilde{y}_{I \cup\{i\}}\right] \stackrel{(c)}{=} \mathbb{E}[\tilde{y}_{I} \cdot \underbrace{\sum_{i \in U_{k}} \tilde{y}_{i}}_{=1}]=\mathbb{E}\left[\tilde{y}_{I}\right]=y_{I} .
$$

Here we apply (c) for index sets $I \cup\{i\}$ where variables in $J:=I$ have been made integral. Note that indeed $|I \cup(I \cup\{i\})| \leq r+1$ as required.
(f) By an inductive argument it suffices to consider the case of $|H|=1$. Let $H=\{k\}$ and set $U:=U_{k}$, i.e. the constraints for polytope $P$ contain the assignment constraint $\sum_{i \in U} x_{i}=1$ and we want to make all variables in $U$ integral while only losing a single round in the hierarchy. Abbreviate $U^{+}:=\left\{i \in U \mid y_{\{i\}}>0\right\}$. For $i \in U^{+}$, define $y^{(i)} \in \mathbb{R}^{\mathcal{P}_{r}([n])}$ to be the vector with $y_{I}^{(i)}:=\frac{y_{I \cup\{i\}}}{y_{i}}$. By (a) we know that $y^{(i)} \in S A_{r-1}(K)$. Moreover $y_{\{i\}}^{(i)}=\frac{y_{\{i\}}}{y_{\{i\}}}=1$. Then the assignment constraint of the LP forces that $y_{\left\{i^{\prime}\right\}}^{(i)}=0$ for $i^{\prime} \in U \backslash\{i\}$. Now we define a probability distribution over vectors $\tilde{y}$ as follows: for $i \in U^{+}$, with probability $y_{i}$ we set $\tilde{y}:=y^{(i)}$. Then (i) and (ii) hold for $\tilde{y}$ as discussed. Property (iii) follows from

$$
\mathbb{E}\left[\tilde{y}_{I}\right]=\sum_{i \in U^{+}} y_{i} y_{I}^{(i)}=\sum_{i \in U^{+}} y_{i} \frac{y_{I \cup\{i\}}}{y_{i}}=\sum_{i \in U^{+}} y_{I \cup\{i\}} \stackrel{(b)}{=} \sum_{i \in U} y_{I \cup\{i\}} \stackrel{(e)}{=} y_{I}
$$

It is known that Theorem 5(f) holds in a stronger form for the SDP-based Lasserre hierarchy. Karlin, Mathieu and Nguyen [KMN11 proved a result that can be paraphrased as follows: if one has any set $J \subseteq[n]$ of variables with the property that there is no LP solution with more than $k$ ones in $J$, then one can make all variables of $J$ integral while losing only $k$ rounds. Interestingly, Karlin, Mathieu and Nguyen prove that this is completely false for Sherali-Adams. In particular, for a Knapsack instance with unit size items and capacity $2-\varepsilon$, the integrality gap is still $2-2 \varepsilon$ after $\Theta_{\varepsilon}(n)$ rounds of Sherali-Adams. In a different setting, Friggstad et al. [FKK ${ }^{+} 14$ ] realized that given a "tree constraint", a Sherali-Adams lift can provide the same guarantees that Rothvoss Rot11 derived from Lasserre. While Friggstad et al. did not state their insight in the generality that we need here, our Lemma (e) + (f) are inspired by their work.

### 2.2 Semimetric Spaces

A semimetric space is a pair $(V, d)$ where $V$ is a finite set (we denote $n:=|V|)$ and $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$ is a semimetric, i.e.

- $d(u, u)=0$ for all $u \in U$.
- Symmetry: $d(u, v)=d(v, u)$ for all $u, v \in V$.
- Triangle inequality: $d(u, v)+d(v, w) \geq d(u, w)$ for all $u, v, w \in V$.

Recall that the more common notion is that of a metric, which additionally requires that $d(u, v)>0$ for $u \neq v$. For a set $U \subseteq V$ we denote the diameter as $\operatorname{diam}(U):=\max _{u, v \in U} d(u, v)$. Our goal is to find a partition $V=V_{1} \dot{\cup} \ldots \dot{U} V_{k}$ such that the diameter of every cluster $V_{i}$ is bounded by some parameter $\Delta$. We denote $d(w, U):=\min \{d(w, u): u \in U\}$ as the distance to the set $U$. Moreover, for $r \geq 0$ and $U \subseteq V$, let $N(U, r):=\{v \in V \mid d(v, U) \leq r\}$ be the distance $r$-neighborhood of $U$.

We use a very influential clustering algorithm due to Calinescu, Karloff and Rabani [CKR04, which assigns each $v \in V$ to a random cluster center $c \in V$ such that $d(u, c) \leq \beta \Delta$. Nodes assigned to the same cluster center form one block $V_{i}$ in the partition. Formally the algorithm is as follows:

```
CKR Clustering algorithm
Input: Semimetric space \((V, d)\) with \(V=\left\{v_{1}, \ldots, v_{n}\right\}\), parameter \(\Delta>0\)
Output: Clustering \(V=V_{1} \dot{\cup} \ldots \dot{\cup} V_{k}\) for some \(k\).
    (1) Pick a uniform random \(\beta \in\left[\frac{1}{4}, \frac{1}{2}\right]\)
    (2) Pick a random ordering \(\pi: V \rightarrow\{1, \ldots, n\}\)
    (3) For each \(v \in V\) set \(\sigma(v):=v_{\ell}\) so that \(d\left(v, v_{\ell}\right) \leq \beta \cdot \Delta\) and \(\pi\left(v_{\ell}\right)\) is minimal
    (4) Denote the points \(v \in V\) with \(\sigma^{-1}(v) \neq \emptyset\) by \(c_{1}, \ldots, c_{k} \in V\) and return clusters
        \(V_{i}:=\sigma^{-1}\left(c_{i}\right)\) for \(i=1, \ldots, k\)
```

Note that the algorithm has two sources of randomness: it picks a random parameter $\beta$, and independently it picks a random ordering $\pi$. Here the ordering is to be understood such that element $v_{\ell}$ with $\pi\left(v_{\ell}\right)=1$ is the "highest priority" element. The original work of Calinescu, Karloff and Rabani CKR04 only provided an upper bound on the probability that a short edge ( $u, v$ ) is separated. Mendel and Naor MN06 note that the same clustering provides the guarantee of $\operatorname{Pr}[N(u, t)$ separated $] \leq 1-O\left(\frac{t}{\Delta} \cdot \ln \left(\frac{|N(u, \Delta)|}{|N(u, \Delta / 8)|}\right)\right)$ for all $u \in V$ and $0 \leq t<\frac{\Delta}{8}$. Mendel and Naor attribute this to Fakcharoenphol, Rao and Talwar [FRT04] (while Fakcharoenphol, Rao and Talwar FRT04 do not state it explicitly in this form and focus on the "local growth ratio" aspect).

We state the formal claim in a form that will be convenient for us. For the sake of completeness, a proof can be found in the Appendix.

Theorem 6 (Analysis of CKR). Let $V=V_{1} \dot{\cup} \ldots \dot{U} V_{k}$ be the random partition of the CKR algorithm. The following holds:
(a) The blocks have $\operatorname{diam}\left(V_{i}\right) \leq \Delta$ for $i=1, \ldots, k$.
(b) Let $U \subseteq V$ be a subset of points. Then

$$
\operatorname{Pr}[U \text { is separated by clustering }] \leq \ln (2|N(U, \Delta / 2)|) \cdot \frac{4 \operatorname{diam}(U)}{\Delta} \leq \ln (2 n) \cdot \frac{4 \operatorname{diam}(U)}{\Delta} .
$$

In the above, separated means that there is more than one index $i$ with $V_{i} \cap U \neq \emptyset$.

## 3 An Approximation for $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\max }$

In this section, we provide an approximation algorithm for scheduling $n$ unit-length jobs $J$ with communication delay $c \in \mathbb{N}$ on an unbounded number of machines so that precedence constraints given by a partial order $\prec$ are satisfied. Instead of working with $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c \mid C_{\text {max }}$ directly, it will be more convenient to consider a slight variant that we call $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid$


Figure 1: Left: example of an instance of $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\text {max }}$ with $c=4$ (where the partial order $\prec$ is the transitive closure of the depicted digraph). Right: a valid schedule in 2 intervals.
$C_{\text {max }}$. This problem variant has the same input but the time horizon is partitioned into time intervals of length $c$, say $I_{s}=[s c,(s+1) c)$ for $s \in \mathbb{Z}_{\geq 0}$. The goal is to assign jobs to intervals and machines. We require that if $j_{1} \prec j_{2}$ then either $j_{1}$ is scheduled in an earlier interval than $j_{2}$ or $j_{1}$ and $j_{2}$ are scheduled in the same interval on the same machine. Other than that, there are no further communication delays. The objective function is to minimize the number of intervals used to process the jobs. In fact we do not need to decide the order of jobs within intervals as any topological order will work. In a more mathematical notation, the problem asks to find a partition $J=\dot{U}_{s \in\{0, \ldots, S-1\}, i \in \mathbb{N}} J_{s, i}$ with $\left|J_{s, i}\right| \leq c$ such that $S$ is minimized and for every $j_{1} \prec j_{2}$ with $j_{1} \in J_{s_{1}, i_{1}}$ and $j_{2} \in J_{s_{2}, i_{2}}$ one has either $s_{1}<s_{2}$ or $\left(s_{1}, i_{1}\right)=\left(s_{2}, i_{2}\right)$. See Figure 1 for an illustration.

