# Minimum Cuts in Directed Graphs via Partial Sparsification* 

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#### Abstract

We give an algorithm to find a minimum cut in an edge-weighted directed graph with $n$ vertices and $m$ edges in $\tilde{O}\left(n \cdot \max \left\{m^{2 / 3}, n\right\}\right)$ time. This improves on the 30 year old bound of $\tilde{O}(n m)$ obtained by Hao and Orlin for this problem. Using similar techniques, we also obtain $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$-time $(1+\epsilon)$-approximation algorithms for both the minimum edge and minimum vertex cuts in directed graphs, for any fixed $\epsilon$. Before our work, no $(1+\epsilon)$-approximation algorithm better than the exact runtime of $\tilde{O}(n m)$ is known for either problem.

Our algorithms follow a two-step template. In the first step, we employ a partial sparsification of the input graph to preserve a critical subset of cut values approximately. In the second step, we design algorithms to find the (edge/vertex) mincut among the preserved cuts from the first step. For edge mincut, we give a new reduction to $\tilde{O}\left(\min \left\{n / m^{1 / 3}, \sqrt{n}\right\}\right)$ calls of any maxflow subroutine, via packing arborescences in the sparsifier. For vertex mincut, we develop new local flow algorithms to identify small unbalanced cuts in the sparsified graph.


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## 1 Introduction

The minimum cut (or mincut) problem is one of the most widely studied problems in graph algorithms. In (edge-)weighted ${ }^{1}$ directed graphs (or digraphs), a mincut is a bipartition of the vertices into two non-empty sets $(S, V \backslash S)$ so that the total weight of edges from $S$ to $V \backslash S$ is minimized. This problem can be solved by solving the $s$-mincut problem (also called rooted mincut), where for a given root vertex $s$, we want to find the minimum weight cut $(S, V \backslash S)$ such that $s \in S$. (We call such cuts minimum $s$-cuts or $s$-mincuts.) This is because the mincut can be computed as the minimum between two $s$-mincuts for an arbitrary vertex $s$ : one with the original edge directions in the input digraph, and the other with the edge directions reversed.

A simple algorithm for $s$-mincut (and thus mincut) on an $m$-edge, $n$-vertex digraph is to use $n-1$ maxflow calls to obtain the minimum $s-t$ cut for every vertex $t \neq s$ in the graph, and return the minimum among these. A beautiful result of Hao and Orlin [HO94] showed that these maxflow calls can be amortized to match the running time of a single maxflow call, provided one uses the push-relabel maxflow algorithm [GT88]. This leads to an overall running time of $\tilde{O}(m n)$. Since their work, better maxflow algorithms have been designed, but the amortization does not work for these algorithms. As a consequence, the Hao-Orlin bound remains the best known for the directed mincut problem almost 30 years after their work.

### 1.1 Our Results

In this paper, we can solve the $s$-mincut - thus the directed mincut problem-by essentially reducing it to $O(\sqrt{n})$ maxflow calls. At first glance, this is worse than the Hao-Orlin algorithm that only uses a single maxflow call. But crucially, while the Hao-Orlin algorithm is restricted to a specific maxflow subroutine and therefore cannot take advantage of faster, more recent maxflow algorithms, our new algorithm treats the maxflow subroutine as a black box, thereby allowing the use of any maxflow algorithm. Using state of the art maxflow algorithms that run in $\tilde{O}\left(m+n^{3 / 2}\right)$ time [vdBLL ${ }^{+} 21$, this already improves on the Hao-Orlin bound. Using some additional ideas, we further reduce to $O\left(\min \left\{n / m^{1 / 3}, \sqrt{n}\right\}\right)$ maxflow calls, which yields our eventual running time of $\tilde{O}\left(n m^{2 / 3}+n^{2}\right)$ :

Theorem 1.1. There is a randomized Monte Carlo algorithm that solves s-mincut (and therefore directed mincut) whp in $\tilde{O}\left(n m^{2 / 3}+n^{2}\right)$ time on an $n$-vertex, $m$-edge (edge-weighted) directed graph.

In fact, our reduction in general implies a running time bound of $\tilde{O}\left(\min \left\{m k, n k^{2}\right\}+\frac{n}{k} \cdot F(m, n)\right)$, where $k$ is a parameter that we can choose and $F(m, n)$ is the time complexity of maxflow (see Theorem 2.1).

Our techniques also yield fast approximations for the mincut problem in directed graphs. In particular, for any $\epsilon \in(0,1)$, we can find a $(1+\epsilon)$-approximate mincut in $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ time:

Theorem 1.2. For any $\epsilon \in(0,1)$, there is a randomized Monte Carlo algorithm that finds a $(1+\epsilon)$ approximate s-mincut (and therefore directed mincut) whp in $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ time on an n-vertex (edgeweighted) directed graph.

Finally, we consider vertex-weighted digraphs. A vertex cut in a digraph is defined as a tripartition of vertices into sets $(L, X, R)$ such that there is no edge from $L$ to $R$. (In other words, removing the vertices in $X$ results in a digraph where the directed cut ( $L, R$ ) is empty.) A minimum vertex cut (or vertex mincut) is a vertex cut $(L, X, R)$ that minimizes the sum of weights of vertices in $X$. We give an algorithm to find a $(1+\epsilon)$-approximate vertex mincut in $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ time:

[^1]Theorem 1.3. For any $\epsilon \in(0,1)$, there is a randomized Monte Carlo algorithm that finds a $(1+\epsilon)$ approximate minimum vertex s-cut and the minimum global vertex cut whp in $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ time on an $n$-vertex (vertex-weighted) directed graph.

To the best of our knowledge, before this work, the fastest algorithms for ( $1+\epsilon$ )-approximation of mincuts in edge or vertex weighted directed graphs were the respective exact algorithms themselves, which obtained a running time of $\tilde{O}(m n)$ [HO94, HRG00].Our approximation results establish a separation between the best exact and ( $1+\epsilon$ )-approximation algorithms for both edge and vertex mincut problems in directed graphs.
Remark: Our results are the first to break the $O(m n)$ barrier for directed mincut problems in general, weighted digraphs. For all values of $m$ except when $m=n^{1+o(1)}$, this is immediate from the above theorems. If $m=n^{1+o(1)}$, we can also break the $O(m n)$ barrier by employing the recent $\tilde{O}\left(m^{1.5-1 / 328}\right)$-time max-flow algorithm of [GLP21] to obtain $O\left(m n^{1-\Omega(1)}\right)$-time algorithms for all problems in Theorems 1.1 to 1.3.
Related Work. The directed (edge) mincut problem has been studied over several decades. Early work focused on unweighted graphs [ET75, Sch79] eventually resulting in an $O(m n)$-time algorithm due to Mansour and Schieber [MS89]. This was matched (up to log factors) in weighted graphs by Hao and Orlin [HO94], whose result we improve in this paper. For unweighted graphs (and graphs with small integer weights), the current record is a recent $\tilde{O}\left(n^{2}\right)$-time algorithm due to Chekuri and Quanrud [CQ21]. A similar story has unfolded for directed vertex mincuts. Early work again focused on unweighted graphs [Pod73, ET75, CR94, Gal80] until the work of Henzinger, Rao, and Gabow [HRG00] who obtained an $\tilde{O}(m n)$-time algorithm for weighted graphs. The current best for directed vertex mincut in unweighted graphs is an $\tilde{O}\left(m n^{11 / 12+o(1)}\right)$-time algorithm due to Li et al. [LNP $\left.{ }^{+} 21\right]$. Faster algorithms are known when the mincut size is small and for $(1+\epsilon)-$ approximations in unweighted digraphs [NSY19, FNY ${ }^{+}$20, CQ21].

### 1.2 Our Techniques

Our results are obtained by solving the $s$-mincut problem. Let us consider the edge-weighted case. Gabow [Gab95] obtained a running time of $\tilde{O}(m \lambda)$ for this problem (assuming integer weights), where $\lambda$ is the size of an $s$-mincut. He did so via arborescense packing: Define an $s$-arborescense to be any spanning tree rooted at $s$ with edges pointing toward the leaves. In $\tilde{O}(m \lambda)$ time, Gabow's algorithm computes $\lambda s$-arborescenses such that an edge $e$ of weight $w(e)$ is contained in at most $w(e)$ arborescenses (this is called arborescense packing). ${ }^{2}$ Gabow's algorithm is at least as fast as that of Hao and Orlin for unweighted simple graphs (since $\lambda \leq n-1$ ), but can be much worse for weighted (or multi) graphs. Nevertheless, Karger [Kar00] gave an interesting approach to use arborescence packing for the mincut problem even with edge weights, but only in undirected graphs. Karger's algorithm had three main steps:
(a) sparsify the input graph $G$ to $H$ by random sampling of edges to reduce the mincut value in $H$ to $O(\log n)$ while guaranteeing that the mincut in $G$ is a $(1+\epsilon)$-approximate mincut in $H$,
(b) pack $O(\log n) s$-arborescences ${ }^{3}$ in the sparsifier $H$, and
(c) find the minimum weight cut among those that have at most two edges in an arboresence using a dynamic program.

[^2]The last step is sufficient because the duality between cuts and arborescences ensures that an $s$-mincut, which is now a $(1+\epsilon)$-approximate mincut after sparsification, has at most two edges in at least one $s$-arborescence. ${ }^{4}$ Karger implements all these steps in $\tilde{O}(m)$ time to obtain an $\tilde{O}(m)$-time mincut algorithm in undirected graphs.

Unfortunately, steps (a) and (c) in Karger's scheme are not valid in a directed graph. To begin with, directed graphs do not admit sparsifiers similar to Karger's sparsifier: while Karger's sparsifier approximately preserves all cuts in an undirected graph (after some scaling), it is well known that a sparsifier with a similar property does not exist in directed graphs (see, e.g., [CCPS21]). This is mainly because we cannot bound the number of mincuts and approximate mincuts in directed graphs while we can do so in undirected graphs.

Partial sparsification. Since it is impossible for a sparsifier to preserve all cuts in directed graphs, it is natural to try to preserve a partial subset of cuts. Suppose we were guaranteed that the $s$-mincut $(S, V \backslash S)$ (recall that $s \in S$ ) is unbalanced in the sense that $|V \backslash S| \leq k$ for some parameter $k$ that we will fix later. Let us randomly sample edges to scale down the value of the mincut to $\tilde{O}(k)$. In an undirected graph, as long as $k=\Omega(\log n)$, all the cuts would converge to their expected values. This is not true in digraphs, but crucially, all the unbalanced cuts converge to their expected values since there are only $n^{\tilde{O}(k)}$ of them. However, it is possible that some balanced cut is (misleadingly) the new mincut of the sampled graph, having been scaled down disproportionately by the random sampling. So, we overlay this sampled graph with a star rooted at $s$, and show that this sufficiently increases the values of all balanced cuts, while only distorting the unbalanced cuts slightly. After this overlay, we can claim that we now have a digraph where $(S, V \backslash S)$ is a $(1+\epsilon)$-approximate mincut. We use one additional idea here. We show that in the sparsifier, every vertex in $V \backslash S$ has only $\tilde{O}(k)$ incoming edges (note that each edge can be weighted) - $\tilde{O}(k)$ edges across the cut from $S$ and $\leq k$ edges from within $V \backslash S$. By contracting all vertices with unweighted in-degree $>\tilde{O}(k)$ into $s$, we reduce the number of edges in the digraph to $\tilde{O}(n k)$.

But, what if our premise that the $s$-mincut is unbalanced does not hold? This case is actually simple. We use a uniform random sample of $\tilde{O}\left(\frac{n}{k}\right)$ vertices, and find $s-t$ mincuts for all vertices $t$ from the sample, using $\tilde{O}\left(\frac{n}{k}\right)$ maxflow calls. It is easy to see that whp, the sample hits $V \backslash S$, and hence, the minimum weighted cut among these $s-t$ mincuts will reveal the $s$-mincut of the graph.

Let us, therefore, return to the case where the $s$-mincut is unbalanced. Recall that we have already sparsified the graph to one that has only $\tilde{O}(n k)$ edges, and where the mincut has weight $\tilde{O}(k)$. The next step is to create a maximum packing of edge-disjoint $s$-arborescences. Because the graph is weighted, instead of using Gabow's algorithm described above, we create a (fractional) packing using a multiplicative weights update procedure (e.g., [You95]). By duality, ${ }^{5}$ these arborescences have the following property: if we sample $O(\log n)$ random $s$-arborescences, then whp there will be at least one arborescence $T$ such that there is exactly one edge in $T$ that goes from $S$ to $V \backslash S$. In this case, we say that the cut $(S, V \backslash S)$ 1-respects the arborescence $T$.

