# A Quantum Advantage for a Natural Streaming Problem 

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#### Abstract

Data streaming, in which a large dataset is received as a "stream" of updates, is an important model in the study of space-bounded computation. Starting with the work of Le Gall [SPAA '06], it has been known that quantum streaming algorithms can use asymptotically less space than their classical counterparts for certain problems. However, so far, all known examples of quantum advantages in streaming are for problems that are either specially constructed for that purpose, or require many streaming passes over the input.

We give a one-pass quantum streaming algorithm for one of the best-studied problems in classical graph streaming - the triangle counting problem. Almost-tight parametrized upper and lower bounds are known for this problem in the classical setting; our algorithm uses polynomially less space in certain regions of the parameter space, resolving a question posed by Jain and Nayak in 2014 on achieving quantum advantages for natural streaming problems.


## 1 Introduction

### 1.1 Streaming Algorithms

Streaming algorithms are a class of algorithms for processing very large datasets that arrive "one piece at a time"-some dataset too large to fit into memory is built up by a series of updates. More formally, a vector $x \in \mathbb{Z}^{N}$ is received as a series of updates $\left(\sigma_{t}\right)_{t=1 \ldots . .}$, where each update $\sigma_{t}=z e_{i}$ consists of adding a scalar $z \in \mathbb{Z}$ to a co-ordinate $i \in[N]$, and the goal of a streaming algorithm is to estimate some statistic of $x=\sum_{t} \sigma_{t}$ in o $(n)$ space $^{1}$.

Streaming algorithms have been studied for a wide variety of problems, such as cardinality estimation [FN85], approximating the moments of a vector [AMS96], and subgraph counting [BKS02]. In this paper we will be concerned with quantum streaming algorithms.

Quantum Streaming The prospect of space-constrained quantum computers has motivated the study of quantum streaming, in which a stream of updates is received by an algorithm that is able to maintain a quantum state and perform quantum operations (including measurements) on this state as it processes the stream ${ }^{2}$.

[^0]The study of quantum streaming algorithms started with [LG06], in which a problem was constructed that exhibits an exponential separation between quantum and classical space complexity. The problem in question is not quite a streaming problem in the sense we defined above, as the function tested depends on the order of the updates of the stream, but in $\left[\mathrm{GKK}^{+} 08\right]$ it was shown that such a separation exists for an update-order-independent function.

This suggests the question, raised in [JN14], of whether it is possible to obtain such separations for "natural" problems. They proposed as a candidate the problem of recognizing the Dyck(2) language in the stream ${ }^{3}$, but while better lower bounds for this problem have since been shown [NT17], better-than-classical upper bounds are still unknown.

When many passes are allowed over the stream, [Mon16, HM19] demonstrate a quantum advantage for the problem of estimating the frequency moments of a vector, giving algorithms for various settings of the problem that can save a $k^{2}$ factor in space complexity when they make $k$ passes, instead of the $k$ factor possible in classical streaming [AMS96, CR11]. But typically in streaming the objective is to make only one pass over the stream, or at most $\mathrm{O}(1)$ passes.

We resolve the question of [JN14], giving a new one-pass streaming algorithm for the triangle counting problem, one of the best-studied problems in graph streaming.

Triangle Counting In the (insertion-only) graph streaming model a graph $G=(V, E)$ is received as a sequence of edges ${ }^{4}\left(\sigma_{t}\right)_{t=1}^{m}$ from $E$. The first problem to be studied in this setting was that of estimating the number of triangles (three-cliques) in $G$ [ BKS 02 ].

All algorithms for this problem are parametrized, as counting triangles requires $\Omega\left(n^{2}\right)$ space if the number of them is sufficiently small [BKS02], and even graphs with $\Omega(m)$ triangles can be hard to distinguish from triangle-free graphs in sufficiently "hard" graphs [BOV13]. The space complexity of such algorithms is therefore typically quoted in terms of these parameters-it will often be unreasonable to assume that the algorithm knows these parameters exactly in advance, but constant factor bounds on them will suffice.

The best-known classical algorithm in this setting is from [JK21], which gives an

$$
\widetilde{\mathrm{O}}\left(\frac{m}{T} \cdot\left(\Delta_{E}+\sqrt{\Delta_{V}}\right) \cdot \frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}\right)
$$

space upper bound for obtaining a $(1 \pm \varepsilon)$-multiplicative approximation with probability ${ }^{5} 1-\delta$ in a graph with $m$ edges, $T$ triangles, and in which no more than $\Delta_{E}$ triangles share an edge and no more than $\Delta_{V}$ share a vertex. This algorithm is known to be classically optimal for this parametrization, up to log factors, as [BOV13] gives a $\Omega\left(\frac{m \Delta_{E}}{T}\right)$ lower bound and [KP17] gives $\Omega\left(\frac{m \sqrt{\Delta_{V}}}{T}\right)$ when $T=\mathrm{O}(m)$.

These two lower bounds are both based on reductions from communication complexity. The first is from the Indexing problem [KNR95], which is as hard for quantum communication as

[^1]it is for classical communication (see [ANTSV02], in which it is called the problem of quantum random access codes) and so the bound directly applies to any quantum streaming algorithm. However, the second is from the Boolean Hidden Matching problem, in particular the variant studied in $\left[\mathrm{GKK}^{+} 08\right]$. This problem is easier in quantum communication, and indeed was already used to prove a quantum-classical streaming separation.

We give a quantum triangle counting algorithm that beats the classical lower bound.
Theorem 1. For any $\varepsilon, \delta \in(0,1]$, there is a quantum streaming algorithm that uses

$$
O\left(\frac{m^{8 / 5}}{T^{6 / 5}} \Delta_{E}^{4 / 5} \log n \cdot \frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}\right)
$$

quantum and classical bits in expectation to return a ( $1 \pm \varepsilon$ )-multiplicative approximation to the triangle count in an insertion-only graph stream with probability $1-\delta$.
$m$ is the number of edges in the stream, $T$ the number of triangles, and $\Delta_{E}$ the greatest number of triangles sharing any given edge.

In particular, this means that when $\Delta_{E}=\mathrm{O}(1), \Delta_{V}=\Omega(T)=\Omega(m)$ (i.e. maximizing the separation, as $T$ must be $O(m)$ if the classical lower bounds are to hold), we require $\widetilde{\mathrm{O}}\left(\mathrm{m}^{2 / 5}\right)$ space instead of the $\Omega(\sqrt{m})$ required by any classical algorithm ${ }^{6}$.

### 1.2 Other Related Work

Other work has investigated streaming problems with quantum inputs [BKCG13, Yu20], as well as quantum versions of models closely related to streaming, such as online algorithms [KKM18], limited-width branching programs [NHK00, SS05, AAKV18, HMWW20], and finite automata [KW97, AF98, MC00, ANTSV02].

Of particular relevance to streaming (see, e.g. [BGW20]) is the coin problem, in which a coin is flipped repeatedly, and the task is to determine whether it is $p$-biased or $(p+\varepsilon)$-biased. This problem actually exhibits arbitrary quantum advantage, as any classical algorithm requires at least $\log (p(1-p))+\log (1 / \varepsilon)$ space, while a quantum algorithm can solve it with a single qubit [AD11], although not if $\varepsilon$ is unknown [KO17].

## 2 Overview of the Algorithm

### 2.1 Classical Triangle Counting

To understand how maintaining a quantum state will help us with triangle counting, we will start by describing an optimal classical algorithm, from [JK21]. For ease of exposition we will consider the related problem of triangle distinguishing-determining whether a graph $G=(V, E)$ has 0 triangles or whether it has at least $T$ triangles. As is often (although not necessarily) the case with subgraph counting problems, converting to a counting algorithm will be almost immediate.

[^2]The classical algorithm is as follows:

1. Sample vertices with probability $p$.
2. Sample edges incident to sampled vertices with probability $q$.
3. Whenever an edge arrives that "completes" a pair of sampled edges incident to some sampled vertex ("a wedge centered at a sampled vertex") we will note that we have found a triangle.