It is rather straightforward to give reductions between $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c \mid C_{\max }$ and $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\max }$ that only lose a small constant factor in both directions. The only subtle point to consider here is that when the optimum makespan for $\mathrm{P} \infty|\mathrm{prec}, c| C_{\text {max }}$ is less than $c$, the problem admits a PTAS; we refer to Section 4 for details.

### 3.1 The Linear Program

Let $m \in \mathbb{N}$ be a parameter defining the number of machines that we admit for the LP. Moreover, let $S \in \mathbb{N}$ be the number of intervals that we allow for the time horizon. To obtain an approximation for $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\text {max }}$ one can set $m:=n$ and perform a binary search to find the minimal $S$ for which the LP is feasible. But we prefer to keep the approach general.

We construct the LP in two steps. First consider the variables

$$
x_{j, i, s}=\left\{\begin{array}{ll}
1 & \text { if } j \text { is scheduled on machine } i \text { in interval } I_{s} \\
0 & \text { otherwise }
\end{array} \quad \forall j \in J, i \in[m], s \in\{0, \ldots, S-1\}\right.
$$

Let $K$ be the set of fractional solutions to the following linear system

$$
\begin{aligned}
\sum_{i \in[m]} \sum_{s \geq 0} x_{j, i, s} & =1 \quad \forall j \in J \\
\sum_{j \in J} x_{j, i, s} & \leq c \quad \forall i \in[m] \forall s \in\{0, \ldots, S-1\} \\
0 \leq x_{j, i, s} & \leq 1 \quad \forall j \in J, i \in[m], s \in\{0, \ldots, S-1\}
\end{aligned}
$$

So far, $K$ simply assigns jobs (fractionally) to intervals and machines without taking any precedence constraints into account. Next, we will use a lift $x \in \operatorname{SA}_{r}(K)$ containing variables $x_{\left(j_{1}, i_{1}, s_{1}\right),\left(j_{2}, i_{2}, s_{2}\right)}$,
which provide the probability for the event that $j_{1}$ is scheduled in interval $s_{1}$ on machine $i_{1}$ and $j_{2}$ is scheduled in interval $s_{2}$ on machine $i_{2}$. We introduce two more types of decision variables:

$$
\begin{aligned}
y_{j_{1}, j_{2}} & = \begin{cases}1 & j_{1} \text { and } j_{2} \text { are scheduled on the same machine in the same interval } \\
0 & \text { otherwise }\end{cases} \\
C_{j} & =\text { index of interval where } j \text { is processed }
\end{aligned}
$$

Let $Q(r)$ be the set of vectors $(x, y, C)$ that satisfy

$$
\begin{aligned}
y_{j_{1}, j_{2}} & =\sum_{s \in\{0, \ldots, S-1\}} \sum_{i \in[m]} x_{\left(j_{1}, i, s\right),\left(j_{2}, i, s\right)} \\
C_{j_{2}} & \geq C_{j_{1}}+\left(1-y_{j_{1}, j_{2}}\right) \quad \forall j_{1} \prec j_{2} \\
C_{j} & \geq 0 \quad \forall j \in J \\
x & \in S A_{r}(K)
\end{aligned}
$$

The analysis of our algorithm will work for all $r \geq 5$ while solving the LP takes time $n^{O(r)}$. Here we make no attempt at optimizing the constant $r$. The main technical contribution of this section is the following rounding result:

Theorem 7. Consider an instance with unit-length jobs $J$, a partial order $\prec$, and parameters $c, S, m \in \mathbb{N}$ such that $Q(r)$ is feasible for $r:=5$. Then there is a randomized algorithm with expected polynomial running time that finds a schedule for $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\max }$ using at most $O(\log m \cdot \log c) \cdot S$ intervals.

We would like to emphasize that we require $\prec$ to be a partial order, which implies that it is transitive. While replacing any acyclic digraph with its transitive closure does not change the set of feasible integral schedules and hence can be done in a preprocessing step, it corresponds to adding constraints to the LP that we rely on in the algorithm and in its analysis.

We will now discuss some properties that are implied by the Sherali-Adams lift:
Lemma 8. Let $(x, y, C) \in Q(r)$ with $r \geq 2$. Then for any set $\tilde{J} \subseteq J$ of $|\tilde{J}| \leq r-2$ jobs, there exists a distribution $\mathcal{D}(\tilde{J})$ over pairs ( $\tilde{x}, \tilde{y})$ such that
(A) $\tilde{x}_{j, i, s} \in\{0,1\}$ for all $j \in \tilde{J}$, all $i \in[m]$ and $s \geq 0$.
(B) $\tilde{y}_{j_{1}, j_{2}}=\sum_{s \geq 0} \sum_{i \in[m]} \tilde{x}_{j_{1}, i, s} \cdot \tilde{x}_{j_{2}, i, s}$ if $\left|\left\{j_{1}, j_{2}\right\} \cap \tilde{J}\right| \geq 1$.
(C) $\tilde{x} \in K, \tilde{y}_{j_{1}, j_{2}}=\sum_{s \in\{0, \ldots, S-1\}} \sum_{i \in[m]} \tilde{x}_{\left(j_{1}, i, s\right),\left(j_{2}, i, s\right)}$ for all $j_{1}, j_{2} \in J$.
(D) $\mathbb{E}\left[\tilde{x}_{j, i, s}\right]=x_{j, i, s}$ and $\mathbb{E}\left[\tilde{y}_{j_{1}, j_{2}}\right]=y_{j_{1}, j_{2}}$ for all $j, j_{1}, j_{2}, i, s$.

Proof. By Theorem[5(f), there is a distribution over $\tilde{x} \in S A_{2}(K)$ which satisfies $(A)$ and has $\tilde{x} \in K$, $\mathbb{E}\left[\tilde{x}_{j, i, s}\right]=x_{j, i, s}$ and $\mathbb{E}\left[\tilde{x}_{\left(j_{1}, i_{1}, s_{1}\right),\left(j_{2}, i_{2}, s_{2}\right)}\right]=x_{\left(j_{1}, i_{1}, s_{1}\right),\left(j_{2}, i_{2}, s_{2}\right)}$, and additionally is integral on variables involving only jobs from $\tilde{J}$, where $|\tilde{J}| \leq r-2$. Here, we crucially use that every job $j \in \tilde{J}$ is part of an assignment constraint $\sum_{i \in[m]} \sum_{s \geq 0} x_{j, i, s}=1$, hence making these variables integral results in the loss of only one round per job. Then, the $y$-variables are just linear functions depending on the $x$-variables, so we can define

$$
\tilde{y}_{j_{1}, j_{2}}:=\sum_{s \in\{0, \ldots, S-1\}} \sum_{i \in[m]} \tilde{x}_{\left(j_{1}, i, s\right),\left(j_{2}, i, s\right)}
$$

and the claim follows.

From the LP solution, we define a semimetric $d$. Here the intuitive interpretation is that a small distance $d\left(j_{1}, j_{2}\right)$ means that the LP schedules $j_{1}$ and $j_{2}$ mostly on the same machine and in the same interval.

Lemma 9. Let $(x, y, C) \in Q(r)$ be a solution to the LP with $r \geq 5$. Then $d\left(j_{1}, j_{2}\right):=1-y_{j_{1}, j_{2}}$ is a semimetric.

Proof. The first two properties from the definition of a semimetric (see Section 2.2) are clearly satisfied. We verify the triangle inequality. Consider three jobs $j_{1}, j_{2}, j_{3} \in J$. We apply Lemma 8 with $\tilde{J}:=\left\{j_{1}, j_{2}, j_{3}\right\}$ and consider the distribution $(\tilde{x}, \tilde{y}) \sim \mathcal{D}(\tilde{J})$. For $j \in \tilde{J}$, define $Z(j)=$ $(\tilde{s}(j), \tilde{i}(s))$ as the random variable that gives the unique pair of indices such that $\tilde{x}_{j, \tilde{i}(j), \tilde{s}(j)}=1$. Then for $j^{\prime}, j^{\prime \prime} \in \tilde{J}$ one has

$$
d\left(j^{\prime}, j^{\prime \prime}\right)=\operatorname{Pr}\left[Z\left(j^{\prime}\right) \neq Z\left(j^{\prime \prime}\right)\right]=\operatorname{Pr}\left[\left(\tilde{s}(j), \tilde{i}\left(j^{\prime}\right)\right) \neq\left(\tilde{s}\left(j^{\prime \prime}\right), \tilde{i}\left(j^{\prime \prime}\right)\right)\right]
$$

Then indeed

$$
\begin{array}{ccl}
d\left(j_{1}, j_{3}\right) & = & \operatorname{Pr}\left[Z\left(j_{1}\right) \neq Z\left(j_{3}\right)\right] \leq \operatorname{Pr}\left[Z\left(j_{1}\right) \neq Z\left(j_{2}\right) \vee Z\left(j_{2}\right) \neq Z\left(j_{3}\right)\right] \\
& \text { union bound } & \operatorname{Pr}\left[Z\left(j_{1}\right) \neq Z\left(j_{2}\right)\right]+\operatorname{Pr}\left[Z\left(j_{2}\right) \neq Z\left(j_{3}\right)\right]=d\left(j_{1}, j_{2}\right)+d\left(j_{2}, j_{3}\right) .
\end{array}
$$

