Cut query algorithms with star contraction

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Abstract

We study the complexity of determining the edge connectivity of a simple graph with cut queries. We show that (i) there is a bounded-error randomized algorithm that computes edge connectivity with O(n) cut queries, and (ii) there is a bounded-error quantum algorithm that computes edge connectivity with $\tilde{O}(\sqrt{n})$ cut queries. To prove these results we introduce a new technique, called star contraction, to randomly contract edges of a graph while preserving non-trivial minimum cuts. In star contraction vertices randomly contract an edge incident on a small set of randomly chosen "center" vertices. In contrast to the related 2-out contraction technique of Ghaffari, Nowicki, and Thorup [SODA'20], star contraction only contracts vertex-disjoint star subgraphs, which allows it to be efficiently implemented via cut queries.

The O(n) bound from item (i) was not known even for the simpler problem of connectivity, and it improves the $O(n\log^3 n)$ upper bound by Rubinstein, Schramm, and Weinberg [ITCS'18]. The bound is tight under the reasonable conjecture that the randomized communication complexity of connectivity is $\Omega(n\log n)$, an open question since the seminal work of Babai, Frankl, and Simon [FOCS'86]. The bound also excludes using edge connectivity on simple graphs to prove a superlinear randomized query lower bound for minimizing a symmetric submodular function. The quantum algorithm from item (ii) gives a nearly-quadratic separation with the randomized complexity, and addresses an open question of Lee, Santha, and Zhang [SODA'21]. The algorithm can alternatively be viewed as computing the edge connectivity of a simple graph with $\tilde{O}(\sqrt{n})$ matrix-vector multiplication queries to its adjacency matrix.

Finally, we demonstrate the use of star contraction outside of the cut query setting by designing a one-pass semi-streaming algorithm for computing edge connectivity in the complete vertex arrival setting. This contrasts with the edge arrival setting where two passes are required.

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1 Introduction and contribution

The minimization of a submodular function is a classic problem in combinatorial optimization. Over a universe V, a submodular function $f: 2^V \to \mathbb{R}$ is a function that satisfies $f(S) + f(T) \ge f(S \cap T) + f(S \cup T)$ for all subsets $S, T \subseteq V$. The submodular function minimization (SFM) problem is the task of computing $\min_{S \subseteq V} f(S)$. The SFM problem generalizes several well known combinatorial optimization problems such as computing the minimum weight of an st-cut in a directed graph and the matroid intersection problem. The SFM problem comes in another flavor when the submodular function f is symmetric, i.e. also satisfies $f(S) = f(V \setminus S)$ for all S. In this case, \emptyset and V are trivial minimizers, so the interesting problem is to compute $\min_{\emptyset \subseteq S \subseteq V} f(S)$, the non-trivial minimum. The global minimum cut problem on an undirected graph is an instance of non-trivial symmetric submodular function minimization, which we denote by sym-SFM.

The size of the truth table of a submodular function is exponential in the size of V, so (sym-)SFM is typically studied in the setting where we have access to an evaluation oracle for f, that is, we can query any $S \subseteq V$ and receive the answer f(S). When |V| = n, Grötschel, Lovász, and Schrijver [GLS88] showed that the ellipsoid method can be used to solve SFM with $\widetilde{O}(n^5)$ oracle queries and overall running time $\widetilde{O}(n^7)$ [McC05, Theorem 2.8]. Since then a long line of work has developed faster and simpler (combinatorial) algorithms for SFM [Sch00, IFF01, Orl09, LSW15, DVZ21, Jia21]. The work of [Jia21] shows that SFM can be solved by a deterministic algorithm making $O(n^2 \log n)$ queries to an evaluation oracle. By the isolating cut lemma of [LP20] this immediately also gives an $\widetilde{O}(n^2)$ query randomized algorithm for sym-SFM [CQ21, MN21]. In the deterministic case, the best upper bound on the number of queries to solve sym-SFM remains the $O(n^3)$ algorithm of Queyranne [Que98].

While sym-SFM is a much more general problem, it has a close relationship with one of its simplest instantiations: the global minimum cut problem. In this problem we are given a weighted and undirected graph G = (V, E, w) and the task is to find the minimum weight of set of edges whose removal disconnects G. For a subset $S \subseteq V$ let $\operatorname{cut}_G(S)$ be the set of edges of G with exactly one endpoint in S. The cut function $f: 2^V \to \mathbb{R}$, where $f(S) = w(\operatorname{cut}_G(S))$ is the total weight of edges in $\operatorname{cut}_G(S)$, is a symmetric submodular function. Evaluation queries in this case are called $\operatorname{cut} queries$ and the goal is to compute $\lambda(G) := \min_{\emptyset \subsetneq S \subsetneq V} w(\operatorname{cut}_G(S))$ with as few cut queries as possible.

Both the best known deterministic and randomized algorithms for sym-SFM use ideas that originated in the study of minimum cuts: Queyranne's algorithm is based on the Stoer-Wagner minimum cut algorithm [SW97], and the best randomized algorithm makes use of the isolating cut lemma originally developed in the context of a deterministic minimum cut algorithm [LP20]. On the lower bound side, the best known bounds on the query complexity of sym-SFM are $\Omega(n)$ in the deterministic case [HMT88, Har08] and $\Omega(n/\log n)$ in the randomized case [BFS86] (see Table 1). Both of these bounds can be shown for the cut query complexity of determining the weight of a minimum cut in a simple graph. The weight of a minimum cut in simple graph G is known as the edge connectivity of G, and is the minimum number of edges whose removal disconnects the graph. The aforementioned lower bounds even hold for the more special case of determining if the edge connectivity is positive, i.e. if the graph is connected or not.

Recent work has given randomized algorithms that can compute $\lambda(G)$ with $O(n \log^3 n)$ cut queries in the case of simple graphs [RSW18] and $n \log^{O(1)}(n)$ cut queries in the case of weighted graphs [MN20]. For the deterministic case, however, the best upper bound remains $O(n^2/\log n)$ [GK00] and proceeds by learning the entire graph. Researchers continue to study the minimum cut problem as a candidate to show superlinear query lower bounds on sym-SFM. Graur, Pollner,

 $^{^{1}}$ We say that a graph is simple if it is undirected and unweighted and contains at most one edge between any pair of vertices.

Ramaswamy and Weinberg [GPRW20] introduced a linear-algebraic lower bound technique known as the *cut dimension* to show a deterministic cut query lower bound of 3n/2 - 2 for minimum cut on weighted graphs. Lee, Li, Santha, and Zhang [LLSZ21] show that the cut dimension cannot show lower bounds larger than 2n - 3, but use a generalization of the cut dimension to show a cut query lower bound of 2n - 2, the current best lower bound known on sym-SFM in general.

Despite this work, showing a superlinear lower bound on the query complexity of sym-SFM remains elusive. In this paper, we show that for randomized algorithms and the special case of edge connectivity there is actually a *linear upper bound*.

Theorem 1.1. There is a randomized algorithm that makes O(n) cut queries and outputs the edge connectivity of a simple input graph G with probability at least 2/3.

In particular one cannot hope to prove superlinear lower bounds on the randomized query complexity of sym-SFM via the edge connectivity problem. It remains open if Theorem 1.1 is tight. The best known lower bound is $\Omega(n \log \log(n)/\log(n))$ which follows from the $\Omega(n \log \log n)$ randomized communication complexity lower bound for edge connectivity by Assadi and Dudeja [AD21]. An o(n) randomized cut query upper bound on edge connectivity would in particular imply a randomized communication complexity protocol for determining if a graph is connected with $o(n \log n)$ bits,² resolving one of the longest standing open problems in communication complexity. Graph connectivity was a focus of many early works on communication complexity [HMT88, BFS86, RS95], and while a deterministic lower bound of $\Omega(n \log n)$ was established early on [HMT88], to this day the randomized communication complexity is only known to be between $\Omega(n)$ and $O(n \log n)$.

Theorem 1.1 even improves the previous best cut query upper bound for deciding if a graph is connected. Harvey [Har08, Theorem 5.10] gave a deterministic $O(n \log n)$ cut query upper bound for connectivity, and we are not aware of any better upper bound in the randomized case. For the case of connectivity we can give a linear upper bound even for zero-error algorithms.

Theorem 1.2. Let G = (V, E) be a simple n-vertex graph. There is a zero-error randomized algorithm that makes O(n) cut queries in expectation and outputs a spanning forest of G.

The best lower bound we are aware of in this case is $\Omega(n \log \log(n)/\log(n))$ which follows from the non-deterministic communication complexity lower bound for connectivity of $\Omega(n \log \log(n))$ by Raz and Spieker [RS95].

A key to both Theorem 1.1 and Theorem 1.2 is to think in terms of matrix-vector multiplication queries. If A is the adjacency matrix of an n-vertex simple graph G, in a matrix-vector multiplication query we can query any vector $x \in \{0,1\}^n$ and receive the answer Ax. If G has maximum degree d, and so A has at most d ones in every row, we can learn the entire graph G with only $O(d \log n)$ matrix-vector multiplication queries—this is one of the key ideas behind compressed sensing. As a single matrix-vector multiplication query can be simulated with O(n) cut queries, this shows that we can learn G with $O(nd \log n)$ cut queries. Grebinski and Kucherov [GK00] show the surprising fact that if G is bipartite with maximum degree d, and the left and right hand sides are roughly the same size, then one can actually learn G with only O(nd) cut queries. This savings of a log n factor over the trivial simulation is key to our improved algorithms.

We use this idea to design a primitive called Recover-k-From-All. Given two disjoint subsets S, T of vertices, with the promise that all vertices in S have at least k neighbors in T, Recover-k-From-All makes O(kn) cut queries and learns at least k neighbors in T of every vertex in S.

²With shared randomness the parties can simulate a randomized cut query algorithm with an $O(\log n)$ multiplicative overhead: whenever the algorithm makes a cut query, the parties communicate the number of cut edges in their part of the graph with $O(\log n)$ bits. By Newman's theorem, this protocol can be simulated without shared randomness (and with only an additive $O(\log n)$ overhead).

	Connectivity		Edge Connectivity	
	Lower	Upper	Lower	Upper
Deterministic	$\Omega(n)$ [HMT88]	$O(n \log n)$ [Har08]	$\Omega(n)$ [HMT88]	$O\left(\frac{n^2}{\log n}\right)$ [GK00]
Zero-error	$\Omega\left(\frac{n\log\log(n)}{\log(n)}\right)$ [RS95]	O(n) (Theorem 1.2)	$\Omega(n)$ [LS21]	$O\left(\frac{n^2}{\log n}\right)$ [GK00]
Bounded-error	$\Omega\left(\frac{n}{\log n}\right)$ [BFS86]	O(n) (Theorem 1.2)	$\Omega\left(\frac{n\log\log(n)}{\log(n)}\right)$ [AD21]	O(n) (Theorem 1.1)
Quantum	$\Omega(1)$	$O(\log^5(n))$ [AL21]	$\Omega(1)$	$\widetilde{O}(\sqrt{n})$ (Theorem 1.3)

Table 1: The cut query complexity of connectivity and edge connectivity on simple graphs in various models. The upper bounds on edge connectivity in the deterministic and zero-error models follow from using [GK00, Section 4.1] to learn the full graph with $O(n^2/\log(n))$ cut queries. The bound in [GK00] is stated for additive queries, but the same argument holds for cut queries: with $O(n/\log(n))$ cut queries we can learn the neighborhood of a vertex by Lemma 5.4.

This routine is the heart of our algorithm for Theorem 1.2, which uses it to implement Borůvka's spanning forest algorithm.

It is less obvious how such a primitive is useful to compute edge connectivity as it only gives us local snapshots of sparse bipartite induced subgraphs. To this end we develop a new technique for edge connectivity called *star contraction*. Star contraction is inspired by the randomized 2-out contraction algorithm of Ghaffari, Nowicki and Thorup [GNT20]. In that algorithm, each vertex independently and uniformly at random selects two incident edges. Ghaffari et al. show that when the selected edges are contracted the resulting graph G' has only $O(n/\delta(G))$ vertices with high probability, where $\delta(G)$ is the minimum degree of G, and further with constant probability no edge of a non-trivial minimum cut³ is contracted. When these good things happen the edge connectivity of G is the minimum of $\delta(G)$ and the edge connectivity of G'.

2-out contraction is not very compatible with our primitive Recover-k-From-All because of the combination of requiring independent sampling and choosing an edge incident to a vertex uniformly at random. Instead, in star contraction we first randomly choose a subset R of size $\Theta(n \log(n)/\delta(G))$. With high probability every vertex in $V \setminus R$ will have a neighbor in R, and in star contraction we only contract edges with an endpoint in R. The fact that the edges that we want to contract are incident on a small number of vertices is a key to the savings of star contraction over 2-out contraction in the cut query model. Further, if for every vertex in $v \in V \setminus R$ we contract an edge connecting it to R then the contracted graph G' will automatically have its size bounded by $|R| = \Theta(n \log(n)/\delta(G))$. While proving the contracted graph has few vertices is the most difficult part of the argument in 2-out contraction, for star contraction it is trivial (although the bound we get is larger by a $\log n$ factor).

The tricky part remaining is how to choose a neighbor in R for each $v \in V \setminus R$ without having too high a probability of choosing a neighbor on the other side of a non-trivial minimum cut. In our main technical contribution, we show that each $v \in V \setminus R$ can learn just a constant number of neighbors in R without too high a fraction of them being on the opposite side of a non-trivial minimum cut. Moreover, we can allow correlations between the neighbors learned for different

³We call a cut *trivial* if it isolates a single vertex.

vertices which allows us to efficiently learn these constant number of neighbors using Recover-k-From-All with constant k.

Not surprisingly, the aforementioned matrix-vector multiplication perspective also leads to efficient randomized algorithms for edge connectivity in the matrix-vector multiplication query model. While this model has been used previously in the study of sequential graph algorithms [OSV12] and (implicitly) in streaming algorithms for graph problems [AGM12], it began to be studied in and of itself relatively recently in the work of [SWYZ21], and has since seen several several follow-ups [CHL21, AL21]. More surprisingly, it turns out that the study of quantum algorithms with cut query access to a graph is also closely related to the matrix-vector multiplication model. This is because with $O(\log n)$ cut queries a quantum algorithm can simulate a restricted form of a matrixvector multiplication query, namely it can compute Ax in the entries where x is zero (this is implicit in [LSZ21] and made explicit in [AL21, Corollary 11]). Lee, Santha and Zhang [LSZ21] used this to show that a quantum algorithm making only $O(\log^6 n)$ cut queries can decide if an n-vertex simple graph is connected [LSZ21, Theorem 44], a nearly exponential speedup over the best possible randomized algorithm. They left it as an open question whether any quantum speedup is possible for the problem of edge connectivity. This problem is particularly interesting because not much is known about the complexity of (sym)-SFM with respect to quantum algorithms, either in terms of upper or lower bounds (some work has been done on approximation algorithms for SFM, see [HRRS19]). The best classical algorithms for (sym)-SFM tend to be highly sequential, a feature which is typically hard to speed up quantumly. Our matrix-vector multiplication perspective leads to a quantum improvement for the cut query complexity of edge connectivity, as shown by the following theorem.

Theorem 1.3. There is a quantum algorithm that makes $O(\sqrt{n})$ cut queries and outputs the edge connectivity of the input simple graph G correctly with high probability. Similarly, there is a randomized algorithm making $O(\sqrt{n})$ matrix-vector multiplication queries to the adjacency matrix of G that outputs the edge connectivity of the input simple graph G correctly with high probability.

The quantum part of this theorem gives a near-quadratic speedup over the best possible randomized algorithm. Moreover, there is a natural bottleneck to giving a $o(\sqrt{n})$ quantum cut query algorithm for edge connectivity, which is that even computing the minimum degree of a graph seems to require $\Omega(\sqrt{n})$ quantum cut queries.⁴ There is a very natural $O(\sqrt{n})$ quantum algorithm for computing the minimum degree: the degree of a single vertex can be computed with one cut query, and one can then use quantum minimum finding [DH96] on top of this to find the minimum degree with $O(\sqrt{n})$ cut queries. We conjecture that this simple algorithm is optimal, which would imply that the quantum statement of Theorem 1.3 is tight up to polylogarithmic factors.