For any triangle in $G$, we will find it iff we sample its "first" vertex, the vertex shared by the first two of its edges to arrive, and then sample both of those edges. We can then distinguish between graphs with 0 triangles and those with $T$ by reporting whether we found a triangle or not (for counting, we count the number of triangles we found and scale it by $p^{-1} q^{-2}$ ). The vertex sampling can be implemented with a pairwise independent hash function, so the expected space needed is $\mathrm{O}(p q m \log n)$ for a graph with $m$ edges and $n$ vertices, as each edge is kept with probability $p q$.

How small can $p q$ be? This is given by the graph parameters $T$ (the number of triangles in the graph), $\Delta_{E}$ (the maximum number of triangles sharing an edge), and $\Delta_{V}$ (the maximum number of triangles sharing a vertex). Ignoring constant factors, if we want to find a triangle with constant probability:

- $p$ must be at least $\Delta_{V} / T$, as the triangles might share as few as $T / \Delta_{V}$ "first" vertices.
- $p q$ must be at least $\Delta_{E} / T$, as there might be a set $S$, containing as few as $T / \Delta_{E}$ edges, such that every triangle has an edge from $S$ as one of its first two edges.
- $p q^{2}$ must be at least $1 / T$, as each triangle will be found with probability $p q^{2}$.

It turns out that these are also sufficient, and subject to them $p q$ is minimized when

$$
p=\frac{\Delta_{V}}{T}, \quad q=\max \left\{\frac{\Delta_{E}}{\Delta_{V}}, \frac{1}{\sqrt{\Delta_{V}}}\right\}
$$

giving an algorithm that uses

$$
\mathrm{O}\left(\frac{m}{T}\left(\Delta_{E}+\sqrt{\Delta_{V}}\right) \log n\right)
$$

space. Lower bounds from [BOV13, KP17] establish that (up to log factors) both the $\Delta_{E}$ and $\Delta_{V}$ terms are necessary. However, while the first of these is based on a reduction from the Indexing problem, which is known to require as much quantum communication as classical communication to solve [ANTSV02], the latter is based on a reduction from the Boolean Hidden Matching problem (in particular, the " $\alpha$-Partial Matching" variant $\alpha \mathrm{PM}_{n}$ of $\left[\mathrm{GKK}^{+} 08\right]$ ), which is known to exhibit an exponential separation between classical and quantum communication.

### 2.2 Quantum Triangle Counting

### 2.2.1 Two Players

In seeking a quantum advantage, we consider the hard instance of [KP17], depicted in Figure 1. This is as follows:

1. $T / \Delta_{V}$ stars of degree $m \Delta_{V} / T$ arrive. We call the central vertices of these "hubs" and their neighboring vertices "spokes".

(a) The first half of the stream is $T / \Delta_{V}$ "hubs", (b) They are followed by $m$ edges, disjoint from each a vertex with edges to $m \Delta_{V} / T$ "spokes". each other but potentially incident to the "spoke" vertices of the hubs.

(c) The union of these two halves may contain 0 or $T$ triangles.

Figure 1: A hard graph for classical algorithms, when $\Delta_{E}=1$ but $\Delta_{V}$ is unrestricted.
2. Another $m$ edges arrive, all disjoint from each other. We have that either, for each hub, $\Delta_{V}$ of these edges form triangles by connecting two spokes of the hub, or none of them do for any hub.

We will start by considering this in the simpler ${ }^{7}$ two-player setting-Alice gets the first half of the stream, Bob gets the second half of the stream, and Alice wants to send Bob a message that he can use with his input to determine whether the graph has 0 or $T$ triangles.

If she wanted to do this by sending some subset of the edges, she would need to send $\mathrm{O}\left(\frac{m \sqrt{\Delta}}{T}\right)$ of them - within any given hub the edges are indistinguishable, so at best she can choose one specific hub and send a $1 / \sqrt{\Delta_{V}}$ fraction of its edges, to have a $\left(1 / \sqrt{\Delta_{V}}\right)^{2}=1 / \Delta_{V}$ chance of finding any given one of its $\Delta_{V}$ triangles ${ }^{8}$. By embedding an instance of $(T / m) \mathrm{PM}_{m \Delta_{V} / T}$ in a hub, and then copying that hub $T / \Delta_{V}$ times, it can be shown that there is no asymptotically better classical message Alice can send.

What if Alice is allowed to send a quantum message? We cannot emulate the $\alpha$ PM protocol

[^3]of $\left[\mathrm{GKK}^{+} 08\right]$ directly, as not all graphs of this form will correspond to an embedding of $\alpha \mathrm{PM}$. Instead, if the set of hub vertices is $H$, with spoke vertices $S_{u}$ for each $u \in H$, Alice may construct the $O(\log m)$-bit quantum state
$$
\frac{1}{\sqrt{m}} \sum_{u \in H} \sum_{v \in S_{u}}|\overrightarrow{u v}\rangle
$$
where $\overrightarrow{u v}$ denotes the directed edge from ${ }^{9} u$ to $v$. As Bob's edges are disjoint, he can then construct an orthonormal basis of $\mathbb{R}^{\left|V^{2}\right|}$ which contains
$$
\frac{|\overrightarrow{w u}\rangle+|\vec{w}\rangle\rangle}{\sqrt{2}}, \frac{|\overrightarrow{w u}\rangle-|\overrightarrow{w v}\rangle}{\sqrt{2}}
$$
for every $w \in V$ and edge $u v$ in his set of $m$ edges. If he measures Alice's state in this basis, he will see:

- Each basis element of the form $\frac{|\overrightarrow{w u}\rangle+|\vec{w}\rangle\rangle}{\sqrt{2}}$ with probability $1 / 2 m$ if Alice has a hub $w$ with exactly one of the spokes $u, v$, and $2 / m$ if Alice has a hub $w$ with both of these as spokes (i.e. if $u v$ completes a triangle in Alice's input).
- Each basis element of the form $\frac{|\overrightarrow{w u}\rangle-|\vec{w}\rangle\rangle}{\sqrt{2}}$ with probability $1 / 2 m$ if Alice has a hub $w$ with exactly one of the spokes $u, v$, and 0 if Alice has a hub $w$ with both of these as spokes.

So if $G$ is triangle-free these are the same, but if $G$ has $T$ triangles they differ by $2 T / m$, and so Bob can work out which situation they are in if Alice sends him $\Theta\left(m^{2} / T^{2}\right)$ copies of this state, at the cost of $\Theta\left(\frac{m^{2}}{T^{2}} \log m\right)$ qubits.

To generalize this technique, we will need to address two questions: how to construct and measure the state in the stream, and what to do when the edges we want to measure by are not disjoint, and therefore do not give an orthonormal basis.

### 2.2.2 Streaming

Constructing the State We start by constructing the superposition

$$
\frac{1}{\sqrt{2 m}} \sum_{i=1}^{2 m}|i\rangle
$$

of $2 m$ "dummy" states. Then, whenever we process the $i^{\text {th }}$ edge $u v$ in the stream, we swap the dummy states $|2 i-1\rangle,|2 i\rangle$ for $|\overrightarrow{u v}\rangle,|\overrightarrow{v u}\rangle$.

Measurements If we could remember all of the measurements we want to make, we could take this state and perform the measurements we made in the two-player game at the end of the stream. However, this would require remembering every edge we see, so instead we perform the measurements one edge at a time.

[^4]When the edge $u v$ arrives, we construct the set of projectors $\mathcal{O}^{u v}=\left\{O_{b}^{u v}: b \in\{-1,0,1\}\right\}$ given by

$$
\begin{aligned}
& O_{1}^{u v}=\frac{1}{2} \sum_{w \in V}(|\overrightarrow{w u}\rangle+|\vec{w}\rangle)(\langle\langle\overrightarrow{w u}|+\langle\overrightarrow{w v}|) \\
& \left.O_{-1}^{u v}=\frac{1}{2} \sum_{w \in V}(|\overrightarrow{w u}\rangle-|\vec{w}\rangle\rangle\right)(\langle\overrightarrow{w u}|-\langle\vec{w}|) \\
& O_{0}^{u v}=I-\sum_{w \in V}(|\overrightarrow{w u}\rangle\langle\overrightarrow{w u}|+|\vec{w}\rangle\langle\overrightarrow{w v}|)
\end{aligned}
$$

and measure with them ${ }^{10}$, with $b$ the outcome corresponding to the operator $O_{b}^{u v}$. If we see +1 or -1 we terminate the algorithm and return with that value, while if we see 0 we continue processing the stream.

