# GROUP ISOMORPHISM IS NEARLY-LINEAR TIME FOR MOST ORDERS 

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#### Abstract

We show that there is a dense set $\Upsilon \subseteq \mathbb{N}$ of group orders and a constant $c$ such that for every $n \in \Upsilon$ we can decide in time $O\left(n^{2}(\log n)^{c}\right)$ whether two $n \times n$ multiplication tables describe isomorphic groups of order $n$. This improves significantly over the general $n^{O(\log n)}$-time complexity and shows that group isomorphism can be tested efficiently for almost all group orders $n$. We also show that in time $O\left(n^{2}(\log n)^{c}\right)$ it can be decided whether an $n \times n$ multiplication table describes a group; this improves over the known $O\left(n^{3}\right)$ complexity. Our complexities are calculated for a deterministic multi-tape Turing machine model. We give the implications to a RAM model in the promise hierarchy as well.


## 1. Introduction

Given a natural number $n$, there are many structures that can be recorded by an $n \times n$ table $T$ taking values $T_{i j}$ in $[n]=\{1, \ldots, n\}$. Isomorphisms of these tables are permutations $\sigma$ on $[n]$ with $T_{\sigma(i) \sigma(j)}=\sigma\left(T_{i j}\right)$ for all $i, j \in[n]$. It is convenient to assign these tables either a geometric or algebraic interpretation. A geometric view treats these as edge colored directed graphs or as the relations of an incidence structure. We will consider the algebraic interpretation where the table describes a binary product $*:[n] \times[n] \rightarrow[n]$.

An upper bound complexity to decide isomorphism can be given by testing all $n$ ! permutations. Better timings arise when we consider subfamilies of structures, for example, by imposing equational laws on the product such as associativity $a *(b * c)=(a * b) * c$ (i.e. semigroups) or the existence of left and right fractions (i.e. quasigroups or latin squares). Booth [23, p. 132] observed that the complexity of isomorphism testing of semigroups is polynomial-time equivalent to the complexity of graph isomorphism. At the time, that complexity was subexponential, but it has since been shown by Babai [1 to be in quasi-polynomial time with a highly inventive algorithm. Meanwhile Miller [22, 23] observed that the complexity of quasigroup isomorphism is in quasi-polynomial time $n^{O(\log n)}$ through an almost brute-force algorithm: since quasigroups of order $n$ are generated by $\log _{2} n$ elements, a brute-force comparison of all $\left\lceil\log _{2} n\right\rceil$-tuples either finds an isomorphism between two quasigroups or determines that no isomorphism exists.

An intriguing bottleneck to further improvement has been the case of groups that have associative products with an identity and left and right inverses. Because these are quasigroups, they have a brute-force isomorphism test with complexity $n^{O(\log n)}$; Miller credited Tarjan for the complexity of group isomorphism. Guralnick and Lucchini [5, Theorem 16.6] showed independently that every finite group of order $n$ can be generated by at most $d(n)$ elements and $d(n) \leqslant \mu(n)+1$, where $\mu(n)$ is the largest exponent of a prime power divisor $p^{\mu(n)}$ of $n$. Thus, the complexity of brute-force group isomorphism testing is more accurately described as $n^{O(\mu(n))}$. Since $\mu(n) \in \Theta(\log n)$ when $n=2^{\ell} m$ with $m \in O(1)$, this does not improve on the generic $n^{O(\log n)}$ bound. Group isomorphism testing seems to be a leading bottleneck to improving the complexity of graph isomorphism, see Babai [1, Section 13.2]. Even so, we prove here that for most orders, group isomorphism is in nearly-linear time compared to the input size; this also shows that groups of general orders are not a bottleneck to graph isomorphism.

[^0]1.1. Current state of group isomorphism. Surprisingly, the brute-force $n^{O(\log n)}$ complexity of group isomorphism has been resilient. Progress has fragmented into work on numerous subclasses $\mathfrak{X}$ of groups; the precise problem studied today is:

## $\mathfrak{X}$-GroupIso

Given a pair $\left(T, T^{\prime}\right)$ of $n \times n$ tables with entries in $[n]$ representing groups in $\mathfrak{X}$,
Decide if the groups are isomorphic.
We follow the convention that an $n \times n$ table with entries in $[n]$ corresponds to the multiplication table of a binary product where the rows and columns are both labelled by $1,2, \ldots, n$. Moreover, for a group multiplication we require that the identity element is denoted " 1 ", that is, the first row and first column must both be $1,2, \ldots, n$; if this is not the case, then we reject the input.

Grochow-Qiao [14] give a detailed survey of recent progress on group isomorphism; here we summarize a few results related to our setting. Iliopoulos [15], Karagiorgos-Poulakis [17], Vikas [35], and Kavitha [18] progressively improved the complexity for the class $\mathfrak{A}$ of abelian groups (where the product satisfies $a * b=b * a$ for all $a, b$ ), resulting in a linear-time algorithm for $\mathfrak{A}$ GroupIso in a RAM model (more on this below). Wagner-Rosenbaum [30] gave an $n^{0.25 \log n+O(1)}$ time algorithm for the class $\mathfrak{N}_{p}$ of groups of order a power of a prime $p$, and later generalized this to the class of solvable groups. Li-Qiao [20] proved an average run time of $n^{O(1)}$ for an essentially dense subclass $\mathfrak{N}_{p, 2} \subset \mathfrak{N}_{p}$. Babai-Codenotti-Qiao [3] proved an $n^{O(1)}$ bound for the class $\mathfrak{T}$ of groups with no nontrivial abelian normal subgroups. Das-Sharma [7] described a nearly-linear time algorithm for the class of Hamiltonian groups, again in a RAM model.

Other research combines results for separate classes by considering isomorphism between groups that decompose into a subgroup in class $\mathfrak{X}$ and a quotient in class $\mathfrak{Y}$, see also Section 2f we call this the $(\mathfrak{X}, \mathfrak{Y})$-extension problem. Le Gall [19] studied $(\mathfrak{A}, \mathfrak{C})$-extensions, where $\mathfrak{C}$ consists of cyclic groups. Grochow-Qiao [14] considered $(\mathfrak{A}, \mathfrak{T})$-extensions, and outlined a general framework for solving extension problems.

A further class of algorithms considers terse input models, such as black-box models or groups of matrices or permutations; we refer to Seress [32, Section 2] for details of those models. In this format, groups can be exponentially larger than the data it takes to specify the group. Using this model, the second author proved an $(\log n)^{O(1)}$-time algorithm for subgroups and quotients of finite Heisenberg groups, and further variations in collaborations with Lewis and Brooksbank-Maglione; see [36] and the references therein. Recently, in [9] the authors proved a polynomial-time isomorphism test for permutation groups of square-free and cube-free orders. These examples demonstrate that input models may have an outsized influence on the complexity of group isomorphism.

Some of the motivation of this and earlier work [9] has been the observation that, in contrast to graph isomorphism, the difficulty of group isomorphism is influenced by the prime power factorization of the group orders $n$. For example, if $n=2^{e} \pm 1$ is a prime, then there is exactly one isomorphism type of groups of prime order $n$ and isomorphism can be tested by comparing orders. Yet, there are $n^{2 e^{2} / 27-O(e)}$ isomorphism types of groups of prime power order $n \mp 1=2^{e}$, see [5], p. 23]. As of today, isomorphism testing of groups of order $2^{e}$ has the worst-case complexity.
1.2. Main results. The main result of this paper is a proof that group isomorphism can be tested efficiently for almost all group orders $n$ in time $O\left(n^{2}(\log n)^{c}\right)$ for some constant $c$, if the groups are input by their Cayley tables, that is, by $n \times n$ tables describing their multiplication maps $[n] \times[n] \rightarrow[n]$. To make "almost all" specific, we define the density of a set $\Omega \subseteq \mathbb{N}$ to be the limit $\delta(\Omega)=\lim _{n \rightarrow \infty}|\Omega \cap[n]| / n$; the set $\Omega$ is dense if $\delta(\Omega)=1$. By abuse of notation, $\Omega$-GroupIso denotes the isomorphism problem for the class of groups whose orders lie in $\Omega$. All our complexities are stated for deterministic multi-tape Turing machines; we give details in Section 2 ,
Theorem 1.1. There is a dense subset $\Upsilon \subset \mathbb{N}$ and a deterministic Turing machine that decides $\Upsilon$-Grouplso for $n \in \Upsilon$ in time $O\left(n^{2}(\log n)^{c}\right)$ for some constant $c$.

We provide a proof in Section 4.1 the set $\Upsilon$ is specified in Definition 2.3 and motivated by the Hardy-Ramanujan Theorem [27, Section 8] and number theory results of Erdős-Pálfy [11]. Since every multiplication table $[n] \times[n] \rightarrow[n]$ can be encoded and recognized from a binary string of length $\Theta\left(n^{2} \log n\right)$, the algorithm of Theorem 1.1 is nearly-linear time in the input size. The dense set $\Upsilon$ is specified in Definition [2.3, here we mention that we can determine in time $O\left(n^{2}(\log n)^{c}\right)$ whether $n \in \Upsilon$, see Remark 4.1, and the complexity for brute-force isomorphism testing of groups of order $n \in \Upsilon$ is $n^{O(\log \log n)}$. Because of this, we would have been content with a polynomial-time bound; being able to prove nearly-linear time bound was a surprise.

Our set $\Upsilon$ excludes an important but difficult class of group orders, specifically orders that have a large power of a prime as a divisor. Theorem 1.1 therefore goes some way towards confirming the expectation that groups of prime power order are the essential bottleneck to group isomorphism testing. Indeed, examples such as provided in [36] show that large numbers of groups of prime power order can appear identical and yet be pairwise non-isomorphic. In fact, known estimates on the proportions of groups show that most isomorphism types of groups accumulate around orders with large prime powers, see [5, pp. 1-2]. So our Theorem 1.1 should not be misunderstood as saying that group isomorphism is efficient on most groups, just on most orders. Even so, we see in results like Li-Qiao [20] and Theorem 1.1 the beginnings of an approach to show that group isomorphism is polynomial-time on average, and we encourage work in this direction.