As a final application, we use our new star contraction technique to obtain a one-pass O(n)space algorithm for computing edge connectivity with high probability in the complete vertex-arrival
streaming model. In this model, the vertices of the graph G arrive in an arbitrary order with all
incident edges. This contrasts with the edge-arrival streaming model, where edges of G arrive
in arbitrary order, for which a $\tilde{\Omega}(n^2)$ lower bound was proven on the space complexity of a onepass algorithm that computes the edge connectivity [Zel11]. This bound can be modified to also
prove an $\Omega(n^2)$ lower bound on the one-pass space complexity of edge connectivity in the more
restrictive explicit vertex-arrival model, where the vertices of G arrive only with the edges incident
on the previously seen vertices, as was considered in e.g. [CDK19]. For completeness, we include
a proof sketch of this lower bound in Appendix C.1. If however the vertices arrive with edges
incident on the previously seen vertices in a random order, then our technique still implies an $\tilde{O}(n)$ -

⁴It is intuitive that computing the edge connectivity of a graph is more difficult than computing the minimum degree, and we formalize this via a simple reduction in Appendix D.

space algorithm. For comparison, we also discuss why it is not clear how to use the related 2-out contraction technique to achieve these results.

2 Technical overview

In the following sections we introduce one of the main tools in this work, star contraction, and give a sketch of the classical and quantum cut query algorithms that make use of star contraction.

2.1 Star contraction

The main workhorse for proving our results is a new technique for randomly contracting edges of a simple graph while preserving a non-trivial minimum cut with constant probability. The idea of contracting edges while preserving non-trivial minimum cuts comes from the celebrated result of Kawarabayashi and Thorup (Fulkerson Prize 2021) [KT19], which gave the first near-linear time deterministic algorithm for computing the edge connectivity of a simple graph. A critical observation in their work is the following: we can contract edges in a simple graph G to get a graph G' so that (i) G' has $\widetilde{O}(n/\delta(G))$ vertices and $\widetilde{O}(n)$ edges, where $\delta(G)$ is the minimum degree of G, and (ii) all non-trivial minimum cuts in G are preserved (i.e., no edge participating in a non-trivial minimum cut is contracted). In particular, if G has a non-trivial minimum cut then $\lambda(G') = \lambda(G)$. Such a contraction is useful for computing edge connectivity since when the edge connectivity is large and there is a non-trivial minimum cut (which is usually harder to handle), the contraction significantly sparsifies and reduces the number of vertices of the graph. We call this type of contraction a KT contraction.

The KT contraction technique has been highly influential, and many works have since used and studied it. The algorithm for KT contraction given in [KT19] takes time $O(m \log^{12} n)$ in the sequential setting when the graph has m edges. This was later improved by Henzinger, Rao, and Wang [HRW20] to $O(m \log^2 n(\log \log n)^2)$. Using an expander decomposition algorithm [CGL+20, SW19, NS17, Wul17] as a black box, Saranurak [Sar21] showed a slower but simpler $\widetilde{O}(m^{1+o(1)})$ time algorithm to compute a KT contraction. All these algorithms are deterministic but rather complicated, making them hard to adapt to other settings. Rubinstein, Schramm, and Weinberg [RSW18] provide a randomized algorithm for computing a KT contraction that is efficient in the cut-query setting, and leads to their aforementioned $O(n \log^3 n)$ cut query algorithm for edge connectivity.

Most relevant for our work is the beautiful 2-out contraction algorithm by Ghaffari, Nowicki, and Thorup [GNT20]. In this algorithm, every vertex independently at random (with replacement) chooses two of its incident edges to contract. Ghaffari et al. show that the resulting contracted graph G' has only $O(n/\delta(G))$ vertices with high probability, and moreover if G has a non-trivial minimum cut then $\lambda(G) = \lambda(G')$ with constant probability. They use this algorithm to get the current fastest randomized algorithm for edge connectivity with runtime⁵ $O(\min\{m+n\log^2 n, m\log n\})$, and they also obtain improved algorithms for edge connectivity in the distributed setting.

Although [GNT20] did not study the cut query model, the 2-out contraction approach gives a simple randomized algorithm for edge connectivity with $O(n \log n)$ cut queries, improving the bound from [RSW18]. As this is very related to our approach, we give an outline of the proof here. First, we can compute $\delta(G)$ with n cut queries by querying $|\text{cut}(\{v\})|$ for every vertex v. The next thing to notice is that for any vertex v we can randomly choose a neighbor of v with $O(\log n)$ cut queries using a randomized version of binary search. This is because with 3 cut queries we can

⁵The stated bound in [GNT20] is $O(\min\{m + n \log^3 n, m \log n\})$, but more recent work on the minimum cut problem by [GMW20] improves it to the bound we quote here.

compute |E(v,S)| for any set $S \subseteq V \setminus \{v\}$ (see Proposition 3.6), and thus can continue searching for a neighbor of v in the set S with probability proportional to this number. Thus with $O(n \log n)$ queries we can perform 2-out contraction and identify the sets of vertices forming the "supervertices" of the contracted graph G'. By the main theorem of [GNT20], with high probability G' will have $O(n/\delta(G))$ supervertices. The remaining task is to compute the edge connectivity of G'. To do this we can make use of a very useful tool developed by Nagamochi and Ibaraki [NI92] called a sparse redge connectivity certificate. Let F be the set of edges found by repeating r times: (i) find a spanning forest of G' and (ii) add the edges of this spanning forest to F and remove them from G'. Then Nagamochi and Ibaraki show that if $|\mathrm{cut}_G(S)| \leq r$ then $\mathrm{cut}_G(S) = \mathrm{cut}_F(S)$. In particular, the edge connectivity of a sparse $\delta(G)$ -edge connectivity certificate of G' will equal $\tau = \min\{\delta(G), \lambda(G')\}$. Contraction cannot decrease edge connectivity, so if $\lambda(G) = \delta(G)$ then τ will always be the correct answer; if $\lambda(G) < \delta(G)$ then it will be correct whenever we do not contract an edge of a non-trivial minimum cut in the 2-out contraction, which happens with constant probability. We can find a single spanning forest of G' deterministically with $O(n \log(n)/\delta(G))$ cut queries [Har08, Theorem 5.10], thus we can find a sparse $\delta(G)$ -edge connectivity certificate of G' with $O(n \log n)$ cut queries overall.

It is not obvious how to independently sample a uniformly random neighbor of every vertex without spending $\Omega(\log n)$ cut queries on average per vertex. Even with the very powerful matrix-vector multiplication queries, where one can learn the entire neighborhood of a vertex with a single query, it is not clear what else one can do to avoid spending $\Omega(1)$ queries per vertex on average to implement 2-out contraction.

We introduce a new graph contraction technique called *star contraction* that allows one to take advantage of the power of cut and matrix-vector multiplication queries to process vertices in parallel. For greater flexibility in the applications to different types of queries, we state this as a general method that can be instantiated in various ways.

Technique 2.1 (Star contraction). Let G = (V, E) be a simple graph and $p \in (0, 1]$ be a probability parameter (think of $p \in \widetilde{\Theta}(1/\delta(G))$).

- 1. Define a set of "center vertices" R where every vertex is put into R independently at random with probability p.
- 2. Define a set of "star edges" X by doing the following for every vertex $v \notin R$: pick a neighbor $c \in R$ (if it exists) and put the edge $\{v,c\}$ into X. The set X is a collection of star subgraphs centered at vertices in R.

Output the graph G' obtained from G by contracting all edges in X.

Note that in item 2 we do not specify how to pick a neighbor in R. The rule for doing this will vary in our applications. The nice thing about the star contraction framework is that no matter what rule is used here, the number of vertices in G' will always be at most |R| plus the number of vertices in $V \setminus R$ that have no neighbor in R. By taking $p = \Theta(\log(n)/\delta(G))$, for example, with high probability all vertices will have a neighbor in R and hence G' will only have $O(n \log(n)/\delta(G))$ many vertices. This leaves one only with the question of choosing a good rule to instantiate item 2 that does not choose an edge of non-trivial minimum cut with too high a probability, and that can be efficiently performed in the query model of interest.

The most natural rule to instantiate item 2 of Technique 2.1 is to have each vertex in $V \setminus R$ independently and uniformly at random choose a neighbor in R. We refer to this instantiation as uniform star contraction. The analysis of this case suffices for our algorithms in the quantum cut query model, the matrix-vector multiplication query model, and the streaming model. As an example, in the matrix-vector multiplication and quantum cut query settings we can learn all the

neighbors of a vertex with $\widetilde{O}(1)$ queries. Hence, we can learn all edges incident on the center vertices R with only $\widetilde{O}(|R|)$ queries, which then allows us to implement uniform star contraction. Now if $\delta(G)$ is large (and we choose $p = \Theta(\log(n)/\delta(G))$ as above) then the query cost $\widetilde{O}(|R|)$ will be small, and we will take advantage of this. This contrasts with the case of 2-out contraction, where in general there does not exist a small set so that all contracted edges are incident on this set (as an example, consider the case where G is the complete graph). Formally, we show the following theorem about uniform star contraction.

Theorem 2.2. Let G = (V, E) be an n-vertex simple graph with $\lambda(G) < \delta(G)$. Then uniform star contraction with $p = \frac{1200 \ln n}{\delta(G)}$ gives G' where

- 1. G' has at most $2400n \ln(n)/\delta(G)$ vertices with probability at least $1-1/n^4$, and
- 2. $\lambda(G') = \lambda(G)$ with probability at least $2 \cdot 3^{-13}$.

We give an overview of the proof here. As mentioned, the number of vertices in the contracted graph G' is at most |R| plus the number of vertices in $V \setminus R$ that have no neighbor in R. By a Chernoff bound, with high probability |R| will be at most twice its expectation, which is $\Theta(n \log(n)/\delta(G))$. Further, the expected number of neighbors of any vertex v among the centers R is $\Omega(\log n)$. Thus, by a Chernoff bound plus a union bound, every vertex in $V \setminus R$ will have a neighbor in R with high probability. This argument nearly trivially bounds the number of vertices in G'. In contrast, bounding the number of vertices in G' is the most complicated part of the proof for the analog of Theorem 2.2 for 2-out contraction, although it must be noted the bound obtained there is better by a factor of $\log n$. Another nice property of star contraction is that each connected component of G' has diameter 2, a property that is useful for designing algorithms in models of distributed computing. Ghaffari et al. show that after 2-out contraction the average diameter of a component is $O(\log \delta(G))$ with high probability [GNT20, Lemma 5.1].

For the second item of the theorem it is useful to first review the proof of the analogous statement for 2-out contraction. Let C be a non-trivial minimum cut of G. Let a random 1-out sample of G be the set of edges obtained by independently and uniformly at random selecting an edge incident to each vertex. A 2-out contraction is exactly the process of independently performing two random 1-out samples of G and contracting all the selected edges. The probability that we contract an edge of G in performing a 2-out contraction is exactly the square of the probability that we select an edge of G in a random 1-out sample.

For every vertex v let d(v) be the degree of v and c(v) be the number of u such that $\{u,v\} \in C$. Let $N(C) = \{v \in V : c(v) > 0\}$ be the set of vertices incident to C. When we take a random 1-out sample of G, the probability that we do not choose an edge of C is exactly

$$\prod_{v \in N(C)} \left(1 - \frac{c(v)}{d(v)} \right) . \tag{1}$$

At first it might seem that this probability could be very small. The key to lower bounding it combines two observations:

$$\frac{c(v)}{d(v)} \le 1/2 \text{ for every } v \in N(C)$$
 (2)

$$\sum_{v \in N(C)} \frac{c(v)}{d(v)} \le 2 \frac{|C|}{\delta(G)} \le 2. \tag{3}$$

⁶Perhaps more comparable to our case, Ghaffari et al. also obtain a worst-case upper bound of $O(\log n)$ on the diameters of the connected components of a contracted graph G' with $O(n\log(n)/\delta(G))$ vertices obtained by only contracting a subset of the edges selected in a 2-out sample [GNT20, Remark 5.3].

The first inequality follows from the fact that C is a non-trivial cut, and if Eq. (2) did not hold then we could move v to the other side and obtain a smaller cut. To derive Eq. (3) we use the fact that $d(v) \geq \delta(G)$ and $|C| \leq \delta(G)$.

How small can Eq. (1) be under the constraints of Eq. (2) and Eq. (3)? In fact it is always at least 1/16: some thought shows that the minimum of Eq. (1) will be achieved at an extreme point of the set of constraints, which is obtained when 4 vertices have c(v)/d(v) = 1/2, thereby saturating both Eq. (2) and Eq. (3).⁷ The lower bound of 1/16 is tight as can be seen by taking a non-trivial minimum cut of the cycle graph. This completes the slick argument that indeed 2-out contraction will not contract an edge of a non-trivial minimum cut with constant probability.

Correctness of our algorithms based on star contraction is proven using analogs of Eq. (2) and Eq. (3) (with slightly worse constants). We again illustrate the correctness proof for the case of uniform star contraction. Let $d_R(v) = |E(v, R \setminus \{v\})|$ and $c_R(v) = |C \cap E(v, R \setminus \{v\})|$. As an analog of Eq. (3), we directly show that for all $v \in V$

$$\mathbb{E}_R \left[\frac{c_R(v)}{d_R(v)} \middle| d_R(v) > 0 \right] = \frac{c(v)}{d(v)} . \tag{4}$$

Therefore by linearity of expectation and Eq. (3) we have $\mathbb{E}[\sum_{v:d_R(v)>0} c_R(v)/d_R(v)] \leq 2$, and by Markov's inequality the sum will not significantly exceed this quantity with constant probability. To prove an analog of Eq. (2) we again use the fact that, with high probability, $d_R(v) = \Omega(\log n)$ for all v. By Eq. (4) we also know that $\mathbb{E}[c_R(v)/d_R(v) \mid d_R(v) > 0] = c(v)/d(v) \leq 1/2$. Thus for $c_R(v)/d_R(v)$ to exceed 2/3 we must have $c_R(v) = \Omega(\log n)$ and $c_R(v)$ exceeding its expected value by a constant factor bigger than 1. We then again use a Chernoff bound to show that for each v individually this does not happen with high probability, and finally apply a union bound over all v.

2.2 Matrix-vector multiplication and quantum cut query algorithm

As a direct application of our uniform star contraction procedure we obtain an algorithm for computing the edge connectivity of a simple graph with $O(\sqrt{n})$ quantum cut queries (i.e., Theorem 1.3) or matrix-vector multiplication queries to the adjacency matrix. We sketch the algorithm here and postpone details to Section 4.3. The algorithm uses the following primitives (which we can run either on the original graph, a vertex-induced subgraph, or a vertex-induced subgraph with an explicit set of edges removed):

- P1. Find all neighbors of a vertex. This can be done with 1 matrix-vector multiplication query to the adjacency matrix (for a vertex v, query $A\chi_v$ with χ_v the standard basis vector corresponding to vertex v) or $O(\log n)$ quantum cut queries (this is implicitly shown in [LSZ21] and made explicit in [AL21, Corollary 11]).
- P2. Construct a spanning forest. This can be done with polylog(n) matrix-vector multiplication queries [AL21] or polylog(n) quantum cut queries [LSZ21].
- P3. Compute the minimum degree. This takes 1 matrix-vector multiplication query (query for the matrix-vector product $A\mathbf{1}$, with $\mathbf{1}$ the all-ones vector) or $O(\sqrt{n})$ quantum cut queries (run quantum minimum finding on the vertex degrees).
- P4. Compute a cut query. This can clearly be done with 1 matrix-vector multiplication or quantum cut query.

The Proposition 4.1 for a proof. One can alternatively obtain a looser bound by using the inequality $1-x \ge \exp(-x/(1-x))$ for 0 < x < 1, as is done in [GNT20].

Uniform star contraction for a given parameter p can be implemented with just the first primitive: (i) pick a random subset of vertices R by including every vertex independently at random with probability p, (ii) for every vertex in R learn all its neighbors, explicitly giving the set $\operatorname{cut}(R)$, and (iii) for every vertex v not in R, select a random edge in $\operatorname{cut}(R)$ incident to v (if it exists). Contracting the resulting star graphs yields (the supervertices of) the contracted graph G'. We only make queries in step (ii). By primitive P1, this can be done with O(|R|) queries, and this is O(np) in expectation.