This gives us a somewhat different result from performing the measurement at the end of the stream, since now when we measure by an edge $u v$ we can only pick up on triangles wuv such that $w u$ and $w v$ appear before $u v$. However, this is not a concern, as for each triangle there will be exactly one final edge that arrives.

Non-Disjoint Edges However, there is a problem with this strategy. The two-player strategy worked because the edges in Bob's set $B$ were disjoint, and so the elements

$$
\left\{\frac{|\overrightarrow{w u}\rangle+|\overrightarrow{w v}\rangle}{\sqrt{2}}, \frac{|\overrightarrow{w u}\rangle-|\overrightarrow{w v}\rangle}{\sqrt{2}}: w \in V, u v \in B\right\}
$$

could be a subset of an orthonormal basis of $\mathbb{R}^{\left|V^{2}\right|}$. But in the general setting, we have to measure by every edge, as we do not know in advance which edges will be "last edges" of triangles.

Now, the individual edge measurements described in the previous section are still valid, but what will be the impact of measuring by them? After measuring with $\mathcal{O}^{u v}$, if we do not terminate the algorithm, then the state has been projected onto $O_{0}^{u v}$ (and re-weighted appropriately). This means that for all $w \in V$, if either $|\overrightarrow{w u}\rangle$ or $|\overrightarrow{w v}\rangle$ were present in the superposition, they will now be gone.

The consequence of this is that instead of an estimator of the number of triangles in the graph, we instead have an estimator of the number of triangles $w u v$ such that $w u$, $w v$ appear in the stream and then no edges incident to $u$ or $v$ arrive before $u v$ does. However, this is not very useful, as this could easily be 0 even in a graph with many triangles.

In order to have at least some chance of finding triangles that do not fit this description, we will refrain from measuring with all of the edges we see. Suppose that when seeing the edge $u v$ we only perform the $\mathcal{O}^{u v}$ measurement with probability $1 / k$, for some $k$. Then, for every triangle $w u v$, if its edges arrive in the order $w u, w v, u v$, the probability that, $|\overrightarrow{w u}\rangle$ and $|\overrightarrow{w v}\rangle$ are still in the state when $u v$ arrives is

$$
(1-1 / k)^{d_{w w v}+d_{w v u}}
$$

where $d_{w u v}$ denotes the degree between $w u$ and $u v$, the number of edges incident to $u$ that arrive in between $w u$ and $u v$. So as the measurement $\mathcal{O}^{u v}$ itself is performed with probability $1 / k$, the

[^5]
(a) $T$ wedges $\left(u w_{i} v_{i}\right)_{i=1}^{T}$ arrive, (b) more edges $\left(u z_{i}\right)_{i=1}^{m}$ arrive (c) $T$ triangles $\left(w_{i} u v_{v} i\right)_{i=1}^{T}$ are all incident to $u$. incident to $u$. formed, each with $d_{w_{i} u v_{i}}^{\rightarrow}=m$.

Figure 2: A hard graph for our quantum estimator, with $d_{w u v}^{\rightarrow}+d_{w v u}^{\vec{~}}$ large for every triangle wuv.
probability that we perform it and both $|\overrightarrow{w u}\rangle$ and $|\overrightarrow{w v}\rangle$ are in the state at the time is

$$
\frac{1}{k}(1-1 / k)^{d_{w u v}+d_{w v u}^{\vec{w}}}
$$

This means that any triangle with $d_{w u v}+d_{w v u} \leq k$ has at least an $\Omega(1 / k)$ probability of contributing to our estimator. More specifically, we can estimate $T^{<k}$, given by down-weighting every triangle by $(1-1 / k)^{d_{\vec{w} v u}+d_{\vec{w} u v}}$, by running $\Theta\left(\left(k m / T^{<k}\right)^{2}\right)$ copies of the estimator in parallel, as now the probability of seeing +1 is only $m / k T^{<k}$ greater than the probability of seeing -1 . If we only care to estimate it to $\varepsilon T$ accuracy for some constant $\varepsilon$ (i.e. we are fine having a poor multiplicative estimate when $T^{<k}$ is small) we can replace this with $\Theta\left((k m / T)^{2}\right)$ copies instead.

However, this is not helpful on its own, as it is possible to have a graph stream where $T^{<k}$ is much smaller than $T$ for any $k \ll m$. Consider the following stream, depicted in Figure 2 :

1. The edges $\left(u w_{i}\right)_{i=1}^{T}$ and $\left(w_{i} v_{i}\right)_{i=1}^{T}$ arrive for some sequences of unique vertices $\left(w_{i}\right)_{i=1}^{m},\left(v_{i}\right)_{i=1}^{m}$ and some unique fixed vertex $u$.
2. The edges $\left(u z_{i}\right)_{i=1}^{m}$ arrive for some sequence of unique vertices $\left(z_{i}\right)_{i=1}^{m}$.
3. The edges $\left(u v_{i}\right)_{i=1}^{T}$ arrive.

Now every triangle $w_{i} u v_{v} i$ in this stream has $d_{w_{i} u v_{i}}^{\rightarrow}=m$, and so $T^{<k}$ will be very small if $k \ll m$.
However, even though this corresponds to a graph where $\Delta_{V}$ is large, it will still be easy for the classical algorithm we described at the start of this section.

Hybrid Quantum-Classical Algorithm In the stream described above, the first two edges of each triangle $w_{i} u v_{i}$ are $w_{i} u, w_{i} v_{i}$, and so if we run the classical algorithm from the start of this section with $q=1$, the triangle will be found if $w_{i}$ is sampled by the vertex sampling stage. As the vertices $\left(w_{i}\right)_{i=1}^{T}$ are disjoint, the classical algorithm can succeed with $p=\Theta(1 / T)$, for a classical space complexity of $\mathrm{O}(m / T)$.

More generally, if $\Omega(m / k)$ triangles $\left(w u_{i} v_{i}\right)_{i=1}^{m / k}$ share the same "first" vertex $w$, with distinct $\left(u_{i}\right)_{i=1}^{m / k},\left(v_{i}\right)_{i=1}^{m / k}$ then

$$
\sum_{i=1}^{m / k}\left(d_{w v_{i} u_{i}}+d_{w u_{i} v_{i}}\right) \leq \sum_{i=1}^{m / k}\left(d_{u_{i}}+d_{v_{i}}\right) \leq 2 m
$$

and so the average value of $\left(d_{w u_{i} v_{i}}+d_{w_{v} u_{i}}\right)$ across these triangles is at most $\mathrm{O}(k)$. This means that, if we consider the set of triangles wuv with $d_{w u v}^{\rightarrow}+d_{w v u}^{\rightarrow} \geq k$, at most $\mathrm{O}(m / k)$ of them can share any one "first" vertex.

This means that a classical algorithm can count them to $\varepsilon T$ accuracy (for some constant $\varepsilon$ ) with $s$ space by sampling vertices with probability $p=m / T k$ and incident edges with probability $\sqrt{k / m}$, for $\mathrm{O}\left(m^{3 / 2} / T \sqrt{k}\right)$ total samples in expectation. Moreover, by maintaining degree counters for the endpoints of each edge it samples, such an algorithm can record $d_{w u v}^{\rightarrow}+d_{w v u}^{\rightarrow}$ for each triangle $w u v$ it samples, and therefore estimate ${ }^{11} T^{>k}=T-T^{<k}$.

In other words, triangles that are hard for the classical algorithm to count are easier for the quantum algorithm to count, and vice-versa. This suggests the following hybrid algorithm:

1. Choose $k$ appropriately.
2. Use the quantum algorithm to estimate $T^{<k}$ to $\varepsilon T / 2$ error.
3. Use the classical algorithm to estimate $T^{>k}=T-T^{<k}$ to $\varepsilon T / 2$ error.
4. Return the sum of the estimates.