Our 1-respecting cut algorithm. Our final task, therefore, is the following: given an arborescence, find the minimum weight cut in the original graph among all those that 1-respect the arborescence $T$. At first sight, this may look similar to part (c) of Karger's algorithm which can be solved by a dynamic program or other techniques (e.g. [MN20, GMW20, GMW21, LMN21]). However, these techniques relied on the fact that if the $s$-mincut $(S, V \backslash S$ ) has a single edge $e$ in $T$, then $S$

[^3]and $V \backslash S$ would be contiguous in an $s$-arborescence $T$ (i.e. removing $e$ from $T$ gives $S$ and $V \backslash S$ as the two connected components). This is not true for a directed graph: While an $s$-arborescence will contain exactly one edge from $S$ to $V \backslash S$, it could contain an arbitrary number of edges in the opposite direction from $V \backslash S$ to $S$, thereby only guaranteeing the contiguity of $V \backslash S$ but not $S$.

One of the main contributions of this paper is to provide an algorithm to solve the above problem using $O(\log n)$ maxflow computations. The main idea is to use a centroid-based recursive decomposition of the arborescence, where in each step, we use a set of maxflow calls that can be amortized on the original graph. The minimum cut returned by all these maxflow calls is eventually returned as the $s$-mincut of the graph.

Approximation Algorithms. The ideas above also lead to a quadratic time approximation algorithm for edge mincuts. At a high level, if we execute each $(s, t)$-max flow in the sparsifier instead of in the input graph (both in the unbalanced and balanced settings, with care), then we obtain a $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ time $(1+\epsilon)$-approximation algorithm instead.

A similar approach can be taken for approximate vertex mincuts. Our partial sparsification technique reduces the graph to $\tilde{O}\left(n k / \epsilon^{2}\right)$ edges while maintaining the vertex $s$-mincut (with some additional adjustments for vertex mincuts). For large $k$, we similarly run ( $s, t$ )-max flow between $\tilde{O}(n / k)$ pairs of vertices on the sparsifier. For small $k$, we design a new local cut algorithm from $\tilde{O}(n / k)$ seeds each of which takes $\tilde{O}\left(k^{3} / \epsilon^{2}\right)$ time. This local algorithm is inspired by local algorithms for unweighted graphs [FNY ${ }^{+}$20], and speeds up the running time by a factor of $k$ by leveraging the structure of our sparsifiers (beyond sparsity). We finally obtain $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ running time by balancing the two cases and calling the max flow algorithm by [vdBLL $\left.{ }^{+} 21\right]$.

Summary. To summarize, our algorithms distinguish between balanced and unbalanced mincuts, solving the former using maxflows on randomly sampled terminals. For unbalanced mincuts, we follow a two-step template. In the first step, we employ partial sparsification to preserve the values of unbalanced cuts approximately, while suppressing balanced cuts using an overlay. In the second step, we design algorithms to find the (edge/vertex) mincut among unbalanced cuts. For edge mincut, the sparsifier allows one to quickly obtain an arborescence that 1-respects the directed mincut. From this arboresence, we obtain the exact minimum cut in $O(\log n)$ max flows via a centroid-based recursive decomposition. For vertex mincut, we develop new local flow algorithms to identify small unbalanced cuts in the sparsified graph.

## 2 Minimum Cut Algorithms in Edge-weighted Directed Graphs

Given a directed graph $G=(V, E)$ with non-negative edge weights $w$ and a fixed root vertex $s$, we consider the problem of finding an $s$-mincut. An $s$-arborescence is a directed spanning tree rooted at $s$ such that all edges are directed away from $s$. For simplicity, we assume that all edge weights $w$ are integers and are polynomially bounded. We denote $\bar{U}=V \backslash U$. Let $\partial^{+}(U)$ be the set of edges from $U$ to $\bar{U}, \partial^{-}(U)$ be the set of edges from $\bar{U}$ to $U$, and let $\delta^{+}(U)$ and $\delta^{-}(U)$ be the weight of the cut, i.e., $\delta^{+}(U)=\sum_{e \in \partial^{+}(U)} w(e), \delta^{-}(U)=\sum_{e \in \partial^{-}(U)} w(e)$. Our goal is to compute the minimum cut $\left(S^{*}, T^{*}\right)$ where $s \in S^{*}=\arg \min _{s \in U \subset V} \delta^{+}(U)$ and $T^{*}=\overline{S^{*}}$. Let $F(m, n)$ denote the time complexity of $s$ - $t$ maximum flow on a digraph with $n$ vertices and $m$ edges. The current record for this bound is $F(m, n)=\tilde{O}\left(m+n^{3 / 2}\right)\left[\operatorname{vdBLL}^{+} 21\right]$. We emphasize that our directed mincut algorithm uses maxflow subroutines in a black box manner and therefore, any maxflow algorithm suffices. Correspondingly, we express our running times in terms of $F(m, n)$.

Theorem 2.1. There is a Monte Carlo algorithm that finds a minimum s-cut whp in $\tilde{O}\left(\min \left\{m k, n k^{2}\right\}+\right.$ $\left.F(m, n) \frac{n}{k}\right)$ time, where $k$ is a parameter and $F(m, n)$ is the time complexity of $s$ - $t$ maximum flow.

This section is devoted to prove Theorem 2.1. If we set $k=m^{1 / 3}+n^{1 / 2}$ and use the $\tilde{O}(m+$ $n^{3 / 2}$ ) max-flow algorithm, the time complexity becomes $\tilde{O}\left(n m^{2 / 3}+n^{2}\right)$, which establishes Theorem 1.1. If we assume an hypothetical $\tilde{O}(m)$-time max-flow algorithm, then our result becomes $\tilde{O}\left(\min \left\{n m^{2 / 3}, m n^{1 / 2}\right\}\right)$ for $k=\min \left\{m^{1 / 3}, n^{1 / 2}\right\}$.

We obtain Theorem 2.1 via a new $s$-mincut algorithm. The algorithm considers the following two cases, computing a $s$-cut for each case and returning the minimum as its final output. The cases are split on $\left|T^{*}\right|$ by a threshold $k>0$.

1. The first case aims to compute the correct mincut in the event that $\left|T^{*}\right|>k$. In this case, if we randomly sample $t \in V$, then with probability at least $1 / k, t \in T^{*}$. Then $T^{*}$ can be obtained via the maxflow from $s$ to $t$. Repeating the sampling $O\left(\frac{n}{k} \log n\right)$ times, we obtain the minimum $s$-cut whp. The total running time for this case is $O\left(F(m, n) \frac{n}{k} \log n\right)$.
2. The second case is for the the event that $\left|T^{*}\right| \leq k$. Let $\lambda$ denote the value of the minimum rooted cut. By enumerating $O(\log n)$ powers of 2 , we can obtain an estimate $\tilde{\lambda}$ such that $\lambda \leq \tilde{\lambda} \leq 2 \lambda$. For each value of $\tilde{\lambda}$, we apply Lemma 2.5 to sparsify the graph in the following manner. First, Lemma 2.5 returns a set of vertices $V_{0} \subseteq V$ such that $s \in V_{0}$ and $T^{*} \subseteq V_{0}$ whp. In particular, one can safely contract any vertex $v \in V \backslash V_{0}$ into $s$ without affecting the minimum $s$-cut. We contract $G$ accordingly and, overloading notation, let $G$ denote the contracted graph with vertex set $V_{0}$ henceforth. Second, Lemma 2.5 returns a graph $G_{0}=\left(V_{0}, E_{0}\right)$ in which $T^{*}$ still induces an $(1+\epsilon)$-approximate $s$-mincut, but the weight of the cut is now reduced to $O(k \log (n))$. We note that $G_{0}$ is not necessarily a subgraph of $G$. We then invoke Lemma 2.6 from Section 2.2 to fractionally pack an approximately maximum amount of $O(k \log n) s$-arboresences in $G_{0}$ in $\tilde{O}\left(m+\min \left\{m k, n k^{2}\right\}\right)$ time. In a random sample of $O(\log n) s$-arboresences from this packing, one of them will 1-respect the $s$-mincut in $G$ (for appropriate $\tilde{\lambda}$ ) whp:

Definition 2.2. A directed $s$-cut $(S, V \backslash S) k$-respects an $s$-arborescence if there are at most $k$ edges in the arborescence from $S$ to $V \backslash S$.

Finally, for each of the $O(\log n) s$-arborescences, the algorithm computes the minimum $s$-cut that 1-respects each arborescence. This algorithm is described in Algorithm 1 and proved in Theorem 2.7 from Section 2.3. It runs in $O((F(m, n)+m) \cdot \log n)$ time for each of the $O(\log n)$ arborescences.

Combining both cases, the total running time becomes $\tilde{O}\left(\min \left\{m k, n k^{2}\right\}+F(m, n) \frac{n}{k}\right)$, which establishes Theorem 2.1.

Fast approximations. The exact algorithm described above can be modified to produce a randomized $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$-time approximation algorithm that computes a $(1+\epsilon)$-approximate minimum $s$-cut (hence also the global cut). With logarithmic overhead, we can obtain a parameter $k$ such that $k / 2 \leq\left|T^{*}\right| \leq k$, where $\partial^{-}\left(T^{*}\right)$ is the minimum $s$-cut. We then follow the same steps as in the exact algorithm, except whenever we compute the max-flow, we compute it in the sparsifier produced by Lemma 2.5 instead. Since the sparsifier has at most $\tilde{O}\left(n k / \epsilon^{2}\right)$ edges, we obtain a running time of the form $\tilde{O}\left(\min \left\{n k^{2} / \epsilon^{2}\right\}+F\left(n k / \epsilon^{2}, n\right)(n / k)\right)$. For $F(m, n)=\tilde{O}\left(m+n^{1.5}\right)$, this gives a running time of $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$.

Theorem 2.3. For $\epsilon \in(0,1)$, an $(1+\epsilon)$-approximate minimum s-cut (hence global minimum cut) can be computed in $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ time whp.

We remark that $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ can also be obtained by using a local connectivity algorithm similar to the approach for vertex mincuts in Section 3, instead of via an arboresence packing. See [Qua21] for details.

Organization of the rest of this section. The following subsections present each step of the algorithm described above. First, we establish the partial sparsification subroutine in Section 2.1. Next, in Section 2.2, we obtain an arborescence packing from which sampling yields an arborescence that is 1 -respected by the mincut. Finally, in Section 2.3, we describe the algorithm to retrieve the mincut among those that 1 -respect a given arborescence.

### 2.1 Partial Sparsification

This section aims to reduce mincut value to $\tilde{O}(k)$ and edge size to $\min \left\{m, O\left(n k \log (n) / \epsilon^{2}\right\}\right)$ while keeping $\partial^{+}\left(S^{*}\right)$ a $(1+\epsilon)$-approximate $s$-mincut for a constant $\epsilon>0$ that we will fix later. Our algorithm in this stage has three steps. First, we use random sampling to discretize and scale down the expected value of all cuts such that the expected value of the mincut $\delta^{+}\left(S^{*}\right)$ becomes $\tilde{O}(k)$. We also claim that $\partial^{+}\left(S^{*}\right)$ remains an approximate mincut among all unbalanced cuts by using standard concentration inequalities. However, since the number of balanced cuts far exceeds that of unbalanced cuts, it might be the case that some balanced cut has now become much smaller in weight than all the unbalanced cuts. This would violate the requirement that $\partial^{+}\left(S^{*}\right)$ should be an approximate mincut in this new graph. This is where we need our second step, where we overlay a star on the sampled graph to raise the values of all balanced s-cuts above the expected value of $\partial^{+}\left(S^{*}\right)$ while only increasing the value of $\partial^{+}\left(S^{*}\right)$ by a small factor. The third step leverages the fact that, after scaling, the minimum edge weight is 1 , and the minimum cut is $O\left(k \log (n) / \epsilon^{2}\right)$. It follows that any vertex with in-degree at least a constant factor greater than $k \log (n) / \epsilon^{2}$ cannot be in the sink component, and can be safely contracted into the root without affecting the $s$-mincut.