We can now easily sketch our algorithm for computing the edge connectivity of the input graph G using the above primitives:

- 1. Compute the minimum degree $\delta(G)$ using primitive P3.
- 2. If $\delta(G) \leq \sqrt{n}$, find a sparse $\delta(G)$ -edge connectivity certificate using primitive P2. Output the edge connectivity of the connectivity certificate.
- 3. If $\delta(G) > \sqrt{n}$, do uniform star contraction with $p \in \Theta(\log(n)/\delta(G))$, resulting in a contracted multigraph G' that has $\widetilde{O}(\sqrt{n})$ vertices with high probability. Run the randomized algorithm from [MN20] (Theorem 3.10) to compute $\lambda(G')$.

Step 1. can be implemented with $O(\sqrt{n})$ matrix-vector multiplication or quantum cut queries by P3. Step 2. costs polylog(n) queries per spanning forest by P2, thus $\widetilde{O}(\delta(G)) \in \widetilde{O}(\sqrt{n})$ queries overall. In step 3. we have $|R| = O(\sqrt{n} \log n)$ with high probability, in which case the star contraction can be done with $O(\sqrt{n} \log n)$ queries by P1. The algorithm of [MN20] can compute the weight of a minimum cut in a weighted N-vertex graph with high probability after $\widetilde{O}(N)$ classical cut queries, thus by primitive P4 we can compute $\lambda(G')$ with $\widetilde{O}(\sqrt{n})$ queries.

2.3 Randomized cut-query algorithm

Finally we describe our randomized O(n) cut query algorithm for edge connectivity. It does not seem possible to achieve this result using uniform star contraction as we did in the quantum cut query and matrix-vector multiplication query case. The reason is that a vertex in $V \setminus R$ could have up to |R| many neighbors in R and it is too expensive to learn all of these neighbors with cut queries. Instead we use another variation on star contraction that we call sparse star contraction. We show that instead of choosing a uniformly random neighbor in R, we can instead first learn a bipartite subgraph between $V \setminus R$ and R where each vertex in $V \setminus R$ has constant degree. We then do 1-out contraction by independently choosing, for each $v \in V \setminus R$, a uniformly random neighbor in this sparse bipartite subgraph. Our main technical contribution is to show that this process can be done while preserving a non-trivial minimum cut with constant probability. We call this technique sparse star contraction as the contraction is performed on a bipartite subgraph with only O(n) edges. To actually learn this sparse bipartite subgraph with O(n) cut queries we use our second main tool, which is the separating matrix framework of Grebinski and Kucherov [GK98, GK00]. We next elaborate on sparse star contraction and the separating matrix framework in more detail.

Sparse star contraction. To put sparse star contraction into context we begin with a more general scenario. We can imagine a general form of a randomized contraction algorithm that first learns a spanning subgraph H = (V, E') of the input graph G = (V, E), and then for each vertex $v \in V$ independently at random selects an edge incident to v in H. Finally, the selected edges are contracted in the original graph G. In our case it will further be useful to think of H as a directed graph, where we will choose a random outgoing edge from each vertex. This point of view gives us more control over which endpoints can contract an edge. We make the following definition.

Definition 2.3 (Directed subgraph, 1-out contraction). Let G = (V, E) be a simple graph. We say that the directed graph H = (V, A) is a directed subgraph of G if every arc $(u, v) \in A$ satisfies $\{u, v\} \in E$. In a random 1-out sample of H, we independently and uniformly at random choose an outgoing edge in H from every vertex that has one. In a random 1-out contraction of H, we take a random 1-out sample of H and output the graph G' obtained by contracting the sampled edges in G.

Note that in a directed subgraph H, for any edge $\{u,v\}$ of G, we can either have both arcs (u,v),(v,u) in H, just one of them, or neither of them. When we speak about doing 1-out contraction on an undirected subgraph it should be interpreted that all edges are oriented in both directions.

With this terminology, uniform star contraction corresponds to doing a random 1-out contraction on the directed subgraph H which is the induced bipartite graph between $V \setminus R$ and R with all edges directed from $V \setminus R$ to R.

There are two properties that we want in a directed subgraph H. The first is that after a random 1-out contraction on H the contracted graph does not have too many vertices. We can automatically guarantee this condition by working in the star contraction framework. The second is that in taking a random 1-out sample of H we do not have too high probability of selecting an edge of a non-trivial minimum cut. We can precisely extract a sufficient condition that makes a directed subgraph H "good for contracting" in this second sense.

Definition 2.4 $((\alpha, \beta)$ -good for contracting). Let G = (V, E) be a simple graph and $C \subseteq E$. Let H = (V, A) be a directed subgraph of G. For every $u \in V$ let $q_u = \Pr_{v:(u,v)\in A}[\{u,v\} \in C]$ if $|\{(u,v)\in A:v\in V\}|>0$ and $q_u=0$ otherwise. We say that H is (α,β) -good for contracting with respect to C if it satisfies the following two conditions

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1. max property: \max_u q_u \leq \alpha, and
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2. sum property: $\sum_{u} q_u \leq \beta$.

An undirected subgraph of G is (α, β) -good for contracting if and only if its directed version where all edges are directed in both directions is.

As an example, it follows from Eq. (2) and Eq. (3) used in the correctness proof of 2-out contraction that G itself is (1/2, 2)-good for contracting for any non-trivial minimum cut C. In Corollary 4.2 we show that if H is (α, β) -good for contracting with respect to C then the probability we do not select an edge of C in a random 1-out sample of H is at least $(1 - \alpha)^{\lceil \beta/\alpha \rceil}$.

In sparse star contraction, we again start out by choosing a random set R by taking each vertex v to be in R with probability p, although we take $p = \Theta(\log(\delta(G))/\delta(G))$ to be slightly smaller than what we used before. With constant probability the number of vertices in $V \setminus R$ with no neighbor in R will be $O(n/\delta(G))$, and R itself will satisfy $|R| = O(n\log(\delta(G))/\delta(G))$. Let H be the induced bipartite subgraph between $V \setminus R$ and R with all edges directed from $V \setminus R$. In uniform start contraction we do a random 1-out contraction on H. For the randomized cut query algorithm we will learn a sparse subgraph H' of H that has the property that every $v \in V \setminus R$ that has an outgoing edge in H also has an outgoing edge in H'. No matter what H' we take with this property we are guaranteed that after 1-out contraction the resulting contracted graph will have $O(n\log(\delta(G))/\delta(G))$ vertices. Our main technical contribution (in particular Lemma 6.1) shows that we can find such an H' that is (α, β) -good for contracting for $\alpha < 1$ and small constant β that has constant degree. As H' only has O(n) edges, we can hope to learn it with O(n) cut queries, and we show this can indeed be done using the separating matrix machinery, described next.

Separating matrices and Recover-k-From-All. The second key tool of our algorithm is the separating matrix machinery. This toolset is best described by first considering an immediate obstacle to our O(n) cut query algorithm for edge connectivity: an O(n) bound is not even known for the simpler problem of determining if a graph is connected. Harvey gave a deterministic $O(n \log n)$ cut query algorithm for connectivity [Har08, Theorem 5.10], and we are not aware of any better result in the randomized case. Besides the fact that connectivity is a special case of edge connectivity, our algorithmic framework will also heavily rely on being able to efficiently find spanning forests to construct sparse r-edge connectivity certificates.

Harvey's connectivity algorithm, which can also find a spanning forest, is an implementation of Prim's spanning forest algorithm in the cut query model. This algorithm can equally well be implemented with a weaker oracle that simply reports whether or not there exists an edge between two disjoint sets S and T (this is known in the literature as a bipartite independent set oracle, see e.g. [BHR⁺20]). Interestingly, Harvey's algorithm is actually optimal if restricted to this type of queries. Indeed, any deterministic algorithm that determines connectivity while making use of an oracle that returns 1 bit of information must make $\Omega(n \log n)$ queries. This follows from the aforementioned deterministic $\Omega(n \log n)$ 2-party communication complexity lower bound for deciding if a graph is connected [HMT88].⁸

The key now to both our zero-error O(n) cut query algorithm for finding a spanning forest and our randomized O(n) cut query algorithm for edge connectivity is to make use of the fact that a cut query actually returns $\Omega(\log n)$ bits of information. This power was first harnessed by Grebinski and Kucherov [GK98, GK00] who studied a related, but more powerful, query known as an additive query. In this model, when the input is a simple n-vertex graph with adjacency matrix A one can query two Boolean vectors $x, y \in \{0, 1\}^n$ and receive the answer x^TAy . Grebinski and Kucherov [GK00] showed the surprising fact that one can learn an n-vertex simple graph with only $O(n^2/\log n)$ additive queries, achieving the information theoretic lower bound. The main tool in the proof of Grebinski and Kucherov is the use of separating matrices: the existence of an $O(n/\log n)$ -by-n matrix B such that $Bx \neq By$ for any two distinct n-dimensional Boolean vectors x and y. In Lemma 5.4 we use the separating matrix framework of Grebinski and Kucherov to show that if S, T are disjoint subsets of V that are polynomially related in size and $d_T(v) \leq \ell$ for every $v \in S$, then we can learn all edges between S and T with only $O(\ell|S|)$ cut queries.

This fact is the heart of the subroutine Recover-k-From-All (Algorithm 5.7) which plays a key role in both the spanning forest and edge connectivity algorithms. The input to this algorithm is two disjoint subsets $S, T \subseteq V$ that are polynomially related in size with the promise that $d_T(v) \ge \ell$ for every $v \in S$. Recover-k-From-All is a zero-error randomized algorithm that can then learn $\min\{k,\ell\}$ neighbors in T for every vertex $v \in S$ and makes O(k|S|) cut queries in expectation. Recover-k-From-All is based on ideas from ℓ_0 sampling (e.g. Theorem 2 of [JST11]), which is similarly used in the connectivity algorithm in the semi-streaming model by Ahn, Guha, and MacGregor [AGM12]. First we put vertices in S into $O(\log n)$ buckets by putting together those vertices with similar values of $d_T(v)$. For the bucket B with degree around v into v0, we randomly subsample a set v1 by putting each vertex of v2 into v3 with probability v4. We call a vertex in the bucket "caught" if v4. We can then learn v6 be the set of caught vertices, we can then learn v7 with v8 we expect to catch a constant fraction of the remaining

⁸We do not know of any such superlinear lower bound for randomized algorithms making bipartite independent set queries or the randomized communication complexity of connectivity. The best known bounded-error randomized communication complexity lower bound of $\Omega(n)$ follows from a reduction from set-disjointness [BFS86, Corollary 7.4] or the inner-product mod 2 function [IKL⁺12, Theorem 1] on $\Theta(n)$ bit inputs.

⁹We assume that B', T' are polynomially related in size for this high level description. Handling smaller B' is a technicality postponed to the full proof.

vertices in a bucket with each iteration, and the complexity of an iteration scales with the number of remaining vertices, one can argue that the expected number of queries overall is O(kn).

Edge connectivity. Now that we described the main tools, we can describe the main algorithm. To make the exposition simpler, we begin with explaining how star contraction and Recover-k-From-All can be used to give a randomized $O(n \log \log n)$ cut query algorithm for edge connectivity. This algorithm is given in Section 6. We then describe the additional trick needed to reduce the query complexity to O(n), which is given in Section 7. The basic algorithm essentially follows the same steps as used in the quantum cut query case.

- 1. Compute the minimum degree $\delta(G)$.
- 2. Perform sparse star contraction:
 - (a) Choose a set R by taking each $v \in V$ to be in R with probability $\Theta(\log(\delta(G))/\delta(G))$. Let H be the directed subgraph obtained by picking every edge between $V \setminus R$ and R and directing it from $V \setminus R$ to R.
 - (b) Use Recover-k-From-All with constant k on $V \setminus R$ and R to learn a sparse subgraph H' of H.
 - (c) Do a random 1-out contraction on H' and let G' be the resulting graph.
- 3. Compute the edge connectivity of G', and output the minimum of this and $\delta(G)$.

As in the proof of Theorem 2.2, we can again argue that with constant probability (i) H is (α, β) -good for contracting for some $\alpha < 1$ and constant β (more specifically, for $\alpha = 3/5$ and $\beta = 8$) with respect to a non-trivial minimum cut, and (ii) that only $O(n/\delta(G))$ vertices in $V \setminus R$ have no neighbor in R. Now, however, it is too expensive to learn the entire subgraph H as in the quantum cut query algorithm, or even to independently sample a uniformly random neighbor in R for each $v \in V \setminus R$ within the O(n) query budget. Instead, we use Recover-k-From-All with constant k to learn a sparse subgraph H' of H, where H' has an outgoing edge for every $v \in V \setminus R$ that has one in H. Moreover, the outgoing neighbors of v in H' are learned from a random set of vertices, conditioned on this set having at least one and not too many neighbors of v. This can be done with O(n) queries. We then do a random 1-out contraction on the explicitly known graph H'. Letting G' be the result of this contraction, we finally compute the edge connectivity of G' and output the minimum of this and $\delta(G)$.

We postpone describing how we compute the edge connectivity of G' and instead focus on showing that H' is still (α', β') -good for contracting for some $\alpha' < 1$ and constant β' . As we use a constant k in Recover-k-From-All, we only expect to find a constant number of neighbors of a particular vertex v. We have to show that, even in this very small sample, not too high a fraction of neighbors are on the opposite side of a non-trivial cut from v, for all vertices v incident on the cut. In this low probability sampling regime, a Chernoff bound can only upper bound the failure probability for a particular vertex by a constant, which is not good enough as we have to union bound over the possibly growing number of vertices incident on the cut.

Instead, in Lemma 6.1, we show the following statement. Let $v \in V \setminus R$ and C be a non-trivial minimum cut of G. Let $R' \subseteq R$ be chosen by putting each vertex of R into R' with probability $p = 2k/d_R(v)$ conditioned on $d_{R'}(v) > 0$. We have already mentioned the fact that $\mathbb{E}_{R'}[c_{R'}(v)/d_{R'}(v) \mid d_{R'}(v) > 0] = c_R(v)/d_R(v)$. We show that as long as $k \ge 10$

$$\Pr_{R'} \left[\frac{c_{R'}(v)}{d_{R'}(v)} \ge \frac{c_R(v)}{d_R(v)} + \frac{1}{10} \mid d_{R'}(v) > 0 \right] \le \frac{200}{k} \frac{c_R(v)}{d_R(v)} \ .$$

We know that $\sum_{v:c_R(v)>0} c_R(v)/d_R(v) \leq 8$ since H is (3/5,8)-good for contracting. Hence, by relating the failure probability to a sum that is bounded, and taking k to be a large constant, we can again use a union bound to argue that H' satisfies the max property with $\alpha' = 7/10$ with constant probability.

After a random 1-out contraction on H' the contracted graph G' has $O(n\frac{\log(\delta(G))}{\delta(G)})$ vertices. Ideally, however, we would like it to only have $O(n/\delta(G))$ vertices. As we describe next, we can then compute $\lambda(G')$ with O(n) queries by finding a sparse $\delta(G)$ -edge connectivity certificate. To further reduce the size of the contracted graph, we use Recover-k-From-All with $k = \Theta(\log(\delta(G)))$ to learn a directed subgraph H_2 of G[R] where all but $O(n/\delta(G))$ vertices have outdegree $h = \Omega(\log(\delta(G)))$. This requires $O(\log(\delta(G))|R|) = O(n)$ cut queries. As was done with H' we can similarly argue that H_2 is (α, β) -good for contracting for yet other $\alpha < 1$ and constant β . We then do a 2-out contraction on H_2 . We can use a lemma of [GNT20, Lemma 2.5] to conclude that 2-out contraction on H_2 reduces the number of vertices in G[R] by a factor of h and thus it becomes $O(n/\delta(G))$. As all but $O(n/\delta(G))$ vertices in $V \setminus R$ are connected to a vertex in R this reduces the number of vertices in G' overall to $O(n/\delta(G))$.