What should $k$ be? We want to minimize

$$
\left(\frac{k m}{T}\right)^{2}+\frac{m^{3 / 2}}{T \sqrt{k}}
$$

Setting $k=T^{2 / 5} / m^{1 / 5}$ gives us a

$$
\widetilde{\mathrm{O}}\left(\frac{m^{8 / 5}}{T^{6 / 5}}\right)
$$

space algorithm, that becomes

$$
\widetilde{\mathrm{O}}\left(\frac{m^{8 / 5}}{T^{6 / 5}} \Delta_{E}^{4 / 5}\right)
$$

when our bounds are modified to account for up to $\Delta_{E}$ triangles sharing an edge. When e.g. $\Delta_{E}=O(1), \Delta_{V}=\Omega(T)$ and $T=\omega\left(m^{6 / 7}\right)$, this is less space than the best possible classical algorithm.

## 3 Preliminaries

### 3.1 General Notation

$k \in[0, m]$ is a parameter shared by the quantum and classical algorithms, to be specified later.
Let $G=(V, E)$ be a graph on $n$ vertices, received as a stream of undirected edges, adversarially ordered. Let $m \leq\binom{ n}{2}$ be an upper bound on the number of edges in the graph (and thus the number of updates in the stream). We write the stream $\sigma=\left(\sigma_{i}\right)_{i=1}^{m}$, for $\sigma \in E$. We will write $\sigma^{\leq t}=\left(\sigma_{i}\right)_{i=1}^{t}$.

[^6]We will write $N(v)$ for the neighborhood of any $v \in V$, and $d_{v}$ for $|N(v)|$.
We will use $\overrightarrow{u v}$ to denote a directed edge from $u$ to $v$, and $u v$ (or $v u$ ) to refer to the undirected edge (and so $u v=v u$ while $\overrightarrow{u v} \neq \overrightarrow{v u}$ ). We will write $\vec{E}$ for the set of directed edges and $E$ for the set of edges.

We will use $\mathbb{I}(p)$ to denote the indicator on whether the predicate $p$ holds.

### 3.2 Triangles

We use $T$ to refer to the number of triangles in $G, \Delta_{E} \geq 1$ to refer to the maximum number of them sharing a single edge (or 1 if $G$ is triangle-free).

Fix any ordering of the stream. For any edges $e, f$, we will write $e \preccurlyeq f$ if $e$ arrives before $f$ in the stream. For any vertices $u, v, w \in V$ such that $u v, v w \in E$ and $u v \preccurlyeq v w$, let the degree between $u v$ and $v w, d_{u v w}$ be the number of edges incident to $v$ that arrive in between $u v$ and $v w$ (not including $u v$ or $v w$ themselves).

For any triple of vertices $(u, v, w) \in V^{3}$ let

$$
t_{u v w}^{<k}= \begin{cases}(1-1 / k)^{d_{u v w}+d_{u w v}} & \text { if }\{u, v, w\} \text { is a triangle in the graph and } u v \preccurlyeq u w \preccurlyeq v w \\ 0 & \text { otherwise } .\end{cases}
$$

Likewise, let

$$
t_{u v w}^{>k}= \begin{cases}1-(1-1 / k)^{d_{u v w}+d_{u w v}} & \text { if }\{u, v, w\} \text { is a triangle in the graph and } u v \preccurlyeq u w \preccurlyeq v w \\ 0 & \text { otherwise. }\end{cases}
$$

We will write $T^{<k}, T^{>k}$ for $\sum_{(u, v, w) \in V^{3}} t_{u v w}^{<k}, \sum_{(u, v, w) \in V^{3}} t_{u v w}^{>k}$, respectively, so that $T=T^{<k}+T^{>k}$.
For any vertex $u \in V$, we will write $T_{u}^{<k}=\sum_{(v, w) \in V^{2}} t_{u v w}^{<k}$ and $T_{u}^{>k}=\sum_{(v, w) \in V^{2}} t_{u v w}^{>k}$, so $\sum_{u \in V} T_{u}^{<k}=T^{<k}$ and $\sum_{u \in V} T_{u}^{>k}=T^{>k}$.

## 4 Quantum Estimator

Each instance of the quantum algorithm will maintain $\beta=2[\log n]+1$ qubits, indexing the set $\vec{E} \cup[2 m]$. We will write the basis states as $|\overrightarrow{u v}\rangle,|t\rangle$ for $\overrightarrow{u v} \in \vec{E}, t \in[2 m]$.

Let $f:[m] \rightarrow\{0,1\}$ be a fully independent hash function such that

$$
f(t)= \begin{cases}1 & \text { with probability } 1 / k \\ 0 & \text { otherwise }\end{cases}
$$

While $f$ is a fully random function, and so would be infeasible to store, our algorithm will only need to query $f(t)$ at the time step $t$, and therefore will not need to store it.

After the $t^{\text {th }}$ update, the algorithm will either terminate or maintain the state

$$
\Sigma_{t}=\frac{\sum_{i=2 t+1}^{2 m}|i\rangle+\sum_{\vec{u} \vec{v} \in S_{t}}|\overrightarrow{u v}\rangle}{\sqrt{2 m-2 t+\left|S_{t}\right|}}
$$

where

$$
S_{t}=\left\{\overrightarrow{u v}: \exists i \in[t], \sigma_{i}=u v, \forall j=i+1, \ldots, t, f(j)=0 \vee v \notin \sigma_{j}\right\} .
$$

That is, $S_{t}$ contains the directed edges $\overrightarrow{v v}$ and $\overrightarrow{v u}$ for every edge $u v$ that has arrived at time $t$, except that whenever an edge $w z$ arrives at time $s$, if $f(s)=1$ all edges directed towards either $w$ or $z$ are removed.

At each step $t$, the algorithm will first apply a unitary transformation, depending on $t$ and the edge $\sigma_{t}$, to take $\Sigma_{t-1}$ to $\Sigma_{t}$, and then, if $f(t)=1$, it will measure $\Sigma_{t}$ with an operator depending only on $\sigma_{t}$. We now define this transformation and measurement operator.

Definition 2. Take $\{|x\rangle: x \in \vec{E} \cup[m]\}$ and extend it to a basis of $\mathbb{R}^{\beta}$. For each $t \in[m]$, uv $\in E$, if $u<v$ the unitary transformation $U_{u v}^{t}$ is given by swapping the basis elements $|(2 t-1)\rangle$ and $|\overrightarrow{u v}\rangle$, and swapping $|2 t\rangle$ and $|\overrightarrow{v u}\rangle$. If $v<u, U_{u v}^{t}=U_{v u}^{t}$.

Definition 3. For any $u v \in E$, the set of measurement operators $\mathcal{O}^{u v}=\left\{O_{b}^{u v}: b \in\{-1,0,1\}\right\}$ is given by

$$
\begin{aligned}
& \left.O_{1}^{u v}=\frac{1}{2} \sum_{w \in V}(|\overrightarrow{w u}\rangle+|\vec{w}\rangle\rangle\right)(\langle\overrightarrow{w u}|+\langle\overrightarrow{w v}|) \\
& \left.O_{-1}^{u v}=\frac{1}{2} \sum_{w \in V}(|\overrightarrow{w u}\rangle-|\vec{w}\rangle\rangle\right)(\langle\overrightarrow{w u}|-\langle\overrightarrow{w v}|) \\
& O_{0}^{u v}=I-\sum_{w \in V}(|\overrightarrow{w u}\rangle\langle\overrightarrow{w u}|+|\vec{w}\rangle\langle\langle\vec{w}|)
\end{aligned}
$$

with $b$ the outcome corresponding to operator $O_{b}^{u v}$.
Note that $O_{1}^{u v} O_{-1}^{u v}=0$, and $O_{0}^{u v}=I-O_{1}^{u v}-O_{-1}^{u v}$, so this is a complete set of orthogonal projectors.