Lemma 10. For every $j_{1} \in J$ one has $\sum_{j_{2} \in J} y_{j_{1}, j_{2}} \leq c$.
Proof. Consider the distribution $(\tilde{x}, \tilde{y}) \sim \mathcal{D}\left(\left\{j_{1}\right\}\right)$. From Lemma 园, (B) we know that $\mathbb{E}\left[\tilde{y}_{j_{1}, j_{2}}\right]=$ $y_{j_{1}, j_{2}}$ and $\tilde{y}_{j_{1}, j_{2}}=\sum_{s \in\{0, \ldots, S-1\}} \sum_{i \in[m]} \tilde{x}_{j_{1}, i, s} \cdot \tilde{x}_{j_{2}, i, s}$. By linearity it suffices to prove that $\sum_{j_{2} \in J} \tilde{y}_{j_{1}, j_{2}} \leq$ $c$ always. Fix a pair $(\tilde{x}, \tilde{y})$. There is a unique pair of indices $\left(i_{1}, s_{1}\right)$ with $\tilde{x}_{j_{1}, i_{1}, s_{1}}=1$. Then

$$
\sum_{j_{2} \in J} \tilde{y}_{j_{1}, j_{2}}=\sum_{s \in\{0, \ldots, S-1\}} \sum_{j_{2} \in J} \sum_{i \in[m]} \underbrace{\tilde{x}_{j_{1} i, s}}_{0 \text { if } i \neq i_{1} \text { or } s \neq s_{1}} \cdot \tilde{x}_{j_{2}, i, s}=\sum_{j_{2} \in J} \tilde{x}_{j_{2}, i_{1}, s_{1}} \leq c .
$$

A crucial insight is that for any job $j^{*}$, few jobs are very close to $j^{*}$ with respect to $d$.
Lemma 11. Fix $j^{*} \in J$ and abbreviate $U:=\left\{j \in J \mid d\left(j, j^{*}\right) \leq \beta\right\}$ for $0<\beta<1$. Then $|U| \leq \frac{c}{1-\beta}$.
Proof. For each $j \in U$ we have $y_{j, j^{*}}=1-d\left(j, j^{*}\right) \geq 1-\beta$. Combining with the last lemma we have $(1-\beta)|U| \leq \sum_{j \in J} y_{j, j^{*}} \leq c$.

### 3.2 Scheduling a Single Batch of Jobs

We now come to the main building block of our algorithm. We consider a subset $J^{*}$ of jobs whose LP completion times $C_{j}$ are very close (within a $\Theta\left(\frac{1}{\log (c)}\right)$ term of each other) and show we can schedule half of these jobs in a single length- $2 c$ interval. The following lemma is the main technical contribution of the paper.

Lemma 12. Let $(x, y, C) \in Q(r)$ with $r \geq 5$ and let $0<\delta \leq \frac{1}{64 \log (4 c)}$ be a parameter. Let $C^{*} \geq 0$ and set $J^{*} \subseteq\left\{j \in J \mid C^{*} \leq C_{j} \leq C^{*}+\delta\right\}$. Then there is a randomized rounding procedure that finds a schedule for a subset $J^{* *} \subseteq J^{*}$ in a single interval of length at most $2 c$ such that every job $j \in J^{*}$ is scheduled with probability at least $1-32 \log (4 c) \cdot \delta \geq \frac{1}{2}$.


Figure 2: Visualization of the partition $V=V_{1} \dot{\cup} \ldots \dot{U} V_{k}$ and the induced sets $V_{\ell}^{\prime} \subseteq V_{\ell}$. Here $\prec$ is the transitive closure of the depicted digraph.

We denote $\Gamma^{-}(j)$ as the predecessors of $j$ and $\Gamma^{+}(j)$ as the successors, and similary $\Gamma^{-/+}\left(J^{\prime}\right)=$ $\left\{j \in J: \exists j^{\prime} \in J^{\prime}\right.$ s.t. $\left.j \in \Gamma^{-/+}\left(j^{\prime}\right)\right\}$. Again, recall that we assume $\prec$ to be transitive. The rounding algorithm is the following:

```
Scheduling a Single Batch
    (1) Run a CKR clustering on the semimetric space \(\left(J^{*}, d\right)\) with parameter \(\Delta:=\frac{1}{4}\)
        and let \(V_{1}, \ldots, V_{k}\) be the clusters.
    (2) Let \(V_{\ell}^{\prime}:=\left\{j \in V_{\ell} \mid \Gamma^{-}(j) \cap J^{*} \subseteq V_{\ell}\right\}\) for \(\ell=1, \ldots, k\).
    (3) Schedule \(V_{\ell}^{\prime}\) on one machine for all \(\ell=1, \ldots, k\).
```

We now discuss the analysis. First we show that no cluster is more than a constant factor too large.

Lemma 13. One has $\left|V_{\ell}^{\prime}\right| \leq 2 c$ for all $\ell=1, \ldots, k$.
Proof. We know by Theorem 6 that $\operatorname{diam}\left(V_{\ell}^{\prime}\right) \leq \operatorname{diam}\left(V_{\ell}\right) \leq \Delta<\frac{1}{2}$ where the diameter is with respect to $d$. Fix a job $j^{*} \in V_{\ell}^{\prime}$. Then we know by Lemma 11 that there are at most $2 c$ jobs $j$ with $d\left(j, j^{*}\right) \leq \frac{1}{2}$ and the claim follows.

Next, we see that the clusters respect the precedence constraints.
Lemma 14. The solution $V_{1}^{\prime}, \ldots, V_{k}^{\prime}$ is feasible in the sense that jobs on different machines do not have precedence constraints.

Proof. Consider jobs processed on different machines, say (after reindexing) $j_{1} \in V_{1}^{\prime}$ and $j_{2} \in V_{2}^{\prime}$. If $j_{1} \prec j_{2}$ then we did not have $\Gamma^{-}\left(j_{2}\right) \subseteq V_{2}^{\prime}$. This contradicts the definition of the sets $V_{\ell}^{\prime}$.

A crucial property that makes the algorithm work is that predecessors of some job $j \in J^{*}$ must be very close in $d$ distance.

Lemma 15. For every $j_{1}, j_{2} \in J^{*}$ with $j_{1} \prec j_{2}$ one has $d\left(j_{1}, j_{2}\right) \leq \delta$.
Proof. We know that

$$
C^{*} \stackrel{j_{1} \in J^{*}}{\leq} C_{j_{1}} \leq C_{j_{1}}+\underbrace{\left(1-y_{\left.j_{1}, j_{2}\right)}\right.}_{=d\left(j_{1}, j_{2}\right)} \stackrel{L P}{\leq} C_{j_{2}} \stackrel{j_{2} \in J^{*}}{\leq} C^{*}+\delta
$$

and so $d\left(j_{1}, j_{2}\right) \leq \delta$.

We will use the three statements above together with Theorem 6 to prove Lemma 12 ,
Proof of Lemma 12. We have already proven that the scheduled blocks have size $\left|V_{\ell}^{\prime}\right| \leq 2 c$ and that there are no dependent jobs in different sets of $V_{1}^{\prime}, \ldots, V_{k}^{\prime}$. To finish the analysis, we need to prove that a fixed job $j^{*} \in J^{*}$ is scheduled with good probability. Consider the set $U:=\left\{j^{*}\right\} \cup\left(\Gamma^{-}\left(j^{*}\right) \cap J^{*}\right)$ of $j^{*}$ and its ancestors in $J^{*}$.

Since the diameter of $U$ is at most $2 \delta$ by Lemma 15, we can use Lemma 11 to see that $|N(U, \Delta / 2)| \leq \frac{c}{1-2 \delta-\Delta}$. For our choice of $\Delta=1 / 4$ and $\delta \leq \frac{1}{64 \log (4 c)},|N(U, 1 / 8)| \leq 2 c$. From Theorem [6, the cluster is separated with probability at most $\log (4 c) \cdot \frac{8 \delta}{\Delta} \leq \frac{1}{2}$.

To schedule all jobs in $J^{*}$, we repeat the clustering procedure $O(\log m)$ times and simply schedule the remaining jobs on one machine.

Lemma 16. Let $(x, y, C) \in Q(r)$ with $r \geq 5$. Let $C^{*} \geq 0$ and set $J^{*} \subseteq\left\{j \in J \mid C^{*} \leq C_{j}<C^{*}+\delta\right\}$. Assume that all jobs in $\Gamma^{-}\left(J^{*}\right) \backslash J^{*}$ have been scheduled respecting precedence constraints. Then there is an algorithm with expected polynomial running time that schedules all jobs in $J^{*}$ using at most $O(\log m)+\frac{\left|J^{*}\right|}{m c}$ many intervals.

Proof. Our algorithm in Lemma 12 schedules each $j \in J^{*}$ in an interval of length $2 c$ with probability at least $1 / 2$. We run the algorithm for $2 \log m$ iterations, where input to iteration $k+1$ is the subset of jobs that are not scheduled in the first $k$ iterations. For $k \in\{1,2, \ldots, 2 \log m\}$, let $J_{k}^{* *}$ denote the subset of jobs that are scheduled in the $k^{t h}$ iteration, and let $J_{k+1}^{*}:=J^{*} \backslash\left\{\bigcup_{k^{\prime}=1}^{k} J_{k^{\prime}}^{* *}\right\}$. In this notation, $J_{1}^{*}:=J^{*}$. Let $\mathcal{S}\left(J_{k}^{* *}\right)$ denote the schedule of jobs $J_{k}^{* *}$ given by Lemma 12, We schedule $\mathcal{S}\left(J_{1}^{* *}\right)$ first, then for $k=2, \ldots, 2 \log m$, we append the schedule $\mathcal{S}\left(J_{k}^{* *}\right)$ after $\mathcal{S}\left(J_{k-1}^{* *}\right)$. Let $J^{\prime}:=J_{2}^{*} \log m+1$ denote the set of jobs that were not scheduled in the $2 \log m$ iterations. We schedule all jobs in $J^{\prime}$ consecutively on a single machine after the completion of $\mathcal{S}\left(J_{2}^{* *} \log m\right)$.