The first two steps described above are implemented in the next lemma, whose proof appears in Appendix A.
Lemma 2.4. Let $G=(V, E)$ be a directed graph with positive edge weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in(0,1)$, let $\lambda>0$, and let $k \in \mathbb{N}$ be given parameters. Suppose there is an s-mincut of the form $\partial^{-}\left(T^{*}\right)$, where $s \notin T^{*}, \lambda / 2 \leq \delta_{G}^{-}\left(T^{*}\right) \leq 2 \lambda$ and $\left|T^{*}\right| \leq k$. In randomized nearly linear time, one can compute a randomized directed and reweighted subgraph $G_{0}=\left(V, E_{0}\right)$, where $V_{0} \subseteq V$ and $s \in V_{0}$ with the following properties.
(i) $G_{0}$ has integral edge weights and the minimum s-cut has weight at most $O\left(k \log (n) / \epsilon^{2}\right)$.
(ii) $\partial_{G_{0}}^{-}\left(T^{*}\right)$ is a $(1+\epsilon)$-approximate minimum s-cut in $G_{0}$.
(iii) Every $\alpha$-approximate minimum s-cut in $G_{0}$ induces an $(1+\epsilon) \alpha$-approximate minimum s-cut in $G$.

The preceding lemma importantly allows us to reduce the weight of the minimum cut to roughly $k$, where $k$ is the number of the vertices in the sink component. However it has not actually reduced the size in the graph, in terms of the number of edges. This is accomplished by our third step, which we formalize in the following lemma that reduces the number of edges to $\tilde{O}(n k)$.

Lemma 2.5. Let $G=(V, E)$ be a directed graph with positive edge weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in(0,1), \lambda>0$, and $k \in \mathbb{N}$ be given parameters. Suppose there is a minimum $s$-cut is of the form $\partial^{-}\left(T^{*}\right)$, where $\lambda / 2 \leq \delta_{G}^{-}\left(T^{*}\right) \leq 2 \lambda$ and $\left|T^{*}\right| \leq k$.

In randomized nearly linear time, one can compute a randomized directed and edge-weighted graph $G_{0}=\left(V_{0}, E_{0}\right)$, where $V_{0} \subseteq V$ and $s \in V_{0}$.
(i) $G_{0}$ has integral edge weights and the s-mincut in $G_{0}$ has weight at most $O\left(k \log (n) / \epsilon^{2}\right)$.
(ii) $G_{0}$ has at most $\left|E_{0}\right|=\min \left\{m, O\left(n k \log (n) / \epsilon^{2}\right)\right\}$ edges.
(iii) $G_{0}$ is a subgraph of the graph obtained by contracting $V \backslash V_{0}$ into $s$ in $G$.
(iv) We have $T^{*} \subseteq V_{0}$, and $\partial_{G_{0}}^{-}\left(T^{*}\right)$ is an $(1+\epsilon)$-approximate minimum s-cut in $G_{0}$.
(v) Every $(1+\epsilon)$-approximate minimum s-cut in $G_{0}$ induces an $(1+\epsilon)^{2}$-approximate minimum s-cut in $G$.

Proof. Consider the reweighted subgraph produced by Lemma 2.4, which we denote by $G_{1}=\left(V, E_{1}\right)$. We claim that every vertex $v \in T^{*}$ has unweighted in-degree at most $O\left(k \log (n) / \epsilon^{2}\right)$. Indeed, at most $k-1$ of these edges are from other vertices in $T$, and the remaining edges must be in $\partial_{G_{1}}^{-}\left(T^{*}\right)$. But $\partial_{G_{1}}^{-}\left(T^{*}\right)$ has at most $\delta_{G_{1}}^{-}\left(T^{*}\right)=O\left(k \log (n) / \epsilon^{2}\right)$ edges by properties (i) and (ii) of Lemma 2.4.

Let $G_{0}=\left(V_{0}, E_{0}\right)$ be the graph obtained from $G_{1}$ by contracting all vertices with unweighted in-degree $\geq c k \log (n) / \epsilon^{2}$ for a sufficiently large constant $c$ that excludes all vertices in $T^{*}$. It is easy to see that $G_{0}$ satisfies the claimed properties, particularly as contractions into $s$ do not decrease the $s$-mincuts, and $\partial_{G_{0}}^{-}\left(T^{*}\right)$ is preserved exactly as in $G_{1}$.

### 2.2 Finding a 1-respecting Arborescence

In this section, we assume that there is an unbalanced $s$-mincut and show how to obtain an $s$ arborescence that 1 -respects the $s$-mincut. More formally, we prove the following:

Lemma 2.6. Given weighted digraph $G$ and a fixed root vertex $s$, suppose the sink side of an $s$ mincut $T^{*}$ has at most $k$ vertices. In $O\left(m \log n+\min \left\{m k \log ^{2} n, n k^{2} \log ^{3} n\right\}\right)$ time, we can find $O(\log n)$ s-arborescences on vertex set $V_{0} \supset T^{*}$, such that whp an s-mincut 1-respects at least one of them.

The idea of this lemma is as follows. First, we apply Lemma 2.5 to our graph $G$ and obtain the graph $G_{0}$. Whp, a minimum $s$-cut $\partial^{-}\left(T^{*}\right)$ in $G$ corresponds to a $(1+\epsilon)$-approximate minimum $s$-cut in $G_{0}$. It remains to find an arborescence in $G_{0}$ that 1-respects $\partial^{-}\left(T^{*}\right)$. To do this, we employ a multiplicative weight update (MWU) framework. The algorithm begins by setting all edge weights to be uniform (say, weight 1). Then, we repeat the following for $O\left(k \log (n) / \epsilon^{2}\right)$ rounds: in each round, we find a minimum weight arborescence in $\tilde{O}(m)$ time and multiplicatively increase the weight of every edge in the arborescence. Using the fact that there is no duality gap between arborescence packing and mincut [Edm73, Gab95] (see Lemma B.3), a standard MWU analysis implies that these arborescences that we found, after scaling, form a ( $1+\epsilon$ )-approximately optimal fractional arborescence packing. So our arborescence crosses $T^{*}$ at most $(1+O(\epsilon))<2$ times on average. Thus, if we sample $O(\log n)$ arborescences from this set, one of them will 1 -respect $T^{*}$ whp. ${ }^{6}$ To obtain the running time bound, we note that each iteration of the MWU framework requires us to find the minimum cost $s$-arborescence, for which an $\tilde{O}(m)$-time algorithm is known [GGST86].

Since the argument above is a standard application of the MWU framework, we give the detailed proof in Appendix B.

[^4]

Figure 1: Construction of auxiliary graph $G_{i}$ in Algorithm 1. Solid lines represent the arborescence $T$. Dashed lines are other edges in the graph. Rectangles are sets formed by the first level of centroid decomposition. Left: The original graph. Right: The part of $G_{1}$ solving the case that the mincut separates root and the centroid of the middle subtree.

### 2.3 Mincut Given 1-respecting Arborescence

We propose an algorithm (Algorithm 1) that uses $O(\log n)$ maxflow subroutines to find the minimum $s$-cut that 1-respects a given $s$-arborescence. The result is formally stated in Theorem 2.7.

Theorem 2.7. Consider a directed graph $G=(V, E)$ with polynomially bounded edge weights $w_{e}>$ 0 . Let $s \in V$ be a fixed root vertex and $S \ni s$ be the source side of a fixed $s$-mincut. Given an s-arborescence $T$ with $\left|T \cap \partial^{+}(S)\right|=1$, Algorithm 1 outputs a s-mincut of $G$ in time $O((F(m, n)+$ $m) \cdot \log n)$.

We first give some intuition for Algorithm 1. Because $s \in S$, if we could find a vertex $t \in \bar{S}$, then computing the $s$ - $t$ mincut using one maxflow call would yield a global mincut of $G$. However, we cannot afford to run one maxflow between $s$ and every other vertex in $G$. Instead, we carefully partition the vertices into $\ell=O(\log n)$ sets $\left(C_{i}\right)_{i=1}^{\ell}$. We show that for each $C_{i}$, we can modify the graph appropriately so that it allows us to (roughly speaking) compute the maximum flow between $s$ and every vertex $c \in C_{i}$ using one maxflow call.

More specifically, Algorithm 1 has two stages. In the first stage, we compute a centroid decomposition of $T$. Recall that a centroid of $T$ is a vertex whose removal disconnects $T$ into subtrees with at most $n / 2$ vertices. This process is done recursively, starting with the root $s$ of $T$. We let $P_{1}$ denote the subtrees resulting from the removal of $s$ from $T$. In each subsequent step $i$, we compute the set $C_{i}$ of the centroids of the subtrees in $P_{i}$. We then remove the centroids and add the resulting subtrees to $P_{i+1}$. This process continues until no vertices remain.

In the second stage, for each layer $i$, we construct a directed graph $G_{i}$ and perform one maxflow computation on $G_{i}$. The maxflow computation on $G_{i}$ would yield candidate cuts for every vertex in $C_{i}$, and after computing the appropriate maximum flow across every layer, we output the minimum candidate cut as the minimum cut of $G$. The details are presented in Algorithm 1.

We first state two technical lemmas that we will use to prove Theorem 2.7.
Lemma 2.8. Recall that $P_{i}$ is the set of subtrees in layer $i$ and $C_{i}$ contains the centroid of each subtree in $P_{i}$. If $C_{j} \subseteq S$ for every $0 \leq j<i$, then $\bar{S}$ is contained in exactly one subtree in $P_{i}$, and consequently, at most one vertex $u \in C_{i}$ can be in $\bar{S}$.
Lemma 2.9. Let $G_{i}$ be the graph constructed in Step 11 of Algorithm 1. Let $f_{i}^{*}$ be a maximum $s-t_{i}$ flow on $G_{i}$ as in Step 15. For any $U \in P_{i}$ with centroid $u$, the amount of flow $f_{i}^{*}$ puts on edge $\left(u, t_{i}\right)$ is equal to the value of the minimum cut from $\bar{U}$ to $u$.

```
Algorithm 1: Finding an \(s\)-mincut.
    Input : An arborescence \(T\) rooted at \(s \in S\) such that \(S\) 1-respects \(T\).
    // Stage I: Build centroid decomposition.
    Let \(C_{0}=\{s\}, P_{1}=\) the set of subtrees obtained by removing \(s\) from \(T\), and \(i=1\).
    while \(P_{i} \neq \varnothing\) do
        Initialize \(C_{i}\) (the centroids of \(P_{i}\) ) and \(P_{i+1}\) as empty sets.
        for each subtree \(U \in P_{i}\) do
            Compute the centroid \(u\) of \(U\) and add it to \(C_{i}\).
            Add all subtrees generated by removing \(u\) from \(U\) to \(P_{i+1}\).
        Set \(\ell=i\) and iterate \(i=i+1\).
    // Stage II: Calculate integrated maximum flow for each layer.
    for \(i=1\) to \(\ell\) do
        Construct a digraph \(G_{i}=\left(V \cup\left\{t_{i}\right\}, E_{1} \cup E_{2} \cup E_{3}\right)\) as follows (see Figure 1):
            1) Add edges \(E_{1}=E \cap \cup_{U \in P_{i}}(U \times U)\) with capacity equal to their original weight.
            2) Add edges \(E_{2}=\left\{(s, v):(u, v) \in E \backslash E_{1}\right\}\) with capacity of \((s, v)\) equal to the
        original weight of \((u, v)\).
            3) Add edges \(E_{3}=\left\{\left(u, t_{i}\right): u \in C_{i}\right\}\) with infinite capacity.
        Compute the maximum \(s-t_{i}\) flow \(f_{i}^{*}\) in \(G_{i}\).
        For each component \(U \in P_{i}\) with centroid \(u\), the value of \(f_{i}^{*}\) on edge \(\left(u, t_{i}\right)\) is a
        candidate cut value, and the nodes in \(U\) that can reach \(u\) in the residue graph is a
        candidate for \(\bar{S}\).
    17 Return the smallest candidate cut value and the corresponding \((S, \bar{S})\) as an \(s\)-mincut.
```

We defer the proofs of Lemmas 2.8 and 2.9, and first use them to prove Theorem 2.7.
Proof of Theorem 2.7. We first prove the correctness of Algorithm 1.
Because $C_{0}=\{s\}$ and $s \in S$, and the $C_{i}$ 's form a disjoint partition of $V$, there must be a layer $i$ such that for the first time, we have a centroid $u \in C_{i}$ that belongs to $\bar{S}$. By Lemma 2.8, we know that $\bar{S}$ must be contained in exactly one subtree $U \in P_{i}$, and hence $u$ must be the centroid of $U$. In summary, we have $u \in \bar{S}$ and $\bar{S} \subseteq U$.