We can now define the algorithm.

```
Algorithm 1 Quantum estimator for \(T^{<k}\)
    procedure QuantumEstimator \((k)\)
        \(t \leftarrow 1\)
        \(\Sigma_{t} \leftarrow \frac{1}{\sqrt{m}} \sum_{i=1}^{m}|i\rangle\)
        for each update \(u v\) do
            \(\Sigma_{t} \leftarrow U_{u v}^{t} \Sigma_{t-1}\)
            if \(f(t)=1\) then \(\quad \triangleright f(t)\) is not re-used so we can generate it here.
                Measure \(\Sigma_{t}\) with the operators \(\mathcal{O}^{u v}\), storing the result in \(\mathbf{b}\).
                if \(\mathbf{b} \neq 0\) then
                return \(b\)
                end if
            end if
            \(t \leftarrow t+1\)
        end for
        \(\mathbf{b} \leftarrow 0\)
        return b
    end procedure
```

Lemma 4. For all $t=0, \ldots, m$, after QuantumEstimator( $k$ ) has processed $t$ updates, either it will have returned or

$$
\Sigma_{t}=\frac{\left.\sum_{i=2 t+1}^{2 m}|i\rangle+\sum_{\vec{u} \in S_{t}}|\vec{u}\rangle\right\rangle}{\sqrt{2 m-2 t+\left|S_{t}\right|}}
$$

where

$$
S_{t}=\left\{\overrightarrow{u v}: \exists i \in[t], u v=\sigma_{i}, \forall j=i+1, \ldots, t, f(j)=0 \vee v \notin \sigma_{j}\right\} .
$$

Proof. We proceed by induction. For $t=0$,

$$
\Sigma_{t}=\frac{1}{\sqrt{m}} \sum_{i=1}^{2 m}|i\rangle
$$

and so the result holds. Now, for any $t \in[m-1]$, suppose that the result holds after $t$ updates. Let $x y$ (with $x<y$ ) be the $(t+1)^{\text {th }}$ update. Then after applying the unitary $U_{x y}^{t+1}$,

$$
\Sigma_{t+1}=\frac{\sum_{i=2 t+3}^{2 m}|i\rangle+\sum_{\vec{u} \vec{v} \in S_{t} \cup\{\vec{x}, \vec{y} \vec{x}\}}|\vec{u} \vec{v}\rangle}{\sqrt{2 m-2 t+\left|S_{t}\right|}}
$$

as $U_{x y}^{t+1}$ swapped the basis vectors $|(2 t+1)\rangle,|(2 t+2)\rangle$ for $|\overrightarrow{x y}\rangle,|\overrightarrow{y x}\rangle$.
So if $f(t+1)=0$, the result continues to hold for $t+1$, as in this case

$$
S_{t+1}=S_{t} \cup\{\overrightarrow{x y}, \overrightarrow{y x}\}
$$

and so $2 m-2 t+\left|S_{t}\right|=2 m-2(t+1)+\left|S_{t+1}\right|$. Now suppose $f(t+1)=1$. Let

$$
W=\left\{\vec{w} \vec{z} \in S_{t}: z \in\{x, y\}\right\} .
$$

Then

$$
S_{t+1}=\{\overrightarrow{x y}, \overrightarrow{y x}\} \cup S_{t} \backslash W
$$

and

$$
\begin{aligned}
U_{x y}^{t+1} \Sigma_{t} & =\frac{\sum_{i=2 t+3}^{2 m}|i\rangle+\sum_{\vec{u} \vec{v} \in S_{t} \cup\{\vec{x}, \overrightarrow{y \vec{y}\}}\}}|\overrightarrow{u \vec{v}\rangle}\rangle}{\sqrt{2 m-2 t+\left|S_{t}\right|}} \\
& =\frac{\sum_{i=2 t+3}^{2 m}|i\rangle+\sum_{\vec{u} \vec{u} \in S_{t+1}}|\vec{u}\rangle}{\sqrt{2 m-2 t+\left|S_{t}\right|}}+\frac{\sum_{\vec{u} \in W}|\overrightarrow{u v}\rangle}{\sqrt{2 m-2 t+\left|S_{t}\right|}} .
\end{aligned}
$$

If QuantumEstimator does not return after this update, that means the measurement operation returned 0 . Therefore, after the measurement,

$$
\begin{aligned}
\Sigma_{t+1} & =\frac{O_{0}^{x y} U_{x y}^{t+1} \Sigma_{t}}{\left\|O_{0}^{x y} U_{x y}^{t+1} \Sigma_{t}\right\|_{2}} \\
& =\frac{\sum_{i=2 t+3}^{2 m}|i\rangle+\sum_{\vec{u} \in S_{t+1}}|\overrightarrow{u v}\rangle}{\| \sum_{i=2 t+3}^{2 m}|i\rangle+\sum_{\vec{u} \in S_{t+1}}|\overrightarrow{u v}\rangle \|_{2}} \\
& =\frac{\sum_{i=2 t+3}^{2 m}|i\rangle+\sum_{\overrightarrow{u v} \in S_{t+1}}|\overrightarrow{u v}\rangle}{\sqrt{2 m-2(t+1)+\left|S_{t+1}\right|}}
\end{aligned}
$$

and so the result holds for all $t \in[m]$.

## Lemma 5.

$$
\mathbb{E}[\mathbf{b}]=\frac{T^{<k}}{k m}
$$

Proof. For $\mathbf{b}$ to take a non-zero value, it must be returned from the measurement step at some $t$ such that $f(t)=1$, and the algorithm must not return before time $t$. Condition on the value of $f$. For any $t$ such that $f(t)=1$, let $x y=\sigma_{t}$, so $S_{t}=\{\overrightarrow{x y}, \overrightarrow{y x}\} \cup S_{t-1} \backslash W_{t}$, where $W_{t}=\left\{\overrightarrow{w z} \in S_{t-1}: z \in\{x, y\}\right\}$.

Suppose that the algorithm has not yet returned a value. By Lemma 4,

$$
\Sigma_{t-1}=\frac{\sum_{i=2 t-1}^{2 m}|i\rangle+\sum_{\vec{u} \vec{b} \in S_{t-1}}|\vec{u} \vec{v}\rangle}{\sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}}
$$

and so

$$
\begin{aligned}
U_{x y}^{t} \Sigma_{t-1}= & \frac{\sum_{i=2 t+1}^{2 m}|i\rangle+\sum_{\vec{u} \in S_{t}}|\overrightarrow{u v}\rangle}{\sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}}+\frac{\sum_{\vec{u} \in W_{t}}|\overrightarrow{u v}\rangle}{\sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}} \\
= & \frac{\sum_{i=2 t+1}^{2 m}|i\rangle+\sum_{\vec{u} \in S_{t}}|\overrightarrow{u v}\rangle}{\sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}}+\frac{\sum_{w \in W_{t}^{+}}(|\overrightarrow{w x}\rangle+|\overrightarrow{w y}\rangle)}{\sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}} \\
& +\frac{\sum_{w \in W_{t}^{-}}(|\overrightarrow{w x}\rangle+|\overrightarrow{w y}\rangle)}{2 \sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}}+\frac{\sum_{w \in W_{t}^{-}}(|\overrightarrow{w x}\rangle-|\overrightarrow{w y}\rangle)}{2 \sqrt{2 m-2(t-1)+\left|S_{t-1}\right|}}(-1)^{\mathbb{I}\left(\overrightarrow{w y} \in W_{t}\right)}
\end{aligned}
$$

where

$$
\begin{aligned}
& W_{t}^{+}=\left\{w \in V:\left|\{\vec{w} \vec{x}, \vec{w} \vec{y}\} \cap W_{t}\right|=2\right\} \\
& W_{t}^{-}=\left\{w \in V:\left|\{\overrightarrow{w x}, \overrightarrow{w y}\} \cap W_{t}\right|=1\right\} .
\end{aligned}
$$