The solutions of $\mathfrak{X}$-GroupIso cited so far deal with the problem in the promise polynomial hierarchy [13] where one promises that inputs are known to be groups and that they lie in $\mathfrak{X}$. To relate those solutions to the usual deterministic polynomial-time hierarchy forces us to consider the complexity of the associated membership problem:

## $\mathfrak{X}$-Group

Given a binary string $T$,
Decide if $T$ encodes the Cayley table of a group contained in $\mathfrak{X}$.
Current methods in the literature seem to require $O\left(n^{3}\right)$ steps to verify that a binary product on $[n]$ is associative, see [12, Chapter 2]. Here we present a method that solves $\mathfrak{G}$-Group for the class $\mathfrak{G}$ of all finite groups in time $O\left(n^{2}(\log n)^{d}\right)$ for some constant $d$. We note that Rajagopalan-Schulman [28, Theorem 5.2] provide an $O\left(n^{2} \log n\right)$ algorithm for this task, but they cost the binary operation as $O(1)$, which gives an upper bound of $O\left(n^{4}(\log n)^{2}\right)$ on a Turing machine model.
Theorem 1.2. There is a deterministic Turing machine that decides in time $O\left(n^{2}(\log n)^{d}\right)$ for some constant $d$ whether an $n \times n$ table with entries in $[n]$ describes a group and, if so, returns a homomorphism $[n] \rightarrow \operatorname{Sym}_{n}$ into the group $\operatorname{Sym}_{n}$ of permutations on $[n]$.

Recall our assumption that the multiplication table arising from the input table has labels $1,2, \ldots, n$ and that " 1 " must be the identity, that is, the first row and column of the table must be $1,2, \ldots, n$; if not, then the input is rejected as it does not represent a group. Relaxing that assumption leads to comparing multiplication tables with independent permutations of the rows, columns, and entries producing what is known as an isotopy (or isotopism) instead of an isomorphism. We consider only isomorphism.

We prove Theorem 1.2 in Appendix 4.2, From that result, Theorem 1.1 can be described as nearly-linear time on all inputs, that is, it properly accepts or rejects all strings without assuming external promises on these inputs (such as that the tables represent groups or groups with some property). Theorem 1.2 also offers a hint that our strategy partly entails working with data structures for permutation groups, instead of working with the multiplication tables directly. This is responsible for much of the nearly-linear time complexity of the various group theoretic routines upon which we build our algorithm for Theorem 1.1,

While we provide a self-contained proof, Theorem 1.2 is an example of a general approach we are developing for shifting promise problems to deterministic problems, see also Section 5romise
problems are especially common whenever inputs are given by compact encodings such as blackbox inputs, see Goldreich [13]. In ongoing work [10], we introduce a more general process for verifying promises by specifying inputs not as strings for a Turing machine, but rather as types in a sufficiently strong Type Theory. Theorem 1.2 can be interpreted as an example of such an input where the rows of the multiplication table are themselves treated as inhabitants of a permutation type. The algorithm then effectively type-checks that these rows satisfy the required introduction rules for a permutation group type. Type-checking is not in general decidable so the effort is to confirm an efficient complexity for specific settings. As a by-product of such models, the subsequent algorithms also profit from using these rich data types; for more details on this topic we refer to [10].

Group isomorphism has been such a tenacious problem that it has benefited from analysis in stronger computational models, such as a RAM model where the data is pre-loaded into registers and operations are costed as $O(1)$-time; cf. Section 2.1. It is also common to use the promise hierarchy where the axioms of a group and any further restrictions on the input are presupposed without being part of the timing. Recasting our result in similar models yields the following.

Corollary 1.3. Working in the promise hierarchy and using a RAM model with $O(1)$-time table look-ups, there is a deterministic algorithm that decides in time $O\left(n^{1+o(1)}\right)$ whether two multiplication tables on $[n]$, with $n \in \Upsilon$, describe isomorphic groups.
1.3. Structure. In Section 2 we introduce relevant notation and state Theorems 2.4 2.6, which are the main ingredients for our proof of Theorem 1.1. Since the proofs of Theorems [2.4] 2.6 are more involved and partly depend on technical Group Theory results, we delay them until Appendix A. In Section 3 we discuss some algorithmic results required for our proof of Theorem 2.6, Proofs for our main results are provided in Section 4. A conclusion and outlook are given Section 5 .

## 2. Notation and preliminary Results

2.1. Computational model. Throughout $n$ is the order of the multiplication tables used as input to programs, so input lengths are $\Theta\left(n^{2} \log n\right)$. We write $\tilde{O}\left(n^{d}\right)$ for $O\left(n^{d}(\log n)^{c}\right)$, where $c$ and $d$ are constants, and we note that $O\left(n(\log n)^{O\left((\log \log n)^{c}\right)}\right) \subset O\left(n^{1+o(1)}\right)$. Computations are carried out on a multi-tape Turing machine (TM) with separate tapes for each input group, an output tape, and a pair of spare tapes of length $O(n \log n)$ to store our associated permutation group representations developed in the course of Theorem 1.2. In this model, carrying out a group multiplication requires one to reposition the tape head to the correct product, at the cost of $O\left(n^{2} \log n\right)$. That will be prohibitive for our given timing, so our first order of business will be to replace the input with an efficient $\tilde{O}(n)$-time multiplication, cf. Remark 4.2. For comparison, work of Vikas [35] and Kavitha [18] provide isomorphism tests for abelian groups using $O(n)$ group operations. Such an algorithm can be considered as linear time in a random access memory (RAM) model where group and arithmetic operations are stated as a unit cost. This is partly how it is possible to produce a running time shorter than the input length. In general, an $f(n)$-time algorithm in a RAM model produces an $O\left(f(n)^{3}\right)$-time algorithm on a TM, see [25, Section 2], although a lower complexity reduction may exist for specific programs. In particular results in [7] [18] are no more than $\tilde{O}\left(n^{3}\right)$-time on a Turing Machine.
2.2. Group theory preliminaries. We follow most common conventions in group theory, e.g. as in [29, 32]. Given a group $G$, a subgroup $H \leqslant G$ is a nonempty subset which is a group with the inherited operations from $G$. Homomorphisms between groups $G$ and $K$ are functions $f: G \rightarrow K$ with $f(x y)=f(x) f(y)$ for all $x, y \in G$. For $S \subseteq G$, let $\langle S\rangle$ be the intersection of all subgroups containing $S$; it is the smallest subgroup of $G$ containing $S$, also called the subgroup generated by $S$. The commutator of group elements $x, y$ is $[x, y]=x^{-1} y^{-1} x y$, and conjugation is written as $x^{y}=x[x, y]$. For $X, Y \subseteq G$, let $[X, Y]=\langle[x, y]: x \in X, y \in Y\rangle$. The number of elements in $G$, the order of $G$, is denoted $|G|$; in this work, $G$ always is a finite group.

Given a set $\pi$ of primes, a subgroup $H \leqslant G$ is a Hall $\pi$-subgroup if every prime divisor of $|H|$ lies in $\pi$, and every prime divisor of the index $|G: H|=|G| /|H|$ does not lie in $\pi$. If $\pi=\{p\}$, then $H$ is a Sylow p-subgroup. A further convention is to let $\pi^{\prime}$ denoted the complement of $\pi$ in the set of all primes and to speak of Hall $\pi^{\prime}$-subgroups. The $\pi$-factorization of an integer $n>1$ is $n=a b$, where every prime divisor of $b$ lies in $\pi$, and no prime divisor of $a$ lies in $\pi$.

A subgroup $B$ is normal in $G$, denoted $B \unlhd G$, if $[G, B] \leqslant B$. The group $G$ is simple if its only normal subgroups are $\{1\}$ and $G$. A composition series for $G$ is a series $G=G_{1}>\ldots>G_{m}=\{1\}$ of subgroups with each $G_{i+1} \unlhd G_{i}$ and each composition factor $G_{i} / G_{i+1}$ is simple. The group $G$ is solvable if every composition factor is abelian.

A normal subgroup $B \unlhd G$ splits in $G$, denoted $G=H \ltimes B$, if there is $H \leqslant G$ with $H \cap B=\{1\}$ and $G=\langle H, B\rangle$. Let $\operatorname{Aut}(B)$ be the set of invertible homomorphisms $B \rightarrow B$. If $G=H \ltimes B$ and $h \in H$, then conjugation $b \mapsto b^{h}$ defines $\theta(h) \in \operatorname{Aut}(B)$, and $\theta: H \rightarrow \operatorname{Aut}(B), h \mapsto \theta(h)$, is a homomorphism. Conversely, given $(H, B, \theta)$ with homomorphism $\theta: H \rightarrow \operatorname{Aut}(B)$, there is a group $H \ltimes_{\theta} B$ on the set $\{(h, b): h \in H, b \in B\}$ with product $\left(h_{1}, b_{1}\right)\left(h_{2}, b_{2}\right)=\left(h_{1} h_{2}, b_{1}^{h_{2}} b_{2}\right)$; here we abbreviate $b_{1}^{h_{2}}=\theta\left(h_{2}\right)\left(b_{1}\right)$. If conjugation in $G=H \ltimes B$ induces $\theta: H \rightarrow \operatorname{Aut}(B)$, then $G$ is isomorphic to $H \ltimes_{\theta} B$. In fact, the following observation holds; we will use this lemma only in the situation that $B$ and $\tilde{B}$ are cyclic groups; this case of Lemma 2.1 is proved in [8, Lemma 2.8].
Lemma 2.1. Let $G=H \ltimes_{\theta} B$ and $\tilde{G}=\tilde{H} \ltimes_{\tilde{\theta}} \tilde{B}$. If $\alpha: H \rightarrow \tilde{H}$ and $\beta: B \rightarrow \tilde{B}$ are isomorphisms such that for all $h \in H$

$$
\begin{equation*}
\tilde{\theta}(\alpha(h))=\beta \circ \theta(h) \circ \beta^{-1} \tag{1}
\end{equation*}
$$

then $(h, b) \mapsto(\alpha(h), \beta(b))$ is an isomorphism $G \rightarrow \tilde{G}$. Conversely, if $G$ and $\tilde{G}$ are isomorphic and $H$ and $B$ have coprime orders, then there is an isomorphism $G \rightarrow \tilde{G}$ of this form.