Consider the graph $G_{i}$ constructed for layer $i$. By Lemma 2.9, based on the flow $f_{i}^{*}$ puts on the edge ( $u, t_{i}$ ), we can recover the value of the minimum cut from $\bar{U}$ to $u$. Because $\bar{S} \subseteq U$ (or equivalently $\bar{U} \subseteq S$ ) and $u \in \bar{S}$, the cut $(S, \bar{S})$ is one possible cut that separates $\bar{U}$ and $u$. Therefore, the flow that $f_{i}^{*}$ puts on the edge $\left(u, t_{i}\right)$ is equal to the $s$-mincut value in $G$.

In addition, the candidate cut value for any other centroid $u^{\prime}$ of a subtree $U^{\prime} \in P_{i}$ must be at least the mincut value between $s$ and $u^{\prime}$. This is because the additional restriction that the cut has to separate $\overline{U^{\prime}}$ from $u^{\prime}$ can only make the mincut value larger, and the value of this cut in $G_{i}$ is equal to the value of the same cut in $G$. Therefore, the minimum candidate cut value in all $\ell$ layers must be equal to the $s$-mincut value of $G$.

Now we analyze the running time of Algorithm 1. We can find the centroid of an $n$-node tree in time $O(n)$ (see e.g., [MTZC81]). The total number of layers $\ell=O(\log n)$ because removing the centroids reduces the size of the subtrees by at least a factor of 2 . Thus, the running time of Stage I of Algorithm 1 is $O(n \log n)$. In Stage II, we can construct each $G_{i}$ in $O(m)$ time and every $G_{i}$ has $O(m)$ edges. Since there are $O(\log n)$ layers and the maximum flow computations take a total of $O(M F(m, n) \cdot \log n)$ time, the overall runtime is $O(n \log n+(M F(m, n)+m) \log n)=$
$O((M F(m, n)+m) \log n)$.
Before proving Lemmas 2.8 and 2.9 we first prove the following lemma.
Lemma 2.10. If $x$ and $y$ are vertices in $\bar{S}$, then every vertex on the (undirected) path from $x$ to $y$ in the arborescence $T$ also belongs to $\bar{S}$.

Proof. Consider the lowest common ancestor $z$ of $x$ and $y$. Because there is a directed path from $z$ to $x$ and a directed path from $z$ to $y$, we must have $z \in \bar{S}$. Otherwise, there are at least two edges in $T$ that go from $S$ to $\bar{S}$.

Because $s \in S$ and $z \in \bar{S}$, there is already an edge in $T$ (on the path from $s$ to $z$ ) that goes from $S$ to $\bar{S}$. Consequently, all other edges in $T$ cannot go from $S$ to $\bar{S}$, which means the entire path from $z$ to $x$ (and similarly $z$ to $y$ ) must be in $\bar{S}$.

Recall that Lemma 2.8 states that if all the centroids in previous layers are in $S$, then $\bar{S}$ is contained in exactly one subtree $U$ in the current layer $i$.

Proof of Lemma 2.8. For contradiction, suppose that there exist distinct subtrees $U_{1}$ and $U_{2}$ in $P_{i}$ and vertices $x, y \in \bar{S}$ such that $x \in U_{1}$ and $y \in U_{2}$.

By Lemma 2.10, any vertex on the (undirected) path from $x$ to $y$ also belongs to $\bar{S}$. Consider the first time that $x$ and $y$ are separated into different subtrees. This must have happened because some vertex on the path from $x$ to $y$ is removed. However, the set of vertices removed at this point of the algorithm is precisely $\bigcup_{0 \leq j<i} C_{j}$, but our hypothesis assumes that none of them are in $\bar{S}$. This leads to a contradiction and therefore $\bar{S}$ is contained in exactly one subtree of $P_{i}$.

It follows immediately that at most one centroid $u \in C_{i}$ can be in $\bar{S}$.
Next we prove Lemma 2.9, which states that the maximum flow between $s$ and $t_{i}$ in the modified graph $G_{i}$ allows one to simultaneously compute a candidate mincut value for each vertex $u \in C_{i}$.

Proof of Lemma 2.9. First observe that the maxflow computation from $s$ to $t_{i}$ in $G_{i}$ can be viewed as multiple independent maxflow computations. The reason is that, for any two subtrees $U_{1}, U_{2} \in P_{i}$, there are only edges that go from $s$ into $U_{1}$ and from $U_{1}$ to $t_{i}$ in $G_{i}$ (similarly for $U_{2}$ ), but there are no edges that go between $U_{1}$ and $U_{2}$.

The above observation allows us to focus on one subtree $U \in P_{i}$. Consider the procedure that we produce $G_{i}$ from $G$ in Steps 12 to 14 of Algorithm 1. The edges with both ends in $U$ are intact (the edge set $E_{1}$ ). If we contract all vertices out of $U$ into $s$, then all edges that enter $U$ would start from $s$, which is precisely the effect of removing cross-subtree edges and adding the edges in $E_{2}$. One final infinity-capacity edge $\left(u, t_{i}\right) \in E_{3}$ connects the centroid of $U$ to the super $\operatorname{sink} t_{i}$.

Therefore, the maximum $s$ - $t_{i}$ flow $f_{i}^{*}$ computes the maximum flow between $\bar{U}$ and $u \in U$ simultaneously for all $U \in P_{i}$, whose value is reflected on the edge $\left(u, t_{i}\right)$. It follows from the maxflow mincut theorem that the flow on edge ( $u, t_{i}$ ) is equal to the mincut value between $\bar{U}$ and $u$ in $G$ (i.e., the minimum value $w(A, \bar{A})$ among all $A \subset V$ with $\bar{U} \subseteq A$ and $u \in \bar{A})$.

## 3 Minimum Cut Algorithms in Vertex-weighted Directed Graphs

In this section we present the approximation algorithm for the minimum rooted and global vertex cut. Similar to Section 2, the main focus is on rooted cuts, and the algorithm is presented in three main parts. All three parts are parameterized by values $\kappa>0$ and $k \in \mathbb{N}$ that, in principle, are meant to be constant factor estimates for the weight and the number of vertices in the sink component of the minimum rooted vertex cut. The first part, in Section 3.1, presents the sparsification lemma
that reduces the number of edges to roughly $n k$ and the rooted mincut to roughly $k$ in a graph with integer weights. This sparsifier is used in the remaining two parts. The second part, in Section 3.2, gives a roughly $n k^{2}$ time approximation algorithm for the minimum rooted cut via a new local flow algorithm. The third part, in Section 3.3, gives a roughly $n^{2}+n^{2.5} / k$ time approximation algorithm via sampling and ( $s, t$ )-flow (as with minimum edge cuts before). Finally, in Section 3.4, we balance terms to obtain the claimed running time for rooted cut. The rooted vertex mincut algorithm then leads to a global vertex mincut algorithm via an argument due to [HRG00] (with some modifications).

### 3.1 Partial Sparsification

The first part is a sparsification lemma that preserves rooted vertex cuts where the number of vertices in the sink component is below some given parameter. It is similar in spirit to Lemma 2.5, but with some necessary changes as we are now preserving the vertex mincut rather than edge mincut. We give a brief overview of the algorithm, highlighting in particular the differences from the partial edge cut sparsifier. The proof and algorithmic details are deferred to Appendix C.

At a high level, the following sparsifier for vertex cuts randomly samples the vertex weights so that the weights are integral, and the weight of the minimum vertex cut becomes $O\left(k \log (n) / \epsilon^{2}\right)$. Similar to the partial edge sparsifier, this rounding is calibrated to preserve $s$-cuts with (roughly) $k$ or fewer vertices in the sink components. To pad the weight of vertex $s$-cuts with large sink components, we add an weighted auxiliary vertex on a short directed path between $s$ and each vertex (as opposed to just adding an edge from $s$, as we did for edge cuts). If a sampled weight of a vertex $v$ is 0 , we cannot simply drop the vertex from the graph (in the way we can drop weight 0 edges) since the vertex may be in the sink component of the min $r$-cut. Instead we remove all outgoing edges from $v$. Also, when we detect that a vertex $v$ cannot be in the sink component (by a similar counting argument as before), rather than contract $v$ into $s$ (which may effect the min vertex $s$-cut), we replace all of the incoming edges to $v$ with a single edge from $s$. The culmination of these modifications is a similar net effect as for edge cuts: a graph with $O\left(n k \log (n) / \epsilon^{2}\right)$ that preserves the sink component of the minimum vertex $s$-cut. That said, the following bounds are more detailed than the bounds for preserving the edge cut in Lemma 2.5. These additional properties play a critical role in the customized local flow algorithms presented later.

In the following, let $N^{+}(v \mid G)$ denote the set of out-neighbors of $v$ in the graph $G$. We omit $G$ and simply write $N^{+}(v)$ when $G$ can be inferred from the context.

Lemma 3.1. Let $G=(V, E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed vertex. Let $k, \kappa>0$ be given parameters. Let $V^{\prime}=V \backslash\left(\{s\} \cup N^{+}(s)\right)$. In randomized linear time, one can compute a randomized directed and vertex-weighted graph $G_{0}=\left(V_{0}, E_{0}\right)$, and a scaling factor $\tau>0$, with the following properties.
(i) $s \in V_{0}$.
(ii) Let $V_{0}^{\prime}=V_{0} \backslash\left(\{s\} \cup N^{+}\left(s \mid G_{0}\right)\right)$. We have $V_{0}^{\prime}=V^{\prime}$.
(iii) $G_{0}$ has integer vertex weights between 0 and $O\left(k \log (n) / \epsilon^{2}\right)$.
(iv) Every vertex $v \in V_{0}$ has at most $O\left(k \log (n) / \epsilon^{2}\right)$ incoming edges.
(v) Every vertex $v$ with weight 0 has no outgoing edges.
(vi) With high probability, for all $S \subseteq V^{\prime}$, the weight of the vertex in-cut induced by $S$ in $G_{0}$ (up to scaling by $\tau$ ) is at least the minimum of the $(1-\epsilon)$ times the weight of the induced vertex
in-cut in $G$ or $c \kappa$ (for any desired constant $c>1$ ), and at most $(1+\epsilon)$ times its weight in $G$ plus $\epsilon \kappa|S| / k$.
(vii) With high probability, for all $S \subseteq V^{\prime}$ such that $|S| \leq k$ and the weight of the induced vertex in-cut is $\leq O(\kappa)$, we have $S \subseteq V_{0}^{\prime}$. (That is, $S$ is still the sink component of an $s$-cut in $G_{0}$.)

In particular, if the minimum vertex s-cut has weight $\Theta(\kappa)$, and the sink component of a minimum vertex s-cut has at most $k$ vertices, then with high probability $G_{0}$ preserves the minimum vertex s-cut up to a $(1+O(\epsilon))$-multiplicative factor.

As stated above, the proof is deferred to Appendix C.

### 3.2 Rooted vertex mincut for small sink components

This section presents an approximation algorithm for rooted vertex mincut for the particular setting where the sink component is small. In particular, we are given an upper bound $k$ on the number of vertices in the sink component, and want to obtain running times of the form $n$ poly $(k)$. When a similar situation arose previously for small integer capacities in [CQ21], [CQ21] modified a local algorithm from [FNY ${ }^{+}$20] which works well for unweighted graphs. Here, while Lemma 3.1 produces relatively sparse graphs with integral vertex capacities, the vertex capacities imply imply that the algorithm from [CQ21, FNY $\left.{ }^{+} 20\right]$ would take roughly $n k^{3} / \epsilon^{5}$ time. This section develops an alternative algorithm that is inspired by these local algorithms for (global and rooted) vertex cuts, but reduces the dependency on $k$ to $k^{2}$. Compared to $\left[\mathrm{FNY}^{+} 20, \mathrm{CQ} 21\right]$, the algorithm here is designed to take full advantage of the properties of the graph produced by Lemma 3.1. These modifications have some tangible benefits. First, it improves the dependency on $k$ and $\epsilon$. Second, the local subroutine here is deterministic whereas before they were randomized. Third and last, as suggested by the better running time and the determinism, the version presented here is arguably simpler and more direct than the previous algorithms (for this setting).

Lemma 3.2. Let $G=(V, E)$ be a directed graph with positive vertex weights. Let $r \in V$ be a fixed root vertex. Let $\epsilon \in(0,1), \kappa>0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized linear time Monte Carlo algorithm that, with high probability, produces a deterministic data structure that supports the following query.

For $t \in V^{\prime} \stackrel{\text { def }}{=} V \backslash\left(\{s\} \cup N^{+}(s)\right)$, let $\kappa_{t, k}$ denote the weight of the minimum $(s, t)$-vertex cut such that the sink component has at most $k$ vertices. Given $t \in V^{\prime}$, deterministically in $O\left(k^{3} \log ^{2}(n) / \epsilon^{4}\right)$ time, the data structure either (a) returns the sink component of a minimum ( $s, t$ )-vertex cut of weight at most $(1+\epsilon) \kappa_{t, k}$, or (b) declares that $\kappa_{t, k}>\kappa$.