Spanning forests and sparse edge connectivity certificates. In order to accomplish step 3. of the algorithm we show that we can construct a sparse r-edge connectivity certificate in a contracted graph with q vertices using $O(n + rq \log(n)/\log(q))$ cut queries. This lets us construct a sparse $\delta(G)$ -edge connectivity certificate of G' with O(n) queries when G' has $O(n/\delta(G))$ vertices. In the final part of this section we give an overview of the key ideas that go into this algorithm and the obvious prerequisite of constructing a spanning forest with O(n) cut queries.

Our spanning forest algorithm follows the framework of Borůvka's spanning forest algorithm as has been used in several works related to matrix-vector multiplication queries [AGM12, LSZ21, AL21]. The application here requires several additional tricks to stay within the O(n) query budget.

The algorithm proceeds in rounds and maintains the invariant that in each round there is a paritition S_1, \ldots, S_t of V and a spanning tree for each S_i in the partition. Initially, each S_i is just a single vertex. In each round, it performs the following two steps:

- 1. For each S_i , it finds a vertex that has at least one neighbor outside S_i . We call such vertices active vertices. Whether or not a vertex is active can be determined with a constant number of cut queries by computing $|E(v, \bar{S}_i)|$ for $v \in S_i$. We go over each S_i looking for an active vertex; once we find an active vertex in S_i we move on to S_{i+1} . The vertices that are discovered to be inactive are ignored for all future rounds of the algorithm.
- 2. Next we randomly bipartition the set of connected components and use Recover-k-From-All with constant k to learn, for each active vertices on one side, a neighbor on the other side. As in the case of Borůvka's algorithm, we then combine the components which are connected by edges we discovered and reduce the number of components by a constant fraction.

Note that, across all iterations of step 1, we make at most n many useless queries (i.e., queries where we find a vertex to be inactive). So we only need to account for the query complexity of step 2. Here we crucially use the fact that we can reduce the number of connected components by a constant factor to show that total number of cut queries required over all invocations of Recover-k-From-All is bounded by O(n).

The next task is to extend the spanning forest algorithm to also construct sparse edge connectivity certificates. For our application we will want to construct a sparse edge connectivity certificate of the contracted graph G', which is an integer weighted graph with $q \in o(n)$ vertices. The most natural idea would be to extend our spanning forest algorithm to construct a spanning forest of

such a graph while making only O(q) cut queries. We could then directly construct a sparse redge connectivity certificate with O(rq) cut queries by iteratively finding one spanning forest at a
time. Unfortunately, we do not know how to find a spanning forest in a contracted graph more
efficiently. The reason is that if the adjacency matrix of the graph has entries with magnitude Mthis introduces an extra $\log(M+1)$ factor into the separating matrix bounds, which we cannot
afford.

Instead we revisit the spanning forest algorithm and further parallelize Borůvka's algorithm by simultaneously building the different forests of the sparse edge connectivity certificate. We also crucially make use of the fact that G' is not an arbitrary integer weighted graph, but the contraction of a simple graph G, and that for our application we can afford an extra additive O(n) term. The fact that G' is a contraction of a simple graph allows us to keep using the separating matrix machinery on Boolean matrices by working on appropriate submatrices of the adjacency matrix of G, and the extra O(n) term is used to identify these submatrices.

Let F_1, \dots, F_r be the spanning forests that we want to compute, recalling that F_i is spanning forest in the graph $G \setminus (\bigcup_{j < i} F_i)$. Initially, each F_i is empty. As before, we use steps 1. and 2. to find edges to extend these spanning tree. However, the crucial difference is the following: We find these edges with respect to the connected components of the last tree F_r , and we add each of these edges into the spanning forest F_i for the least value of i where it does not create a cycle. It is not hard to see that the set of connected components $\{S_1^{(i)}, \dots, S_{t_i}^{(i)}\}$ of F_i for different $i \in [r]$ form a laminar family: $\{S_1^{(i+1)}, \dots, S_{t_{i+1}}^{(i+1)}\}$ is a refinement of $\{S_1^{(i)}, \dots, S_{t_i}^{(i)}\}$. However, we cannot expect that the number of connected components of F_r will decrease by a constant factor in each round as before. We can however show that it happens within O(r) rounds. This, together with a similar accounting of cut queries as before, leads to the following theorem.

Theorem 2.5 (Informal version, see Theorem 5.9). Let G = (V, E) be an n-vertex simple graph, and let G' = (V', E') be a contraction of G with q supervertices, for q sufficiently large. There is a zero-error randomized algorithm that makes $O(n + rq \log(n)/\log(q))$ cut queries in expectation and outputs a sparse r-edge connectivity certificate for G'.

2.4 Open questions

Our work raises some open questions that concern both upper bounds and lower bounds.

Lower bounds. The tightness of a number of our algorithms hinges on a positive answer to the following questions.

• Can we show a lower bound of $\Omega(n \log n)$ for the randomized two-party communication complexity of edge connectivity? The current best known bound in the randomized case is $\Omega(n \log \log n)$ [AD21], while the deterministic communication complexity of this problem is known to be $\Omega(n \log n)$ [HMT88]. A positive answer to this question implies an $\Omega(n)$ lower bound on the randomized cut query complexity of edge connectivity, showing that Theorem 1.1 is tight. On the flip side, a randomized algorithm for edge connectivity making o(n) cut queries would imply a negative answer to this question. It is reasonable to think that a lower bound of $\Omega(n \log n)$ for the randomized two-party communication complexity should hold even for the simpler problem of deciding if graph is connected, and we conjecture

¹⁰Nagamochi-Ibaraki [NI92] also construct all spanning forests of the sparse edge connectivity certificate in parallel. They iterate over each edge of the graph and place it in the first spanning forest in which it does not create a cycle. We cannot afford to iterate over all edges and instead modify Borůvka's algorithm to build the spanning forests in parallel.

this to be true. Proving this would resolve the randomized communication complexity of connectivity, which has remained open since the work of Babai, Frankl, and Simon [BFS86].

• Does computing the minimum degree of a simple graph indeed require $\Omega(\sqrt{n})$ quantum cut queries? As mentioned before, quantum minimum finding gives a simple $O(\sqrt{n})$ upper bound. By a reduction from minimum degree to edge connectivity (Appendix D), a positive answer would imply that our $\widetilde{O}(\sqrt{n})$ quantum cut query algorithm for edge connectivity is tight (up to polylogarithmic factors).

Upper bounds. Our algorithms could give rise to new algorithms and upper bounds in a number of ways.

- Weighted graphs: Can we find a minimum cut in a weighted graph with O(n) cut queries? This would not violate any currently known lower bound, and it would improve on the $O(n \operatorname{polylog} n)$ cut query algorithm from [MN20]. Similarly, can we find a minimum cut in a weighted graph with o(n) quantum cut queries or matrix-vector queries?
- (Approximate) edge connectivity with polylog(n) queries: We mentioned that a key bottleneck for edge connectivity with quantum cut queries is computing the minimum degree, which might require $\Omega(\sqrt{n})$ quantum cut queries. In contrast, we can approximate the minimum degree with polylog(n) quantum cut queries [LSZ21]. The possibility hence remains of approximating the edge connectivity with polylog(n) quantum cut queries. We also note that the minimum degree can be computed with exactly 1 matrix-vector query to the adjacency matrix. Hence, computing the edge connectivity with polylog(n) matrix-vector queries is also an interesting open question.
- **Zero-error and deterministic:** Can we make our zero-error O(n) cut query algorithm for connectivity deterministic? Alternatively, could we make our randomized O(n) cut query algorithm for edge connectivity zero-error? Both improvements would lead to tight algorithms (see Table 1).

2.5 Organization

In Section 3, we define and state the necessary preliminary results that we need to state the technical details in the subsequent sections. In Section 4, we give a formal proof of the correctness of uniform star contraction and provide three direct applications: in Section 4.3, we design efficient quantum cut query and matrix-vector multiplication query algorithms (thereby proving Theorem 1.3), in Section 4.4, we provide one-pass semi-streaming algorithms for edge connectivity in the complete and random vertex arrival model, and in Section 4.5, we show a linear-time sequential algorithm for edge connectivity for dense graphs. Next, in Section 5, we show a zero-error algorithm for computing a spanning forest with O(n) cut queries. In Section 6 we give a randomized $O(n \log \log n)$ cut query algorithm for edge connectivity which combines many of the ideas from the previous sections. Finally, in Section 7, we add one additional trick to give a randomized O(n) cut query algorithm for edge connectivity, thereby proving Theorem 1.1.

3 Preliminaries

Notation (strings, sets and matrices). For a string $x \in \{0,1\}^n$ we use |x| for the number of ones in x. For a positive integer n we let $[n] = \{1, \ldots, n\}$. For a set $S \subseteq [n]$ we use \bar{S} for the complement of S and |S| for its cardinality. Given a matrix M of dimension k-by- ℓ , we denote

the i^{th} row of M as M(i,:). Note that M(i,:) is a vector of dimension ℓ . Given a set of column indices $R \subseteq [\ell]$, we define M(i,R) to be the subvector of M(i,:) that has entries restricted to the indices in R. Clearly M(i,R) has dimension |R|. For a subset $R \subseteq [n]$, we let $\chi_R \in \{0,1\}^n$ be the characteristic vector of R, that is, $\chi_R(i) = 1$ if $i \in R$ and $\chi_R(i) = 0$ otherwise.

Notation (graphs). Let V be a finite set and $V^{(2)}$ the set of all 2-element subsets of V. We represent a weighted graph G by the triple G = (V, E, w), where $E \subseteq V^{(2)}$ is the set of edges and $w : E \to \mathbb{R}_{>0}$ assigns a positive weight to each edge. For a subset of edges $F \subseteq E$ we let $w(F) = \sum_{f \in F} w(f)$. When the weight of every edge is 1, we call the graph unweighted. We will consider two kinds of unweighted graphs, multigraphs and simple graphs. In a multigraph E is allowed to be a multiset of $V^{(2)}$ while in a simple graph it is simply a subset. In both cases we drop the weight function and write the graph as G = (V, E). The inputs to our algorithms will always be simple graphs, and multigraphs only arise by considering contractions of a simple graph, described below. We typically denote the number of vertices |V| = n and the number of edges |E| = m.

Let G = (V, E, w) be a weighted graph. For disjoint sets $S, T \subseteq V$ we let $E_G(S, T) = \{e \in E : |e \cap S| = |e \cap T| = 1\}$. As shorthand, we use $\operatorname{cut}_G(S) = E_G(S, \overline{S})$ for the *cut* defined by S, the set of edges with exactly one endpoint in S. We will drop the subscript when the graph is clear from context. We let $\lambda(G) = \min_{\emptyset \neq S \subseteq V} w(\operatorname{cut}_G(S))$ be the weight of a minimum cut of G. When G is unweighted we call $\lambda(G)$ the *edge connectivity* of G. The degree of a vertex v is denoted by $d(v) = |\operatorname{cut}(\{v\})|$ and the number of edges from v into a subset R is $d_R(v) = |E(\{v\}, R \setminus \{v\})|$. Finally, the minimum degree of a graph is denoted by $\delta(G) = \min_{v \in V} d(v)$.

We will also consider directed graphs H=(V,A) where $A\subseteq\{(u,v):u,v\in V,u\neq v\}$ is set of directed edges or arcs. Directed graphs in this work will always arise by taking a (subgraph of a) simple graph and orienting the edges, possibly in both directions. We will use analogous notations with arrows on top of them for directed graphs. For disjoint sets S,T we let $\overrightarrow{E}_H(S,T)=\{(u,v)\in A:u\in S,v\in T\}$ for the set of arcs directed from S to T and use the shorthand $\overrightarrow{\operatorname{cut}}_H(S)=\overrightarrow{E}_H(S,\overline{S})$ for the set of arcs leaving S. Again we drop the subscript when the graph is clear from context.

Finally, in our algorithms we will look at contractions of simple graphs. For a simple graph G = (V, E) and a subset of edges $F \subseteq E$, the multigraph G' = (V', E') formed by contracting the edges F in G is defined as follows. V' is the set of connected components of F. We will sometimes refer to the vertices of G' as supervertices as they are sets of vertices of G. For $S, T \in V'$ the number of edges between S, T in G' is $|E_G(S,T)|$. We will sometimes instead view a contraction of a simple graph as an integer weighted graph, as a cut query cannot distinguish between these representations.

Useful bounds. Next, we define some useful algebraic bounds which we use in the analysis of our algorithms.

Claim 3.1. Let b_1, \ldots, b_t satisfy $\sum_{i=1}^t b_i = n$ and $b_i \ge 1$ for all $i = 1, \ldots, t$. Then

$$\sum_{i=1}^{t} \frac{b_i}{\log(2b_i)} \le \frac{3n}{\log(2n/t)} .$$

Proof. Let e = 2.718... be Euler's constant. On the interval $[1, \infty)$ the function $x/\log(e^2x)$ is

concave and satisfies $x/\log(2x) \leq 3x/\log(e^2x)$. Therefore by Jensen's inequality

$$\sum_{i=1}^{t} \frac{b_i}{\log(2b_i)} \le 3 \sum_{i=1}^{t} \frac{b_i}{\log(e^2b_i)}$$

$$\le \frac{3n}{\log(e^2n/t)} . \square$$

Claim 3.2. Let b_1, \ldots, b_k satisfy $\sum_i b_i = n$. Then $\sum_{i=1}^k \log(b_i) \le k \log(n/k)$.

Proof. By concavity of log(x) we have

$$\frac{1}{k} \sum_{i=1}^{k} \log b_i \le \log \left(\sum_{i=1}^{k} b_i / k \right) = \log(n/k) ,$$

which gives the claim.

We will frequently make use of the following versions of the Chernoff bound.

Lemma 3.3. Let X_1, \ldots, X_n be independent random variables taking values in $\{0, 1\}$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$. Then

$$\Pr[X \le (1 - \delta)\mu] \le \exp(-\delta^2 \mu/2) \text{ for any } 0 \le \delta \le 1$$
 (5)

$$\Pr[X \ge (1+\delta)\mu] \le \exp(-\delta^2 \mu/(2+\delta)) \text{ for any } 0 \le \delta$$
 (6)

In particular, if $\mu \ge 10$ then $\Pr[\mu/2 \le X \le 2\mu] \ge 1/2$.

3.1 Edge connectivity certificates

Here we mention a few known results regarding so-called connectivity certificates (or, in our case, edge connectivity certificate), which we define next.

Definition 3.4 (sparse edge connectivity certificate). Let G = (V, E) be an n-vertex unweighted multigraph. A sparse r-edge connectivity certificate for G is a subgraph $\hat{H} = (V, F)$ with at most rn edges and with the property that $|\text{cut}_{\hat{H}}(S)| = \min\{r, |\text{cut}_{G}(S)|\}$ for any $S \subseteq V$.

A famous result of Nagamochi and Ibaraki [NI92] gives a recipe for computing a sparse edge connectivity certificate efficiently by packing edge disjoint spanning forests.

Theorem 3.5 ([NI92]). Let G = (V, E) be an unweighted multigraph with m edges and for $i \in [r]$ let F_i be a spanning forest of $(V, E \setminus \bigcup_{j=1}^{i-1} F_j)$. Then $\hat{H} = (V, \bigcup_{j=1}^r F_j)$ is a sparse r-edge connectivity certificate for G. In addition, this sparse certificate can be constructed by a deterministic algorithm in time O(m).

3.2 Cut query primitives

In this section, we state a few primitives in the cut query model that we use in various places of our algorithms. Most of these results appear in similar guise in [RSW18, MN20]. We mention them here for completeness.

Proposition 3.6 (Claim 5.1 in [MN20]). Let G = (V, E, w) be a weighted graph and $S, T \subseteq V$ be disjoint sets. The quantity w(E(S,T)) can be computed with 3 cut queries.

Proof. Let A be the adjacency matrix of G, i.e. $A(u,v) = w(\{u,v\})$. Then

$$w(E(S,T)) = \chi_S^T A \chi_T = \frac{1}{2} \left(\chi_{\bar{S}}^T A \chi_S + \chi_{\bar{T}}^T A \chi_T - \chi_{\overline{S \cup T}}^T A \chi_{S \cup T} \right) .$$

The proposition follows by noting that $\chi_{\bar{R}}^T A \chi_R = w(\text{cut}(R))$ can be evaluated with a single cut query, for any $R \subseteq [n]$.