A variation of this observation also occurs in [4, Section 4.2] and in [14, Section A.4.1]. Note that in the last part of the lemma (when $H$ and $B$ have coprime orders), the group $B$ is generated by all elements that have order coprime to $|H|$; this makes $B$ a characteristic Hall subgroup.
2.3. Number theory preliminaries. Our algorithm for Theorem 1.1 depends on crucial number theoretic observations. For integers $n$, we characterize a family of prime divisors we call strongly isolated, and we show that any group $G$ of order $n$ in our dense set $\Upsilon$ decomposes as $G=H \ltimes B$ such that the prime factors of $|B|$ are exactly the strongly isolated prime divisors of $n$ that are larger than $\log \log n$; we also prove that $B$ is cyclic. This reduces our isomorphism test to considering the data ( $H, B, \theta$ ) and Lemma [2.1, Not just isomorphism testing, but many natural questions of finite groups reduce to properties of $(H, B, \theta)$, and so this decomposition has interesting implications for computing with groups generally. Note that for a group $G$ with order $n \in \Upsilon$, the decomposition described above defines integers $a=|H|$ and $b=|B|$ that depend only on $n$. Our definition of $\Upsilon$ will also imply that $b$ is square-free, and if a prime power $p^{e}$ with $e>1$ divides $n$, then $p^{e} \leqslant \log n$ and $p^{e} \mid a$. With $B$ being cyclic, the group theory of $B$ is elementary, and while the group theory of $H$ can be quite complex, we will see in Theorem [2.5 that $H$ has relatively small size, meaning that brute-force becomes an efficient solution. The following definition is central for our work.

Definition 2.2. Let $n \in \mathbb{N}$. Write $2^{\nu_{2}(n)}$ for the largest 2-power dividing $n$. A prime $p \mid n$ is isolated if $k=0$ for every prime power $q^{k}$ with $q^{k} \mid n$ and $p \mid\left(q^{k}-1\right)$. If, in addition, $p \nmid|T|$ for every non-abelian simple group $T$ of order dividing $n$, then $p$ is strongly isolated. We write $\pi^{\text {si }}(n)$ for the set of strongly isolated prime divisors of $n$; the subset of 'large' prime divisors is

$$
\pi^{\mathrm{lsi}}(n)=\left\{p \in \pi^{\mathrm{si}}(n): p>\log \log n\right\} .
$$

For example, 31 is isolated in $2^{4} \cdot 5^{2} \cdot 31$ but not in $2^{5} \cdot 5^{2} \cdot 31$ or in $2^{4} \cdot 5^{3} \cdot 31$. As indicated above, the properties of our dense set $\Upsilon$ are a critical ingredient in our algorithm for Theorem 1.1] we are now in the situation to give the formal definition, cf. [11].

Definition 2.3. Let $\Upsilon \subseteq \mathbb{N}$ be the set of all integers $n$ that factor as $n=a b$ such that:
a) if $p \mid a$ is a prime divisor, then $p \leqslant \log \log n$ and, if $p^{e} \mid a$, then $p^{e} \leqslant \log n$;
b) if $p \mid b$ is a prime divisor, then $p>\log \log n$ and $p \mid n$ is isolated;
c) the factor $b$ is square-free, that is, if $p^{e} \mid b$ is a prime power divisor, then $e \in\{0,1\}$;
d) the factor $b$ has at most $2 \log \log n$ prime divisors.

Theorem 2.4. The set $\Upsilon$ is a dense subset of $\mathbb{N}$.
A proof of Theorem 2.4 and more details on $\Upsilon$ are given in Appendix A. 1 and Section 5 ,
2.4. Splitting results. As mentioned in Section 2.3, the properties of $n \in \Upsilon$ impose limits on the structure of groups of order $n$; we prove the next theorem in Appendix A. 2 ,

Theorem 2.5. Every group $G$ of order $n \in \Upsilon$ has a unique Hall $\pi^{\text {lsi }}(n)$-subgroup B, which is cyclic, and $G=H \ltimes B$ for some subgroup $H \leqslant G$ of small order $|H| \in(\log n)^{O\left((\log \log n)^{c}\right)}$ for some $c$.

Remark 2.1. In fact, our proof of Theorem 2.5 also shows the following result for any group $G$ of any order $n \in \mathbb{N}$ : If $G$ is solvable and $p \mid n$ is an isolated prime, or if $G$ is non-solvable, $p \mid n$ is a strongly isolated prime, and $p>\nu_{2}(n)$, then $G$ has a normal Sylow $p$-subgroup $S \unlhd G$; in both cases, $G=H \ltimes S$ for some $H \leqslant G$ by the Schur-Zassenhaus Theorem [29, (9.1.2)]. Property d) of Definition 2.3 is solely required to bound the size of $H$.

Remark 2.2. Theorem 2.5 shows that every group of order $n \in \Upsilon$ has a 'large' normal cyclic Hall subgroup that admits a 'small' complement subgroup. This situation is covered by the main results of [4] (stated for groups with abelian Sylow towers), which deal with the case that a normal Hall subgroup is abelian and the complement subgroups admit polynomial-time isomorphism tests, see the discussion around [4, Theorem 1.2]. In conclusion, 4] together with our Theorem [2.5 already prove a polynomial-time isomorphism test for group of order $n \in \Upsilon$. A significant part of this paper is devoted to proving Theorem [2.5, and a careful analysis provides the nearly-linear time complexity stated in Theorem 1.1.

The next theorem shows that we can construct generators for the decomposition in Theorem 2.5 , we discuss the proof of Theorem 2.6 in Appendix A.2,
Theorem 2.6. There is an $\tilde{O}\left(n^{1+o(1)}\right)$-time algorithm that, given a group $G$ of order $n \in \Upsilon$, returns generators for $H, B \leqslant G$ such that $G=H \ltimes B$ and $B$ is a Hall $\pi^{\text {lsi }}(n)$-subgroup.

## 3. Algorithmic preliminaries: PRESENTATIONS AND COMPLEMENTS

We assume now that our input has been pre-processed and confirmed to be a group by our algorithm for Theorem 1.2. In so doing, we also produce a series of important data types and accompanying routines, including the following for each input group of order $n$ :

- a permutation group representation,
- a generating set of size $O(\log n)$,
- an algorithm to write group elements as a product of the generators in time $\tilde{O}(n)$,
- an algorithm to multiply two group elements in time in $(\log n)^{O(1)}$, and
- an algorithm to test equality of elements in the group in time $\tilde{O}(n)$.

Remark 4.2 explains some of these routines in more detail. The advantage is that we can now multiply group elements without moving the Turing machine head over the original Cayley tables which would cost us $\tilde{O}\left(n^{2}\right)$ steps for each group operation.

Many of the following observations are variations on classical techniques designed originally for permutation groups, e.g. as in [21, 32, 33]. We include proofs here to demonstrate nearly-linear time when applied to the Cayley table model. For simplicity we assume that all generating sets contain 1. All our lists have size $O(n)$, so all searches can be done in nearly-linear time.

A membership test for a subgroup $B \leqslant G$ is a function that, given $g \in G$, decides whether $g \in B$. For example, a membership test for the center $Z(G) \leqslant G$ is to report the outcome of the test whether $g \in G$ satisfies $g * s=s * g$ for all $s \in S$. This test defines $Z(G)$ without having specified a generating set for it.

The next result discusses the construction of the Schreier coset graph $\operatorname{SCG}(S, B)$ describing the action of $G=\langle S\rangle$ on cosets of $B \leqslant G$ : this is a labeled graph with vertices $\{x B: x \in G\}$, and an edge $(u B, v B)$ exists if and only if $v B=s u B$ for some $s \in S$. One such $s$ is chosen as the label of $(u B, v B)$, and we denote such a labeled edge by $(u B, s v B ; s)$. We note that a coset $u B$ is stored by a chosen representative $u$. Note that if $B=1$, then this graph is the usual Cayley graph with respect to the given generating set.

## SchreierGraph

$\left\lvert\, \begin{array}{ll}\text { Given } & \text { a group } G \text { generated by } S \subseteq G \text { and a membership test for a subgroup } B, \\ \text { Return } & \text { the Schreier coset graph } \operatorname{SCG}(S, B)=\{(x B, \text { sxB; } s) \mid s \in S, x \in G\} \text { and } \\ \text { together with a spanning tree (a so-called Schreier tree) } .\end{array}\right.$
The next proposition uses well-known ideas that can already be found in [21, 32]; we provide a proof to justify our complexity statements.

Proposition 3.1. Let $G=\langle S\rangle$ with $|S| \in O(\log n)$ be a group of order $n$ and let $B \leqslant G$ with $[G: B] \in(\log n)^{O\left((\log \log n)^{c}\right)}$ be given by an $\tilde{O}\left(n^{1+o(1)}\right)$-time membership test. SchreierGraph can be solved in time $\tilde{O}\left(n^{1+o(1)}\right)$. Once solved, the following hold: There is an $\tilde{O}\left(n^{1+o(1)}\right)$-time algorithm that, given $g \in G$, finds a word $\bar{g}$ in $S$ with $\bar{g} B=g B$. There is also an an $\tilde{O}\left(n^{1+o(1)}\right)$ time algorithm to compute a generating set for $B$ of size $(\log n)^{O\left((\log \log n)^{c}\right)}$.