Proof. Given $s, \kappa$, $k$, and $\epsilon$, let $\epsilon^{\prime}=c \epsilon$ for a sufficiently small constant $c>0$. We first apply Lemma 3.1 to $G$ with root $s$ and parameters $\kappa, k$, and $\epsilon^{\prime}$. This produces a vertex capacitated graph $G_{0}=\left(V_{0}, E_{0}\right)$ with $V \subset V_{0}$. We highlight the features that we leverage. All new vertices (in $\left.V_{0} \backslash V\right)$ are in $N^{+}\left(s \mid G_{0}\right)$; that is, $V^{\prime}$ equals $V_{0}^{\prime} \stackrel{\text { def }}{=} V_{0} \backslash\left(\{s\} \cup N^{+}\left(s \mid G_{0}\right)\right)$. Put alternatively, none of the new vertices is in the sink component of any $s$-cut. The vertex weights are integers between 0 and $O\left(k \log (n) / \epsilon^{2}\right)$. Every vertex has unweighted in-degree at most $O\left(k \log (n) / \epsilon^{2}\right)$. Every vertex with weight 0 has no outgoing edges.

With high probability, we have the following guarantees on the vertex $s$-cuts of $G_{0}$. The vertex weights in $G_{0}$ are scaled so that a weight of $\kappa$ in $G$ corresponds to weight $O\left(k \log (n) / \epsilon^{2}\right)$ in $G_{0}$. Modulo scaling, every vertex $s$-cut in $G_{0}$ has weight no less than the minimum of its weight in $G$ and $2 \kappa$. Additionally, modulo scaling, for every vertex $s$-cut in $G$ with capacity at most $\kappa$ and at most $k$ vertices in the sink component, the corresponding vertex cut in $G_{0}$ has weight at most a
$c_{0} \epsilon \kappa$ additive factor larger than in $G$, for any desired constant $c_{0}>0$. We consider the algorithm to fail if the cuts are not preserved in the sense described above.

Given $t \in V$, the data structure will search for a small $(s, t)$-cut in $G_{0}$ via a customized, edgecapacitated flow algorithm. This algorithm may or may not return the sink component of $(s, t)$-cut. If the search does return a sink component, and the corresponding vertex in-cut in $G_{0}$ has weight that, upon rescaling back to the scale of the input graph $G$, is at most $(1+\epsilon / 2) \kappa$, the data structure returns it. Otherwise the data structure indicates that $\kappa_{t, k}>\kappa$.

Proceeding with the flow algorithm, let $G_{\text {rev }}$ be the reverse of $G_{0}$, and let $G_{\text {split }}$ be the standard "split-graph" of $G_{\text {rev }}$ modeling vertex capacities with edge capacities. We recall that the split graph splits each vertex $v$ into an auxiliary "in-vertex" $v^{-}$and an auxiliary "out-vertex" $v^{+}$. For each $v$ there is a new edge $\left(v^{-}, v^{+}\right)$with capacity equal to the vertex capacity of $v$. Each edge $(u, v)$ is replaced with an edge $\left(u^{+}, v^{-}\right)$with capacity ${ }^{7}$ equal to the vertex capacity of $u$. Every $(s, t)$-vertex cut in $G_{0}$ maps to a ( $t^{+}, s^{-}$)-edge cut in $G_{\text {rev }}$ with the same capacity. Any $\left(t^{+}, s^{-}\right.$)-edge capacitated cut maps to a $(s, t)$-vertex cut in $G_{0}$ (with negligible overhead in the running time). Now, recall that for each $v \in V^{\prime}$, the sparsification procedures introduces an auxiliary path $\left(s, a_{v}, s^{-}\right)$where $a_{v}$ is was given weight $\Theta(\epsilon \kappa / k)$. It is convenient to replace the corresponding auxiliary path $\left(v^{+}, a_{v}^{-}, a_{v}^{+}, s^{-}\right)$ in $G_{\text {rev }}$ with a single edge $\left(v^{+}, s^{-}\right)$with capacity equal to the weight of $a_{v}$. This does not effective the weight of the minimum $\left(t^{+}, s^{-}\right)$-edge cut for any $t \in V^{\prime}$. This adjustment can be easily made within the allotted preprocessing time.

In this graph, given $t \in V^{\prime}$, we run a specialization of the Ford-Fulkerson algorithm [FF56] that either computes a minimum $\left(t^{+}, s^{-}\right)$-cut or concludes that the minimum $\left(t^{+}, s^{-}\right)$-cut is at least $O\left(k \log (n) / \epsilon^{2}\right)$ (which corresponds to $O(\kappa)$ in $\left.G\right)$ after $O\left(k \log (n) / \epsilon^{2}\right)$ iterations. To briefly review, each iteration in the Ford-Fulkerson algorithm searches for a path from $t$ to $s$ in the residual graph. If such a path is found, then it routes one unit of flow along this path, and updates the residual graph by reversing (one unit capacity) of each edge along the path. After $\ell$ successful iterations we have a flow of size $\ell$. If, after $\ell$ iterations, there is no path in the residual graph from $t^{+}$to $s^{-}$, then the set of vertices reachable from $t$ gives a minimum $\left(t^{+}, s^{-}\right)$-cut of size $\ell$. Observe that updating the residual graph along a $\left(t^{+}, s^{-}\right)$-path preserves the weighted in-degree and out-degree of every vertex except $t^{+}$and $s^{-}$. The weighted out-degree of $t^{+}$decreases by 1 and the weighted in-degree of $s^{-}$changes by 1 . Moreover, updating the residual graph along a path increases the unweighted out-degree of any vertex by at most one, since a path contains at most one edge going into any single vertex. Since every vertex initially has unweighted out-degree at most $O\left(k \log (n) / \epsilon^{2}\right)$ in $G_{\text {rev }}$ (reversing the upper bound on the unweighted in-degrees in $G_{0}$ ), and the flow algorithm updates the residual graph along at most $O\left(k \log (n) / \epsilon^{2}\right)$ paths before terminating, the maximum unweighted out-degree over all vertices never exceeds $O\left(k \log (n) / \epsilon^{2}\right)$.

Within the Ford-Fulkerson framework, we give a refined analysis that takes advantages of the auxiliary $\left(v^{+}, s^{-}\right)$edges. Call an out-vertex $v^{+}$saturated if the auxiliary edge $\left(v^{+}, s^{-}\right)$is saturated; that is, if $\left(v^{+}, s^{-}\right)$is not in the residual graph. Call an in-vertex $v^{-}$saturated if the edge $\left(v^{-}, v^{+}\right)$is saturated and $v^{+}$is not saturated. (A vertex $v^{+}$or $v^{-}$is called unsaturated if it is not saturated.) We modify the search for an augmenting path to effectively end when we first visit an unsaturated vertex $v^{+}$or an unsaturated $v^{-}$. If we visit an unsaturated $v^{-}$, then we automatically complete a path to $s^{-}$via $v^{+}$. If we find an unsaturated $v^{+}$, then we automatically complete a path to $s^{-}$via the edge $\left(v^{+}, s^{-}\right)$. It remains to bound the running time of this search. We first bound the number of saturated $v^{+}$s.

Claim 1. There are at most $O(k / \epsilon)$ saturated $v^{+}$'s.

[^5]Indeed, each saturated $v^{+}$implies $O(\log (n) / \epsilon)$ units of flow along $\left(v^{+}, s^{-}\right)$, and the flow is bounded above $O\left(k \log (n) / \epsilon^{2}\right)$.

Note that Claim 1 also implies there are at most $O(k / \epsilon) v^{-}$s such that $v^{+}$is saturated. The next claim bounds the total out-degree of saturated $v^{-}$'s.

Claim 2. The sum of out-degrees of saturated $v^{-}$'s is at most the amount of flow that has been routed to $s^{-}$.

Indeed, the out-degree of a $v^{-}$in the residual graph is bounded above by the amount of flow through $\left(v^{-}, v^{+}\right)$, since initially $\left(v^{-}, v^{+}\right)$is the only outgoing edge from $v^{-}$. Recall that if $v^{-}$is saturated, then by definition $v^{+}$is unsaturated. As long as $v^{+}$is unsaturated, each unit of flow through $\left(v^{-}, v^{+}\right)$goes directly to $s^{-}$via the edge $\left(v^{+}, s^{-}\right)$, and can be charged to the total flow.

We now apply the above two claims to bound the total running time for each search, as follows.
Claim 3. Every (modified) search for an augmenting path traverses at most $O\left(k^{2} \log (n) / \epsilon^{2}\right)$ edges.
We first observe that every vertex visited in the search, except the unsaturated vertex terminating the search, is either (a) a saturated $v^{-}$, (b) a saturated $v^{+}$, or (c) an unsaturated $v^{-}$such that $v^{+}$ is saturated. We will upper bound the number of edges traversed in each iteration based on the type of vertex at the initial point of that edge. First, the amount of time spent exploring edges leaving (a) a saturated $v^{-}$is, by Claim 2, at most the size of the flow at that point, which is at most $O\left(k \log (n) / \epsilon^{2}\right)$. Second, consider the time spent traversing edges leaving either (b) a saturated $v^{+}$ or (c) an unsaturated $v^{-}$such that $v^{+}$is saturated. By Claim 1, there are at most $O(k / \epsilon)$ such vertices, and each has out-degree at most $O\left(k \log (n) / \epsilon^{2}\right)$. Thus we spend $O\left(k^{2} \log (n) / \epsilon^{2}\right)$ time traversing such edges. All together, we obtain an upper bound of $O\left(k^{2} \log (n) / \epsilon^{2}\right)$ total edges per search.

Claim 3 also bounds the running time for each iteration. The algorithm runs for at most $O\left(k \log (n) / \epsilon^{2}\right)$ iterations before either finding an $\left(t^{+}, s^{-}\right)$-cut or concluding that the weight of the minimum $\left(t^{+}, s^{-}\right)$-cut, rescaled to the input scale of $G$, is at least a constant factor greater than $\kappa$. The total running time follows.

We now present the overall algorithm for finding vertex $s$-cuts with small sink components. The algorithm combines Lemma 3.2 with randomly sampling for a vertex $t$ in the sink component of an approximately minimum $s$-cut. In the following, we let $\operatorname{deg}^{+}(s)$ denote the unweighted out-degree pf $s$ in $G$.

Lemma 3.3. Let $G=(V, E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in(0,1), \kappa>0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized algorithm that runs in $O\left(m+\left(n-\operatorname{deg}^{+}(s)\right) k^{2} \log ^{3}(n) / \epsilon^{4}\right)$ time and has the following guarantee. If there is a vertex s-cut of capacity at most $\kappa$ and where the sink component has at most $k$ vertices, then with high probability, the algorithm returns a vertex $(s, t)$-cut of capacity at most $(1+\epsilon) \kappa$.

Proof. Let $T^{*}$ be the sink component of the minimum vertex $s$-cut subject to $\left|T^{*}\right| \leq k$. Assume the capacity of the vertex in-cut of $T^{*}$ is at most $\kappa$ (since otherwise the algorithm makes no guarantees). Let $V^{\prime}=V \backslash\left(\{s\} \cup N^{+}(s)\right)$ and note that $\left|V^{\prime}\right|=n-1-\operatorname{deg}^{+}(s)$.

Suppose we had a factor-2 overestimate $\ell \in\left[\left|T^{*}\right|, 2\left|T^{*}\right|\right]$ of the number of vertices in $T^{*}$. We apply Lemma 3.2 with upper bounds $\kappa$ on the size of the cut and $\ell$ on the number of vertices in the sink component, which returns a data structure that, with high probability, is correct for all queries. Let us assume the data structure is correct (and otherwise the algorithm fails). We randomly sample $O\left(\left(n-\operatorname{deg}^{+}(s)\right) \log (n) / \ell\right)$ vertices from $V^{\prime}$. For each sampled vertex $t$, we query
the data structure from Lemma 3.2. Observe that if $t \in T^{*}$, then the query for $t$ returns an $s$-cut with capacity at most $(1+\epsilon) \kappa$. With high probability we sample at least one vertex from $T^{*}$, which produces the desired $s$-cut. By Lemma 3.2, the total running time to serve all queries is $O\left(m+\left(n-\operatorname{deg}^{+}(s)\right) \ell^{2} \log ^{3}(n) / \epsilon^{4}\right)$.