The following two corollaries follow easily from Proposition 3.6.

Corollary 3.7. Let G = (V, E, w) be a weighted graph. For disjoint subsets $S, T \subset V$, let $G' = (S \cup T, E(S, T), w')$ be the induced bipartite subgraph between S and T, where w' is the weight function w restricted to E(S, T). Let $A_{G'} \in \mathbb{R}^{|S| \times |T|}$ be the bipartite adjacency matrix of G'. We can simulate a query of the form $x^T A_{G'} y$ for $x \in \{0, 1\}^{|S|}$ and $y \in \{0, 1\}^{|T|}$ with 3 cut queries to G.

Proof. Let $S_x \subseteq S$ and $T_y \subseteq T$ denote the supports of x and y, respectively. Then the corollary follows by noting that $x^T A_{G'} y = w(E(S_x, T_y))$ and using Proposition 3.6.

Corollary 3.8. Let G = (V, E) be an unweighted multigraph. Let subsets $V' \subseteq V$ and $X \subseteq E$ be given explicitly, and define the graph G' = (V', E') by setting $E' = \{\{u, v\} \in E \setminus X \mid u, v \in V'\}$. Then we can simulate a cut query to G' with 3 cut queries to G.

Proof. Consider $\operatorname{cut}_{G'}(S)$ in G' for a vertex set $S \subset V'$. It is easy to observe that $|\operatorname{cut}_{G'}(S)| = |E_G(S, V' \setminus S) \setminus X|$. The subscripts G and G' clarify the corresponding graph associated with the edge set. We can compute $|E_G(S, V' \setminus S)|$ by making 3 cut queries to G using Proposition 3.6. As we know the set X explicitly, we can compute $|E_G(S, V' \setminus S) \cap X|$ without queries. Subtracting the latter from the former we get the value of $|\operatorname{cut}_{G'}(S)|$.

Proposition 3.9 (Randomized binary search). Let G = (V, E) be a simple graph. Let $v \in V$ and let $R \subseteq V \setminus \{v\}$. There is a randomized algorithm that, if $d_R(v) > 0$, can output a uniformly random neighbor of v in R with $O(\log n)$ cut queries.

Proof. This is a simple variation on [RSW18, Corollary 2.2] which we describe for completeness.

Use a single cut query to check whether v has a neighbor in R. If so, do the following recursively, until we end up with a single neighbor of v: split R into two sets R_1 and R_2 with $|R_1| = \lceil |R|/2 \rceil$ and $|R_2| = \lfloor |R|/2 \rfloor$, and learn the number of neighbors $d_{R_1}(v)$ and $d_{R_2}(v)$ of v in R_1 resp. R_2 . Each of these can be computed with 3 cut queries by Proposition 3.6. Now pick R_1 with probability $d_{R_1}(v)/(d_{R_1}(v)+d_{R_2}(v))$ and otherwise pick R_2 . If the chosen set has a single vertex then return that vertex, otherwise recurse on the chosen set.

Finally, a much more involved primitive is the following randomized cut query algorithm for finding a minimum cut in a weighted graph by Mukhopadhyay and Nanongkai [MN20].

Theorem 3.10 ([MN20, Section 5.1]). Let G = (V, E, w) be a weighted graph with n vertices. There is a randomized algorithm that computes $\lambda(G)$ with high probability after $O(n \log^8 n)$ cut queries.¹¹

4 Star contraction and direct applications

In this section we give the full proof of the uniform star contraction theorem stated in the introduction (Theorem 2.2). We then derive several direct applications of it.

¹¹The bound is stated as $\widetilde{O}(n)$ in [MN20] but we need a concrete exponent in the polylog term to make our algorithm explicit. We take $\log^8(n)$ as a conservative over-estimate; the true cost of the algorithm in [MN20] is likely smaller.

4.1 (α, β) -good for contracting

We prove some preliminaries to set the scene. First, we verify that if a directed subgraph H is (α, β) -good for contracting for C (Definition 2.4) then in taking a random 1-out sample of H we do not have too high a probability of selecting an edge of C. Let G = (V, E) be a simple graph and $C \subseteq E$. Let H = (V, A) be a directed subgraph of G. For every $v \in V$ let $q_v = \Pr_{u:(v,u)\in A}[\{u,v\}\in C]$ if $|\{(v,u)\in A: u\in V\}|>0$ and $q_v=0$ otherwise. The probability we do not choose an edge of C when taking a random 1-out sample on H is exactly

$$\prod_{v \in V} (1 - q_v)$$

We can lower bound this probability via the next proposition. A very similar statement is given in [GNT20, Lemma 2.7]; we give an alternative analysis that improves the bound by a constant factor.

Proposition 4.1. Let n be a positive integer, $0 \le \alpha < 1$, and $\beta \ge 1$. Define

$$F(\alpha, \beta) = \underset{x \in \mathbb{R}^n}{minimize} \quad \prod_i (1 - x_i)$$

$$subject \ to \quad \sum_i x_i = \beta,$$

$$0 \le x_i \le \alpha$$

Then $F(\alpha, \beta) \ge (1 - \alpha)^{\lceil \beta/\alpha \rceil}$.

Proof. We instead analyze $\ln(F(\alpha,\beta))$ whose objective function is $\sum_i \ln(1-x_i)$. The problem then becomes the minimization of a continuous concave function over a compact convex set K. By the Krein-Milman theorem [Rud91, Section 3.23], K is the closed convex hull of its extreme points and therefore by Jensen's inequality a global optimum will occur at an extremal point. It is clear that $F(\alpha,\beta) \geq F(\alpha,\beta')$ for $\beta \leq \beta'$. Therefore we instead lower bound $F(\alpha,\beta')$ with $\beta' = \alpha \lceil \beta/\alpha \rceil$. The extremal points in the set $0 \leq x_i \leq \alpha, \sum_i x_i = \beta'$ have β'/α non-zero entries which are all equal to α . Plugging this in gives the bound.

Corollary 4.2. If H is (α, β) -good for contracting with respect to C then the probability an edge of C is not selected in taking a random 1-out sample of H is at least $(1-\alpha)^{\lceil \beta/\alpha \rceil}$.

Finally, we note a simple property of an (α, β) -good for contracting subgraph.

Proposition 4.3. Let G = (V, E) be a simple graph, $C \subseteq E$ a set of edges, and H a directed subgraph of G that is (α, β) -good for contracting with respect to C. For $v \in V$ if we form the graph H' from H by removing all outgoing edges of v in H then H' is also (α, β) -good for contracting with respect to C.

With these preliminaries in hand to set the scene, we next give the proof of the uniform star contraction theorem (Theorem 2.2). There a subset $R \subseteq V$ is chosen randomly and H is taken to be $(V, \overrightarrow{\text{cut}}(V \setminus R))$, where $\overrightarrow{\text{cut}}(V \setminus R)$ denotes the edges in $\text{cut}(V \setminus R)$ directed from $V \setminus R$ to R.

4.2 Full proof of uniform star contraction

In this section we prove Theorem 2.2, which for convenience is restated here.

Theorem 2.2. Let G = (V, E) be an n-vertex simple graph with $\lambda(G) < \delta(G)$. Then uniform star contraction with $p = \frac{1200 \ln n}{\delta(G)}$ gives G' where

- 1. G' has at most $2400n \ln(n)/\delta(G)$ vertices with probability at least $1-1/n^4$, and
- 2. $\lambda(G') = \lambda(G)$ with probability at least $2 \cdot 3^{-13}$.

We first show two propositions that will help to prove Theorem 2.2. In uniform star contraction we do a random 1-out contraction on the graph $H = (V, \overline{\operatorname{cut}}(V \setminus R))$ where R is chosen randomly by putting each vertex into R with probability p. As in doing 1-out contraction on H we choose a neighbor of every vertex in $V \setminus R$ that has one, the total number of vertices in G' can be upper bounded by |R| plus the number of vertices in $V \setminus R$ that have no neighbor in R. The next proposition bounds the probability that these values are much larger than their expectations.

Proposition 4.4. Let G = (V, E) be an n-vertex graph. Randomly choose a set of vertices R by putting each vertex into R independently at random with probability p. Then

- 1. $\Pr_R[|R| \ge 2pn] \le \exp(-pn/3)$.
- 2. $\Pr_R[d_R(v) \le 0.9pd(v)] \le \exp(-pd(v)/200)$ for any $v \in V$.

Proof. Both items follows directly from a Chernoff bound. For the first item we use Eq. (6) and for the second Eq. (5).

Proposition 4.4 will handle item 1 of Theorem 2.2. We next show that $H = (V, \overrightarrow{\operatorname{cut}}(V \setminus R))$ is (2/3, 8)-good for contracting with probability at least 2/3. This will imply item 2 by Corollary 4.2. We single out showing that H has the "sum property" (item 2 of Definition 2.4) with constant probability in the next proposition. We go ahead and prove a slightly more general statement than is needed for Theorem 2.2, but which will be used in the O(n) randomized cut query algorithm for edge connectivity. It is also interesting to note that this statement holds for any sampling probability p.

Proposition 4.5. Let G = (V, E) be a simple n-vertex graph and let $C \subseteq E$. Let $0 < f \le g \le n$ be positive integers. Choose a set R by putting each vertex of V into R independently at random with probability p. Then for any $v \in V$

$$\mathbb{E}_R\left[\frac{c_R(v)}{d_R(v)} \middle| f \le d_R(v) \le g\right] = \frac{c(v)}{d(v)} \tag{7}$$

Proof. Let d be the degree of v and c be the number of edges of C incident to it. The proposition is equivalent to the following purely probabilistic statement. Let $X_1, \ldots, X_c, Z_1, \ldots, Z_{d-c}$ be independent and identically distributed Bernoulli random variables that are 1 with probability p. $X_i = 1$ represents the event that the ith edge of C incident to v is selected, and $Z_i = 1$ the event that the ith non-edge of C incident to v is selected. Then $X = \sum_{i=1}^{c} X_i$ is the random variable for the total number of edges of C incident to v selected and $V = X + \sum_{i=1}^{d-c} Z_i$ is the random variable for the total number of edges incident to v selected. We want to show $\mathbb{E}[X/Y \mid f \leq Y \leq g] = c/d$.

For $0 < b \le d$ let us first compute $\mathbb{E}[X/Y \mid Y = b] = (1/b)\mathbb{E}[X \mid Y = b]$. We claim that $\mathbb{E}[X \mid Y = b] = cb/d$. By linearity of conditional expectation, $\mathbb{E}[X \mid Y = b] = \sum_{i=1}^{c} \mathbb{E}[X_i \mid Y = b]$. As each X_i and Z_j are identically distributed, $\mathbb{E}[X_i \mid Y = b]$ is independent of i and also equal to $\mathbb{E}[Z_j \mid Y = b]$ for $j = 1, \ldots, d - b$. Thus $b = \sum_{i=1}^{c} \mathbb{E}[X_i \mid Y = b] + \sum_{j=1}^{d-c} \mathbb{E}[Z_j \mid Y = b] = d \cdot \mathbb{E}[X_i \mid Y = b]$ for any i. This implies $\mathbb{E}[X_i \mid Y = b] = b/d$ and so $\mathbb{E}[X \mid Y = b] = cb/d$.

As $\mathbb{E}[X \mid Y = b] = cb/d$ for any $0 < b \le d$, we directly obtain $\mathbb{E}[X/Y \mid f \le Y \le g] = c/d$ for any integers $0 < f \le g$.

Lemma 4.6. Let G = (V, E) be a simple graph and C be a non-trivial minimum cut of G. Choose a set R by putting each $v \in V$ into R independently at random with probability $p \ge 1200 \ln(n)/\delta(G)$. Then the directed subgraph $H = (V, \overrightarrow{cut}(V \setminus R))$ is (2/3, 8)-good for contracting with respect to C with probability at least 2/3.

Proof. As C is a non-trivial minimum cut we know that $\sum_{v:c(v)>0} c(v)/d(v) \leq 2$ (see Eq. (3) and following discussion). Together with Proposition 4.5, linearity of expectation, and Markov's inequality gives

$$\Pr_{R} \left[\sum_{v: c_{R}(v) > 0} \frac{c_{R}(v)}{d_{R}(v)} \ge 8 \right] \le 1/4 ,$$

showing the "sum property" of Definition 2.4 holds except with probability 1/4. Note that we did not have to use the fact that $p \ge 1200 \ln(n)/\delta(G)$ here, this holds for any p.

It remains to show the "max property" of Definition 2.4, where we do use the assumption that p is not too small. For any $v \in V$ we have $d_R(v) \geq 0.9pd(v)$ except with probability at most $\exp(-6 \ln n)$ by item 2 of Proposition 4.4. Thus except with probability n^{-5} this will hold for all v by a union bound. Let us add this to our error probability and assume this holds in the rest of the proof. The expected value of $c_R(v)$ is pc(v) and we know that $c(v)/d(v) \leq 1/2$ because C is non-trivial. Thus $\mathbb{E}[c_R(v)] \leq pd(v)/2$ and to have $c_R(v)/d_R(v) > 2/3$ we must have $c_R(v) > 1.2\mathbb{E}[c_R(v)]$ and $c_R(v) > 720 \ln n$, the latter because we are in the case $d_R(v) \geq 0.9 \cdot 1200 \ln n$. A Chernoff bound thus gives the probability this happens is at most n^{-8} . Hence by a union bound this will be true for all v except with probability at most n^{-7} .

This shows that $H = (V, \overrightarrow{\operatorname{cut}}(V \setminus R))$ is (2/3, 8)-good for contracting with respect to C except with probability at most $1/4 + n^{-5} + n^{-7} \le 1/3$ over the choice of R.

With these tools in hand, we can now prove Theorem 2.2.

Proof of Theorem 2.2. Let us first show item 1. The number of vertices in G' can be upper bounded by |R| plus the number of vertices in $V \setminus R$ that have no neighbor in R. By item 1 of Proposition 4.4 |R| will be at most twice its expected value, which is $2400n \ln(n)/\delta(G)$, except with probability at most n^{-400} . By item 2 of Proposition 4.4 the probability that a vertex has no neighbor in R is at most $\exp(-6 \ln n)$. Thus by a union bound, except with probability n^{-5} , every vertex in $V \setminus R$ will have a neighbor in R. Both items will hold with probability at least $1 - 1/n^4$. This completes the proof of item 1.

For item 2, let C be a non-trivial cut realizing $\lambda(G)$. With the choice of p in the theorem we know that $H = (V, \operatorname{cut}(V \setminus R))$ is (2/3, 8)-good for contracting with respect to C with probability at least 2/3 over the choice of R by Lemma 4.6. When this happens, we do not select an edge of C in doing a random 1-out sample of H with probability at least 3^{-12} by Corollary 4.2. Not selecting an edge of C to contract implies that in the contracted graph G', for every super-vertex S, all vertices in S will lie on the same side of the cut C. This in turn gives $\lambda(G) = \lambda(G')$. Thus overall item 2 of the theorem holds with probability at least $2 \cdot 3^{-13}$.

4.3 Quantum cut queries and matrix-vector multiplication queries

In this subsection we consider randomized algorithms that can make matrix-vector multiplication queries to the adjacency matrix of a graph and quantum algorithms with cut queries to a graph. In the former, if A is the adjacency matrix of an n-vertex simple graph G, one can query a vector $x \in \{0,1\}^n$ and receive the answer Ax.

As described in Section 2.2, our proof works for any query model that has the following primitives, which we now state more formally. We say that an algorithm that can perform these

operations in the stated costs has MDCP access to the graph, which stands for "Minimum Degree and Cut Product." The origin of the name is explained below in Remark 4.12.

Definition 4.7 (MDCP access). Let G = (V, E) be a simple graph. We say a query model has MDCP access to G if it can execute the following query operations with high probability in the specified cost.