Proof. As a pre-processing step, we can assume that for each of the $O(\log n)$ generators in $S$ we have also stored its inverse; the inverse of $s \in S$ can be computed as $s^{n-1}$ using fast exponentiation. We initialize graphs $C$ and $T$, both with vertex set $V=\{B\}$ and empty edge sets, and proceed as follows: While there is $s \in S$ and a vertex $x B \in V$ such that $s x B \notin V$, add $s x B$ to $V$ and add $(x B, s x B ; s)$ to the edge sets of $C$ and $T$. Otherwise, $s x B \in V$ and we add $(x B, s x B ; s)$ only to the edge set of $C$. This algorithm terminates after $O(|S||G: B|)$ steps, and then $|V|=|G: B|$ and $V=\operatorname{SCG}(S, B)$ with Schreier tree $T$; this follows from a discussion of the orbit-stabiliser algorithm, for example in [32, Section 4].

This iterative process guarantees that every vertex $g B$ we consider in the algorithm (that is, $x B$ or $s x B$ with $x B \in V$ and $s \in S$ ) is represented by a product of generators in $S$, say $g B=s_{i_{t}} \cdots s_{i_{1}} B$ with each $s_{i_{j}} \in S$. In particular, we can assume that, along the way, we have also stored the product $s_{i_{1}}^{-1} \cdots s_{i_{t}}^{-1}$ with the vertex. This allows us to compare $g B$ with $y B \in V$ by deciding whether the product of $\left(s_{i_{1}}^{-1} \cdots s_{i_{t}}^{-1}\right)$ with $y$ lies in $B$ via the membership test. This shows that the algorithm described so far requires $O\left(|S||G: B|^{2}\right)$ group multiplications and membership tests.

Note that we add a vertex $y B$ to $V$ if and only if we add an edge to the Schreier tree $T$. By construction, the representative chosen to store $y B$ is the product of the labels $s_{i_{t}}, \ldots, s_{i_{1}}$ of the unique path $B \xrightarrow{s_{i_{1}}} s_{i_{1}} B \xrightarrow{s_{i_{2}}} \ldots \xrightarrow{s_{i_{C}}} s_{i_{r}} \cdots s_{i_{1}} B$ from $B$ to $y B$ in the Schreier tree. In particular, it follows that $y B=\bar{y} B$ where $\bar{y}=s_{i_{1}} \cdots s_{i_{t}}$. Now if $g \in G$ is given, then we first run over $V$ to identify the vertex $y B \in V$ with $g B=y B$; as shown above, since we assume that inverses of vertex representatives are stored, we can find $y B$ by using $O(|G: B|)$ group multiplications and membership test applications. Subsequently, we look for the path from $B$ to $y B$ in the Schreier tree; this can be done in time $O(|G: B|)$. The labels of this path express the element $\bar{y}$ as a product of the generators; by construction, $g B=y B=\bar{y} B$, and we define $\bar{g}=\bar{y}$.

It remains to prove the last claim. Let $\mathcal{T}$ be the set of representatives in $G$ describing the vertices in $V$; note that $\mathcal{T}=\{\bar{y}: y B \in V\}$. This is is a transversal for $B$ in $G$, that is, $G$ is the disjoint union of all $t B$ with $t \in \mathcal{T}$. Schreier's lemma [32, Lemma 4.2.1] shows that $B$ is generated by the set of all $(\overline{s t})^{-1} s t$ where $s \in S$ and $t \in \mathcal{T}$. The previous paragraph shows that for each such $s$ and
$t$ we can compute $\overline{s t}$ in time $O(|G: B|)$. Since $\overline{s t}$ is the stored representative of the vertex $s t B$, we have already stored its inverse $(\overline{s t})^{-1}$. This shows that we can compute a set of Schreier generators for $B$ using $O\left(|S||G: B|^{2}\right)$ group multiplications and membership tests.

Corollary 3.2. Let $G=\langle S\rangle$ be a group of order $n \in \Upsilon$ with $|S| \in O(\log n)$. If $G$ has a normal Hall subgroup $B$ with $|G / B| \in(\log n)^{O\left((\log \log n)^{2}\right)}$, then there is an $\tilde{O}\left(n^{1+o(1)}\right)$-time algorithm that returns generators for $B$, and a membership test for $B$ that decides in time $\tilde{O}(n)$.

Proof. Since a normal Hall subgroup is characteristic, hence unique, $g \in G$ lies in $B$ if and only if the order of $g$ divides $b=|B|$; therefore we may test membership in $B$ by testing if $g^{b}=1$. This can be done in $O(\log b)$ group products using fast exponentiation, followed by a comparison with the identity 1. From our forgoing assumptions on $G$, this can be done in time $\tilde{O}(n)$. Finally, as $|S| \in O(\log n)$, generators of $B$ can be obtained from $\tilde{O}\left(|G: B|^{2}\right)$ group products and membership tests, using $\operatorname{SchreierGraph}(S, B)$; all this can be done in time $\tilde{O}\left(n^{1+o(1)}\right)$.

We now introduce a tool that lets us find a complement to a normal Hall $\pi^{\operatorname{lsi}}(n)$-subgroup.

## BigSplit

```
Given a group \(G\) of order \(n \in \Upsilon\),
    Return a subgroup \(H \leqslant G\) such that \(G=H \ltimes B\), where \(B\) is the Hall \(\pi^{\operatorname{lsi}}(n)\) -
        subgroup.
```

To solve BigSplit we need a brief detour into a generic model for encoding groups via presentations, cf. [33, Section 1.4]: The free group $F[X]$ on a given alphabet $X$ is formed by creating a disjoint copy $X^{-}$of the alphabet and treating the elements of $F[X]$ as words over the disjoint union $X \sqcup X^{-}$, including the empty word $1 \notin X \sqcup X^{-}$. Formally, one can replace the given set $X$ by $\{(x, 1): x \in X\}$ and then defines $X^{-}=\{(x,-1): x \in X\}$; to keep the notation simply, we identify $x=(x, 1)$ and $x^{-}=(x,-1)$. The empty word serves as the identity and word concatenation is the group product; to impose the existence of inverses we apply rewriting rules $x x^{-} \rightarrow 1$ and $x^{-} x \rightarrow 1$ for each $x \in X$ and corresponding $x^{-} \in X^{-}$. For a set $M$ we denote by $M^{X}$ the set of maps from $X$ to $M$, whose elements are naturally represented as tuples $\mathbf{m}=\left(\mathbf{m}_{x}\right)_{x \in X}$. For a group $G$, tuple $\mathbf{g} \in G^{X}$, and word $w \in F[X]$, we assign an element $w(\mathbf{g}) \in G$ by replacing each variable $x^{ \pm}$in $w$ with the value $\mathbf{g}_{x} \in G$ and $\mathbf{g}_{x}^{-1} \in G$, respectively, and then evaluating the corresponding product in $G$. The mapping $w \mapsto w(\mathbf{g})$ is a homomorphism $\hat{\mathbf{g}}: F[X] \rightarrow G$, whose kernel $\operatorname{ker} \hat{\mathbf{g}}=\{w \in F[X]: w(\mathbf{g})=1\}$ is a normal subgroup of $F[X]$. If $G$ is generated by the image $S=\left\{\mathbf{g}_{x}: x \in X\right\}$ and $R$ generates ker $\hat{\mathbf{g}}$ as a normal subgroup, then the pair $\langle S \mid R\rangle$ is a presentation of $G$, where $R$ is a set of relations for $G$ relative to $S$. Note that $\langle S \mid R\rangle$ carries all the information necessary to describe $G$ up to isomorphism; however, in such an encoding isomorphism testing may be even become undecidable, see [33, Section 1.9].

Our interest in presentations is to produce a relatively small number of equations whose solutions help to solve BigSplit; for that purpose the following will suffice.

Proposition 3.3. Let $G$ be a group of order $n \in \Upsilon$ with Hall $\pi^{\mathrm{lsi}}(n)$-subgroup B. There is an $\tilde{O}\left(n^{1+o(1)}\right)$-time algorithm to compute a presentation $\langle S \mid R\rangle$ of $G / B$ such that $|S| \in O(\log n)$ and $|R| \in(\log n)^{O\left((\log \log n)^{2}\right)}$, and each $w \in R$ is a word in $S$ of length $O(\log n)$.

Proof. As shown above, we find $G=\langle S\rangle$ with $|S| \in O(\log n)$. Use SchreierGraph and Corollary 3.2 to get a transversal $\mathcal{T}$ for $B$ in $G$, generators for $B$, and a rewriting algorithm that given $g \in G$, finds a word $\bar{g}$ in $S$ with $\bar{g} B=g B$. Choose a set $X=\left\{x_{g}: g \in S\right\}$ of variables, and for each $g \in G$ define $w_{g} \in F[X]$ as the word in $X \cup X^{-}$produced by replacing each $u \in S$ in $\bar{g}$ with $x_{u}$. Now $R=\left\{w_{t} x_{s} w_{t s}^{-1}: t \in \mathcal{T}, s \in S\right\}$ is a set of relations for $G / B$ relative to $\{s B: s \in S\}$, cf. [32, Exercise 5.2]. Note that $|R| \leqslant|G: B| \cdot|S|$, and $|G: B| \in(\log n)^{O\left((\log \log n)^{2}\right)}$ by Theorem [2.5,

Also computing $w_{g}$ is dominated by the time $\tilde{O}\left(|G: B|^{2}\right)$ it takes to compute $\bar{g}$. We do this on $O(|S||G: B|)$ elements for a total time of $(\log n)^{O\left((\log \log n)^{2}\right)} \in O\left(n^{o(1)}\right)$.