A factor- 2 overestimate $\ell$ can be obtained by enumerating all powers of 2 between 1 and $2 k$. One of these choices of $\ell$ will be accurate and produce the minimum $s$-cut with high probability. Note that the sum of $O\left(\left(n-\operatorname{deg}^{+}(s)\right) \ell^{2} \log ^{3}(n) / \epsilon^{4}\right)$ over this range of $\ell$ is dominated by the maximum $\ell$. The claimed running time follows.

### 3.3 Rooted vertex mincut for large sink components

The third and final part (before the overall algorithm) is an approximation for the rooted vertex cut that is well-suited for large sink components.

Lemma 3.4. Let $G=(V, E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in(0,1), \kappa>0$, and $k \in \mathbb{N}$ be given parameters. There is a randomized algorithm that runs in $\tilde{O}\left(m+\left(n-\operatorname{deg}^{+}(s)\right)\left(n / \epsilon^{2}+n^{1.5} / k\right)\right)$ time and has the following guarantee. If there is a vertex s-cut of capacity at most $\kappa$ and where the sink component has at most $k$ vertices, then with high probability, the algorithm returns a vertex $(s, t)$-cut of capacity at most $(1+\epsilon) \kappa$.

Proof. Let $T^{*}$ be the sink component of the minimum $s$-cut subject to $\left|T^{*}\right| \leq k$. We assume the capacity of the $s$-cut induced by $T^{*}$ is at most $\kappa$. (Otherwise the output is not well-defined.) Let $V^{\prime}=V \backslash\left(\{s\} \cup N^{+}(s)\right)$ and note that $\left|V^{\prime}\right|<n-\operatorname{deg}^{+}(s)$.

We apply Lemma 3.1 to produce the graph $G_{0}$. Lemma 3.1 succeeds with high probability and for the rest of the proof we assume it was successful. (Otherwise the algorithm fails.) We sample $O\left(\left(n-\operatorname{deg}^{+}(s)\right) \log (n) / k\right)$ vertices $t \in V^{\prime}$. For each sampled $t$, we compute the minimum $(s, t)-$ vertex cut in $G_{0}$. With high probability, some $t$ will be drawn from the sink component of the true minimum $s$-cut, in which case the minimum $(s, t)$-cut in $G_{0}$ gives an $(1+\epsilon)$-approximate $s$-cut in $G$ (by Lemma 3.1). We use the $\tilde{O}\left(m+n^{1.5}\right)$ time vertex-capacitated flow algorithm [vdBLN $\left.{ }^{+} 20\right]$. By Lemma 3.1, we have $m=O\left(n k \log (n) / \epsilon^{2}\right)$. This gives the total running time.

### 3.4 Approximating the rooted and global vertex mincut

We now combine the two parameterized approximation algorithms for rooted vertex mincut to give the following overall algorithm for rooted vertex mincut. This establishes one part of 1.3 concerning approximate rooted vertex cuts.

Theorem 3.5. Let $\epsilon \in(0,1)$, let $G=(V, E)$ be a directed graph with polynomially bounded vertex weights, and let $s \in V$ be a fixed root. A $(1+\epsilon)$-approximate minimum vertex s-cut can be computed with high probability in $\tilde{O}\left(m+n\left(n-\operatorname{deg}^{+}(s)\right) / \epsilon^{2}\right)$ randomized time.

Proof. The high-level approach is similar to Theorem 1.2 for edge mincut - we are balancing two algorithms for rooted vertex mincut, where one is better suited for small sink components, the second is better suited for large sink components. Both leverage the randomized sparsification lemma. As before, with polylogarithmic overhead, we can assume access to values $\kappa$ and $k$ that are within a factor 2 of the weight of the minimum $s$-cut and the number of vertices in the sink component of the minimum $s$-cut, respectively. For a fixed choice of $k$ and $\kappa$, we run the faster of two randomized algorithms, both of which would succeed with high probability when $k$ and $\kappa$ are (approximately) correct. The first option, given by Lemma 3.3, runs in $\tilde{O}\left(\left(n-\operatorname{deg}^{+}(s)\right) k^{2} / \epsilon^{3}\right)$. The second option, given by Lemma 3.4, runs in $\tilde{O}\left(n\left(n-\operatorname{deg}^{+}(s)\right) / \epsilon^{2}+n^{1.5}\left(n-\operatorname{deg}^{+}(s)\right) / k\right)$. The overall running
time is obtained by choosing $k$ to balance the running times. For $k=\epsilon \sqrt{n}$, we obtain the claimed running time.

Next we use the algorithm for rooted vertex mincut to obtain an algorithm for global vertex mincut and establish Corollary 3.6. [HRG00] showed that running times of the form $\left(n-\mathrm{deg}^{+}(s)\right) T$ for rooted mincut from a root $s$ imply a randomized $n T$ expected time algorithm for global vertex mincut. Theorem 3.5 gives a $\tilde{O}\left(m+n\left(n-\operatorname{deg}^{+}(s)\right) / \epsilon^{2}\right)$ running time, so some modifications have to be made to address the additional $\tilde{O}(m)$ additive factor. This establishes the remaining part of 1.3.

Corollary 3.6. For all $\epsilon \in(0,1)$, a $(1+\epsilon)$-approximate minimum weight global vertex cut in a directed graph with polynomially bounded vertex weights can be computed with high probability in $\tilde{O}\left(n^{2} / \epsilon^{2}\right)$ expected time.

Proof. Let $T=\tilde{O}\left(n / \epsilon^{2}\right)$. Let $w: V \rightarrow \mathbb{R}_{>0}$ denote the vertex weights, and let $W=\sum_{v \in V} w(v)$ be the total weight of the graph. Let $\kappa$ denote the weight of the minimum global vertex cut. The algorithm samples $L=O(W \log (n) /(W-\kappa))$ vertices $s$ in proportion to their weight, and - morally, but not actually - computes the minimum $s$-vertex cut for each sampled vertex $s$ via Theorem 3.5. It returns the smallest cut found.

For the sake of running time, we adjust the algorithm from Theorem 3.5. Recall that for a fixed root $s$, and for each of a logarithmic number of values for $k$ and $\kappa$, the algorithm from Theorem 3.5 applies Lemma 3.1 which reduces the graph to having $\tilde{O}\left(n k / \epsilon^{2}\right)$ edges and rooted mincut $\tilde{O}\left(k / \epsilon^{2}\right)$. For fixed $k$ and $\kappa$, rather than rerun Lemma 3.1 entirely for each $s$ we sample, we execute most of it just once for all $s$, and make local modifications for each different root $s$. Referring to the algorithm given in the proof of Lemma 3.1, observe that the only step that directly mentions $s$ is step 3 , which adds auxiliary vertices between $s$ and each other vertex $v$. We move this step to the very end of the algorithm. (Here the vertex weight of auxiliary vertices is scaled down appropriately.) It is easy to see that the proof of Lemma 3.1 still goes through (with minor rearrangement in the argument). The advantage is that, over all $L$ roots $s$, we now spend a total of $O(m+n L)$ time, rather than $O(m L)$. Thereafter, the rest of the rooted mincut algorithm takes $\tilde{O}\left(\left(n-\operatorname{deg}^{+}(s)\right) T\right)$ per root $s$. Note that $\tilde{O}\left(\left(n-\operatorname{deg}^{+}(s)\right) T\right)$ dominates the $O(n)$ time required to complete the sparsification for each root.

Consider a single root $s$ sampled from $V$ in proportion to its weight. The expected running time to compute the minimum $s$-cut is

$$
\mathbf{E}\left[\left(n-\operatorname{deg}^{+}(s)\right) T\right]=n T-\frac{T}{W} \sum_{v \in V} \operatorname{deg}^{+}(v) w(v) \stackrel{(\mathrm{a})}{=} n T-\frac{T}{W} \sum_{v \in V} \sum_{x \in N^{-}(v)} w(x) \stackrel{(\mathrm{b})}{\leq} n T\left(1-\frac{\kappa}{W}\right)
$$

Here, in (a), $N^{-}(v)$ denotes the in-neighborhood of $v$. The equality is obtained by implicitly interchanging sums. (b) is because for each $v$, the $\operatorname{sum} \sum_{x \in N^{-}(v)} w(x)$ is the weighted in-degree of $v$, and at least $\kappa$. Thus The overall expected running time over all the sampled roots is

$$
O\left(\left(\frac{W \log (n)}{W-\kappa}\right) \mathbf{E}\left[\left(n-\operatorname{deg}^{+}(s)\right) T\right]\right)=O\left(\left(\frac{W \log (n)}{W-\kappa}\right) \cdot n T\left(1-\frac{\kappa}{W}\right)\right)=\tilde{O}\left(n^{2} / \epsilon^{2}\right)
$$

Meanwhile, when we sample $O(W \log (n) /(W-\kappa))$ vertices in proportion to their weight, then with high probability, at least one sampled vertex lies outside the minimum global vertex cut. Such a vertex then leads to the minimum global vertex cut with high probability.

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## References

[CCPS21] Ruoxu Cen, Yu Cheng, Debmalya Panigrahi, and Kevin Sun. Sparsification of balanced directed graphs. ICALP, 2021. 3
$\left[\mathrm{CLN}^{+} 21\right]$ Ruoxu Cen, Jason Li, Danupon Nanongkai, Debmalya Panigrahi, and Thatchaphol Saranurak. Minimum cuts in directed graphs via $\sqrt{ } \mathrm{n}$ max-flows. CoRR, abs/2104.07898, 2021.
[CQ21] Chandra Chekuri and Kent Quanrud. Faster algorithms for rooted connectivity in directed graphs. CoRR, abs/2104.07205, 2021. To appear in ICALP, 2021. 2, 12
[CR94] Joseph Cheriyan and John H. Reif. Directed $s-t$ numberings, rubber bands, and testing digraph $k$-vertex connectivity. Comb., 14(4):435-451, 1994. 2
[Edm73] Jack Edmonds. Edge-disjoint branchings. Combinatorial algorithms, 1973. 7
[ET75] Shimon Even and Robert Endre Tarjan. Network flow and testing graph connectivity. SIAM J. Comput., 4(4):507-518, 1975. 2
[FF56] L. R. Ford and D. R. Fulkerson. Maximal flow through a network. Canadian Journal of Mathematics, 8:399-404, 1956. 13
[FNY ${ }^{+}$20] Sebastian Forster, Danupon Nanongkai, Liu Yang, Thatchaphol Saranurak, and Sorrachai Yingchareonthawornchai. Computing and testing small connectivity in nearlinear time and queries via fast local cut algorithms. In Shuchi Chawla, editor, Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020, pages 2046-2065. SIAM, 2020. 2, 4, 12
[Gab95] H.N. Gabow. A matroid approach to finding edge connectivity and packing arborescences. Journal of Computer and System Sciences, 50(2):259-273, 1995. 2, 7, 21
[Gal80] Zvi Galil. Finding the vertex connectivity of graphs. SIAM J. Comput., 9(1):197-199, 1980. 2
[GGST86] Harold Gabow, Zvi Galil, Thomas Spencer, and Robert Tarjan. Efficient algorithms for finding minimum spanning tree in undirected and directed graphs. Combinatorica, 6:109-122, 06 1986. 7, 22
[GLP21] Yu Gao, Yang P. Liu, and Richard Peng. Fully dynamic electrical flows: Sparse maxflow faster than goldberg-rao. CoRR, abs/2101.07233, 2021. 2
[GMW20] Pawel Gawrychowski, Shay Mozes, and Oren Weimann. Minimum cut in o(m $\left.\log ^{2} n\right)$ time. In ICALP, volume 168 of LIPIcs, pages 57:1-57:15. Schloss Dagstuhl - LeibnizZentrum für Informatik, 2020. 3
[GMW21] Pawel Gawrychowski, Shay Mozes, and Oren Weimann. A note on a recent algorithm for minimum cut. In SOSA, pages 74-79. SIAM, 2021. 3
[GT88] Andrew V. Goldberg and Robert Endre Tarjan. A new approach to the maximum-flow problem. J. ACM, 35(4):921-940, 1988. 1
[HO94] Jianxiu Hao and James B. Orlin. A faster algorithm for finding the minimum cut in a directed graph. J. Algorithms, 17(3):424-446, 1994. 1, 2
[HRG00] Monika Rauch Henzinger, Satish Rao, and Harold N. Gabow. Computing vertex connectivity: New bounds from old techniques. J. Algorithms, 34(2):222-250, 2000. 2, 11, 16
[Kar00] David R Karger. Minimum cuts in near-linear time. Journal of the ACM (JACM), 47(1):46-76, 2000. 2
[LMN21] Andrés López-Martínez, Sagnik Mukhopadhyay, and Danupon Nanongkai. Workoptimal parallel minimum cuts for non-sparse graphs. SPAA, 2021. 3
[LNP ${ }^{+}$21] Jason Li, Danupon Nanongkai, Debmalya Panigrahi, Thatchaphol Saranurak, and Sorrachai Yingchareonthawornchai. Vertex connectivity in poly-logarithmic max-flows, 2021. 2
[MN20] Sagnik Mukhopadhyay and Danupon Nanongkai. Weighted min-cut: sequential, cutquery, and streaming algorithms. In STOC, pages 496-509. ACM, 2020. 3
[MS89] Yishay Mansour and Baruch Schieber. Finding the edge connectivity of directed graphs. Journal of Algorithms, 10(1):76-85, 1989. 2
[MTZC81] N. Megiddo, Arie Tamir, Eitan Zemel, and Ramaswamy Chandrasekaran. An $o\left(n \log ^{2} n\right)$ algorithm for the k th longest path in a tree with applications to location problems. SIAM Journal on Computing, 10, 05 1981. 9
[NSY19] Danupon Nanongkai, Thatchaphol Saranurak, and Sorrachai Yingchareonthawornchai. Breaking quadratic time for small vertex connectivity and an approximation scheme. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019, pages 241-252, 2019. 2
[Pod73] V. D. Podderyugin. An algorithm for finding the edge connectivity of graphs. Vopr. Kibern., 2:136, 1973. 2
[Qua21] Kent Quanrud. Fast approximations for rooted connectivity in weighted directed graphs. CoRR, abs/2104.06933, 2021. , 6
[Sch79] Claus-Peter Schnorr. Bottlenecks and edge connectivity in unsymmetrical networks. SIAM Journal on Computing, 8(2):265-274, 1979. 2
[vdBLL ${ }^{+}$21] Jan van den Brand, Yin Tat Lee, Yang P. Liu, Thatchaphol Saranurak, Aaron Sidford, Zhao Song, and Di Wang. Minimum cost flows, mdps, and $\ell_{1}$-regression in nearly linear time for dense instances. CoRR, abs/2101.05719, 2021. 1, 4
[vdBLN ${ }^{+}$20] Jan van den Brand, Yin Tat Lee, Danupon Nanongkai, Richard Peng, Thatchaphol Saranurak, Aaron Sidford, Zhao Song, and Di Wang. Bipartite matching in nearlylinear time on moderately dense graphs. In 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020, pages 919-930. IEEE, 2020. 15
[You95] Neal E. Young. Randomized rounding without solving the linear program. In Proceedings of the Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '95, page 170-178, USA, 1995. 3, 21