Therefore, conditioned on $f$ and the algorithm not having returned yet, after the measurement (breaking the 1 case into two for clarity)

$$
\mathbf{b}= \begin{cases}1 & \text { with probability } \frac{2\left|W_{t}^{+}\right|}{2 m-2(t-1)+\left|S_{t-1}\right|} \\ 1 & \text { with probability } \frac{\left|W_{t}^{-}\right|}{\frac{1\left(2 m-2(-1)+\left|S_{t-1}\right|\right.}{}} \\ -1 & \text { with probability } \frac{\left|W_{t}^{-}\right|}{\left.\frac{1}{2\left(2 m-2(-1)+\left|S_{t-1}\right|\right.}\right)} \\ 0 & \text { with probability } \frac{2 m-2 t+\left|S_{t}\right|}{2 m-2(t-1)+\left|S_{t-1}\right|} .\end{cases}
$$

Now, let $t_{i}$ be the $i^{\text {th }} t$ such that $f(t)=1$, and let $l=|\{t \in[m]: f(t)=1\}|$. For each $i \in[l]$, let $H_{i}^{f}$ be the expected value of $\mathbf{b}$ when it is returned, conditioned on $f$ and on the algorithm not returning at a time step before $t_{i}$. Then, by the above we have

$$
H_{i}^{f}=\frac{2\left|W_{t_{i}}^{+}\right|}{2 m-2\left(t_{i}-1\right)+\left|S_{t_{i}-1}\right|}+\frac{2 m-2 t_{i}+\left|S_{t_{i}}\right|}{2 m-2\left(t_{i}-1\right)+\left|S_{t_{i}-1}\right|} H_{i+1}^{f}
$$

with $H_{l+1}^{f}$ defined to be 0 . We will prove by reverse induction on $i$ that

$$
H_{i}^{f}=\frac{2 \sum_{j=i}^{l}\left|W_{t_{j}}^{+}\right|}{2 m-2\left(t_{i}-1\right)+\left|S_{t_{i}-1}\right|}
$$

for all $i \in[l+1]$. We have $H_{l+1}^{f}=0$ by definition. Now, for any $i<l+1$, suppose

$$
H_{i+1}^{f}=\frac{2 \sum_{j=i+1}^{l}\left|W_{t_{j}}^{+}\right|}{2 m-2\left(t_{i+1}-1\right)+\left|S_{t_{i+1}-1}\right|} .
$$

Then

$$
\begin{aligned}
H_{i}^{f} & =\frac{2\left|W_{t_{i}}^{+}\right|}{2 m-2\left(t_{i}-1\right)+\left|S_{t_{i}-1}\right|}+\frac{2 m-2 t_{i}+\left|S_{t_{i}}\right|}{2 m-2\left(t_{i}-1\right)+\left|S_{t_{i}-1}\right|} \frac{2 \sum_{j=i+1}^{l}\left|W_{t_{j}}^{+}\right|}{2 m-2\left(t_{i+1}-1\right)+\left|S_{t_{i+1}-1}\right|} \\
& =\frac{2 \sum_{j=i}^{l}\left|W_{t_{j}}^{+}\right|}{2 m-2\left(t_{i}-1\right)+\left|S_{t_{i}-1}\right|}
\end{aligned}
$$

as $\left|S_{t_{i+1}-1}\right|=\left|S_{t_{i}}\right|+2\left(t_{i+1}-t_{i}-1\right)$, as $f(s)=0$ when $t_{i}<s<t_{i+1}$, and so the set grows by two edges at each such $s$.

So we now have

$$
\begin{aligned}
\mathbb{E}[\mathbf{b} \mid f] & =H_{1}^{f} \\
& =\frac{1}{m} \sum_{j=1}^{l}\left|W_{t_{j}}^{+}\right| \\
& =\frac{1}{m} \sum_{t=1}^{m}\left|W_{t}^{+}\right|
\end{aligned}
$$

by defining $W_{t}^{+}$to be the empty set when $f(t)=0$. And so

$$
\mathbb{E}[\mathbf{b}]=\frac{1}{m} \sum_{t=1}^{m} \mathbb{E}\left[\left|W_{t}^{+}\right|\right]
$$

Now, recall that, when $f(t)=1, W_{t}^{+}$consists of the vertices $w$ in $V$ such that, if $u v$ is the edge that arrives at time $t, w u$ and $w v$ arrive at times $s_{1}, s_{2}<t$, and there is no time $s_{1}^{\prime} \in\left(s_{1}, t\right) \cap[m]$ such that $f\left(s_{1}^{\prime}\right)=1$ and $\sigma_{s^{\prime}}$ is incident to $u$, nor time $s_{2}^{\prime} \in\left(s_{2}, t\right) \cap[m]$ such that $f\left(s_{2}^{\prime}\right)=1$ and $\sigma_{s_{2}^{\prime}}$ is incident to $v$.

If both $w u$ and $w v$ arrived before $u v$, the probability that this happens conditioned on $f(t)=1$ is $(1-1 / k)^{d_{u v w}+d_{u w v}}=t_{w u v}^{<k}+t_{w v u}^{<k}$.

So

$$
\mathbb{E}\left[\left|W_{t}^{+}\right| \mid f(t)=1\right]=\sum_{w \in V}\left(t_{w u v}^{<k}+t_{w v u}^{<k}\right)
$$

where $u v=\sigma_{t}$ and so

$$
\mathbb{E}\left[\left|W_{t}^{+}\right|\right]=\frac{1}{k} \sum_{w \in V}\left(t_{w u v}^{<k}+t_{w v u}^{<k}\right) .
$$

As $t_{x y z}^{<k}$ is 0 whenever $y z$ is not an edge that appears in the stream, this gives us

$$
\begin{aligned}
\mathbb{E}[\mathbf{b}] & =\frac{1}{k m} \sum_{\substack{t=1 \\
u v:=\sigma_{t}}}^{m} \sum_{w \in V}\left(t_{w u v}^{<k}+t_{w v u}^{<k}\right) \\
& =\frac{1}{k m} \sum_{(u, v, w) \in V^{3}} t_{u v w}^{<k} \\
& =\frac{T^{<k}}{k m}
\end{aligned}
$$

completing the proof.

## Lemma 6.

$$
\operatorname{Var}(\mathbf{b}) \leq 1
$$

Proof. This follows immediately from the fact that $|\mathbf{b}|$ is at most 1.
Lemma 7. For any $\varepsilon, \delta \in(0,1]$, there is a quantum streaming algorithm, using

$$
O\left(\left(\frac{k m}{T}\right)^{2} \log n \frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}\right)
$$

quantum and classical bits, that estimates $T^{<k}$ to $\varepsilon T$ precision with probability $1-\delta$.
Proof. By taking the average of $\Theta\left((k m / T \varepsilon)^{2}\right)$ copies of QuantumEstimator $(k)$ and multiplying by $k m$, we obtain an estimator with expectation $T^{<k}$ and variance at most $\varepsilon^{2} T^{2} / 4$. So by Chebyshev's inequality the estimator will be within $\varepsilon T$ of $T^{<k}$ with probability $3 / 4$. We can then repeat this $O\left(\log \frac{1}{\delta}\right)$ times and take the median of our estimators to estimate $T^{<k}$ to $T \varepsilon$ precision with probability $1-\delta$.

Each copy of QuantumEstimator uses $2\lceil\log n\rceil+1$ qubits and $\mathrm{O}(\log n)$ classical bits, and so the result follows.