From our construction, the length of a schedule for $J^{*}$, which is a random variable, is at most $O(\log m)+\left\lceil\frac{\left|J^{\prime}\right|}{c}\right\rceil$ many intervals. For $k \in\{1,2, \ldots, 2 \log m\}$, Lemma 12 guarantees that each job $j \in J_{k}^{*}$ gets scheduled in the $k^{\text {th }}$ iteration with probability at least $1 / 2$. Therefore, the probability that $j \in J^{\prime}$, i.e., it does not get scheduled in the first $2 \log m$ iterations, is at most $\frac{1}{2 m}$. This implies that $\mathbb{E}\left[\left|J^{\prime}\right|\right] \leq \frac{\left|J^{*}\right|}{2 m}$. By Markov's inequality $\operatorname{Pr}\left[\left|J^{\prime}\right|>\frac{\left|J^{*}\right|}{m}\right] \leq \operatorname{Pr}\left[\left|J^{\prime}\right|>2 \cdot \mathbb{E}\left[\left|J^{\prime}\right| \mid\right] \leq 1 / 2\right.$. Hence we can repeat the described procedure until indeed we have a successful run with $\left|J^{\prime}\right| \leq \frac{\left|J^{*}\right|}{m}$ which results in the claimed expected polynomial running time.

Let us now argue that the schedule of $J^{*}$ is feasible. For $k \in\{1,2, \ldots, 2 \log m\}$ and any two jobs $j, j^{\prime} \in \mathcal{S}\left(J_{k}^{* *}\right)$, Lemma 12 guarantees that precedence and communication constraints are satisfied. Furthermore, Lemma 12 also ensures that there cannot be jobs $j, j^{\prime}$ such that $j \in \mathcal{S}\left(J_{k}^{* *}\right)$, $j^{\prime} \in \mathcal{S}\left(J_{k^{\prime}}^{* *}\right)$ and $j^{\prime} \prec j$ and $k^{\prime}>k$. Finally note that every length- $2 c$ interval can be split into 2 length- $c$ intervals. The claim follows.

### 3.3 The Complete Algorithm for $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c$-intervals $\mid C_{\max }$

Now we have all the pieces to put the rounding algorithm together and prove its correctness. We partition the jobs into batches, where each batch consists of subset of jobs that have $C_{j}$ very close to each other in the LP solution. The complete algorithm is given below.

## The Complete Algorithm

(1) Solve the LP and let $(x, y, C) \in Q(r)$ with $r \geq 5$.
(2) For $\delta=\frac{1}{64 \log (4 c)}$ and $k \in\left\{0,1,2 \ldots \frac{S-1}{\delta}\right\}$, define
$J_{k}=\left\{j \in J: k \cdot \delta \leq C_{j}<(k+1) \cdot \delta\right\}$
(3) FOR $k=0$ TO $\frac{S-1}{\delta}$ DO
(4) Schedule the jobs in $J_{k}$ using the algorithm in Subsection 3.2,

Now we finish the analysis of the rounding algorithm.
Proof of Theorem [7, Let us quickly verify that the schedule constructed by our algorithm is feasible. For jobs $j_{1} \prec j_{2}$ with $j_{1} \in J_{k_{1}}$ and $j_{2} \in J_{k_{2}}$, the LP implies that $C_{j_{1}} \leq C_{j_{2}}$ and so $k_{1} \leq k_{2}$. If $k_{1}<k_{2}$, then $j_{1}$ will be scheduled in an earlier interval than $j_{2}$. If $k_{1}=k_{2}=k$, then Lemma 16 guarantees that precedence constraints are satisfied.

It remains to bound the makespan of our algorithm. Lemma 16 guarantees that for $k \in$ $\left\{0,1,2 \ldots \frac{S-1}{\delta}\right\}$, the jobs in $J_{k}$ are scheduled using at most $O(\log m)+\frac{\left|J_{k}\right|}{c m}$ many intervals. Then the total number of intervals required by the algorithm is bounded by

$$
\frac{S}{\delta} \cdot O(\log m)+\sum_{k=0}^{\frac{S-1}{\delta}} \frac{\left|J_{k}\right|}{c m}=O(\log m \cdot \log c) \cdot S+\frac{|J|}{c m} \leq O(\log m \cdot \log c) \cdot S
$$

Here we use that $|J| \leq S \cdot \mathrm{~cm}$ is implied by the constraints defining $K$.
Remark 17. We note that it is possible to reverse-engineer our solution and write a more compact LP for the problem, enforcing only the necessary constraints such as those given by Lemmas 9 and 10. Such an LP would be simpler and could be solved more efficiently. However, we feel that the Sherali-Adams hierarchy gives a more principled and intuitive way to tackle the problem and explain how the LP arises, and hence we choose to present it that way.

## 4 Reductions

We now justify our earlier claim: the special case $\mathrm{P} \infty \mid \mathrm{prec}, p_{j}=1, c$-intervals $\mid C_{\text {max }}$ indeed captures the full computational difficulty of the more general problem $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$. The main result for this section will be the following reduction:

Theorem 18. Suppose there is a polynomial time algorithm that takes a solution for the LP $Q(r)$ with parameters $m, c, S \in \mathbb{N}$ and $r \geq 5$ and transforms it into a schedule for $\mathrm{P} \infty \mid \mathrm{prec}, p_{j}=$ $1, c$-intervals $\mid C_{\max }$ using at most $\alpha \cdot S$ intervals. Then there is a polynomial time $O(\alpha)$-approximation for $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$.

For the reduction we will make use of the very well known list scheduling algorithm by Graham Gra66 that can be easily extended to the setting with communication delays. Here the notation $\sigma(j)=\left(\left[t, t+p_{j}\right), i\right)$ means that the job $j$ is processed in the time interval $\left[t, t+p_{j}\right)$ on machine $i \in[m]$.


Figure 3: Analysis of Graham's algorithm with communication delay $c$.

```
Graham's List Scheduling
    (1) Set \(\sigma(j):=\emptyset\) for all \(j \in J\)
    (2) FOR \(t=0 \mathrm{TO} \infty \mathrm{DO}\) FOR \(i=1 \mathrm{TO} m \mathrm{DO}\)
            (3) Select any job \(j \in J\) with \(\sigma(j)=\emptyset\) where every \(j^{\prime} \prec j\) satisfies the
            following:
                    - If \(j^{\prime}\) is scheduled on machine \(i\) then \(j^{\prime}\) is finished at time \(\leq t\)
                    - If \(j^{\prime}\) is schedule on machine \(i^{\prime} \neq i\) then \(j^{\prime}\) finished at time \(\leq t-c\)
            (4) Set \(\sigma(j):=\left(\left[t, t+p_{j}\right), i\right)\) (if there was such a job)
```

For example, for the problem $\mathrm{P} \mid$ prec $\mid C_{\text {max }}$, Graham's algorithm gives a 2-approximation. The analysis works by proving that there is a chain of jobs covering all time units where not all machines are busy. Graham's algorithm does not give a constant factor approximation for our problem with communication delays, but it will still be useful for our reduction.

Recall that a set of jobs $\left\{j_{1}, \ldots, j_{\ell}\right\} \subseteq J$ with $j_{\ell} \prec j_{\ell-1} \prec \ldots \prec j_{1}$ is called a chain. We denote $\mathcal{Q}(J)$ as the set of all chains in $J$ w.r.t. precedence order $\prec$.

Lemma 19. Graham's list scheduling on an instance of $\mathrm{P}|\mathrm{prec}, c| C_{\max }$ results in a schedule with makespan at most $\frac{1}{m} \sum_{j \in J} p_{j}+\max _{Q \in \mathcal{Q}(J)}\left\{\sum_{j \in Q} p_{j}+c \cdot(|Q|-1)\right\}$.

Proof. We will show how to construct the chain $Q$ that makes the inequality hold. Let $j_{1}$ be the job which finishes last in the schedule produced by Graham's algorithm and let $t_{j_{1}}$ be its start time. Let $j_{2}$ be the predecessor of $j_{1}$ that finishes last. More generally in step $i$, we denote $j_{i+1}$ as the predecessor of $j_{i}$ that finishes last. The construction finishes with a job $j_{\ell}$ without predecessors. Now let $Q$ be the chain of jobs $j_{\ell} \prec j_{\ell-1} \prec \ldots \prec j_{1}$. The crucial observation is that for any $i \in\{1, \ldots, \ell-1\}$, either all machines are busy in the time interval $\left[t_{j_{i+1}}+p_{j_{i+1}}+c, t_{j_{i}}\right)$ or this interval is empty. The reason is that Graham's algorithm does not leave unnecessary idle time and would have otherwise processed $j_{i}$ earlier. It is also true that all $m$ machines are busy in the time interval $\left[0, t_{j_{\ell}}\right)$. The total amount of work processed in these busy time intervals is

$$
L:=m \cdot\left(t_{j_{\ell}}+\sum_{i=1}^{\ell-1} \max \left\{t_{j_{i}}-\left(c+p_{j_{i+1}}+t_{j_{i+1}}\right), 0\right\}\right) \leq \sum_{j \in J} p_{j}-\sum_{i=1}^{\ell} p_{j_{i}} .
$$

Then any time between 0 and the makespan falls into at least one of the following categories: (a) the busy time periods described above, (b) the times that a job of the chain $Q$ is processed, (c) the


Figure 4: Splitting jobs into chains of unit-length jobs.
interval of length $c$ following a job in the chain $Q$. Thus, we see that the makespan from Graham's list scheduling is at most

$$
t_{j_{1}}+p_{t_{j_{1}}} \leq \frac{L}{m}+\sum_{j \in Q} p_{j}+c \cdot(|Q|-1) \leq \frac{1}{m} \sum_{j \in J} p_{j}+\left(1-\frac{1}{m}\right) \sum_{j \in Q} p_{j}+c \cdot(|Q|-1) .
$$

It will also be helpful to note that the case of very small optimum makespan can be well approximated:

Lemma 20. Any instance for $\mathrm{P} \mid$ prec, $c \mid C_{\max }$ with optimum objective function value at most $c$ admits a PTAS.