Our relators have length $O(|G: B|)$, but [32, Lemma 4.4.2] yields an $\tilde{O}(n)$-time algorithm to replace such relators with ones of length $O(\log n)$, by constructing a shallow Schreier tree and short Schreier generators. The analysis in the proof of [32, Lemma 4.4.2] shows that all this can be done using $\tilde{O}\left(|G: B|^{2}\right)$ group multiplications and membership tests; the claim follows. For more details justifying our complexity statement we refer to our proof of Theorem 1.2; there we provide additional details on an explicit application of [32, Lemma 4.4.2].

Remark 3.1. Babai-Luks-Seress, Kantor-Luks-Marks and others (see the bibliography in 16 and [32, Section 6]) developed various algorithms to construct presentations of (quotient) groups of permutations. Their complexities range from polynomial-time, to polylogarithmic-parallel (NC), to Monte-Carlo nearly-linear time, and they produce presentations that can be considerably smaller than what we obtain in Proposition 3.3. Though it is not necessary for our complexity goals, we expect that a better analysis and a better performing implementation would use such methods instead of our above brute-force approach.
Proposition 3.4. BigSplit is in time $\tilde{O}\left(n^{1+o(1)}\right)$.
Proof. BigSplit is solved via the function Complement discussed in [16, Section 3.3]; we briefly sketch the approach. Let $G=\langle S\rangle$ with $S=\left\{s_{1}, \ldots, s_{d}\right\}$ and $d \in O(\log n)$. Use the algorithm of Proposition 3.3 to get a presentation $\left\langle x_{1}, \ldots, x_{d} \mid R\right\rangle$ for $G / B$, such that each $x_{i}=x_{s_{i}}$ as defined in the proof of Proposition 3.3. Every complement $H$ to $B$, if it exists, is generated by $\left\{s_{1} m_{1}, \ldots, s_{d} m_{d}\right\}$ for some $m_{1}, \ldots, m_{d} \in B$, and such a generating set satisfies the relations in $R$, cf. [29, (2.2.1)]. We attempt to compute $m_{1}, \ldots, m_{d}$ by solving the system of equations resulting from $w\left(s_{1} m_{1}, \ldots, s_{d} m_{d}\right)=1$ with $w$ running over $R$ : recall that $w\left(s_{1} m_{1}, \ldots, s_{d} m_{d}\right) \in G$ is defined as $w(\mathbf{g})$ with $\mathbf{g}=\left(s_{1} m_{1}, \ldots, s_{d} m_{d}\right)$, see the remarks above Proposition 3.3, A complement to $B$ exists if and only if this equation system has a solution. Since each $w\left(s_{1}, \ldots, s_{d}\right)$ lies in the finite cyclic group $B$, this system can be described by an integral matrix with $\log n$ variables and $(\log n)^{O\left((\log \log n)^{2}\right)}$ equations; using the algorithms of [34], it can be solved via Hermite Normal Forms in time $(\log n)^{O\left((\log \log n)^{2}\right)}$.

## 4. Proofs of the main results

### 4.1. Proof of Theorem 1.1; isomorphism testing.

Proof of Theorem [1.1. Given two binary maps $[n] \times[n] \rightarrow[n]$, we decide that $n \in \Upsilon$ in time $O(n)$, and we use Theorem 1.2 to decide in time $\tilde{O}\left(n^{2}\right)$ whether these maps describe Cayley tables. If so, we have been given two groups $G$ and $\tilde{G}$ of order $n \in \Upsilon$, and we can use Theorem 2.6 to find, in time $\tilde{O}\left(n^{1+o(1)}\right)$, generators for subgroups $H, B \leqslant G$ and $\tilde{H}, \tilde{B} \leqslant \tilde{G}$ with $G=H \ltimes B$ and $\tilde{G}=\tilde{H} \ltimes \tilde{B}$. Having generators of these subgroups, we can define homomorphisms $\theta$ and $\tilde{\theta}$ such that $G=H \ltimes_{\theta} B$ and $\tilde{G}=\tilde{H} \ltimes_{\tilde{\theta}} \tilde{B}$. Since $n \in \Upsilon$, we know that $B$ and $\tilde{B}$ are cyclic, hence $B$ and $\tilde{B}$ are isomorphic if and only if $|B|=|\tilde{B}|$. Moreover, since $|H|,|\tilde{H}| \leqslant(\log n)^{O\left((\log \log n)^{c}\right)}$ we can test isomorphism $H \cong \tilde{H}$ using brute-force methods in time $(\log n)^{O\left((\log \log n)^{d}\right)} \subseteq \tilde{O}\left(n^{1+o(1)}\right)$. If $H \cong \tilde{H}$ and $B \cong \tilde{B}$ is established, then we can identify $H=\tilde{H}$ and $B=\tilde{B}$, and test $G \cong \tilde{G}$ by using Lemma 2.1] since $B$ is cyclic, $\operatorname{Aut}(B)$ is abelian, and so Condition (1) reduces to $\tilde{\theta}(\alpha(h))=\theta(h)$ for all $h \in H$, which we can test by enumerating $\operatorname{Aut}(H)$ and looking for a suitable $\alpha$; since $|H|$ is small, such a brute-force enumeration is efficient and in $\tilde{O}\left(n^{1+o(1)}\right)$.

Remark 4.1. We briefly explain how to decide $n \in \Upsilon$ on a Turing machine. Computing the table of primes between 1 and $n$ on a Turing machine can be done in time $O\left(n(\log n)^{2} \log \log n\right)$, see [31, p. 227], and the number of primes is clearly bounded by $n$. Approximating an upper bound
for $\log n$ can be done in $O(n)$ by counting the bits that are required to represent $n$. That integer arithmetic for integers $n$ can be done in $(\log n)^{O(1)}$ on multi-tape Turing machines follows from [31, Chapter III.6]. It follows that $n \in \Upsilon$ can be decided in time $\tilde{O}\left(n^{2}\right)$.
4.2. Proof of Theorem 1.2; recognizing groups. Similar to [12, Chapter 2], our strategy for recognizing groups uses Cayley's Theorem [29, (1.6.8)]. The latter implies that the rows of an $n \times n$ group table can be interpreted as permutations which form a regular permutation group on $[n]$, that is, the group is transitive on $[n$ ] and has trivial point-stabilizers, see [32, Section 1.2.2].

A new idea is to exploit that groups of order $n$ can be specified by generating sets of size $\log n$, so some $\log n$ rows determine the entire table. Once the input is verified to be a latin square, our approach is to define a permutation group generated by $O(\log n)$ rows, and then compare its Cayley table with the original table. In more abstract terms, our algorithm creates an instance of an abstract permutation group data type, as defined in [32, Section 3]. That data type is guaranteed to be a group and so the promise is converted into a computable type-check: We confirm that the group we create in this new data type is the one specified by the original table; the proof given below makes this argument specific. This methodology of removing a promise by appealing to a type-checker generalizes; we refer to our forthcoming work 10 for more details.

Proof of Theorem 1.2. Let $*:[n] \times[n] \rightarrow[n]$ be the multiplication defined by the table $T$; recall our assumption that rows and columns are both labelled by $1,2, \ldots, n$. If the table is not reduced, that is, if the first row or the first column do not contain $1,2, \ldots, n$ in order, then " 1 " is not an identity and we return false. Running over the table, we can check in time $\tilde{O}(n)$ whether any particular row contains distinct symbols only, see for example [26, p. 434]; once this is confirmed for all $n$ rows, the rows of $T$ describe permutations of $[n]$. All this preprocessing can be done in time $\tilde{O}\left(n^{2}\right)$. We verify in the proof below that also each column of $T$ contain distinct symbols only (and return false if this is not the case); then $L=([n], *)$ describes a loop with identity 1 ; recall that a loop is a quasigroup (that is, a set with a binary operation whose multiplication table is a Latin square) that has an identity.

For $i \in[n]$ denote by $\lambda_{i} \in \operatorname{Sym}_{n}$ the map $[n] \rightarrow[n]$ defined by left multiplication $\lambda_{i}(a)=i * a$; since $T$ is reduced, $\lambda_{i}$ is given by the $i$-th row of $T$. We define

$$
\Lambda=\left\langle\lambda_{i}: i \in[n]\right\rangle ;
$$

since each $\lambda_{i}(1)=i$, this is a transitive subgroup of $\mathrm{Sym}_{n}$. Note that if one of the columns of $T$ contains duplicate symbols, then $\Lambda$ is not regular. It follows that $L$ is a group if and only if $\Lambda$ is a regular permutation group on [ $n$ ], if and only if $\lambda_{i} \lambda_{j}=\lambda_{i * j}$ for all $i, j \in[n]$, see [12, Theorems 2.16 $\& 2.17]$. Since $\Lambda$ is transitive, it is regular if and only if the stabiliser $\Lambda_{1}$ of $1 \in[n]$ in $\Lambda$ is trivial. We now show how to find a generating set of $\Lambda$ of size $O(\log n)$ and prove that $\Lambda$ is regular, or establish that $L$ is not a group and return false.

We first introduce some notation. For a subset $S \subseteq[n]$ define

$$
\Lambda(S)=\left\langle\lambda_{i}: i \in S\right\rangle \leqslant \Lambda .
$$

Let $\Lambda(S)(1)=\{\lambda(1): \lambda \in \Lambda(S)\}$ and $\Lambda(S)_{1}=\{\lambda \in \Lambda(S): \lambda(1)=1\}$ be the orbit and stabiliser of 1 in $\Lambda(S)$, respectively. We describe how to use the orbit-stabiliser algorithm (cf. the proof of Proposition (3.1) to get $\Lambda(S)(1)$, a generating set for $\Lambda(S)_{1}$ (so-called Schreier generators), and for each $x \in \Lambda(S)(1)$ an transversal element $\lambda_{(x)} \in \Lambda(S)$ with $\lambda_{(x)}(1)=x$. We proceed in two steps.