## 5 Classical Estimator

For this algorithm we will use a hash function $g: V \rightarrow\{0,1\}$ such that

$$
\mathbb{E}[g(v)]=1 / \sqrt{k m}
$$

for each $v \in V$. We will need to store this function, but instead of making it fully independent we will make it pairwise independent, so this will only require $O(\log n)$ bits.

```
Algorithm 2 Classical estimator for \(T^{>k}\).
    procedure ClassicalEstimator \((k)\)
        \(S \leftarrow \emptyset\)
        \(D \leftarrow\) the empty map from \(S\) to \(\mathbb{N}\).
        \(\mathbf{X}=0\)
        for each update \(u v\) do
            for \(w \in V\) do
                    if \(\overrightarrow{w u}, \overrightarrow{w v} \in S\) then
                    \(\left.\mathbf{X} \leftarrow \mathbf{X}+1-(1-1 / k)^{D[\vec{w}]}\right]+D[\vec{w}] \quad \triangleright \operatorname{Add} t_{w u v}^{>k}+t_{w v u}^{>k}\) to \(\mathbf{X}\).
                    end if
            end for
            for \(\overrightarrow{x y} \in S\) do
                    if \(y \in\{u, v\}\) then
                    \(D[\overrightarrow{x y}] \leftarrow D[\overrightarrow{x y}]+1\)
            end if
        end for
        if \(g(u)=1\) then
            Add \(\overrightarrow{u v}\) to \(S\) with probability \(\sqrt{k / m}\).
            If \(\overrightarrow{u v}\) was added, set \(D[\overrightarrow{u v}]=0\).
        end if
        if \(g(v)=1\) then
            Add \(\overrightarrow{v u}\) to \(S\) with probability \(\sqrt{k / m}\).
            If \(\overrightarrow{v u}\) was added, set \(D[\overrightarrow{v u}]=0\).
        end if
        end for
        return \(X\).
    end procedure
```

Lemma 8. The expected space usage of CLASSICALESTIMATOR is $O(\log n)$ bits.
Proof. For each edge in $G$, ClassicalEstimator will keep it with probability $\frac{1}{\sqrt{k m}} \times \sqrt{k / m}=\frac{1}{m}$ for each of its endpoints. So the algorithm keeps $O(1)$ edges in expectation, along with a counter (of size at most $m$ ) for each edge. The edge and the counter can each be stored in $O(\log n)$ bits, and so the result follows.

## Lemma 9.

$$
\mathbb{E}[\mathbf{X}]=\frac{T^{>k} \sqrt{k}}{m^{3 / 2}}
$$

Proof. For any triangle $u v w$ such that $u v \preccurlyeq u w \preccurlyeq v w, t^{>k}$ will be added to $\mathbf{X}$ iff $g(u)=1$ (which happens with probability $1 / \sqrt{k m}$ ) and both $u v$ and $u w$ are then kept by the algorithm (which happens with probability $(\sqrt{k / m})^{2}$ conditioned on $g(u)=1$ ).

Lemma 10.

$$
\operatorname{Var}(\mathbf{X}) \leq 4 \frac{T^{>k} \sqrt{k}}{m^{3 / 2}} \Delta_{E}
$$

Proof. For each $(u, v, w) \in V^{3}$, let $\mathbf{X}_{u v w}$ be the contribution to $\mathbf{X}$ from (possibly) adding $t_{u v w}^{>k}$. For each $u \in V$, let $\mathbf{X}_{u}=\sum_{(v, w) \in V^{2}} X_{u v w}$. Then $\mathbf{X}=\sum_{u} \mathbf{X}_{u}$ and the $\mathbf{X}_{u}$ are all independent, as each depends on $g(u)$ and the independent edge addition events. So $\operatorname{Var}(\mathbf{X})=\sum_{u \in V} \operatorname{Var}\left(\mathbf{X}_{u}\right)$.

Now, for each $u \in V$,

$$
\begin{aligned}
& \operatorname{Var}\left(\mathbf{X}_{u}\right) \leq \mathbb{E}\left[\mathbf{X}_{u}^{2}\right] \\
& =\mathbb{E}\left[\left(\sum_{(v, w) \in V^{2}} \mathbf{X}_{u v w}\right)^{2}\right] \\
& =\mathbb{E}\left[\left(\sum_{\substack{(v, w) \in N(u)^{2}: \\
v w \in E \\
u v \preccurlyeq u w \preccurlyeq v w}} \mathbf{X}_{u v w}\right)^{2}\right]
\end{aligned}
$$

We will bound each of these three terms in turn. First, as the probability that we add $t_{u v w}^{>k}$ to $\mathbf{X}$ is $\frac{1}{\sqrt{k m}} \times(\sqrt{k / m})^{2}$, and $0 \leq t_{u v w}^{>k} \leq 1$,

$$
\sum_{\substack{(v, w) \in N(u)^{2}: \\ v w \in E \\ u v \preccurlyeq u w \preccurlyeq v w}} \mathbb{E}\left[\mathbf{X}_{u v w}^{2}\right] \leq \sum_{\substack{(v, w) \in N(u)^{2}: \\ v w \in E \\ u \preccurlyeq u w \preccurlyeq v w}} \mathbb{E}\left[\mathbf{X}_{u v w}\right]
$$

Next, each triangle shares an edge with at most $\Delta_{E}$ other triangles, and $\mathbf{X}_{u v w} \mathbf{X}_{u x y}>0$ only if $g(u)=1$ and all of $u v, u w, u x, u y$ are kept by ClaSSICALESTIMATOR, which occurs with probability $\frac{1}{\sqrt{k m}} \times(\sqrt{k / m})^{3}=k / m^{2}$ when there are exactly three distinct vertices among $v, w, x, y$. So again using the fact that $0 \leq t_{u v w}^{>k} \leq 1$,

$$
\begin{align*}
\sum_{\substack{(v, w, x, y) \in N(u))^{4}: \\
v w, x y \in E \\
u \preccurlyeq \rightsquigarrow w \preccurlyeq v \\
u x \preccurlyeq y \preccurlyeq x y \\
|\{v, w\} \cap\{x, y\}|=1}} \mathbb{E}\left[\mathbf{X}_{u v w} \mathbf{X}_{x y y}\right] & \leq \sum_{\substack{(v, w) \in N(u)^{2}: \\
v w \in E \\
u v \preccurlyeq u w \preccurlyeq v w}} \Delta_{E} t_{u v w}^{>k} \frac{k}{m^{2}} \\
& =\frac{T_{u}^{>k} k}{m^{2}} \Delta_{E} .
\end{align*}
$$

For the final term, we will need the fact that $t_{u v w}^{>k} \leq \frac{d_{v}+d_{w}}{k}$. If $k \leq d_{v}+d_{w}$, this follows immediately
from the fact that $t_{u v w}^{>k} \leq 1$. Otherwise, if $t_{u v w}^{>k} \neq 0$,

$$
\begin{aligned}
t_{u v w}^{>k} & =1-(1-1 / k)^{d_{\overrightarrow{u v}}+d_{u w}} \\
& \leq 1-(1-1 / k)^{d_{v}+d_{w}} \\
& =-\sum_{i=1}^{d_{v}+d_{w}}\binom{d_{v}+d_{w}}{i}(-1 / k)^{i} \\
& =\frac{d_{v}+d_{w}}{k}-\sum_{i=2}^{d_{v}+d_{w}}\binom{d_{v}+d_{w}}{i}(-1 / k)^{i} \\
& \leq \frac{d_{v}+d_{w}}{k}
\end{aligned}
$$

as the terms of $\sum_{i=2}^{d_{v}+d_{w}}\left(\underset{i}{d_{v}+d_{w}}\right)(-1 / k)^{i}$ alternate between positive and negative, and their magnitude is decreasing in $i$ ( as $k>d_{v}+d_{w}$ ), and they start positive, so the sum is non-negative.