Proof. Let $J$ be the jobs in the instance and let $\mathrm{OPT}_{m} \leq c$ be the optimum value. Consider the undirected graph $G=(J, E)$ with $\left\{j_{1}, j_{2}\right\} \in E \Leftrightarrow\left(\left(j_{1} \prec j_{2}\right)\right.$ or $\left.\left(j_{2} \prec j_{1}\right)\right)$. Let $J=J_{1} \dot{\cup} \ldots \dot{U} J_{N}$ be the partition of jobs into connected components w.r.t. graph $G$. We abbreviate $p\left(J^{\prime}\right):=\sum_{j \in J^{\prime}} p_{j}$. The assumption guarantees that the optimum solution cannot afford to pay the communication delay and hence there is a length- $c$ schedule that assigns all jobs of the same connected component to the same machine. If we think of a connected component $J_{\ell}$ as an "item" of size $p\left(J_{\ell}\right)$, then for any fixed $\varepsilon>0$ we can use a PTAS for $\mathrm{P} \| C_{\max }$ (i.e. makespan minimization without precedence constraints) to find a partition of "items" as $[N]=I_{1} \cup \dot{\cup} \dot{\cup} I_{m}$ with $\sum_{\ell \in I_{i}} p\left(J_{\ell}\right) \leq(1+\varepsilon) \cdot \mathrm{OPT}_{m}$ in polynomial time HS87]. Arranging the jobs $\bigcup_{\ell \in I_{i}} J_{\ell}$ on machine $i$ in any topological order finishes the argument.

Additionally, it is a standard argument to convert an instance with arbitrary $p_{j}$ to an instance where all $p_{j} \leq n / \varepsilon$, while only losing a factor of $(1+2 \varepsilon)$ in the approximation. For $p_{\max }:=\max _{j} p_{j}$, we simply scale the job lengths and communication delay down by a factor of $\frac{n}{\varepsilon p_{\text {max }}}$ then round them to the nearest larger integer. This results in at most a $2 \varepsilon$ fraction of the optimal makespan being rounded up and all job sizes are integral and at most $n / \varepsilon$.

Now we can show the main reduction:
Proof of Theorem 18. Consider an instance of $\mathrm{P} \mid$ prec, $c \mid C_{\text {max }}$ with $p_{j}, c \in \mathbb{N}$. Let $J$ denote its job set with precedence constraints, and $\mathrm{OPT}_{m}(J)$ denote its optimal value where $m$ is the number of available machines. By the previous argument, we may assume that $p_{j} \leq 2 n$ for all $j \in J$. Moreover, by Lemma [20 we only need to focus on the case where $\operatorname{OPT}_{m}(J)>c$. We may guess the optimum value of $\operatorname{OPT}_{m}(J)$ as $\mathrm{OPT}_{m}(J) \in\left\{1, \ldots, 2 n^{2}\right\}$.

Let $J^{\prime}$ denote the job set obtained by splitting each job $j \in J$ into a chain of $p_{j}$ unit sub-jobs $j^{(1)} \prec \cdots \prec j^{\left(p_{j}\right)}$. Moreover, precedence constraints in $J$ are preserved in $J^{\prime}$ as we set all predecessors of $j$ to be predecessors of $j^{(1)}$ and all successors of $j$ to be successors of $j^{\left(p_{j}\right)}$, see Figure 4 . We note that $\operatorname{OPT}_{m}\left(J^{\prime}\right) \leq \mathrm{OPT}_{m}(J)$ as splitting does not increase the value of the optimum. Let $\mathcal{S}_{m}\left(J^{\prime}\right)$ be a schedule achieving the value of $\mathrm{OPT}_{m}\left(J^{\prime}\right)$. Next, observe that $\mathcal{S}_{m}\left(J^{\prime}\right)$ implies an integral solution


Figure 5: Transformation of the schedule $\mathcal{S}_{\infty, \text { int }}\left(J^{\prime}\right)$ (top) to $\mathcal{S}_{\infty}(J)$ (bottom), where $\mathcal{S}_{\infty}(J)$ is compressed by a factor of 4 . Here a "private" machine for a job $j$ means the machine never processes any job other than $j$.
for $Q(r)$ with parameters $m, c, S$ where $S:=\left\lceil\operatorname{OPT}_{m}(J) / c\right\rceil$ and $r:=5$. In particular here we use that if jobs $j_{1} \prec j_{2}$ are scheduled on different machines by $\mathcal{S}_{m}\left(J^{\prime}\right)$, then their starting times differ by at least $c+1$ and hence they are assigned to different length $-c$ intervals.

Now we execute the assumed $\alpha$-approximate rounding algorithm and obtain a schedule $\mathcal{S}_{\infty, \text { int }}\left(J^{\prime}\right)$ that uses $T \leq \alpha S$ many intervals. We will use this solution $\mathcal{S}_{\infty, \text { int }}\left(J^{\prime}\right)$ to construct a schedule $\mathcal{S}_{\infty}(J)$ for $\mathrm{P} \infty \mid$ prec, $c \mid C_{\max }$ with job set $J$ by running split sub-jobs consecutively on the same processor. This will use $4 T$ time intervals in total. Recall that $I_{s}$ denotes the time interval $[s c,(s+1) c)$. The rescheduling process is as follows:

For a fixed job $j \in J$, let $I_{s_{1}}$ be the time interval where $j^{(1)}$ is scheduled in $\mathcal{S}_{\infty, \text { int }}\left(J^{\prime}\right)$. Then all other sub-jobs of $j$ should be either scheduled in $I_{s_{1}}$ or the time intervals after $I_{s_{1}}$.

- Case 1: Some sub-job of $j$ is not scheduled in $I_{s_{1}}$.

Schedule job $j$ at the beginning of time interval $I_{4 s_{1}+1}$ on a new machine. If $j$ is a short job, then it will finish running by the end of the interval. Otherwise $j$ is a long job. Let $I_{s_{2}}$ be the last time interval where a sub-job of $j$ is scheduled in $\mathcal{S}_{\infty, \text { int }}\left(J^{\prime}\right)$. Then, the length satisfies $p_{j} \leq c \cdot\left(s_{2}-s_{1}+1\right)$, which implies that the job finishes by time $c \cdot\left(4 s_{1}+1\right)+p_{j} \leq c \cdot\left(4 s_{2}-1\right)$.

- Case 2: All sub-jobs of $j$ are scheduled in $I_{s_{1}}$.

Simply schedule job $j$ during time interval $I_{4 s_{1}}$ on the same machine as in $\mathcal{S}_{\infty, \text { int }}\left(J^{\prime}\right)$.
See Figure 5 for a visualization. Then $\mathcal{S}_{\infty}(J)$ is a valid schedule for $\mathrm{P} \infty \mid$ prec, $c \mid C_{\text {max }}$, with makespan $\leq 4 c \cdot T$. Moreover, $\mathcal{S}_{\infty}(J)$ satisfies the following:
(a) A short job is fully contained in some interval $I_{s}$.
(b) A long job's start time is at the beginning of some interval $I_{s}$.

For $\mathcal{S}_{\infty}(J)$, define a new job set $H$. Every long job $j$ becomes an element of $H$ with its original running time $p_{j}$. Meanwhile, every set of short jobs that are assigned to the same machine in one
time interval becomes an element of $H$, with running time equal to the sum of running times of the short jobs merged. To summarize, a new job $h \in H$ corresponds to a set $h \subseteq J$ and $p_{h}=\sum_{j \in h} p_{j}$.

We define the partial order $\tilde{\prec}$ on $H$ with $h_{1} \tilde{\prec} h_{2}$ if and only if there are $j_{1} \in h_{1}$ and $j_{2} \in h_{2}$ with $j_{1} \prec j_{2}$. One can check that this partial order is well defined. Moreover, by the fact that jobs assigned to the same interval but different machines do not have precedence constraints, the length of the longest chain in $(H, \tilde{\chi})$ in terms of the number of elements is bounded by the number of intervals that are used, which is at most $4 T$.

Now run Graham's list scheduling on the new job set $H$ with order $\tilde{\imath}$ and $m$ machines. By Lemma 19, the makespan of the list scheduling is bounded by $\frac{1}{m} \sum_{h \in H} p_{h}+\max _{Q \in \mathcal{Q}(H)}\left\{\sum_{h \in Q} p_{h}+\right.$ $c \cdot|Q|\}$. As the total sum of the processing times does not change from $J$ to $H$, we see that $\frac{1}{m} \sum_{h \in H} p_{h} \leq \operatorname{OPT}_{m}(J)$. Moreover, for any chain $Q \in \mathcal{Q}(H), \sum_{h \in Q} p_{h}$ is no greater than the makespan of $\mathcal{S}_{\infty}(J)$, which is $4 c T$. Finally, as argued earlier, the chain has $|Q| \leq 4 T$ elements. Above all,

$$
\begin{aligned}
\frac{1}{m} \sum_{h \in H} p_{h}+\max _{Q \in \mathcal{Q}(H)}\left\{\sum_{h \in Q} p_{h}+c \cdot|Q|\right\} & \leq \operatorname{OPT}_{m}(J)+4 c T+4 c T \\
& \leq \operatorname{OPT}_{m}(J)+8 \alpha \cdot c S \\
& \leq \operatorname{OPT}_{m}(J)+16 \alpha \cdot \mathrm{OPT}_{m}(J) \\
& =O(\alpha) \cdot \operatorname{OPT}_{m}(J)
\end{aligned}
$$

## 5 Minimizing Weighted Sum of Completion Times

To illustrate the generality of our framework we show that it can be extended to handle different objective functions, in particular we can minimize the weighted sum of completion times of the jobs. Here we restrict to the simplest case where jobs have unit length and an unbounded number of machines are available. In the 3 -field notation, this problem is denoted by $\mathrm{P} \infty \mid$ prec, $p_{j}=1, c \mid$ $\sum_{j} w_{j} C_{j}$. The input for this problem is the same as for the makespan minimization problem except that each job $j$ now has a weight $w_{j} \geq 0$. The goal is to minimize the objective function $\sum_{j} w_{j} C_{j}$, where $C_{j}$ is the completion time of $j$, which is defined as the time slot in which job $j$ is scheduled.