- 1. Minimum degree queries: One can query mindeg(G) and receive as answer the minimum degree of G. The cost is $O(\sqrt{n} \log n)$.
- 2. Neighborhood queries: For any vertex $v \in V$, the characteristic vector $nbh(v) \in \{0,1\}^n$ of v can be computed with cost $O(\log n)$.
- 3. Spanning forest queries: For any subset $E' \subseteq E$ known to the algorithm one can compute a spanning forest $\operatorname{spf}(G, E')$ of $G' = (V, E \setminus E')$. The cost is $O(\log^6 n)$.
- 4. Cut queries: For any $S \subseteq V$ one can compute |cut(S)| with cost O(1).

The formulation of the spanning forest query may seem unusual. It is formulated in this way to allow the computation of sparse edge connectivity certificates using the technique of Nagamochi-Ibaraki given in Theorem 3.5. Further, both matrix-vector multiplication queries and quantum cut queries allow one to compute a spanning forest in a polylogarithmic number of queries, as shown later in this section (Lemma 4.11).

Proposition 4.8. Let G = (V, E) be an n-vertex simple graph and $0 < r \le n$ an integer. There is a deterministic algorithm with MDCP access to G that computes a sparse r-edge connectivity certificate for G with cost $O(r \log^6 n)$.

Proof. This follows directly from the construction of an r-sparse edge connectivity certificate by Nagamochi and Ibaraki [NI92] given in Theorem 3.5. Let $F_0 = \emptyset$. For i = 1, ..., r one computes $F_i = \operatorname{spf}(G, \bigcup_{j=0}^{i-1} F_j)$ with a query of cost $O(\log^6 n)$. At stage i the edges in $\bigcup_{j=0}^{i-1} F_j$ are all known to the algorithm from previous queries so this is a valid query. The total cost is $O(r \log^6 n)$ and a sparse r-edge connectivity certificate is given by $\bigcup_{j=1}^r F_j$.

Theorem 4.10. Let G be an n-vertex simple graph. There is a randomized algorithm with MDCP access to G that outputs the edge connectivity of G with high probability and has total query cost of $\widetilde{O}(\sqrt{n})$.

Proof. The algorithm is given in Algorithm 4.9. We go through the steps to describe in more detail their implementation and give their cost.

In the first step we compute the minimum degree δ of the graph, which can be done with cost $O(\sqrt{n} \log n)$. The rest of the algorithm breaks down into two cases depending on δ .

If $\delta < \sqrt{n}$ then $\lambda(G) < \sqrt{n}$ as well. Thus by definition, if F is a sparse \sqrt{n} -edge connectivity certificate for G we will have $\lambda(G) = \lambda(F)$ and the algorithm will output correctly in line 4. The cost of computing a sparse \sqrt{n} -edge connectivity certificate is $O(\sqrt{n}\log^6 n)$ by Proposition 4.8. Thus overall in the low degree case the query cost is $O(\sqrt{n}\log^6 n)$.

Let us now consider the case where $\delta \geq \sqrt{n}$. We first describe the implementation of each step in the for loop beginning on line 7 and its cost. In each iteration of the for loop we do uniform star contraction with $p = 1200 \ln(n)/\delta$ as detailed in lines 8–14. By item 1 of Theorem 2.2 the probability that we abort in line 10 or line 15 is n^{-4} , thus the probability we abort in any iteration of the for loop is $O(n^{-3})$. We add this to our error bound and henceforth assume this does not happen. The cost of line 11 is thus $O(n \log^2(n)/\delta)$ as we are assured in this step that $|R| \leq 2400 \ln(n)/\delta$.

Algorithm 4.9 MDCP query algorithm for edge connectivity

```
Input: MDCP query access to a simple graph G
    Output: \lambda(G)
 1: \delta \leftarrow \operatorname{mindeg}(G).
 2: if \delta < \sqrt{n} then
        Compute a sparse \sqrt{n}-edge connectivity certificate F of G.
        Output \lambda(F).
 4:
 5: else
        \mathsf{best} \leftarrow \infty
 6:
        for all i = 1, \ldots, \lceil 100 \log n \rceil do
 7:
             R \leftarrow \emptyset
 8:
 9:
            for all v \in V do Put v in R at random with probability 1200 \ln(n)/\delta.
            if |R| > 2400n \ln(n)/\delta then Abort.
10:
            for all c \in R do Query nbh(c).
11:
12:
            for all v \in V \setminus R do Choose a neighbor u \in R of v uniformly at random (if it exists)
13:
    and add \{u,v\} to X.
            Let G' be the multigraph formed from G by contracting all edges in X.
14:
            if G' has more than 2400n \ln(n)/\delta vertices then Abort.
15:
            if \lambda(G') < \text{best then best} \leftarrow \lambda(G').
16:
        Output the minimum of \delta and best.
17:
```

Once we know all neighbors of all vertices in R, for each $v \in V \setminus R$ we can compile a list B(v) of its neighbors in R. We then use these lists to execute line 13.

Next we argue that MDCP access to G gives us cut query access to the multigraph G' constructed in line 14. We can compute the connected components of the set of edges X. Say that this gives the partition $\mathcal{P} = \{A_1, \ldots, A_t\}$ of V. Then \mathcal{P} is the vertex set of G' and by Theorem 2.2 we have $|\mathcal{P}| = O(n \log(n)/\delta)$ with high probability. To execute the cut query $|\operatorname{cut}_{G'}(T)|$ for $T \subseteq \mathcal{P}$, note that $|\operatorname{cut}_{G'}(T)| = |\operatorname{cut}_G(S)|$ where $S = \bigcup_{A \in T} A$. Thus with MDCP access to G we can answer cut queries to G'. Therefore to execute line 16 we can run the randomized cut query algorithm of [MN20] quoted in Theorem 3.10 to compute $\lambda(G')$ with high probability. The cost of this step is $O(n \log^9(n)/\delta)$. Thus over all $O(\log n)$ iterations of the for loop the total cost is $O(n \log^{10}(n)/\delta)$. As $\delta \geq \sqrt{n}$ the total cost in this case is $O(\sqrt{n} \log^{10}(n))$.

Now let us argue correctness in the high degree case. Even when item 2 of Theorem 2.2 does not hold in the star contraction, the graph G' is always a contraction of G and thus $\lambda(G') \geq \lambda(G)$. As we compute $\lambda(G')$ correctly with high probability in line 16, with high probability we have best $\geq \lambda(G)$ and so the algorithm will output correctly if $\lambda(G) = \delta$. If $\lambda(G) < \delta$ by Theorem 2.2 we will have $\lambda(G') = \lambda(G)$ with probability at least $2 \cdot 3^{-13}$ in each iteration of the for loop. Thus in this case, as we repeat the star contraction $100 \log n$ times and take the minimum result, with high probability we will have best $= \lambda(G)$, and the algorithm will output correctly.

Next we show that with both quantum cut queries and matrix-vector multiplication queries to the adjacency matrix of G we can simulate MDCP access to G.

Lemma 4.11. Let G be a simple graph. A quantum algorithm with cut query access to G and a randomized algorithm with matrix-vector multiplication queries to the adjacency matrix of G both have MDCP access to G.

Proof. Let A be the adjacency matrix of G = (V, E). For two vectors $x, y \in \mathbb{R}^n$ let $x \circ y \in \mathbb{R}^n$ be their entrywise product, and let 1 be the n-dimensional all one vector. Lee, Santha, and Zhang [LSZ21] use a generalization of the Bernstein-Vazirani algorithm [BV97] to show that for any $x \in \{0, 1\}^n$ a quantum algorithm can with certainty compute $Ax \circ (1-x)$ with $O(\log n)$ cut queries. This power was stated explicitly in [AL21, Corollary 11]. One can also clearly compute $Ax \circ (1-x)$ with one matrix-vector query to the adjacency matrix. We postpone item 1 for now and first discuss items 2 and 3 together for the two models in terms of $Ax \circ (1-x)$ queries.

For item 2 note that we can learn the neighborhood of v via the query $Ae_v \circ (1 - \chi_{\{v\}})$.

For item 3, Auza and Lee [AL21, Theorem 8], in simplifying and quantitatively improving the results of [LSZ21], show that one can compute a spanning forest of G with high probability with $O(\log^5 n)$ many $Ax \circ (\mathbf{1} - x)$ queries. Say that one explicitly knows a set of edges $E' \subseteq E$. Let B be the adjacency matrix of the graph (V, E') and A' be the adjacency matrix of the graph $G' = (V, E \setminus E')$. Thus A' = A - B and we can compute $A'x \circ (\mathbf{1} - x) = Ax \circ (\mathbf{1} - x) - Bx \circ (\mathbf{1} - x)$ from $Ax \circ (\mathbf{1} - x)$ as $Bx \circ (\mathbf{1} - x)$ is known explicitly. This allows us to also compute spanning forests of G' with high probability in $O(\log^5 n)$ many $Ax \circ (\mathbf{1} - x)$ queries as well.

For item 4 it is clear that a quantum cut query algorithm can compute a classical cut query. Since $(\mathbf{1} - x)^T A x = (\mathbf{1} - x)^T (A x \circ (\mathbf{1} - x))$, one can also compute a cut query with one $A x \circ (\mathbf{1} - x)$ query.

Item 1 is where the arguments diverge. It is not obvious how to compute the minimum degree with o(n) queries of the form $Ax \circ (\mathbf{1} - x)$. However, the minimum degree can be computed with one matrix-vector multiplication query as all degrees are given by $A\mathbf{1}$.

For the quantum case, note that the degree of one vertex can be computed with a single classical cut query. Thus we can find the minimum degree with high probability in $O(\sqrt{n} \log n)$ quantum cut queries using the quantum minimum finding routine of Dürr and Høyer [DH96, Theorem 1].

Remark 4.12. As can be seen from the proof of Lemma 4.11, to simulate MDCP access it suffices to be able to

- 1. Compute mindeg(G) with high probability in cost $O(\sqrt{n}\log(n))$.
- 2. Compute $Ax \circ (\mathbf{1} x)$ with high probability in cost $O(\log n)$ for any $x \in \{0, 1\}$, where A is the adjacency matrix of G. We call this a cut product.

This is a smaller set of primitives that can still be used in Theorem 4.10 to give a $\widetilde{O}(\sqrt{n})$ cost algorithm for edge connectivity, and is the origin of the name Minimum Degree and Cut Product. We chose to define MDCP access with a more verbose but less mysterious set of primitives for greater clarity.

4.4 One-pass semi-streaming algorithms

Next, we consider applications of star contraction for edge connectivity computation in various settings of the streaming model of computation. Specifically, we consider the following settings.

- 1. Explicit vertex arrivals, in which vertices appear in an arbitrary order, along with all edges incident to previously seen vertices.
- 2. Complete vertex arrivals, in which vertices appear in an arbitrary order, along with all incident edges.
- 3. Random vertex arrivals, in which vertices appear in a random order, along with all edges incident to previously seen vertices.

For a related model of streaming computation where the edges arrive in arbitrary order, [Zel11] showed a $\Omega(n^2)$ space lower bound against any one-pass randomized algorithm that correctly computes edge connectivity. We observe that this lower bound construction also gives a lower bound of the same strength for the explicit vertex arrival setting.¹² We include a proof sketch in Appendix C.1.

Observation 4.13. [Follows from [Zel11]] Any one-pass streaming algorithm computing the edge connectivity of a simple graph in the explicit vertex arrival setting requires $\Omega(n^2)$ memory.

Next, we employ uniform star contraction to prove that the random vertex arrival setting allows to circumvent the aforementioned lower bound. In contrast, we discuss in Remark 4.17 that it is not clear how to use the related 2-out contraction technique for this purpose. Formally, we prove the following.

Theorem 4.14. There is a one-pass streaming algorithm, using $\widetilde{O}(n)$ memory, that given a simple graph G = (V, E) in the random vertex arrival setting, computes the edge connectivity of G with high probability.

Proof. We run in parallel $\lceil \log(n) \rceil$ independent instances of an algorithm, each of which uses a different estimate $d = 2^{\ell}$ for the minimum degree $\delta(G)$, with $\ell = 0, 1, 2, \ldots, \lceil \log(n) \rceil - 1$. Each algorithm aborts if it uses more than $\widetilde{O}(n)$ memory, and we will show that if $d = 2^{\ell}$ is such that $d \leq \delta(G) < 2d$ then with high probability the corresponding algorithm will not abort and will correctly output $\lambda(G)$. Since we know $\delta(G)$ exactly by the end of the stream (we can keep track of all degrees with $\widetilde{O}(n)$ memory), we can filter out the correct outcome at the end of the algorithm.

In the remainder we describe the algorithm for the value d satisfying $d \leq \delta(G) < 2d$. It will be clear that the algorithms for different estimates are independent, and hence can be run in parallel. In a single pass, the algorithm will perform uniform star contraction on G. Simultaneously, it will construct a sparse 2d-edge connectivity certificate on the contracted graph and compute the edge connectivity of this certificate. Finally, in order to boost the constant success probability of uniform star contraction, we will run $r \in \Theta(\log n)$ parallel repetitions of this. This last step will require some care, as every instance uses the same randomness from the input stream, and this needs to be done appropriately to ensure independence.

The key idea to simulate a single implementation of star contraction in the random vertex arrival model is the following: because the vertices arrive in a random order, we can select the first $\Theta(n\log(n)/d)$ vertices as the set of centers R, and put all other vertices in $V \setminus R$. Because each vertex $v \in V \setminus R$ comes with all edges incident to R, we can for each such vertex v sample a uniform and independent neighbor in R in a single pass, thereby performing uniform star contraction. Dealing with r parallel repetitions requires a slightly more complicated approach, as we need to ensure independence and hence cannot reuse the same set of centers. Nonetheless, we can still assume that all sampled centers come at the start of the stream. This is captured by the following lemma, whose proof we postpone to Appendix C.2.

Lemma 4.15. There is a sampling procedure that operates within $\widetilde{O}(n)$ space and, given a stream S of n vertices, outputs $Y_1, \ldots, Y_r \subseteq [n]$ after reading the first $|\bigcup_{i=1}^r Y_i|$ vertices. The distribution D on (Y_1, \ldots, Y_r) defined by the procedure admits the following property. For every $R_1, \ldots, R_r \subseteq [n]$:

$$\Pr_{S \sim S_n, (Y_1, \dots, Y_r) \sim D} [Y_1 = R_1, \dots, Y_r = R_r] = \prod_{i=1}^r \Pr_{X_i \sim B([n], p)} [X_i = R_i] .$$

¹²This does not follow as a black box reduction. Rather, the edge stream admitted by the lower bound construction of [Zel11] is actually an explicit vertex arrival stream. Hence the same argument provides an $\Omega(n^2)$ space lower bound.

In each independent repetition of star contraction, we want to sample a subset R by choosing every vertex with probability $p = \frac{1200 \ln n}{d}$ under the additional condition that $|R| \leq 2pn$. This is done by applying Lemma 4.15 to obtain the subsets R_1, \ldots, R_r , where R_i is the subset that should be used in the i^{th} repetition. Let $B = \bigcup_{i=1}^r R_i$. By the properties of the sampling procedure, the subsets are generated after reading the first |B| vertices from the stream. Then, the i^{th} repetition uses R_i as the set of centers and performs uniform star contraction for the subsequent vertices in the input stream (i.e., those that follow the first |B| vertices). For each of these, we choose a uniformly random edge towards R_i and contract it in G_i' . All vertices in B, and the subsequent vertices in the input stream with no edge towards R_i , are simply kept in the contracted graph G_i' . We will argue that the number of vertices in G_i' is still $\widetilde{O}(n/d)$ with high probability, even though (as opposed to the original uniform star contraction) we never contract any of the vertices of B in G_i' . In parallel, we build a 2d-edge connectivity certificate $F_i^i \cup \cdots \cup F_{2d}^i$ of G_i' , where the F_k^i 's are edge disjoint spanning forests of G_i' (as in the Nagamochi-Ibaraki certificate, Theorem 3.5). As an invariant throughout the stream, we will have that $F_1^i \cup \cdots \cup F_{2d}^i$ is a 2d-edge connectivity certificate of the (potentially contracted) subgraph G_i' seen so far. Since this subgraph will only have $\widetilde{O}(n/d)$ vertices throughout the stream, the certificate will only contain $\widetilde{O}(n)$ edges.