## A Proof of Lemma 2.4

Proof. For ease of notation, we prove Lemma 2.4 except with $(1+\epsilon)$ replaced by $(1+c \epsilon)$, for a fixed constant $c>1$, in properties (ii) and (iii). The constant factor can then be removed by decreased $\epsilon$ by a constant factor in the construction.

Let $\tau=c_{\tau} \epsilon^{2} \lambda /(k \log n)$ for a sufficiently small constant $c_{\tau}>0$. Decreasing $c_{\tau}$ by at most a constant factor, we can assume that $\lambda$ and $\epsilon \lambda / 2 k$ are integer multiples of $\tau$. Let $G_{0}=\left(V, E_{0}\right)$ be the reweighted subgraph obtained as follows.

1. Randomly round the weight of each edge independently up or down to the nearest multiple $\tau$, preserving the weight of each edge in expectation. ${ }^{8}$ Drop any edges with weight rounded down to 0 .
2. Add an edge of weight $\epsilon \lambda / 2 k$ from the root $s$ to every vertex $v \in V-s$.
3. Scale down all the edge weights by $\tau$.

For each set $T \subseteq V$, let $w_{1}(T)=\delta_{G}^{-}(T)$ denote the weight of the in-cut at $T$ in $G$. Let $w_{2}(T)$ denote the randomized weight of the in-cut after randomly rounding in step 1 . Let $w_{3}(T)$ denote the randomized weight of the in-cut of $T$ after adding the auxiliary edges in 2 . The first claim analyzes the concentration of $w_{2}(T)$ for all sets $T \subseteq V$.

Claim 1. With high probability, for all $T \subseteq V$,

$$
\left|w_{2}(T)-w_{1}(T)\right| \leq \epsilon w_{1}(T)+\frac{\epsilon \lambda|T|}{2 k} .
$$

The above claim consists of an upper and lower bound on $w_{2}(T)$ for all $T$. We first show the lower bound on $w_{2}(T)$ holds for all $T$ with high probability. Fix $T \subseteq V . w_{2}(T)$ is an independent sum with expected value $w_{1}(T)$ and where each term in the sum is nonnegative and varies by at most

[^6]$\tau$. By a variation of standard Chernoff inequalities ${ }^{9}$, for any $\gamma \geq 0$, we have
$$
\mathbf{P}\left[w_{2}(T) \leq(1-\epsilon) w_{1}(T)-\gamma\right] \leq e^{-\epsilon \gamma / \tau}=n^{-\gamma k \log (n) / c_{\tau} \epsilon \lambda},
$$

In particular, for $\gamma=\epsilon \lambda|T| / 2 k$, the RHS is at most $n^{-c_{0}|T|}$ where $c_{0}$ is a constant under our control (via $c_{\tau}$ ). For large enough $c_{0}$, we can take the union bound over all sets of vertices. This establishes that the lower bounds for $w_{2}(T)$ hold for all $T$ with high probability. The upper bounds also hold with high probability by a symmetric argument.

Now we analyze the in-cuts after step 2. Recall that for $T \subseteq V, w_{3}(T)$ denotes the weight of the in-cut of $T$ after adding the auxiliary edges in 2 .

Claim 2. With high probability, for all $T \subseteq V-s$, we have

$$
(1-\epsilon) w_{1}(T) \leq w_{3}(T) \leq(1+\epsilon) w_{1}(T)+\epsilon \lambda|T| / k .
$$

Indeed, we have $w_{3}(T)=w_{2}(T)+\epsilon \lambda|T| / 2 k$ for all $T \subseteq V-s$. The additive term introduced by $h$ offsets the additive error in the lower bound on $w_{2}(T)$ in Claim 1. This term also adds on to the additive error in the upper bound of Claim 1 for a total of $\epsilon \lambda|T| / k$. Thus in the high probability event of Claim 1, we have the bounds described by Claim 2 for all $T$.

Henceforth, let us assume that the high probability event in Claim 2 holds. (Otherwise the algorithm fails.) We now argue that Claim 2 implies that properties (i), (ii) and (iii) hold whp.
(ii): Since the weight of each $s$-cut decreases by at most an $(1-\epsilon)$-multiplicative factor, the minimum $s$-cut $T^{*}$ has weight at least $(1-\epsilon) \delta_{G}^{-}\left(T^{*}\right)$. Meanwhile $\delta^{-}\left(T^{*}\right)$ increases by at most a $(1+2 \epsilon)$-multiplicative factor. Thus $\delta^{-}\left(T^{*}\right)$ remains a $(1+O(\epsilon))$-approximate minimum $r$-cut. This establishes property (ii).
(iii): Since again the weight of any $s$-cut decreases by at most an $(1-\epsilon)$-multiplicative factor, any $\alpha$-approximate minimum cut in the randomly reweighed graph $G_{0}$ is an $(1+O(\epsilon)) \alpha$ approximate min $s$-cut in $G$. This establishes property (iii).
(i): Finally, we observe that before scaling in step 3, all edge weights are at least $\tau$, so after scaling, all edge weights are at least 1 . Meanwhile, after scaling, the minimum $r$-cut has weight at most $O\left(\delta^{-}(T) / \tau\right)=O\left(k \log (n) / \epsilon^{2}\right)$. This establishes property (i) and completes the proof.

## B Proof of Lemma 2.6

We start by defining the fractional arborescence packing problem. Let $A_{1}, A_{2}, \ldots, A_{N}$ be the set of all $s$-arborescences. We represent a fractional packing of arborescences as a vector $x \in \mathbb{R}^{N}$, where $x_{i}$ represents the fractional contribution of arborescence $A_{i}$ in the packing. Define the value of the arborescence packing as the sum of the coordinates of $x$, i.e., $\operatorname{val}(x):=\sum_{i=1}^{N} x_{i}$. The goal is to maximize $\operatorname{val}(x)$, under the constraint that each edge $j$ can be fractionally used only up to its

[^7]weight $w(j)$. This definition generalizes integral arborescence packing, where we pack a multiset of arborescences, under the constraint that each edge $j$ is used at most $w(j)$ times, and the value is the cardinality of the multiset.

It will be convenient to state the fractional arborescence packing problem in the framework of a standard packing problem, defined below:

Definition B. 1 (Packing problem [You95]). For convex set $P \subseteq \mathbb{R}^{n}$ and nonnegative linear function $f: P \rightarrow \mathbb{R}^{m}$, the packing problem aims to find $\gamma^{*}=\min _{x \in P} \max _{j \in[m]} f_{j}(x)$, i.e., the solution in $P$ that minimizes the maximum value of $f_{j}(x)$ over all $j$. The width of the packing problem $(P, f)$ is defined as $\omega=\max _{j \in[m], x \in P} f_{j}(x)-\min _{j \in[m], x \in P} f_{j}(x)$.

For the fractional arborescence packing problem, we have the following: $P=\left\{x \in \mathbb{R}^{N}: \operatorname{val}(x)=\right.$ $1, x \geq 0\}$ is the convex hull of all arborescences. The function $f: P \rightarrow \mathbb{R}^{m}$ defines the total usage of each edge $j$ in the arboresence packing normalized by the edge weight, i.e., $f_{j}(x)=\frac{\sum_{i \in[N]: j \in A_{i}} x_{i}}{w(j)}$ for $x \in P$ and any edge $j \in[m]$; we call this the load of arborescence packing $x$ on edge $j$. The packing problem has width

$$
\begin{equation*}
\omega=\max _{j \in[m], x \in P} f_{j}(x)-\min _{j \in[m], x \in P} f_{j}(x) \leq \max _{j \in[m]} \frac{1}{w_{j}}-0=\frac{1}{w_{\text {min }}}, \tag{1}
\end{equation*}
$$

where $w_{\min }$ is the minimum edge weight. The objective function is to minimize the maximum load: $\gamma^{*}=\min _{x \in P} \max _{j \in[m]} f_{j}(x)$.

For any $x \in P$ with maximum load $\gamma=\max _{j \in[m]} f_{j}(x)$, we can multiply it by $1 / \gamma$ to get an arborescence packing with value $1 / \gamma$ and maximum load 1 . Conversely, for any fractional arborescence packing $x \in \mathbb{R}^{N}$ with $\operatorname{val}(x)=v$ where $f_{j}(x) \leq 1$ for all edges $j$, we have $x / v \in P$. Therefore, it suffices to look for the vector $x \in P$ achieving the optimal value $\gamma^{*}$, and then scale the vector up by $1 / \gamma^{*}$ to obtain the maximum arborescence packing. If the value of maximum $s$-arborescence packing is $\lambda^{*}$, then the optimal value of the packing problem is:

$$
\begin{equation*}
\gamma^{*}=\frac{1}{\lambda^{*}} . \tag{2}
\end{equation*}
$$

Next we describe the packing algorithm (Figure 2 of [You95]). Maintain a vector $y \in \mathbb{R}^{m}$, initially set to $y=\mathbf{1}$. In each iteration, find $x=\arg \min _{x \in P} \sum_{j} y_{j} f_{j}(x)$, and then add $x$ to set $S$ and replace $y$ by the vector $y^{\prime}$ defined by $\left.y_{j}^{\prime}=y_{j}\left(1+\epsilon f_{j}(x)\right) / \omega\right)$. After a number of iterations, return $\bar{x} \in P$, the average of all the vectors $x$ over the course of the algorithm. The lemma below upper bounds the number of iterations that suffice:
Lemma B. 2 (Corollary 6.3 of [You95]). After $\left\lceil\frac{(1+\epsilon) \omega \ln m}{\left.\gamma^{*}(1+\epsilon) \ln (1+\epsilon)-\epsilon\right)}\right\rceil$ iterations of the packing algorithm, $\bar{\gamma}:=\max _{j} f_{j}(\bar{x}) \leq(1+\epsilon) \gamma^{*}$.

We will also make use of the (exact) duality between $s$-arborescence packing and minimum $s$-cut:
Lemma B. 3 (Corollary 2.1 of [Gab95]). The value of maximum s-arborescence packing is equal to the value of s-mincut.