The first step is to find a subset $S \subseteq[n]$ of size $O(\log n)$ such that $\Lambda(S)$ is transitive on $[n]$. We start by choosing a subset $S \subseteq[n]$ of size $O(\log n)$ and by copying $\left\{\lambda_{i}: i \in S\right\}$ to a separate tape; we can assume that $1 \in S$. In the following we will mainly work with this short tape; scanning it takes time $\tilde{O}(n)$. With this assumption, we can compute $\Lambda(S)(1)$ in time $\tilde{O}\left(n^{2}\right)$ by using the usual orbit enumeration of the orbit-stabiliser algorithm, see the proof of Proposition 3.1. If $\Lambda(S)(1) \neq[n]$, then we find $j \in[n] \backslash \Lambda(S)(1)$, replace $S$ by $S \cup\left\{\lambda_{j}\right\}$, and iterate. By construction, $\lambda_{j}(1)=j$, so
the new orbit and hence the new group $\Lambda(S)$ have increased in size. We can increase $|\Lambda(S)|$ only $O(\log n)$ times, so we repeat the work of the previous paragraph $O(\log n)$ times. In conclusion, in time $\tilde{O}\left(n^{2}\right)$ we can find $S \subseteq[n]$ of size $O(\log n)$ with $\Lambda(S)$ transitive on [n].

The second step is to use the method described in the proof of Proposition 3.3 (based on [32, Lemma 4.4.2]) to replace $S$ by a new set $S$ (also of size $O(\log n)$ ) that yields short transversal elements (each of length $O(\log n)$ ) and short Schreier generators. To describe the construction of [32, Lemma 4.4.2], we need more notation. For elements $g_{1}, \ldots, g_{k} \in \Lambda(S)$ with $k \in O(\log n)$ we define $C_{k}=\left\{g_{k}^{e_{k}} \ldots g_{1}^{e_{1}}\right.$ : each $\left.e_{i} \in\{0,1\}\right\}$ and the corresponding orbit $O_{k}=\left(C_{k} C_{k}^{-1}\right)(1)$, and call $C_{k}$ nondegenerate if $\left|C_{k}\right|=2^{k}$. As explained on [32, p. 65], the set $O_{k}$ can be constructed by starting with $\Delta_{0}=\{1\}$ and recursively defining $\Delta_{i}=\Delta_{i-1} \cup\left\{h_{i}(c): c \in \Delta_{i-1}\right\}$, where $h_{i}$ is the $i$-th element in the list $g_{k}^{-1}, \ldots, g_{1}^{-1}, g_{1}, \ldots, g_{k}$. With this construction, $O_{k}=\Delta_{2 k}$. For every $x \in O_{k}$ constructed this way we also store one tuple $\left[x ; j_{1}, \ldots, j_{\ell}\right]$, meaning that $\lambda_{(x)}=h_{j_{1}} \cdots h_{j_{\ell}}$ was one element we used to construct $x=\lambda_{(x)}(1)$ in $O_{k}$. These elements $\lambda_{(x)}$ will be the new short transversal elements; recall that $\ell \in O(\log n)$ by construction. If the elements $g_{1}, \ldots, g_{k}, g_{1}^{-1}, \ldots, g_{k}^{-1}$ are stored on a separate tape (which we can scan in time $\tilde{O}(n)$ ), the set $O_{k}$ can be constructed in time $\tilde{O}\left(n^{2}\right)$; note, however, that we have not yet computes $\lambda_{(x)}$, but rather stored the tuple $\left[x ; j_{1}, \ldots, j_{\ell}\right]$ describing it.

Now we follow the proof of [32, Lemma 4.4.2] to find $g_{1}, \ldots, g_{k} \in \Lambda(S)$ with $k \in O(\log n)$ such that $C_{k}$ is nondegenerate and $\left|O_{k}\right|=n$. We start by choosing a non-identity generator $g_{1} \in S$. If $g_{1}, \ldots, g_{i}$ are found such that $C_{i}$ is nondegenerate, but $\left|O_{i}\right|<n$, then there exists $j \in O_{i}$ and $s_{r} \in S$ such that $s_{r}(j) \notin O_{i}$. Now define $g_{i+1}=s_{r} \lambda_{(j)}$ (multiplied in time $\tilde{O}\left(n^{2}\right)$ ), where $\lambda_{(j)}$ is the stored transversal element for $j \in O_{i}$, and iterate until all required conditions are met. We have to do at $\operatorname{most} O(\log n)$ iterations, and correctness follows from the proof of [32, Lemma 4.4.2]. In conclusion, in time $\tilde{O}\left(n^{2}\right)$ we have found a new generating set $\left\{g_{1}, \ldots, g_{k}, g_{1}^{-1}, \ldots, g_{k}^{-1}\right\}$ of size $O(\log n)$ of the original group $\Lambda(S)$, and we have also constructed for each $x \in[n]$ a tuple $\left[x ; j_{1}, \ldots, j_{\ell}\right]$ of length $O(\log n)$ encoding a short transversal element $\lambda_{(x)}$ mapping 1 to $x$. Let $\mathcal{T}$ be the set of all these tuples, and note that we can scan $\mathcal{T}$ in time $\tilde{O}(n)$.

To simplify the notation, in the following we assume that the new generating set is again stored on a separate short tape and indexed by $S \subseteq[n]$ so that $\left\{g_{1}, \ldots, g_{k}, g_{1}^{-1}, \ldots, g_{k}^{-1}\right\}=\left\{\lambda_{s}: s \in S\right\}$ : using the short tape, we evaluate each $g_{i}^{ \pm}(1)=u$ and then scan the original table $T$ in time $\tilde{O}\left(n^{2}\right)$ to confirm $g_{i}^{ \pm}(1)=\lambda_{u}$ and to add $u$ to $S$; if we find out that $g_{i}^{ \pm}(1) \neq \lambda_{u}$, then $\Lambda(S)$ is not regular and we return false.

We now construct a set $R$ of relations that encode Schreier generators for the stabiliser $\Lambda(S)_{1}$. For each $\left[x ; j_{1}, \ldots, j_{\ell}\right] \in \mathcal{T}$ and for each $r \in S$ evaluate $y=\lambda_{r}(x)$ and scan $\mathcal{T}$ for the entry $\left[y ; v_{1}, \ldots, v_{m}\right]$; this means that $\lambda_{v_{1}} \cdots \lambda_{v_{m}}(1)=y=\lambda_{r} \lambda_{j_{1}} \ldots \lambda_{j_{\ell}}(1)$, and we add $\left[r, j_{1}, \ldots, j_{\ell} ; v_{1}, \ldots, v_{m}\right]$ to $R$. The whole of $R$ can be construced in time $\tilde{O}\left(n^{2}\right)$; moreover, $R$ has $\tilde{O}(n)$ entries of length $O(\log n)$ each.

Next, we use $R$ to check that the stabiliser $\Lambda(S)_{1}$ is trivial. It follows from Schreier's Lemma [32, Lemma 4.2.1] that $\Lambda(S)_{1}$ is generated by the Schreier generators encoded by the relations stored in $R$; more precisely, the stabiliser is trivial if and only if for each of the $\tilde{O}(n)$ relations $\left[j_{1}, \ldots, j_{\ell} ; v_{1}, \ldots, v_{m}\right]$ in $R$ we have $\lambda_{j_{1}} \cdots \lambda_{j_{\ell}}=\lambda_{v_{1}} \cdots \lambda_{v_{m}}$. (At this stage we only know that both sides of this equation map 1 to the same element.) Note that by construction $\ell, m \in O(\log n)$ and each generator $\lambda_{s}$ is stored on a short tape that we can scan in time $\tilde{O}(n)$. We now argue that we can multiply two permutations given by $n$-tuples in time $\tilde{O}(n)$ : recall that we represent a permutation $a \in \operatorname{Sym}_{n}$ as the $n$-tuple $\left[1^{a}, \ldots, n^{a}\right]$; labelling these entries by $1, \ldots, n$, and applying the aforementioned $\tilde{O}(n)$-time sorting algorithm (see [26, p. 434]), we can compute the following transformation (where the top row indicates the labels):

$$
a=\left[\begin{array}{c|cccc}
i & 1, & 2, & \ldots, & n \\
i^{a} & 1^{a}, & 2^{a}, & \ldots, & n^{a}
\end{array}\right] \rightarrow a=\left[\begin{array}{c|cccc}
i & 1^{a^{-1}}, & 2^{a^{-1}}, & \ldots, & n^{a^{-1}} \\
i^{a} & 1, & 2, & \ldots, & n
\end{array}\right]
$$

this is done by simply sorting $\left[1^{a}, \ldots, n^{a}\right]$ and keeping track of the labels. Now if $b$ is a second permutation represented as $\left[1^{b}, \ldots, n^{b}\right]$, then $a b$ is the permutation represented by

$$
a b=\left[\begin{array}{c|cccc}
i & 1^{a^{-1}}, & 2^{a^{-1}}, & \ldots, & n^{a^{-1}} \\
i^{a} & 1, & 2, & \ldots, & n
\end{array}\right] \cdot\left[\begin{array}{c|cccc}
i & 1, & 2, & \ldots, & n \\
i^{b} & 1^{b}, & 2^{b}, & \ldots, & n^{b}
\end{array}\right]=\left[\begin{array}{cccc}
1^{a^{-1}}, & 2^{a^{-1}}, & \ldots, & n^{a^{-1}} \\
1^{b}, & 2^{b}, & \ldots, & n^{b}
\end{array}\right] ;
$$

sorting the right hand array by its first row (again in time $\tilde{O}(n)$ ), we obtain the $n$-tuple describing $a b$. In conclusion, in time $\tilde{O}(n)$ we can compute the elements $\lambda_{j_{1}} \cdots \lambda_{j_{\ell}}$ and $\lambda_{v_{1}} \cdots \lambda_{v_{m}}$, and check for equality. Since $|R| \in \tilde{O}(n)$, checking that $\Lambda(S)_{1}$ is trivial can be done in time $\tilde{O}\left(n^{2}\right)$. If $\Lambda(S)_{1}$ is not trivial, then $\Lambda(S)$ is not regular and we return false. Otherwise, we have proved that $\Lambda(S)$ is a regular permutation group on $[n]$, contained in $\Lambda$.