We summarize the i^{th} parallel repetition in full detail. The set R_i can be accessed after reading the first |B| vertices from the input stream, and it is computed together with all sets R_1, \ldots, R_r using Lemma 4.15 globally, outside of the i^{th} repetition itself. Recall that we abort each repetition as soon as its memory usage exceeds $\widetilde{O}(n)$, in which case we set the i^{th} outcome to be $\lambda_i = \infty$.

- 1. Initialize F_1^i, \ldots, F_{2d}^i as empty forests and set $R_i = \emptyset$. Initialize a mapping $r_i : V \to V$ to be the identity (through the stream this will keep track of the contracted vertices).
- 2. For the j^{th} vertex arrival v with edges $e_1, ..., e_\ell$ between v to previously seen vertices, the following is done:
 - (a) Uniform star contraction: If $j \leq |B|$, do nothing. If j > |B|, pick a uniformly random center w from the center neighborhood $N_{R_i}(v)$ (if it exists) and set $r_i(v) = w$. This amounts to contracting the edge $\{v, w\}$. For each e_t among e_1, \ldots, e_ℓ , except for the contracted edge which is discarded, change the endpoints of $e_t = \{v, u\}$ to be $\{r_i(v), r_i(u)\}$, discard any self loops. At the end of the stream, the vertices with the same $r_i(\cdot)$ values constitute a vertex in G'_i .
 - (b) Maintaining of 2d-edge connectivity certificate: For each (relabelled) incident edge e_t among e_1, \ldots, e_ℓ , add e_t to F_k^i where k is the minimal index for which $F_k^i \cup \{e_t\}$ contains no cycles. If there is no such k, discard the edge.
- 3. If the repetition did not abort by the end of the stream, we compute the edge connectivity of the connectivity certificate $\lambda(G'_i)$ and set $\lambda_i = \lambda(G'_i)$. Note that $\lambda_i \geq \lambda(G)$.

Finally, we combine the r parallel repetitions by outputting $\min\{\delta(G), \lambda_1, \dots, \lambda_r\}$.

Analysis. As mentioned before, it suffices to prove correctness and a O(n) memory bound only for the algorithm that has an estimate d such that $d \leq \delta(G) < 2d$. By Lemma 4.15, a run of the whole algorithm for such an estimate is equivalent to $r \in O(\log n)$ independent repetitions of a variant of the uniform star contraction with $p = \frac{1200 \ln n}{d}$, except that vertices from B remain in the contracted graph.

A single repetition can be analysed as follows. Sample each vertex with probability $p = \frac{1200 \ln n}{d}$ to obtain the set of centers R. Then, for some set of vertices B such that $R \subseteq B$ and $|B| \le 2rpn$, proceed as follows. For any vertex in $[n] \setminus B$, choose a uniform random edge towards R (if it exists)

and contract it. This results in a contracted graph G'. By item 2 of Proposition 4.4 and $d \leq d(v)$, the probability that a vertex in $[n] \setminus B$ has no neighbor in R is at most n^{-6} . Hence, by a union bound, |G'| = |B| except with probability at most n^{-5} over the choice of R. Next, we want to lower bound the probability that $\lambda(G) = \lambda(G')$, assuming that $\lambda(G') < \delta(G)$. Let C be a non-trivial minimum cut of G. By Lemma 4.6, $H = (V, \overrightarrow{\text{cut}}(V \setminus R))$ is (2/3, 8)-good for contracting with respect to C with probability at least 2/3 over the choice of R. We perform a random 1-out contraction on a subgraph of H, so the probability of not contracting an edge of C is at least as large as when performing a random 1-out contraction on the whole H, which is at least 3^{-12} by Corollary 4.2 over the choice of R. Thus $\lambda(G') = \lambda(G)$ with probability at least $2/3 \cdot 3^{-12}$ over the choice of R.

By union bound, the overall probability of error is at most the sum of probabilities that some repetition aborts due to using too much memory and the probability that $\lambda(G'_i) > \lambda(G)$ holds for every $i = 1, \ldots, r$, both over the choice of R_1, \ldots, R_r . By item 1 of Proposition 4.4, $|R_i| \leq 2rpn$ except with probability n^{-400} , thus $|B| \leq 2rpn$ except with probability $r \cdot n^{-400}$. For each $i = 1, \ldots, r$ we have $|G'_i| = |B|$ except with probability at most n^{-5} . Overall, no repetition aborts except with probability $r \cdot n^{-400} + r \cdot n^{-5} = O(n^{-4})$. For each $i = 1, \ldots, r$ independently, we have $\lambda(G'_i) = \lambda(G)$ with probability at least $2/3 \cdot 3^{-12}$ over the choice of R_i . Thus, at least one repetition correctly determines $\lambda(G)$ except with probability at most $(1 - 2/3 \cdot 3^{-12})^{\Theta(\log n)} = n^{-\Omega(1)}$.

We can prove a similar result for the *complete vertex arrival* setting, in which the vertices arrive in an arbitrary order with all edges incident on them. The proof follows along the same lines, but is simpler in this case because we can randomly sample sets of centers offline before the stream begins. As a vertex arrives with all its edges, we can immediately randomly choose an edge incident on the set of centers to implement star contraction. In parallel we also construct a sparse edge connectivity certificate. Due to the similarities with Theorem 4.14, we move the proof to Appendix C.3.

Theorem 4.16. There is a one-pass streaming algorithm, using $\widetilde{O}(n)$ memory, that given a simple graph G = (V, E) in the complete vertex arrival setting, computes the edge connectivity of G with high probability.

Remark 4.17. While we cannot rule it out, it is not obvious how to prove either Theorem 4.14 or Theorem 4.16 using 2-out contraction [GNT20]. We crucially use two features of star contraction in the proofs. The first is that when we see a vertex in the stream we can immediately choose its edges to be contracted, which allows us to contract the graph "on-the-go." In the random vertex arrival model we cannot randomly choose 2 edges incident to a vertex when the vertex arrives, as at that point we have not seen all of its neighbors. To naively implement 2-out this means contracting edges has to be delayed until the end of the stream which prohibits constructing the sparse edge connectivity certificate with $\widetilde{O}(n)$ memory.

The difficulty of using 2-out contraction to obtain the complete vertex arrival result is more subtle. Here we do see all incident edges to a vertex when the vertex arrives, thus we can immediately select two random incident edges to contract. However, in this case it is not clear that the (partly) contracted graph throughout the stream has no more than $\widetilde{O}(n/\delta(G))$ components. This property is needed to ensure that we can keep a sparse $\delta(G)$ -edge connectivity certificate of the contracted graph throughout the stream using a memory of size only $\widetilde{O}(n)$. For star contraction, this issue is resolved by a second key feature of star contraction that we make use of in our proofs. At any point in the stream, the contracted graph on the vertices seen so far has size at most the number of centers, and thus its size can be bounded by $\widetilde{O}(n/\delta(G))$ in the branch of the computation with the correct degree estimate.

4.5 Linear time sequential algorithm for slightly dense graphs

In this section, we give another illustration on how to use the star contraction algorithm. Similar to the approach in the state-of-the-art $O(m + n \log^2 n)$ algorithm [GNT20] for computing the edge connectivity of a simple graph, we use it to obtain a simple $O(m+n \operatorname{polylog}(n))$ sequential algorithm. Formally, we prove the following theorem.

Theorem 4.18. There is a randomized algorithm with running time $O(m + n \log^3 n)$ that computes the edge connectivity of a simple graph G = (V, E) with success probability at least 2/3.

The proof of this theorem is achieved via an application of uniform star contraction as in Theorem 2.2. The naive implementation requires O(m) time by sampling vertices independently into R in O(n) time, and then choosing a random neighbor in R for each $v \notin R$ in $O(\sum_{v \in V} d(v)) = O(m)$ time. We combine this with the connectivity certificate algorithm from Theorem 3.5, whose implementation in the sequential setting also requires O(m) time [NI92], and the following recent result.

Theorem 4.19 ([GMW20]). There is an algorithm that with high probability computes the weight of a minimum cut in a weighted graph G = (V, E, w) with n vertices and m edges in time $O(m \log^2 n)$.

Proof of Theorem 4.18. The algorithm proceeds as follows: Given G = (V, E), compute $\delta(G)$ in O(m) time. If $\lambda(G) \geq \delta(G)$ then this corresponds to the edge connectivity. Now assume $\lambda(G) < \delta(G)$. We perform uniform star contraction as in Theorem 2.2 on G to obtain G' with $O(\frac{n \log n}{\delta(G)})$ vertices, and with constant probability we have that $\lambda(G) = \lambda(G')$. Construct in time O(m) a sparse $\delta(G)$ -edge connectivity certificate (Theorem 3.5) G'' of G' with $O(n \log n)$ edges. Finally, apply Theorem 4.19 on G'' to compute $\lambda(G'') = \lambda(G')$ with witness $S \subseteq V$. Since $|E(G'')| = O(n \log n)$ this takes $O(n \log^3 n)$ time. If we output $\min\{\lambda(G'), \delta(G)\}$ then this yields the correct output with constant probability. We can boost the success probability to above 2/3 by repeating this a constant number of times, and outputting the smallest value achieved.

5 Finding a spanning forest with O(n) cut queries

In this section we describe an O(n) cut query algorithm for constructing a spanning forest of a simple graph. This proves that the cut query complexity of graph connectivity is O(n), which was not known before. We also describe a variation of the algorithm for constructing a connectivity certificate, which is a key building block of the edge connectivity algorithm described in the next section.

Let us first describe a simple algorithm to find a spanning forest of a graph using $O(n \log n)$ cut queries given by Harvey [Har08, Theorem 5.10]. For the application to finding a sparse edge connectivity certificate it will be useful to define the algorithm more generally to work on a contraction of a graph.

Lemma 5.1 (Simple spanning forest algorithm). Let G = (V, E) be an n-vertex simple graph. Let G' be a contraction of G with q many supervertices, which are given explicitly as the partition $\mathcal{P} = \{A_1, \ldots, A_t\}$ of V. There is a deterministic algorithm that outputs a set of edges $F \subseteq E$ that form a spanning forest of G' and makes $O(q \log n)$ cut queries to G.

Proof. We follow the plan of Prim's spanning forest algorithm. We begin at an arbitrary supervertex A of G' and initialize $F = \emptyset$. We want to find an edge $\{u, v\} \in E$ with $u \in A$ and $v \notin A$. We first identify a vertex $u \in V$ which has an edge leaving A by doing binary search with queries of the form $|E(S, \bar{A})|$ for $S \subseteq A$. This takes at most $\log n$ many cut queries. Then we want to find one

of the neighbors $v \in \bar{A}$ of u by doing binary search asking queries of the form |E(v,S)| for $S \subseteq \bar{A}$. We add $\{u,v\}$ to F and let G'' be the graph G' with supervertex A merged with the supervertex containing v. Let A' be the name of this new supervertex. We then repeat this procedure on G'' and A'. We keep repeating this procedure until we find a supervertex with no outgoing edges. This supervertex represents a connected component in G'. If the supervertex is not all of V then we arbitrarily choose another supervertex of the graph and begin the procedure again. There are at most q iterations and each iteration costs $O(\log n)$ cut queries, thus the total number of cut queries is $O(q \log n)$.

An important insight now is that in the proof of Lemma 5.1 we did not use the full power of cut queries. For the binary search routines in the proof we might as well have used bipartite independent set queries, which return just a single bit telling if |E(S,T)| is zero or positive for two disjoint sets S and T. Notice that the (deterministic) algorithm is optimal for that type of queries. Indeed, by the $\Omega(n \log n)$ deterministic communication complexity lower bound for connectivity [HMT88], any deterministic algorithm for connectivity must make $\Omega(n \log n)$ such 1-bit queries.

Crucially, the situation is different with cut queries, which return $\Omega(\log n)$ bits of information per query in a simple graph. Taking advantage of this additional information suggests an avenue towards saving a $\log n$ factor over the simple algorithm, and this is the approach we follow to give a zero-error randomized algorithm to compute a spanning forest of a simple graph with O(n) cut queries.

5.1 Separating matrices and learning buckets

The fact that a cut query returns $\Omega(\log n)$ bits allows us to use the remarkable result that there is a matrix $A \in \{0,1\}^{k \times n}$ with $k = O(n/\log n)$ such that one can recover any Boolean vector $x \in \{0,1\}^n$ given the product Ax. At a very high level, this $\log n$ factor reduction in the size of k over the obvious bound is the key that allows us to save a $\log n$ factor in the spanning forest computation. Such a matrix A is called a separating matrix and formally defined next.

Definition 5.2 (Separating matrix). A k-by-n Boolean matrix B is called a separating matrix for the set $S \subseteq \{0, 1, ..., d\}^n$ if for all $x, y \in S$ with $x \neq y$ it holds that $Bx \neq By$.

Theorem 5.3 ([GK98, Theorem A.1], [GK00, Theorem 1]). There exists a k-by-n separating matrix for

- 1. The set $\{0, 1, \dots, d\}^n$ with $k < 8\lceil \log(d+1) \rceil n / \log(2n)$.
- 2. The set $S_{\ell} = \{x \in \{0,1\}^n : |x| \le \ell\}$ with $k = O(\ell \log(2n)/\log(2\ell))$.

Grebinski and Kucherov [GK00, Theorem 5] use separating matrices to show that if A is the adjacency matrix of a simple n-vertex graph G with maximum degree ℓ , then one can learn G with $O(\ell n)$ queries of the form x^TAy for Boolean vectors $x,y \in \{0,1\}^n$. We will use separating matrices for a very similar application, although we focus on bipartite graphs where the left hand side has bounded degree ℓ , and want to express the complexity in terms of the number of vertices on the left hand side. To this end, let M be an m-by-n Boolean matrix and suppose that every row has at most ℓ ones. Suppose that we have x^TMy query access to M for Boolean vectors x,y. One should imagine M being a bipartite adjacency matrix of a bipartite subgraph of G, in which case we can simulate x^TMy with cut queries to G, and think of ℓ as constant or slowly growing in a typical application. The next lemma shows that if m and n are polynomially related then we can learn M with $O(\ell m)$ queries. A very similar theorem is shown by Grebinski and Kucherov [GK00, Theorem 4] for the case m = n.

Lemma 5.4. Let $M \in \{0,1\}^{m \times n}$ be an m-by-n matrix with at most ℓ non-zero entries per row. There is a deterministic algorithm that learns M with $O\left(\frac{\ell m \log(2n)}{\log(2m)}\right)$ many queries of the form $x^T M y$ with Boolean vectors x, y.

Proof. Let Y be a separating matrix with $O(\ell \log(2n)/\log(2\ell))$ rows for the set $S_{\ell} \subseteq \{0,1\}^n$ of Boolean vectors with at most ℓ non-zero entries, which exists by item 2 of Theorem 5.3. From MY^T we can recover M. Every column of MY^T has integer entries of magnitude at most ℓ . Let X be a separating matrix with $8m\lceil\log(\ell+1)\rceil/\log(2m)$ rows for the set $\{0,1,\ldots,\ell\}^m$ which exists by item 1 of Theorem 5.3. We can recover MY^T from XMY^T . Putting it together we can compute XMY^T with a number of x^TMy queries of order

$$\frac{\ell m \log(2n) \lceil \log(\ell+1) \rceil}{\log(2m) \log(2\ell)} \le \frac{\ell m \log(2n)}{\log(2m)} . \square$$

Using this lemma we can describe a key subroutine LearnBucket[M](r,k) that will be used both in the spanning forest algorithm and in the edge connectivity algorithm. In this algorithm we have oracle access to a matrix M via x^TMy queries for Boolean vectors x and y. The oracle access to M is indicated by having M in brackets in the call of the algorithm. The promise is that every row of M has between r and 2r many ones—in our applications this arises from "bucketing" together vertices with similar degrees, hence the name. The number k is a parameter indicating how many ones we want to learn from each row of M—the algorithm will learn $\min\{k,r\}$ many ones from each row. In the application to finding a spanning forest we just need to find a single one in every row and we take k to be a large constant. In the edge connectivity algorithm we will also apply LearnBucket where k is growing. For the application to edge connectivity, we will require that the found neighbors are selected in a sufficiently random fashion, as recorded in the "further" statement of the theorem. This statement is not needed for the application to finding a spanning forest and can be skipped on a first reading.