Note that the LP $Q(r)$ has variables $C_{j}$ that denote the index of the length- $c$ interval where $j$ is being scheduled. A natural approach would be to interpret $c \cdot C_{j}$ as the completion time of job $j$ and minimize $\sum_{j \in J} w_{j} \cdot c \cdot C_{j}$ over $Q(r)$. Then the rounding algorithm from Section 3 will indeed schedule each job $j$ so that the completion time is at most $\left(O(\log c \cdot \log n) \cdot C_{j}+\Theta(\log n)\right) \cdot c$. We can observe that if a $O(\log c \cdot \log n)$ approximation is the goal, then this argument suffices for all jobs $j$ where the LP solution has $C_{j} \geq \Omega\left(\frac{1}{\log c}\right)$ - but it fails for jobs with $0 \leq C_{j} \ll 1$.

### 5.1 The linear program

In order to address this case, we first start with a more general LP relaxation compared to the makespan result which tracks the actual time slot where the jobs are processed, rather than just the interval. Again, we use the parameter $m \in \mathbb{N}$ to denote the number of machines that we allow the LP to use (one can set $m:=n$ ) and the parameter $S \in \mathbb{N}$ to denote the number of intervals that we allow for the time horizon. We abbreviate $T:=S \cdot c$ as the number of time slots. Note that $T \leq n c$ always suffices for any non-idling schedule. We index time slots as $[T]:=\{1, \ldots, T\}$ and consider an interval as a discrete set of slots $I_{s}:=\{c s+1, \ldots, c(s+1)\}$ where $s \in\{0, \ldots, S-1\}$.

Recall that in the makespan result $x_{j, i, s}$ variables indicated if job $j$ got scheduled on machine $i$ in the interval $s$. Here, we introduce additional variables of the form $z_{j, i, t}$ which indicate if job $j$ is scheduled on machine $i$ at time $t \in[T]$. The variables $x_{j, i, s}$ are fully determined by summing over appropriate variables $z_{j, i, t}$, but we retain them for notational convenience. Further, similar to our makespan result, we impose an interval structure on the optimal solution and lose an $O(1)$ factor in the approximation ratio.

Let $\tilde{K}$ be the set of fractional solutions to the following LP.

$$
\begin{aligned}
\sum_{i \in[m]} \sum_{t \in[T]} z_{j, i, t} & =1 \quad \forall j \in J \\
\sum_{j \in J} z_{j, i, t} & \leq 1 \quad \forall i \in[m] \quad \forall t \in[T] \\
\sum_{t^{\prime}<t} \sum_{i \in[m]} z_{j_{1}, i, t^{\prime}} & \geq \sum_{t^{\prime} \leq t} \sum_{i \in[m]} z_{j_{2}, i, t^{\prime}} \quad \forall j_{1} \prec j_{2} \quad \forall t \in[T] \\
\sum_{t \in I_{s}} z_{j, i, t} & =x_{j, i, s} \quad \forall j \in J \quad \forall s \in\{0, \ldots, S-1\} \\
0 \leq z_{j, i, t} & \leq 1 \quad \forall j \in J, i \in[m], t \in[T]
\end{aligned}
$$

Similar to the makespan LP, let $\tilde{Q}(r)$ be the set of feasible solutions $(x, y, z, C)$ to the following LP:

$$
\begin{aligned}
& \text { Minimize } \sum_{j \in J} w_{j} \cdot C_{j} \\
& y_{j_{1}, j_{2}}=\sum_{s \in\{0, \ldots, S-1\}} \sum_{i \in[m]} x_{\left(j_{1}, i, s\right),\left(j_{2}, i, s\right)} \\
& C_{j_{2}} \geq C_{j_{1}}+\left(1-y_{j_{1}, j_{2}}\right) \cdot c \quad \forall j_{1} \prec j_{2} \\
& C_{j}= \sum_{i \in[m]} \sum_{t \in[T]} z_{j, i, t} \cdot t \quad \forall j \in J \\
&(x, z) \in S A_{r}(\tilde{K})
\end{aligned}
$$

Note that the $C_{j}$ variables in this LP relaxation denote the actual completion time of $j$ unlike their role in the makespan result, where they were used to indicate the interval in which $j$ was scheduled. The main technical result for this section is the following:

Theorem 21. Consider an instance for $\mathrm{P} \infty\left|\mathrm{prec}, p_{j}=1, c\right| \sum_{j} w_{j} C_{j}$ and a solution $(x, y, z, C) \in$ $\tilde{Q}(r)$ with $r \geq 5$. Then there is a randomized algorithm with expected polynomial running time that finds a feasible schedule so that (i) $\mathbb{E}\left[C_{j}^{A}\right] \leq O(\log c \cdot \log n) \cdot C_{j}$ and (ii) $C_{j}^{A} \leq O(\log c \cdot \log n) \cdot C_{j}+$ $O(\log n) \cdot c$ for all $j \in J$, where $C_{j}^{A}$ is the completion time of $j o b j$.

We briefly describe how Theorem 21 implies the approximation algorithm promised in Theorem 3:

Proof of Theorem 3. Note that strictly speaking $\tilde{Q}(r)$ is not actually a relaxation of $\mathrm{P} \infty \mid$ prec, $p_{j}=$ $1, c \mid \sum_{j} w_{j} C_{j}$. However one can take an optimum integral schedule and insert $c$ idle time slots every $c$ time units and obtain a feasible solution for $\tilde{Q}(r)$. This increases the completion time of any job by at most a factor of 2 . Then we set $r:=5$ and $m:=n$ and solve the LP $\tilde{Q}(r)$ in time polynomial
in $n$. Now consider the randomized schedule from Theorem 21 with completion times $C_{j}^{A}$. Then the expected objective function is $\mathbb{E}\left[\sum_{j \in J} w_{j} \cdot C_{j}^{A}\right] \leq O(\log n \cdot \log c) \cdot\left(\sum_{j \in J} w_{j} \cdot C_{j}\right)$. Markov's inequality guarantees that we can find in expected polynomial time a schedule that satisfies this inequality if we increase the right hand side by a constant factor. This completes the proof.

### 5.2 The Rounding Algorithm

Let $(x, y, z, C)$ be an optimal solution to the LP relaxation $\tilde{Q}(r)$ with $r \geq 5$. It remains to show Theorem [21. We partition the jobs based on their fractional completion times. For $\delta=\frac{c}{64 \log (4 c)}$ and $k \geq 0$, let $J_{k}:=\left\{j \in J: k \cdot \delta \leq C_{j}<k \cdot \delta\right\}$.

We give a separate algorithm for scheduling jobs in $J_{0}$ within an interval of length at most $O(\log n) \cdot c$. Now consider the remaining jobs. For $k=1,2, \ldots$, we schedule jobs in the set $J_{k}$ using the algorithm from Section 3.3, inserting $c$ empty time slots between the schedule of jobs in the set $J_{k}$ and $J_{k+1}$. Let $C_{j}^{A}$ denote the completion time of job $j$ in our algorithm.

Lemma 22. For $k \geq 1$, consider a job $j \in J_{k}$. Then deterministically $C_{j}^{A} \leq O(\log n \cdot \log c) \cdot C_{j}$.
Proof. The claim follows from repeating the arguments in Lemma 16, so we only give a sketch here. Fix $k$ and consider scheduling the jobs in the set $J_{k}$ using the procedure described in Lemma 16, where we repeat the CKR clustering algorithm for $k=\{1,2, \ldots 2 \log n\}$ iterations. Then the expected number of jobs that did not get scheduled in the first $2 \log n$ iterations is at most $\frac{\left|J_{k}\right|}{n^{2}}<1$. Therefore, in expected polynomial time we can find a schedule such that $C_{j}^{A} \in[2 \log n \cdot O(c)$. $k, 2 \log n \cdot O(c) \cdot(k+1)]$. From the definition of set $J_{k}$, the fractional completion time $C_{j}$ of every job $j$ in $J_{k}$ is at least $k \cdot \frac{c}{64 \log (4 c)}$ in the LP solution. This completes the proof.