It remains to show that $\Lambda \leqslant \Lambda(S)$. For this it is sufficient to show that each $\lambda_{x} \in \Lambda(S)$; because $\Lambda(S)$ is regular, the latter holds if and only if $\lambda_{x}=\lambda_{(x)}$ is the constructed transversal element. For this we first sort $\mathcal{T}$ such that its elements correspond to $\lambda_{(1)}, \ldots, \lambda_{(n)}$; note that $\mathcal{T}$ has $n$ entries of length $O\left((\log n)^{2}\right)$ bits, so we can sort $\mathcal{T}$ in time $\tilde{O}\left(n^{2}\right)$. Once sorted, we replace each entry $\left[x ; j_{1}, \ldots, j_{\ell}\right]$ by the permutation $\lambda_{(x)}$; the latter has already been constructed above by multiplying $\lambda_{j_{1}} \cdots \lambda_{j_{\ell}}$. Lastly, in time $\tilde{O}\left(n^{2}\right)$ we scan the original table $T$ and compare each $\lambda_{x}=\lambda_{(x)}$.

Some parts of the latter proof are similar to the proof of Proposition 3.1. One difference is that in the proof of Proposition 3.1 checking whether an element $x B$ lies in the orbit $V$ requires $|V|$ group multiplications and membership tests; in the proof above, this only requires running over a list $\mathcal{T}$ of size $n$ and comparing numbers. In Proposition 3.1 the orbit is small of size $|G: B|$, but applying the membership is more expensive; in the proof above, the orbit is large of size $n$, but checking equality is more efficient.

Remark 4.2. Having converted our Cayley table into a regular permutation group $G$, we now have a generating set $S$ of size $O(\log n)$ and a corresponding Schreier tree; both are stored in a separate tape of length $\tilde{O}(n)$. This is the tape that is used whenever we operate in the group; the original input tapes will never be revisited. As mentioned in the proof of Theorem [1.2, we may assume that the Schreier trees are shallow, that is, they have depth bounded by $O(\log n)$, see 32, Lemma 4.4.2]. The algorithms we now use are as follows. Each $g \in G$ is a node in the Schreier tree and there is a unique path from the origin to $g$; if $g_{1}, \ldots, g_{k} \in S$ are the labels of that path, then $g=g_{k} \cdots g_{1}$; note that $k \in O(\log n)$ since the Schreier tree is shallow. We now describe how to compute the product of these labels. Recall that $g_{1}, \ldots, g_{k}$ are permutations on $[n]$, so we can compute the image of $1 \in[n]$ under $g_{k} \cdots g_{1}$, by looking up $i_{1}=g_{1}(1), i_{2}=g_{2}\left(i_{1}\right)$, etc, until we obtain $u=g_{k} \cdots g_{1}(1)$. This scan occurs on the short tape in time $\tilde{O}(n)$. Since the group is regular, there is a unique $g \in G$ with $u=g(1)$, which determines $g=g_{k} \cdots g_{1}$. To multiply elements $g_{k} \cdots g_{1}$ and $g_{j}^{\prime} \cdots g_{1}^{\prime}$, we merely concatenate the generators, and continue with this word, yielding a $(\log n)^{O(1)}$-time multiplication. We note that none of our product lengths exceed $(\log n)^{O(1)}$. To compare $g_{k} \cdots g_{1}$ and $g_{j}^{\prime} \cdots g_{1}^{\prime}$ with $k, j \in O(\log n)$, we determine and compare $g_{k} \cdots g_{1}(1)$ and $g_{j}^{\prime} \cdots g_{1}^{\prime}(1)$ in time $\tilde{O}(n)$. More details of these methods are given in [32, p. 85-86].

## 5. Conclusion and outlook

We have shown that when restricted to a dense set $\Upsilon$ of group orders, testing isomorphism of groups of order $n \in \Upsilon$ given by Cayley tables can be done in time $\tilde{O}\left(n^{2}\right)$; this significantly improves the known general bound of $n^{O(\log n)}$. We note that $\left|\Upsilon \cap\left\{1,2, \ldots, 10^{k}\right\}\right| / 10^{k}$ is approximately 0.723 , 0.732 , and 0.786 for $k=8,9,10$, respectively; to determine whether $n \in v$, all logarithms have been computed with respect to the basis 2 . Recall that one can decide if $n \in \Upsilon$ in time $\tilde{O}\left(n^{2}\right)$, see Remark 4.1. We note that our work [9] considers isomorphism testing of cube-free groups, but under the assumption that groups are given as permutation groups.

We have proved that groups of these orders admit a computable factorisation $G=H \ltimes B$ with the following useful property: firstly, the $\boldsymbol{H}$ ard group theory of $G$ is captured in $H$, but $|H|$ is small compared to $|G|$ so brute-force methods can be applied to $H$; secondly, the Big number theory of $|G|$ is captured by $|B|$, but $B$ is cyclic, hence its group theory is easy. These decompositions exist for a dense set of group orders, so we expect this will be useful for other computational tasks as well. In fact, we will exploit properties of these decompositions in future work: This paper is part of our program to enhance group isomorphism, see [8, 9 for recent work, and we plan to extend the present results to other input models. Specifically, in our current work [10 we develop a new black-box input model for groups (based on Type Theory) that does not need a promise that the input really encodes a group, so algorithms for this model can be implemented within the usual polynomial-time hierarchy. Due to Theorem [1.2, the algorithms presented here do also not require a promise that the input tables describe groups. We conclude by mentioning that our algorithm for isomorphism testing can be adapted to find a single isomorphism, generators for the set of all isomorphisms, or to prescribe a canonical representative of the isomorphism type of a single group.

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## Appendix A. Proofs of Theorems [2.4] [2.6

## A.1. Number theory: Proof of Theorem 2.4.

Proof of Theorem 2.4. Erdős-Pálfy [11, Lemma 3.5] showed that almost every $n \in \mathbb{N}$ has the property that if a prime $p>\log \log n$ divides $n$, then $p^{2} \nmid n$; thus, $n=p_{1}^{e_{1}} \ldots p_{k}^{e_{k}} b$ with $b$ square-free, every prime divisor of $b$ is greater than $\log \log n$, and $p_{1}, \ldots, p_{k} \leqslant \log \log n$ are distinct primes. Let $x>0$ be an integer. We now compute an estimate for the number $N(x)$ of integers $0<n \leqslant x$ which are divisible by a prime $p \leqslant \log \log n$ such that the largest $p$-power $p^{e}$ dividing $n$ satisfies $p^{e}>\log n$. We want to show that $N(x) / x \rightarrow 0$ for $x \rightarrow \infty$; this proves that for almost all integers $n$, if $p^{e} \mid n$ with $p \leqslant \log \log n$, then $p^{e} \leqslant \log n$. To get an upper bound for $N(x)$, we consider integers between $\sqrt{x}$ and $x$ with respect to the above property, and add $\sqrt{x}$ for all integers between 1 and $\sqrt{x}$. Note that if $p^{e} \geqslant \log n$, then $e \geqslant \log \log n / \log p$. Since we only consider $\sqrt{x} \leqslant n \leqslant x$, this yields $e \geqslant c(x)$ where $c(x)=\log \log \sqrt{x} / \log \log \log x$. Note that $c(x) \rightarrow \infty$ if $x \rightarrow \infty$, thus

$$
\begin{aligned}
N(x) & \leqslant \sqrt{x}+\sum_{k=2}^{\lfloor\log \log \sqrt{x}\rfloor} \frac{x}{k^{c(x)}} \leqslant \sqrt{x}+x \int_{2}^{\log \log \sqrt{x}} \frac{1}{y^{c(x)}} \mathrm{d} y \\
& =\sqrt{x}+\frac{x}{1-c(x)}\left[\frac{1}{(\log \log \sqrt{x})^{c(x)-1}}-\frac{1}{2^{c(x)-1}}\right] .
\end{aligned}
$$

Since $1 /(1-c(x)) \rightarrow 0$ from below, we can estimate:

$$
N(x) \leqslant \sqrt{x}+x\left|\frac{1}{1-c(x)}\right|\left[-\frac{1}{(\log \log \sqrt{x})^{c(x)-1}}+\frac{1}{2^{c(x)-1}}\right] \leqslant \sqrt{x}+x\left|\frac{1}{1-c(x)}\right|\left[\frac{1}{2^{c(x)-1}}\right] ;
$$

thus $N(x)=o(x)$, since $N(x) / x \leqslant \sqrt{x} / x+\left|1 /(1-c(x)) 2^{c(x)-1}\right| \rightarrow 0$ if $x \rightarrow \infty$. This proves that the set $\Upsilon_{1}$ of all positive integers satisfying conditions a,c) in Definition [2.3 is dense. By [11, Lemmas $3.5 \& 3.6]$, the set $\Upsilon_{2}$ of positive integers $n$ satisfying conditions b,c) is dense as well. An inclusion-exclusion argument proves that $\Upsilon_{3}=\Upsilon_{1} \cap \Upsilon_{2}$ is dense. The Hardy-Ramanujan Theorem [27. Section 8] proves that the set $\Upsilon_{4}$ of integers $n$ that have at most $2 \log \log n$ distinct prime
divisors is dense, and now the intersection $\Upsilon_{3} \cap \Upsilon_{4}$ is dense as well. Since $\Upsilon_{3} \cap \Upsilon_{4} \subseteq \Upsilon$, the claim follows.