We now prove Lemma 2.6.
Proof of Lemma 2.6. First, construct $G_{0}$ according to Lemma 2.5, with edge size $m_{0}=\min \left\{m, O\left(n k \log n / \epsilon^{2}\right)\right\}$. Let $\lambda_{0}$ denote the minimum s-cut value on $G_{0}$. Lemma 2.5 guarantees

$$
\begin{equation*}
w_{\min } \geq 1 \text { and } \lambda_{0}=O\left(k \log n / \epsilon^{2}\right) \tag{3}
\end{equation*}
$$

```
Algorithm 2: Sparsifying a graph to preserve the rooted vertex mincut
1 Let \(\tau=c_{\tau} \epsilon^{2} \kappa / k \log (n)\) and let \(\Delta=c_{\Delta} k \log (n) / \epsilon^{2}\), where \(c_{\tau}>0\) is a sufficiently small
    constant and \(c_{\Delta}>0\) is a sufficiently large constant.
2 Randomly round each vertex weight to (nearest) multiples of \(\tau\).
3 For each vertex \(v\), introduce an auxiliary vertex \(a_{v}\) with weight \(\epsilon \kappa / 2 k\). Add edges from the
    \(s\) to \(a_{v}\), and from \(a_{v}\) to \(v\).
    /* Decreasing \(c_{\tau}\) and \(\epsilon\) as needed, we assume that \(\kappa\) and \(\epsilon \kappa / 2 k\) are multiples of
        \(\tau\).
4 Remove all outgoing edges from any vertex with weight 0 .
5 Scale down all vertex weights by \(\tau\) (which makes them integers).
6 Truncate all vertex weights to be at most \(c_{w} k \log (n) / \epsilon^{2}\) for a sufficiently large constant \(c_{w}>0\).
7 For all \(v\) with unweighted in-degree at least \(\Delta\), replace all incoming edges to \(v\) with a single edge from \(s\).
```

By the duality (Lemma B. 3 and (2)), the optimal value of the packing problem formulated above is $\gamma^{*}=\frac{1}{\lambda_{0}}$. Run the aforementioned arborescence packing algorithm. Lemma B.2, (1),(2) and (3) guarantee that after

$$
O\left(\frac{\omega \ln m}{\gamma^{*}}\right)=O\left(\frac{\lambda_{0} \ln m}{w_{\min }}\right)=O\left(\lambda_{0} \ln m\right)=O\left(k \log ^{2} m\right)
$$

iterations (with constant $\epsilon$ ) we have

$$
\begin{equation*}
\bar{\gamma} \leq(1+\epsilon) \gamma^{*} \tag{4}
\end{equation*}
$$

Then $\bar{x} / \bar{\gamma}$ is an arborescence packing with value $1 / \bar{\gamma} \geq \frac{1}{1+\epsilon} \lambda_{0}$. Consider sampling a random $s-$ arborescence $A$ from distribution $\bar{x}$, so we choose arborescence $A_{i}$ with probability $\bar{x}_{i}$. Since $\delta_{G_{0}}^{-}\left(T^{*}\right) \leq(1+\epsilon) \lambda_{0}$ by Lemma 2.5, the expected number of edges in $A \cap \partial_{G_{0}}^{-}\left(T^{*}\right)$ is at most $(1+\epsilon) \lambda_{0} /\left(\frac{1}{1+\epsilon} \lambda_{0}\right)=(1+\epsilon)^{2} \leq 1+3 \epsilon$ for small enough constant $\epsilon$. Since we always have $\left|A \cap \partial_{G_{0}}^{-}\left(T^{*}\right)\right| \geq 1$, by Markov's inequality $\mathbf{P}\left[\left|A \cap \partial_{G_{0}}^{-}\left(T^{*}\right)\right|-1 \geq 1\right] \leq 3 \epsilon \leq 1 / 2$ for small enough constant $\epsilon$. Therefore, if we uniformly sample $\Theta(\log n)$ arborescences from the distribution $\bar{x}$, at least one of the arborescences is 1-respecting whp.

It remains to compute $x=\arg \min _{x \in P} \sum_{j} y_{j} f_{j}(x)$ on each iteration. Since $\sum_{j} y_{j} f_{j}(x)$ is linear in $x$, the minimum must be achieved by a single arborescence. So the task reduces to computing the minimum cost spanning $s$-arborescence, which can be done in $O\left(m_{0}+n \log n\right)$ [GGST86]. The total time complexity, over all iterations, becomes $O\left(\left(m_{0}+n \log n\right) \lambda_{0} \log n\right)=$ $O\left(\min \left\{m k \log ^{2} n, n k^{2} \log ^{3} n\right\}\right)$. The construction of Lemma 2.5 costs another $O(m \log n)$ time.

## C Proof of Lemma 3.1

Proof. Consider the Algorithm 2 applied to the input graph $G$. Let $G_{0}$ be the graph obtained at the end of the algorithm. Properties (i) through (v) follow directly from the construction. The remaining proof is dedicated to proving the high probability bounds of (vi) and (vii). We first show that the initial steps Line 1 to 3 - before rescaling - preserves the minimum weight rooted vertex-cut approximately in the sense of (vi) (sans scaling). We then analyze the remaining steps. For each set $S$, let $w_{1}(S)$ denote the weight of the vertex in-cut of $S$. Let $w_{2}(S)$ denote the randomized weight
of the vertex in-cut after step 2 . Let $w_{3}(S)$ denote the randomized weight of the vertex in-cut after step 3.

Claim 1. With high probability, for all $S \subseteq V$, we have

$$
\left|w_{2}(S)-w_{1}(S)\right| \leq \epsilon w_{1}(S)+\frac{\epsilon \kappa|S|}{2 k} .
$$

The claim and proof are similar to Claim 1 in the proof of Lemma 2.5. The claim consists of an upper bound and a lower bound on $w_{2}(S)$ for all $S$ and we first show the lower bound holds with high probability. Fix any set $S . w_{2}(S)$ is an independent sum with expected value $w_{1}(S)$ and where each term in the sum is nonnegative and varies by at most $\tau$. Concentration bounds (see footnote 9 on page 20) imply that for any $\gamma \geq 0$, we have

$$
\mathbf{P}\left[w_{2}(S) \leq(1-\epsilon) w_{1}(S)-\gamma\right] \leq e^{-\epsilon \gamma / \tau}=n^{-\gamma k \log (n) / c_{\tau} \epsilon \kappa}
$$

In particular, for $\gamma=\epsilon \kappa|S| / 2 k$, the RHS is at most $n^{-c_{0}|S|}$ where $c_{0}>0$ is again a constant under our control (via $c_{\tau}$ ). For sufficiently large $c_{0}$, we can take the union bound over all sets $S$, establishing the high probability lower bound. The high probability upper bound follows by a symmetric argument.

Now we analyze the vertex $s$-cuts after step 3 . Recall that for $S \subseteq V, w_{3}(S)$ denotes the weight of the in-cut of $S$ after adding auxiliary vertices in step 3.

Claim 2. For all $S \subseteq V^{\prime}$, we have

$$
(1-\epsilon) w_{1}(S) \leq w_{3}(S) \leq(1+\epsilon) w_{1}(S)+\frac{\epsilon \kappa|S|}{k} .
$$

This claim and its proof is similar to Claim 2 in Lemma 2.5. We have $w_{3}(S)=w_{2}(S)+\epsilon \kappa|S| / 2 k$ for all $S \subseteq V^{\prime}$. The additive factor of $\epsilon \kappa|S| / 2 k$ combine with the high-probability additive error in Claim 1 to establish the claim.

We point out that Claim 2 implies that, with high probability after step 3, the weights of all the vertex $s$-cuts are preserved in the approximate sense described by (vi) (without the scaling). Henceforth we assume that the high probability event of Claim 2 holds. Now, after step 3, all the weights are integer multiples of $\tau$. We have $\kappa / \tau=O\left(k \log (n) / \epsilon^{2}\right)$. After scaling down by $\tau$ in step 5 , we still preserve the $s$-cuts per property (vi). Truncating weights in $G_{0}$ to $O(\kappa / \tau)$ decreases the weight of some cuts, but to no less than $O(\kappa / \tau)$ (which maps to $O(\kappa)$ when rescaled back to the scale of $G$ ). Removing the outgoing edges of vertices with weight 0 also has no impact on the weight of any vertex $s$-cut. The final step adding edges from $s$ only eliminates some of the vertex $s$-cuts from consideration and does not impact the weight of the remaining vertex $s$-cuts. This establishes (vi).

It remains to prove (vii) and in particular we must show that it is not impacted by the last step, 7. Recall that step 7 replaces the incoming edges to any vertex $v$ with unweighted in-degree greater than $\Delta=O\left(k \log (n) / \epsilon^{2}\right)$ with a single edge in $s$. In particular, this edge places $v$ in $N^{+}(s)$ and destroys all $s$-cuts where the sink component contains $v$. Let $T \subset V^{\prime}$ be the sink component of a vertex $s$-cut in $G$ where the capacity of the cut is at most $\kappa$, and $|T| \leq k$. We want to show that all vertices in $T$ have in-degree less than $\Delta$, in which case the extra edges in 7 have no impact on $T$. The vertex in-cut induced by $T$ has weight at most $(1+2 \epsilon) \kappa / \tau$ in the randomized graph before 7 (per (vi)). Fix any $v \in T$ and consider the edges going into $v$. At most $k-1$ of those edges can come from another vertex in $T$, since $T$ has at most $k$ vertices. The remaining edges must be from
vertices in the vertex in-cut of $T$. Each of these vertices have weight at least 1 , and by (vi) the in-cut has weight at most $(1+2 \epsilon) \kappa / \tau$, so there are at most $O(\kappa / \tau)$ of these vertices. This gives a maximum total of less than $\Delta$ edges incident to $v$, as desired. In conclusion, for any vertex $v$ that is the endpoint to at least $\Delta$ edges, it is safe to replace all of $v$ 's incoming edges with a single edge from the root, without violating (vii). This establishes (vii) and completes the proof.


[^0]:    ${ }^{*}$ This paper combines, and improves on, two independent manuscripts by Quanrud [Qua21] and the other authors $\left[\mathrm{CLN}^{+} 21\right]$.

[^1]:    ${ }^{1}$ We assume throughout that edge/vertex weights are polynomially bounded integers.

[^2]:    ${ }^{2}$ Gabow actually constructs a directionless spanning tree packing, which is a relaxation of an arborescence packing, but we ignore this technical detail here since it is not relevant to our eventual algorithm.
    ${ }^{3}$ Since Karger's algorithm considered undirected graphs, the arborescences are simply spanning trees.

[^3]:    ${ }^{4}$ The duality implies that if the undirected mincut is $\lambda$, then we can pack $\lambda$ spanning trees where every edge appears in at most two arborescences.
    ${ }^{5}$ By duality, we have that if the directed mincut is $\lambda$, then we can pack $\lambda$ arborescenses where each edge appears in at most one arborescense. Note that this is different from the case of undirected graphs where each edge appears in at most two arborescences.

[^4]:    ${ }^{6}$ This should be compared with Karger's mincut algorithm in the undirected case, where there is a factor 2 gap, and hence Karger can only guarantee a 2 -respecting tree in the undirected case.

[^5]:    ${ }^{7}$ Usually, this edge is set to capacity $\infty$, but either the weight of $u$ or the weight of $v$ are also valid.

[^6]:    ${ }^{8}$ That is, if the weight of an edge $e$ in $G$ is $x \cdot \tau+y$ for an integer $x \geq 0$ and any $0 \leq y<\tau$, then the weight of $e$ is $(x+1) \cdot \tau$ with probability $y / \tau$ and $x \cdot \tau$ otherwise.

[^7]:    ${ }^{9}$ Here we apply the following bounds (appropriately rescaled) which follow from the same proof as the standard multiplicative Chernoff bound.

    Let $X_{1}, \ldots, X_{n} \in[0,1]$ independent random variables. Then for all $\epsilon>0$ sufficiently small and all $\gamma>0$,

    $$
    \mathbf{P}\left[X_{1}+\cdots+X_{n} \leq(1-\epsilon) \mathbf{E}\left[X_{1}+\cdots+X_{n}\right]-\gamma\right] \leq e^{-\epsilon \gamma}
    $$

    and

    $$
    \mathbf{P}\left[X_{1}+\cdots+X_{n} \geq(1+\epsilon) \mathbf{E}\left[X_{1}+\cdots+X_{n}\right]+\gamma\right] \leq e^{-\epsilon \gamma}
    $$