The only new ingredient for the completion time result is scheduling the jobs in the set $J_{0}$. For $j \in J_{0}$, let $t_{j}^{*}$ denote the earliest time instant $t$ at which the job is scheduled to a fraction of at least $1-\varepsilon$ in the LP solution. Here $0<\varepsilon<1$ is a small constant that we determine later. In scheduling theory this time is also called $\alpha$-point with $\alpha=1-\varepsilon$. Formally

$$
\begin{equation*}
t_{j}^{*}:=\min \left\{t^{\prime} \in[T]: \sum_{i=1}^{m} \sum_{t=1}^{t^{\prime}} z_{j, i, t} \geq 1-\varepsilon\right\} \tag{1}
\end{equation*}
$$

We use the same semimetric $d\left(j_{1}, j_{2}\right):=1-y_{j_{1}, j_{2}}$ as in Section 3 and schedule jobs in $J_{0}$ using the following procedure.

```
Schedule For \(J_{0}\)
    (1) Run a CKR clustering on the semimetric space \(\left(J_{0}, d\right)\) with parameter \(\Delta:=\frac{1}{12}\)
        and let \(V_{1}, \ldots, V_{k}\) be the clusters.
    (2) Let \(V_{\ell}^{\prime}:=\left\{j \in V_{\ell} \mid\left(\Gamma^{-}(j) \cap J_{0}\right) \subseteq V_{\ell}\right\}\) for \(\ell=1, \ldots, k\).
    (3) For all \(\ell=1, \ldots, k\) assign jobs in \(V_{\ell}^{\prime}\) on a single machine and schedule them in
        the increasing of order of \(t_{j}^{*}\) values breaking ties in an arbitrary manner.
    (4) Insert a gap of \(c\) time slots.
    (5) Let \(J_{0}^{\prime} \subseteq J_{0}\) be the set of jobs that did not get scheduled in steps (1) - (3). Use
        Lemma 16 to schedule \(J_{0}^{\prime}\).
```

Lemma 23. For a job $j_{1} \in J_{0}$, the probability that $j_{1}$ gets scheduled in step (5) of the algorithm, i.e., $j_{1} \in J_{0}^{\prime}$, is at most $O(\log c) \cdot \frac{C_{j_{1}}}{c}$.

Proof. The arguments are a slight refinement of Lemma 12, Consider the set $U:=\left\{j_{1}\right\} \cup\left(\Gamma^{-}\left(j_{1}\right) \cap J_{0}\right)$ of $j_{1}$ and its ancestors. If $j_{0} \prec j_{1}$, then $0 \leq C_{j_{0}}+c \cdot d\left(j_{0}, j_{1}\right) \leq C_{j_{1}}$ by the LP constraints and so $d\left(j_{0}, j_{1}\right) \leq \frac{C_{j_{1}}}{c}$. Then the diameter of $U$ with respect to semimetric $d$ is bounded by $2 C_{j_{1}}$ and hence by Theorem 6 (b) the probability that $U$ is separated is bounded by $\ln (2|N(U, \Delta / 2)|) \cdot \frac{4 \text { diam }(U)}{\Delta} \leq$ $O(\log c) \cdot \frac{C_{j_{1}}}{c}$.

The next lemma follows from repeating the arguments in Lemma 22.
Lemma 24. For a job $j \in J_{0}$, condition on the event that $j \in J_{0}^{\prime}$. Then, $\left.C_{j}^{A}\right|_{\left(j \in J_{0}^{\prime}\right)} \leq O(\log n) \cdot c$.
We can now prove that every cluster $V_{\ell}^{\prime}$ can be scheduled on one machine so that the completion time of any job is at most twice the LP completion time.

Lemma 25. For a small enough constant $\varepsilon>0$ ( $\varepsilon=\frac{1}{12}$ suffices) the following holds: Let $U \subseteq J$ be a set of jobs with $\operatorname{diam}(U) \leq \varepsilon$ w.r.t. distance d. Define $t_{j}^{*}$ as in $E q$ (11) and schedule the jobs in $U$ in increasing order of $t_{j}^{*}$ on one machine and denote the completion time of $j$ by $C_{j}^{A}$. Then, in expectation $C_{j}^{A} \leq 2 t_{j}^{*}$ for every $j \in U$.
Proof. Let us index the jobs in $U=\left\{j_{1}, \ldots, j_{|U|}\right\}$ so that $t_{j_{1}}^{*} \leq \ldots \leq t_{j_{|U|} \mid}^{*}$. Suppose for the sake of contradiction that there is some job $j_{N}$ with $C_{j_{N}}^{A}>2 t_{j_{N}}^{*}$. Abbreviate $U^{*}:=\left\{j_{1}, \ldots, j_{N}\right\}$ and $\theta^{*}:=t_{j_{N}}^{*}$ so that $1 \leq t_{j}^{*} \leq \theta^{*}$ for $j \in U^{*}$. We observe that $\left|U^{*}\right|=\sum_{j \in U^{*}} p_{j}>2 \theta^{*}$. Then we have
(A) $\sum_{j \in U^{*}} \sum_{i \in[m]} \sum_{t=1}^{\theta^{*}} z_{j, i, t} \geq(1-\varepsilon)\left|U^{*}\right|$,
(B) $\sum_{j \in U^{*}} y_{j, j_{N}} \geq(1-\varepsilon)\left|U^{*}\right|$,
(C) $\sum_{i \in[m]} \sum_{t=1}^{\theta^{*}} z_{j_{N}, i, t} \geq 1-\varepsilon$
where $(A)$ and $(C)$ are by definition of $t_{j}^{*}$ and $(B)$ follows from $\operatorname{diam}\left(U^{*}\right) \leq \operatorname{diam}(U) \leq \varepsilon$. Intuitively, this means that we have $\left|U^{*}\right|>2 \theta^{*}$ many jobs that the LP schedules almost fully on slots $\left\{1, \ldots, \theta^{*}\right\}$ while (B) means that the jobs are almost fully scheduled on the same machine. As before, we will use the properties of the Sherali-Adams hierarchy to formally derive a contradiction. We know by Lemma that there is a distribution $(\tilde{x}, \tilde{z}, \tilde{y}) \sim \mathcal{D}\left(j_{N}\right)$ so that $\mathbb{E}\left[\tilde{x}_{j, i, s}\right]=x_{j, i, s}, \mathbb{E}\left[\tilde{z}_{j, i, t}\right]=z_{j, i, t}$ and $\mathbb{E}\left[\tilde{y}_{j_{1}, j_{2}}\right]=y_{j_{1}, j_{2}}$ while the variables involving job $j_{N}$ are integral, i.e. $\tilde{x}_{j_{N}, i, s}, \tilde{z}_{j_{N}, i, t} \in\{0,1\}$. Consider the three events

$$
\left(A^{\prime}\right) \sum_{j \in U^{*}} \sum_{i \in[m]} \sum_{t=1}^{\theta^{*}} \tilde{z}_{j, i, t} \geq(1-3 \varepsilon)\left|U^{*}\right|, \quad\left(B^{\prime}\right) \sum_{j \in U^{*}} \tilde{y}_{j, j_{N}} \geq(1-3 \varepsilon)\left|U^{*}\right|, \quad\left(C^{\prime}\right) \sum_{i \in[m]} \sum_{t=1}^{\theta^{*}} \tilde{z}_{j_{N}, i, t}=1 .
$$

Then by Markov inequality $\operatorname{Pr}\left[A^{\prime}\right] \geq \frac{2}{3}, \operatorname{Pr}\left[B^{\prime}\right] \geq \frac{2}{3}$ and $\operatorname{Pr}\left[C^{\prime}\right] \geq 1-\varepsilon$, and so by the union bound $\operatorname{Pr}\left[A^{\prime} \wedge B^{\prime} \wedge C^{\prime}\right]>0$, assuming $\varepsilon<\frac{1}{3}$. Fix an outcome for $(\tilde{x}, \tilde{y}, \tilde{z})$ where the events $A^{\prime}, B^{\prime}, C^{\prime}$ happen and let $i_{N} \in[m], t_{N} \in\left\{1, \ldots, \theta^{*}\right\}$ be the indices with $\tilde{z}_{j_{N}, i_{N}, t_{N}}=1$. Then the interval index with $t_{N} \in I_{s_{N}}$ satisfies $\tilde{x}_{j_{N}, i_{N}, s_{N}}=1$. Hence

$$
\begin{aligned}
&(1-3 \varepsilon)\left|U^{*}\right| \stackrel{\left(B^{\prime}\right)}{\leq} \sum_{j \in U^{*}} \tilde{y}_{j, j_{N}} \stackrel{L P}{=} \sum_{j \in U^{*}} \sum_{i \in[m]} \sum_{s \in\{0, \ldots, S-1\}} \tilde{x}_{(j, i, s),\left(j_{N}, i, s\right)} \stackrel{\tilde{x}_{j_{N}, i_{N}, s_{N}}=1}{=} \sum_{j \in U^{*}} \tilde{x}_{j, i_{N}, s_{N}} \\
& \stackrel{L P}{\leq} \underbrace{\sum_{j \in U^{*}} \sum_{t=1}^{\theta^{*}} \tilde{z}_{j, i_{N}, t}}_{\leq \theta^{*} \text { by LP }}+\underbrace{\sum_{j \in U^{*}} \sum_{t>\theta^{*}} \tilde{z}_{j, i_{N}, t}}_{\leq 3 \varepsilon\left|U^{*}\right| \text { by }\left(A^{\prime}\right)} \leq \theta^{*}+3 \varepsilon\left|U^{*}\right|
\end{aligned}
$$

[^2]Rearranging gives $\left|U^{*}\right| \leq \frac{1}{1-6 \varepsilon} \theta^{*}$, which is a contradiction for $\varepsilon \leq \frac{1}{12}$.
Lemma 26. For a job $j \in J_{0}$, condition on the event that it got scheduled in the step (3) of the algorithm. Then, $\left.C_{j}^{A}\right|_{\left(j \notin J_{0}^{\prime}\right)} \leq O\left(C_{j}\right)$.

Proof. Consider a job $j \in J_{0} \backslash J_{0}^{\prime}$. Then $j \in V_{\ell}^{\prime}$ and by construction, the set $V_{\ell}^{\prime}$ has diameter at most $\Delta=\frac{1}{12}$ w.r.t. $d$. Then Lemma 25 guarantees that the completion time is $C_{j}^{A} \leq 2 t_{j}^{*}$ where we set $\varepsilon=\frac{1}{12}$. Finally note that an $\varepsilon$-fraction of $j$ was finished at time $t_{j}^{*}$ or later and hence $C_{j}=\sum_{i \in[m]} \sum_{t \in[T]} z_{j, i, t} \cdot t \geq \varepsilon \cdot t_{j}^{*}$. Putting everything together we obtain $C_{j}^{A} \leq \frac{2}{\varepsilon} C_{j}$.