## A.2. Splitting theorems: Proofs of Theorems 2.5 \& 2.6.

Proof of Theorem 2.5. Let $G$ be a group of order $n \in \Upsilon$; we first show that $G$ has a normal Hall $\pi^{\text {lsi }}(n)$-subgroup, and if $G$ is solvable, then there is a normal Hall $\pi^{\text {si }}(n)$-subgroup.

First, let $G$ be solvable. We show that $G$ has a normal Sylow $p$-subgroup for every $p \in \pi^{\text {si }}(n)$. Let $q \neq p$ be a prime dividing $n$, and let $A$ be a Hall $\{p, q\}$-subgroup of $G$ of order $p^{e} q^{f}$; see [29, Section 9.1]. The Sylow Theorem [29, (1.6.16)] shows that the number $h_{p}$ of Sylow $p$-subgroups of $A$ divides $q^{f}$ (and hence $n$ ) and $p \mid\left(h_{p}-1\right)$. Since $p \mid n$ is isolated, we have $h_{p}=1$ and $A$ has a normal Sylow $p$-subgroup. Now fix a Sylow basis $\mathcal{P}=\left\{P_{1}, \ldots, P_{s}\right\}$ for $G$, that is, a set of Sylow subgroups, one for each prime dividing $n$, such that $P_{i} P_{j}=P_{j} P_{i}$ for all $i$ and $j$; see [29, Section 9.2]. Let $P=P_{u}$ be the Sylow $p$-subgroup for $G$ in $\mathcal{P}$. Since $G=P_{1} \cdots P_{s}$, every $g \in G$ can be written as $g=g_{1} \ldots g_{s}$ with each $g_{j} \in P_{j}$. Since $P P_{j}=P_{j} P$, the group $P P_{j}$ is a Hall $\left\{p, p_{j}\right\}$-subgroup. As shown above, $P \unlhd P P_{j}$, so each $g_{j} P=P g_{j}$. Thus, $g P=g_{1} \ldots g_{s} P=P g_{1} \ldots g_{s}=P g$, so $P \unlhd G$.

Second, suppose that $G$ is non-solvable. We show that $G$ has a normal Sylow $p$-subgroup for every $p \in \pi^{\text {lsi }}(n)$. Being non-solvable, $G$ has a non-abelian simple composition factor, so $|G|$ is divisible by 4, see [6, p. 155]. Since $n \in \Upsilon$, Definition [2.3k) implies that $2^{\nu_{2}(n)} \leqslant \log n$, so $\nu_{2}(n) \leqslant \log \log n<p$ for every $p \in \pi^{\text {lsi }}(n)$.

In the following we use the Babai-Beals filtration $G \geqslant \operatorname{PKer}(G) \geqslant \operatorname{Soc}^{*}(G) \geqslant O_{\infty}(G) \geqslant 1$, see [2. Section 1.2]: here $O_{\infty}(G)$ is the largest normal subgroup of $G$ and $\operatorname{Soc}^{*}(G) / O_{\infty}$ is the socle of $G / O_{\infty}(G)$, which is defined to be the subgroup generated by all minimal normal subgroups. This socle decomposes as $T_{1} \times \cdots \times T_{\ell}$ where each $T_{i}$ is a non-abelian simple normal subgroup of $G / O_{\infty}(G)$, see also [32, pp. 157-159]. The group $\operatorname{PKer}(G) / O_{\infty}(G)$ is the kernel of the permutation representation $G / O_{\infty}(G) \rightarrow \operatorname{Sym}_{\ell}$ induced by the conjugation action on $\left\{T_{1}, \ldots, T_{\ell}\right\}$.

First, we claim that $p \nmid\left|G: O_{\infty}(G)\right|$. Note that $p \nmid\left|T_{i}\right|$ for each $i$ since $p \in \pi^{\text {lis }}(n)$. By the Classification of Finite Simple Groups, a prime $r$ divides $\left|\operatorname{Aut}\left(T_{i}\right)\right|$ only if $r$ divides $\left|T_{i}\right|$, as seen from the list of known order ${ }^{11}$ of simple groups and their outer automorphism groups. Note that $\operatorname{PKer}(G) / O_{\infty}(G)$ embeds into $\operatorname{Aut}\left(T_{1}\right) \times \ldots \times \operatorname{Aut}\left(T_{\ell}\right)$. Assume, for a contradiction, that $p$ divides $\left|G: O_{\infty}(G)\right|$. By assumption, $p \nmid\left|\operatorname{Aut}\left(T_{i}\right)\right|$ for each $i$, which forces $p||G: \operatorname{PKer}(G)|$ and $p \leqslant \ell$. Every $T_{i}$ has even order by the Odd-Order Theorem, see [37, p. 2], so $2^{\ell} \mid n$ and $\nu_{2}(n) \geqslant \ell$; now $p \leqslant \ell$ contradicts $p>\nu_{2}(n)$, which we have shown above. This forces $p \nmid\left|G: O_{\infty}(G)\right|$, so the Sylow $p$-subgroup $P$ lies in $O_{\infty}$. Since $p$ is also isolated in $\left|O_{\infty}(G)\right|$, the proof of the solvable case shows $P \unlhd O_{\infty}(G)$. Since $O_{\infty}(G)$ is characteristic in $G$, we know that $P$ is normal in $G$.

In conclusion, for every $p \in \pi^{\text {lsi }}(n)$ there is a normal Sylow $p$-subgroup $G_{p}$ in $G$. Since $n \in \Upsilon$ and $p>\log \log n$, this subgroup is cyclic of size $p$. If $p, q \in \pi^{\text {lsi }}(n)$ are distinct, then $G_{p}, G_{q} \unlhd G$ implies that $G_{q} G_{p} \cong G_{p} \times G_{q} \cong C_{p q}$ is cyclic of order $p q$. Thus, $G$ has a normal cyclic Hall $\pi^{\text {lsi }}(n)$-subgroup $B$, and $G=H \ltimes B$ for some $H \leqslant G$ by the Schur-Zassenhaus Theorem [29, (9.1.2)].

It remains to prove that $|H| \leqslant(\log n)^{O\left((\log \log n)^{c}\right)}$ for some $c$. Recall from Definition 2.3 that $n=a b$ such that $a \leqslant(\log n)^{\log \log n}$ and if $p \in \pi^{\operatorname{lsi}}(n)$, then $p \mid b$ and $p^{2} \nmid n$; moreover, $b$ is squarefree and has at most $2 \log \log n$ prime divisors. We have $|H|=a b / z$, where $z$ is the product of the primes in $\pi^{\mathrm{lsi}}(n)$. It remains to show that $b / z \leqslant(\log n)^{O\left((\log \log n)^{c}\right)}$. If $p$ is a prime divisor of $b / z$, then $p>$ $\log \log n$ and $p \notin \pi^{\text {lsi }}(n)$, that is, $p^{2} \nmid n$ and $p \mid n$ is isolated, but not strongly isolated. By definition, this means that there is some non-abelian simple group $T$ of order dividing $n$ with $p||T|$. We

[^1]show below that each such $T$ satisfies $|T| \leqslant(\log n)^{O((\log \log n))}$, in particular, $p \leqslant(\log n)^{O((\log \log n))}$. Since $b / z$ has at most $2 \log \log n$ different prime divisors, we deduce $b / z \leqslant(\log n)^{O\left((\log \log n)^{2}\right)}$; this then completes the proof of the theorem.

One can see from the known factorized orders of the finite non-alternating non-abelian simple groups that every such group $T$ has a distinguished prime power divisor $r^{m}| | T \mid$ with $m>1$ and $|T| \leqslant\left(r^{m}\right)^{O(m)}$ : This is trivially true for the 26 sporadic groups; for the other non-alternating simple groups this follows because they are representable as quotients of groups of $d \times d$ matrices over a field of order $r^{e}$, and then $m=d e$. Now if the order of $T$ divides $n \in \Upsilon$, then $m>1$ forces $r \leqslant \log \log n$ and $r^{m} \leqslant \log n$, hence $m \leqslant \log \log n$, and $|T| \leqslant(\log n)^{O((\log \log n))}$, as claimed.

If $T \cong \mathrm{Alt}_{k}$ is alternating of order $k!/ 2 \leqslant k^{k}=2^{k \log k}$, then the distinguished prime power divisor is $2^{\nu_{2}(k!)-1}$. Legendre's formula [24, Theorem 2.6.4] shows that $\nu_{2}(k!)=k-s_{2}(k)$ where $s_{2}(k) \leqslant \log k$ is the number of 1 's in the 2 -adic representation of $k$. Since $n \in \Upsilon$, we have $2^{\nu_{2}(k!)-1} \leqslant \log n$, so $k-\log (k)-1 \leqslant \nu_{2}(k!)-1 \leqslant \log \log n$. This shows that $2^{k} \leqslant 2 k \log n$, and so $|T| \leqslant(2 k \log n)^{\log k}$. Note that $|T|=k!/ 2$ divides $n$, and so Stirling's formula $\ln (k!)=k \ln k-k+O(\ln k)$ shows that $k \leqslant \log 2 n$ for large enough $k$. This yields $|T| \leqslant(\log n)^{O(\log \log n)}$, as claimed.

Proof of Theorem 2.6. Let $G$ be a group of order $n \in \Upsilon$. By Theorem 2.5 and Corollary [3.2, we can construct generators and a membership test for the cyclic Hall $\pi^{\text {lsi }}(n)$-subgroup $B$ of $G$. With this we can use Proposition 3.3 to construct a complement $H$, thus $G=H \ltimes B$. The complexity statement follows from the results we have used.

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    Key words and phrases. group isomorphism, complexity.

[^1]:    ${ }^{1}$ The finite simple groups (Classification Theorem of Finite Simple Groups) are listed in [37, Section 1.2]; the orders of these groups and the orders of their automorphism groups are described in various places in said book. Simply for the convenience of the reader, we refer to en.wikipedia.org/wiki/List_of_finite_simple_groups\#Summary for a concise list of these orders.