Therefore, as for disjoint $\{v, w\},\{x, y\}$, if $t_{u v w}^{>k} t_{u x y}^{>k}>0$, the probability that $\mathbf{X}_{u v w} \mathbf{X}_{u x y} \neq 0$ is $\frac{1}{\sqrt{k m}} \times(\sqrt{k / m})^{4}=\frac{k^{3 / 2}}{m^{5 / 2}}$,

$$
\begin{aligned}
& \leq \sum_{\substack{(v, w) \in N(u)^{2}: \\
v w \in \in E \\
u \preccurlyeq u w \preccurlyeq v w}} \frac{k^{3 / 2}}{m^{5 / 2}} \sum_{\begin{array}{c}
(x, y) \in N(u)^{2}: \\
x y \in \in E \\
u x \preccurlyeq u y \preccurlyeq x y
\end{array}} \frac{d_{x}+d_{y}}{k}
\end{aligned}
$$

and, as for each $x \in N(u)$ there are at most $\Delta_{E}$ elements $y$ of $N(u)$ such that $u x y$ is a triangle, we have

$$
\begin{aligned}
\sum_{\substack{(x, y) \in N(u)^{2}: \\
y x \in E \\
u x \preccurlyeq y \preccurlyeq x y}} \frac{d_{x}+d_{y}}{k} & =\sum_{x \in N(u)} \sum_{\substack{y \in N(u): \\
x y \in E \\
u x \preccurlyeq x y \\
u y \preccurlyeq x y \\
d_{y} \leq d_{x}}} \frac{d_{x}+d_{y}}{k} \\
& \leq \sum_{x \in N(u)} \frac{2 d_{x}}{k} \Delta_{E} \\
& \leq \frac{2 m}{k} \Delta_{E}
\end{aligned}
$$

so

$$
\begin{align*}
& \sum_{(v, w, x, y) \in N(u)^{4}:} \mathbb{E}\left[\mathbf{X}_{u v w} \mathbf{X}_{x y y}\right] \leq \sum_{(v, w) \in N(u)^{2}:} \frac{2 \sqrt{k}}{m^{3 / 2}} \Delta_{E} \\
& v w, x y \in E \\
& u v \preccurlyeq u w \preccurlyeq v w \\
& u x \preccurlyeq u y \preccurlyeq x y \\
& \{v, w\} \cap\{x, y\}=\emptyset \\
& =2 \frac{T_{u}^{>k} \sqrt{k}}{m^{3 / 2}} \Delta_{E} . \tag{3}
\end{align*}
$$

Therefore, by adding (1), (2), and (3),

$$
\begin{aligned}
\operatorname{Var}\left(\mathbf{X}_{u}\right) & \leq \frac{T_{u}^{>k} \sqrt{k}}{m^{3 / 2}}+\frac{T_{u}^{>k} k}{m^{2}} \Delta_{E}+2 \frac{T_{u}^{>k} \sqrt{k}}{m^{3 / 2}} \Delta_{E} \\
& \leq 4 \frac{T_{u}^{>k} \sqrt{k}}{m^{3 / 2}} \Delta_{E}
\end{aligned}
$$

as $\Delta_{E} \geq 1$ and $k \leq m$. The result then follows from summing over all $u \in V$.
Lemma 11. For any $\varepsilon, \delta \in(0,1]$, there is a classical streaming algorithm, using

$$
O\left(\frac{m^{3 / 2}}{T \sqrt{k}} \Delta_{E} \log n \frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}\right)
$$

bits of space in expectation, that estimates $T^{>k}$ to $\varepsilon T$ precision with probability $1-\delta$.
Proof. As $T^{>k} \leq T$, we can take the average of $\Theta\left(\frac{1}{\varepsilon^{2}} \frac{m^{3 / 2}}{T \sqrt{k}} \Delta_{E}\right)$ copies of CLASSICALEstimator $(k)$ and multiply by $\frac{m^{3 / 2}}{\sqrt{k}}$ to obtain an estimator with expectation $T^{>k}$ and variance at most $\varepsilon^{2}\left(T^{>k}\right)^{2} / 4 \leq$ $\varepsilon^{2} T^{2} / 4$. So by Chebyshev's inequality the estimator will be within $\varepsilon T$ of $T^{>k}$ with probability $3 / 4$. We can then repeat this $O\left(\log \frac{1}{\delta}\right)$ times and take the median of our estimators to estimate $T^{>k}$ to $T \varepsilon$ precision with probability $1-\delta$.

By Lemma 8, each copy of ClassicalEstimator will require $\mathrm{O}(\log n)$ bits of space in expectation, and so the result follows.

## 6 Hybrid Quantum-Classical Algorithm

By combining our quantum and classical estimators, we may now prove Theorem 1.
Theorem 1. For any $\varepsilon, \delta \in(0,1]$, there is a quantum streaming algorithm that uses

$$
O\left(\frac{m^{8 / 5}}{T^{6 / 5}} \Delta_{E}^{4 / 5} \log n \cdot \frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}\right)
$$

quantum and classical bits in expectation to return a $(1 \pm \varepsilon)$-multiplicative approximation to the triangle count in an insertion-only graph stream with probability $1-\delta$.
$m$ is the number of edges in the stream, $T$ the number of triangles, and $\Delta_{E}$ the greatest number of triangles sharing any given edge.

Proof. Let

$$
k=\frac{T^{2 / 5}}{m^{1 / 5}} \Delta_{E}^{2 / 5}
$$

By Lemmas 7 and 11, there are algorithms for estimating each of $T^{<k}, T^{>k}$ to precision $\varepsilon T / 2$ with probability $1-\delta / 2$, using

$$
O\left(\frac{m^{8 / 5}}{T^{6 / 5}} \Delta_{E}^{4 / 5} \log n \frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}\right)
$$

quantum and classical bits in expectation. If we then sum these estimators they will be within $\varepsilon T$ of $T$ with probability $1-\delta$, by taking a union bound.

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[^0]:    ${ }^{1}$ Space has been the primary object of study in the theory of streaming algorithms. Update and pre- and postprocessing time are typically, although not necessarily, manageable if the space required by the algorithm is small.
    ${ }^{2}$ Note that, despite the deferred measurement principle, we may want to perform measurements between updates, as the measurements we make may depend on the updates we see, and therefore we cannot automatically push them to the end of the stream.

[^1]:    ${ }^{3}$ They use the broader definition of streaming that encompasses update-order-dependent functions, but any separation for a function that does not depend on the order would also be one in that model.
    ${ }^{4}$ In our vector model described earlier, this corresponds to receiving the adjacency matrix of the graph as a series of positive updates to individual co-ordinates. Other models of graph streaming, in which edges can be deleted as well as added, also exist.
    ${ }^{5}$ For the remainder of this discussion we will assume $\varepsilon, \delta$ are constant. Most algorithms for this problem have a $\frac{1}{\varepsilon^{2}} \log \frac{1}{\delta}$ dependence, which comes from taking the average of $\Theta\left(\frac{1}{\varepsilon^{2}}\right)$ constant-variance estimators to obtain a ( $1 \pm \varepsilon$ )multiplicative approximation with $2 / 3$ probability, then repeating $\Theta\left(\log \frac{1}{\delta}\right)$ times and taking the median in order to amplify the success probability to $1-\delta$.

[^2]:    ${ }^{6}$ It remains open whether this is the best separation possible - the $\Omega\left(\frac{m \Delta_{E}}{T}\right)$ lower bound from Indexing disappears with these parameter settings and so it is possible that even exponential advantages can be achieved.

[^3]:    ${ }^{7}$ Any streaming algorithm for this problem immediately gives a one-way protocol for the two-player version, with message size equal to the space needed by the algorithm. Alice can run the streaming algorithm on her input, send the algorithm's state to Bob, and then he can initialize it with that state and run it on his input.
    ${ }^{8}$ This is essentially identical to running the algorithm of [JK21] on the input.

[^4]:    ${ }^{9}$ Of course, in the general case, we won't know which vertices are hubs and which are spokes, so each edge $u v$ will need to be included as both $\overrightarrow{u v}$ and $\overrightarrow{v u}$.

[^5]:    ${ }^{10}$ If performing all the measurements at the end of the stream is desired, an alternative with the same outcome (and using only a constant factor extra qubits) is to perform the unitary swapping $\frac{|\overrightarrow{w u}\rangle+|\overrightarrow{w v}\rangle}{2}$ with $|\overrightarrow{w u v}+\rangle$ and $\frac{|\overrightarrow{w u}\rangle-|\overrightarrow{w v}\rangle}{2}$ with $|\overrightarrow{w u v}-\rangle$ for each $w$, and then measuring in the standard basis at the end of the stream.

[^6]:    ${ }^{11}$ Technically the algorithm described here only allows estimating the sum of $1-(1-1 / k)^{d_{w u v}+d_{w v u}}$ over triangles where $d_{w u v}+d_{w v u} \geq k$. But by analyzing the variance of the estimator more carefully it is possible to replace this with the sum of that over all triangles, i.e. $T-T^{<k}$.