We have everything to finish the proof of the completion time result.
Proof of Theorem [21. From Lemma [22, for $k \geq 1$ and $j \in J_{k}$, we have deterministically $C_{j}^{A} \leq$ $O(\log n \cdot \log c) \cdot C_{j}$. Now consider a job $j \in J_{0}$. Then,

$$
\begin{aligned}
\mathbb{E}\left[C_{j}^{A}\right] & =\mathbb{E}\left[C_{j}^{A} \mid\left(j \notin J_{0}^{\prime}\right)\right] \cdot \operatorname{Pr}\left[\left(j \notin J_{0}^{\prime}\right)\right]+\mathbb{E}\left[C_{j}^{A} \mid\left(j \in J_{0}^{\prime}\right)\right] \cdot \operatorname{Pr}\left[\left(j \in J_{0}^{\prime}\right)\right] \\
& \leq O\left(C_{j}\right)+O(\log c) \cdot \frac{C_{j}}{c} \cdot O(\log n) \cdot c \quad(\text { from Lemmas [23|24|26) } \\
& \leq O(\log n \cdot \log c) \cdot O\left(C_{j}\right)
\end{aligned}
$$

Finally note that the completion time of a job $j \in J_{0}$ is always bounded by $C_{j}^{A} \leq O(\log n) \cdot c$. The claim follows.

## Discussion and Open Problems

We gave a new framework for scheduling jobs with precedence constraints and communication delays based on metric space clustering. Our results take the first step towards resolving several important problems in this area. One immediate open question is to understand whether our approach can yield a constant-factor approximation for $\mathrm{P}|\mathrm{prec}, c| C_{\text {max }}$. A more challenging problem is to handle non-uniform communication delays in the problem $\mathrm{P}\left|\mathrm{prec}, c_{j k}\right| C_{\max }$, where $c_{j k}$ is the communication delay between jobs $j \prec k$.

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Figure 6: Visualization of CKR analysis

## Appendix: The analysis of the CKR clustering

In this section we reprove the statement of Theorem 6. The claim from Theorem 6](a) is easy to show as

$$
\operatorname{diam}\left(V_{i}\right)=\max _{u, v \in V_{i}} d(u, v) \leq 2 \max _{u \in V_{i}} \underbrace{d\left(u, c_{i}\right)}_{\leq \beta \Delta} \leq \Delta
$$

The tricky part is to show Theorem 66.(b). The following definition and lemma are needed.
Definition 27. Let us say that a node $w$ is a separator for $U$, if
(A) $\sigma(u)=w$ for at least one $u \in U$
(B) $\sigma(u) \neq w$ for at least one $u \in U$

Moreover, if the set of separators of $U$ is non-empty, then we call the separator that comes first in the order $\pi$ the first separator.

Next, we show that nodes that are closer to the set $U$ are the most likely to be the first separator:
Lemma 28. Let $w_{1}, \ldots, w_{n}$ be the nodes sorted so that $d\left(w_{1}, U\right) \leq \ldots \leq d\left(w_{n}, U\right)$. Then $\operatorname{Pr}\left[w_{s}\right.$ is the first separator for $\left.U\right] \leq \frac{4}{s} \cdot \frac{\operatorname{diam}(U)}{\Delta}$.
Proof. Let $u_{\min }:=\operatorname{argmin}\left\{d\left(u, w_{s}\right): u \in U\right\}$ and $u_{\max }:=\operatorname{argmax}\left\{d\left(u, w_{s}\right): u \in U\right\}$ be the closest and furthest point from $w_{s}$. We claim that in order for $w_{s}$ to be the first separator, both of the following conditions must hold:
(i) $d\left(w_{s}, u_{\min }\right) \leq \beta \cdot \Delta<d\left(w_{s}, u_{\max }\right)$
(ii) The order selects $w_{s}$ as the first node among $w_{1}, \ldots, w_{s}$.

We assume that $w_{s}$ is the first separator, and suppose for the sake of contradiction that either (i) or (ii) (or both) are not satisfied. We verify the cases:

- Case: $\beta \Delta<d\left(w_{s}, u_{\min }\right)$. Then no point will be assigned to $w_{s}$ and $w_{s}$ is not a separator at all.
- Case: $\beta \Delta \geq d\left(w_{s}, u_{\max }\right)$. As $w_{s}$ is a separator, there are nodes $u_{1}, u_{2} \in U$ with $\sigma\left(u_{1}\right)=w_{s}$ and $\sigma\left(u_{2}\right) \neq w_{s}$. Then $\sigma\left(u_{2}\right)$ has to come earlier in the order $\pi$ as $d\left(w_{s}, u_{2}\right) \leq \beta \Delta$. Hence $w_{s}$ is not the first separator.
- Case: $w_{s}$ is not first among $w_{1}, \ldots, w_{s}$ with respect to $\pi$. By assumption there is an index $1 \leq s_{2}<s$ such that $\pi\left(w_{s_{2}}\right)<\pi\left(w_{s}\right)$. As $w_{s}$ is a separator, there is a $u_{1} \in U$ with $\sigma\left(u_{1}\right)=w_{s}$. Let $u_{2}:=\operatorname{argmin}\left\{d\left(u, w_{s_{2}}\right): u \in U\right\}$ be the point in the set $U$ that is closest to $w_{s_{2}}$. Then $d\left(u_{2}, w_{s_{2}}\right)=d\left(w_{s_{2}}, U\right) \leq d\left(w_{s}, U\right) \leq d\left(w_{s}, u_{1}\right) \leq \beta \Delta$. Hence $u_{2}$ would be assigned to a point of order at most $\pi\left(w_{s^{\prime}}\right)<\pi\left(w_{s}\right)$, and therefore $w_{s}$ is not the first separator.

Now we estimate the probability that $w_{s}$ is the first separator. The parameter $\beta$ and the permutation are chosen independently, so $(i)$ and $(i i)$ are independent events. Clearly $\operatorname{Pr}[(i i)]=\frac{1}{s}$. Moreover

$$
\operatorname{Pr}[(i)]=\frac{\left|\left[d\left(w_{s}, u_{\min }\right), d\left(w_{s}, u_{\max }\right)\right] \cap\left[\frac{\Delta}{4}, \frac{\Delta}{2}\right]\right|}{\Delta / 4} \leq \frac{4 d\left(u_{\min }, u_{\max }\right)}{\Delta} \leq \frac{4 \operatorname{diam}(U)}{\Delta}
$$

where we have used the triangle inequality and the notation $|[a, b]|=b-a$ for the length of an interval.

Now we can finish the proof of Theorem 6.
Proof of Theorem 6. As in Lemma [28, let $w_{1}, \ldots, w_{n}$ be an order of nodes such that $d\left(w_{1}, U\right) \leq$ $\ldots \leq d\left(w_{n}, U\right)$. Note that $L:=\left|N\left(U, \frac{\Delta}{2}\right)\right| \leq n$ is the maximal index with $d\left(w_{L}, U\right) \leq \frac{\Delta}{2}$. If $U$ is separated, then there has to be a first separator. Therefore, the following holds:

$$
\begin{aligned}
\operatorname{Pr}[U \text { is separated }] & \leq \sum_{s=1}^{L} \operatorname{Pr}\left[w_{s} \text { is first separator for } U\right] \\
& \stackrel{\operatorname{Lem}}{\leq} \underbrace{28} \underbrace{\sum_{s=1}^{L} \frac{1}{s}}_{\leq \ln (2 L)} \cdot \frac{4 \operatorname{diam}(U)}{\Delta} \leq \ln (2 L) \cdot \frac{4 \operatorname{diam}(U)}{\Delta}
\end{aligned}
$$

Here we use that $\operatorname{Pr}\left[w_{s}\right.$ is first separator for $\left.U\right]=0$ for $s>L$ since a node $w_{s}$ that has a distance bigger than $\frac{\Delta}{2}$ to $U$ will never be a separator.


[^0]:    *University of Washington, Seattle. Email: \{daviess,rothvoss,yihaoz93\}@uw.edu. Thomas Rothvoss is supported by NSF CAREER grant 1651861 and a David \& Lucile Packard Foundation Fellowship.
    ${ }^{\dagger}$ Microsoft Research, Redmond. Email: \{jakul, jatarnaw\}@microsoft.com.
    ${ }^{1}$ Throughout the paper we use the standard scheduling three-field notation GLLK79, VLL90. The respective fields denote: (1) number of identical machines: $P \infty$ : unlimited; $P$ : number $m$ of machines given as input; $\mathrm{P} m$ : constant number $m$ of machines, (2) job properties: prec: precedence constraints; $p_{j}=1$ : unit-size jobs; $c$ : communication delays of length $c$ (can be $c_{j k}$ if dependent on jobs $j \prec k$ ); $c$-intervals: see Section 3, dup: allowed duplication of jobs, (3) objective: $C_{\text {max }}$ : minimize makespan; $\sum w_{j} C_{j}$ : minimize weighted sum of completion times.

[^1]:    ${ }^{2}$ Papadimitriou and Yannakakis PY90 claim a 2-hardness for $\mathrm{P} \infty\left|\mathrm{prec}, p_{j}=1, c\right| C_{\max }$, but give no proof. Schuurman and Woeginger SW99 remark that "it would be nice to have a proof for this claim".

[^2]:    ${ }^{3}$ Strictly speaking, Lemma 8 describes the SA properties for LP $Q(r)$, but an absolutely analogous statement holds for $\tilde{Q}(r)$.