Algorithm 5.5 LearnBucket[M](r,k)

Input: $x^T M y$ query access to a Boolean matrix $M \in \{0,1\}^{m \times n}$, a natural number r with the promise that all rows of M have at least r and at most 2r ones, and a parameter k.

Output: The output consists of a list Z[i] for each $i \in [m]$ where M(i, Z[i][j]) = 1 for all i, j and each Z[i] has at least min $\{k, r\}$ many elements.

```
1: B = [m]
2: while B is non-empty do
        Choose Q \subseteq [n] by putting each a \in [n] into Q independently with probability q =
    \min\{\frac{2k}{r},1\}.
        for j \in B do
4:
            ones(j) \leftarrow \chi_{\{j\}}^T M \chi_Q.
                                                                      \triangleright ones(j) is number of ones in M(j,Q)
        Set K \leftarrow \{j \in B : \min\{r, k\} \le \text{ones}(j) \le 8k\}.
6:
        if |K| > 0 then
7:
            Learn the submatrix M(K,Q) by Lemma 5.4 and populate Z[i] for all i \in K.
8:
            B \leftarrow B \setminus K.
10: Return all lists Z[i].
```

Lemma 5.6. Let m, n be positive integers and $M \in \{0, 1\}^{m \times n}$ be a Boolean matrix where every row has at least r and at most 2r ones. Let $k \ge 10$ and $\ell = \min\{r, k\}$. Suppose we can query $x^T M y$

for any $x \in \{0,1\}^m$, $y \in \{0,1\}^n$. There is a zero-error randomized algorithm, LearnBucket[M](r, k) given in Algorithm 5.5, that makes

$$O\left(m + \frac{km\log(n)}{\log(2m)}\right)$$

queries in expectation and for each $i \in [m]$ outputs a list Z[i] such that M(i, Z[i][j]) = 1 for all $i \in [m]$ and j, and each Z[i] contains at least ℓ many elements. Let d(i) be the number of ones in row i. Further, Z[i] contains all the ones of M(i,:) contained in a set Q chosen by putting each $j \in [n]$ into Q independently at random with probability $q \ge \min\{2k/d(i),1\}$, conditioned on M(i,Q) having at least f and at most g ones, where $0 < f \le qd(i)/2$ and $g \ge 2qd(i)$.

Proof. The algorithm is given by Algorithm 5.5. We first consider the (trivial) case where $2k \geq r$, in which case the sampling probability q = 1. In this case, in the first iteration of the while loop K = [m] and we learn the entire matrix M deterministically via Lemma 5.4. The cost of this is $O(2rm\log(n)/\log(2m)) = O(km\log(n)/\log(2m))$ and we learn at least r ones in each row as desired.

Now consider the case that 2k < r, in which case the goal is to learn the positions of k ones in every row of M. We first show correctness. At the start of the while loop B = [m], and we only remove a row index j from B if it is in the set K processed in line 8. On this line we deterministically learn the entire submatrix M(K,Q). Further we are guaranteed that each row of M(K,Q) has at least k ones by the definition of K on line 6. Thus when j is removed from B we are guaranteed that we have learned the positions of at least k ones in row j. This process continues until B is empty, thus the algorithm is correct with zero error.

Let us now argue about the complexity. In any iteration of the while loop, for each $j \in B$ the expected value of ones(j) is in the interval [2k,4k]. Therefore by a Chernoff bound (Lemma 3.3) using the fact that $k \geq 10$, for any $j \in B$ we have $k \leq \operatorname{ones}(j) \leq 8k$ with probability at least 1/2. Thus $\mathbb{E}[|K|] \geq |B|/2$, and letting b_i be a random variable for the size of B at the start of the i^{th} iteration of the while loop we have $\mathbb{E}[b_{i+1} \mid b_i = s] \leq s/2$. From this it follows that the expected number of iterations of the while loop is at most $2(\log(m) + 1)$ (see [DJW12, Theorem 3]). The fact that $\mathbb{E}[b_{i+1} \mid b_i = s] \leq s/2$ also implies $\mathbb{E}[b_{i+1}] \leq \mathbb{E}[b_i]/2$ and so the expected number of queries from line 5 is

$$\mathbb{E}\left[\sum_{i} b_{i}\right] = \sum_{i} \mathbb{E}[b_{i}] \leq \sum_{i} \frac{m}{2^{i}} \leq 2m .$$

We next turn to queries made in line 8. By Lemma 5.4 the number of queries in an execution of this line is $O(k|K|\log(|Q|)/\log(2|K|)) \in O(k|K|\log(n)/\log(2|K|))$. Let Y be a random variable for the number of times line 8 is executed. As the sum of |K| over all executions of line 8 is at most m, we must have $Y \leq m$. Further $\mathbb{E}[Y] \leq 2\log m + 1$ as it is at most the total expected number of iterations of the while loop. By Claim 3.1, when Y = t the overall number queries from line 8 is $O(km\log(n)/\log(2m/t))$. Thus the expected number of queries from line 8 overall is of order

$$\sum_{t=1}^{m} \frac{km \log(n)}{\log(2m/t)} \Pr[Y = t] \le \sum_{t=1}^{\sqrt{m}} \frac{2km \log(n)}{\log(2m)} \Pr[Y = t] + km \log(n) \Pr[Y > \sqrt{m}] . \tag{8}$$

By Markov's inequality $\Pr[Y > \sqrt{m}] \le 2(\log(m) + 1)/\sqrt{m}$. Hence we can upper bound Eq. (8) by

$$\frac{2km\log(n)}{\log(2m)} + \frac{2(\log(m)+1)}{\sqrt{m}}km\log(n) \in O\left(\frac{km\log(n)}{\log(2m)}\right).$$

The "further" statement follows from the definition of Q on line 3. This set is taken by putting each $i \in [n]$ into Q independently at random with probability $q = \min\{2k/r, 1\} \ge \min\{2k/d(i), 1\}$

as $d(i) \ge r$. Z[i] contains exactly the ones in M(i,Q) for the first such Q chosen that has at least $f = \min\{k,r\} \le qd(i)/2$ and at most $g = 8k \ge 2qd(i)$ ones by line 8.

LearnBucket is restricted in that it requires the matrix to have approximately uniform row sums. In the next algorithm, Recover-k-From-All, we use LearnBucket as a subroutine to locate the position of $\min\{k,d\}$ ones in each row of a matrix with the weaker promise that every row has at least d ones. With respect to its goal, Recover-k-From-All is very similar to the primitive RecoverOneFromAll introduced by [AL21] in the study of connectivity algorithms with matrix-vector multiplication queries. RecoverOneFromAll was in turn inspired by the Recover primitive used by [ACK21] for connectivity algorithms with linear and OR queries.

We follow the same algorithmic plan used by [AL21] in RecoverOneFromAll, which is to count the number of ones in each row, bucket rows together with similar number of ones, and then operate on each bucket separately. The main difference is in the implementation of learning the position of ones for each row in a bucket. We use separating matrices for this in LearnBucket while the technique in [AL21] is based on combinatorial group testing algorithms.

Algorithm 5.7 Recover-k-From-All[M](k)

Input: $x^T M y$ query access to a Boolean matrix $M \in \{0,1\}^{m \times n}$ and a parameter k.

Output: Let d be the minimum number of ones in a row of m. For $\ell = \min\{d, k\}$ the output is a list Z[i] for each $i \in [m]$ such that Z[i] has at least ℓ elements and M(i, Z[i][j]) = 1 for all $i \in [m]$ and j.

```
1: for j \in [m] do
2: d(j) \leftarrow \chi_{\{j\}}^T M \mathbf{1}. \triangleright d(j) is the number of ones in row j
3: d \leftarrow \min_j d(j).
4: for a = 0 to \lceil \log(n/d) \rceil do
5: B_a \leftarrow \{j \in [m] : d(j) \in [d2^a, d2^{a+1})\}.
6: Z_a \leftarrow \text{LearnBucket}[M(B_a,:)](d2^a, k).
7: Output all adjacency lists Z_a.
```

Lemma 5.8. Let $M \in \{0,1\}^{m \times n}$ be a Boolean matrix where every row has at least d > 0 ones. Let $k \geq 10$ and $\ell = \min\{k,d\}$. Suppose we can query $x^T M y$ for any $x \in \{0,1\}^m$, $y \in \{0,1\}^n$. There is a zero-error randomized algorithm that outputs a list Z[i] with at least ℓ elements for each $i \in [m]$ satisfying M(i,Z[i][j]) = 1 for all $i \in [m]$ and j, and makes

$$O\left(m + \frac{km\log(n)}{\log(2m/\log(n))}\right)$$

queries in expectation. Let d(i) be the number of ones in row i. Further, Z[i] contains all the ones of M(i,:) contained in a set Q chosen by putting each $j \in [n]$ into Q independently at random with probability $q \ge \min\{2k/d(i), 1\}$, conditioned on M(i, Q) having at least f and at most g ones, where $0 < f \le qd(i)/2$ and $g \ge 2qd(i)$.

Proof. The algorithm is given in Algorithm 5.7. In line 2 we compute the number of ones in each row of the matrix and then bucket the vertices accordingly in line 5. Thus in the call to LearnBucket($d2^i, k$) for those rows in B_i the promise that each row has number of ones in $[d2^i, d2^{i+1})$ will hold and LearnBucket will return the positions of $\min\{d2^i, k\} \ge \min\{d, k\}$ ones for each row of B_i by Lemma 5.6. This shows correctness.

Let us now examine the complexity. There are m queries made in line 2. The rest of the queries are made in the for loop. In the execution of the for loop on B_a we make $O(|B_a| +$

 $k|B_a|\log(n)/\log(2|B_a|)$) queries in expectation by Lemma 5.6. Thus by Claim 3.1 the total number of queries is at most $O\left(m + km \frac{\log(n)}{\log(2m/\log(n))}\right)$.

The "further" statement follows immediately from the "further" statement of Lemma 5.6.

5.2 Spanning forest algorithm

Now we are ready to describe a zero-error randomized algorithm to compute a spanning forest of a simple n-vertex graph with O(n) cut queries in expectation. Compared to the simple spanning forest algorithm, the first high level idea is to switch from a Prim style spanning forest algorithm to one based on Borůvka's algorithm, which is known to work well in parallel settings. Here the basic task it to find an outgoing edge from each of the connected sets S_1, \ldots, S_t . The second idea is to use Lemma 5.8 to do this in parallel and save a $\log n$ factor compared to the naive sequential computation.

Theorem 1.2. Let G = (V, E) be a simple n-vertex graph. There is a zero-error randomized algorithm that makes O(n) cut queries in expectation and outputs a spanning forest of G.

Proof. We will follow Borůvka's spanning forest algorithm. The algorithm proceeds in rounds and maintains the invariant that in each round there is a paritition S_1, \ldots, S_t of V and a spanning tree for each S_i in the partition. Initially, each S_i is just a single vertex.

In a generic round the goal is to find an outgoing edge from each S_i that is not already a connected component. To help with this, we will label every vertex $v \in V$ as Active or Inactive. Initially, all vertices are marked Active. If in any round we learn that $v \in S_i$ has no edge going outside of S_i then we mark it as Inactive. An inactive vertex is not useful to the algorithm because it will not have an edge leaving its component in any future round of the algorithm. We will similarly call a set S_i Active if and only if it contains an active vertex, and Inactive otherwise. A set that is Inactive is a connected component.

Once we have found an outgoing edge from each S_i that has one, we select a subset of these edges that is cycle free with respect to the partition S_1, \ldots, S_t . These edges are used to merge the corresponding sets of the partition and update the spanning trees accordingly. If t' sets among S_1, \ldots, S_t are Active, then the cycle free subset of edges will have size at least t'/2, and every edge added reduces the number of active sets by at least 1. It follows that the number of Active sets decreases by a factor of at least two in each round. We will crucially use this geometric decrease in our analysis of the algorithm. Once the number of Active sets falls below $n/\log(n)$ we switch to the simple spanning forest algorithm from Lemma 5.1 to finish finding a spanning forest.

We now formally describe the actions of the algorithm in a generic round where we have sets S_1, \ldots, S_t and a spanning tree for each S_i . There are two main steps to a round.

Step 1: For each active S_i find a $v \in S_i$ that has a neighbor outside of S_i . Let t be the number of sets that were Active at the end of the previous round, and say without loss of generality these are the sets S_1, \ldots, S_t . For each $i = 1, \ldots, t$ we do the following. We query $|E(v, \bar{S}_i)|$ for each Active vertex $v \in S_i$ until we find a vertex with $|E(v, \bar{S}_i)| > 0$. In such case we mark v as the representative of S_i and move on to S_{i+1} without any further queries in S_i . For all vertices in S_i with $|E(v, \bar{S}_i)| = 0$ we mark v as Inactive. If all vertices in S_i become Inactive then S_i becomes Inactive: it is a connected component and we do not need to process it in future rounds.

As we only make queries in S_i until we find an Active vertex, the total number of queries in a round is O(t+w), where w is the number of vertices that become Inactive in the round. These vertices will never be queried again, so the term for Inactive vertices will only contribute O(n) queries over all the rounds.

Step 2: Learn an outgoing edge from a constant fraction of the representatives. Let t' be the number of Active sets after Step 1 (so we have already determined that $S_{t'+1}, \ldots, S_t$ are connected components). Equivalently, t' denotes the number of representatives found in step 1, and say without loss of generality these are from the sets $S_1, \ldots, S_{t'}$.

For i = 1, ..., t' we color each S_i red or blue independently at random with equal probability. All vertices in S_i are given the color of S_i . For each red representative we then count how many neighbors it has colored blue. This can be done with O(t') cut queries. Let W be the set of red representatives that have a blue neighbor, and consider the submatrix M of the adjacency matrix with rows labeled by elements of W and columns labeled by vertices colored blue. By Lemma 5.8 with k = 10, for every element of W we can learn the name of a blue neighbor with $O(t' + t' \log(n)/\log(2t'/\log n))$ cut queries.

Now let us compute the expected number of components at the end of the round. In expectation, 1/2 of the representatives labeled red will have a neighbor colored blue. We learn one edge crossing the red-blue cut from each red representative. This set of edges is necessarily cycle free with respect to $S_1, \ldots, S_{t'}$. Thus by this process in expectation we will find a cycle free set of edges of size at least t'/4. As any cycle free set of edges is of size at most t', this means that by a reverse Markov inequality we will find a cycle free set of edges of size at least t'/8 with probability at least 1/7. Hence with probability at least 1/7 the number of Active sets at the start of the next round is at most 7t'/8.

Total number of queries. When there are t active sets remaining at the end of the previous round, then we have seen that the number of queries made in the current round is $O(t + t \log(n)/\log(2t/\log(n))) \in O(t \log(n)/\log(2t/\log(n)))$, plus a term which is O(n) over the course of the algorithm. We have also argued that with probability at least 1/7 the number of active components in the following round is at most 7t/8.

Let T(t) denote the expected number of queries made by the algorithm starting from when there are t active sets remaining. As T(t) is monotonically increasing in t, we have that

$$T(t) \le f(t) + \frac{6}{7}T(t) + \frac{1}{7}T(7t/8)$$
,

where $f(t) = O(t \log(n)/\log(2t/(\log n)))$. Equivalently, $T(t)/7 \le f(t) + T(7t/8)/7$, and letting c = 7/8 we have that for any j we can bound

$$\frac{1}{7}T(n) \le f(n) + f(cn) + f(c^2n) + \dots + f(c^jn) + \frac{1}{7}T(c^{j+1}n).$$

Now notice that we switch to the algorithm in Lemma 5.1 once the number of active sets falls below $n/\log(n)$. In that case the remaining query complexity is O(n) and hence $T(s) \in O(n)$ for $s \le n/\log(n)$. So it remains to bound $f(n) + \cdots + f(c^j n)$ for $c^j \ge 1/\log(n)$. By the definition of f(t) this is of order

$$n\log(n)\left(\frac{1}{\log(2n/\log n)} + \frac{c}{\log(2cn/\log n)} + \dots + \frac{c^j}{\log(2c^jn/\log n)}\right).$$

Using that $c^{j} \geq 1/\log(n)$, for $0 \leq i \leq j$ we can bound all denominators by

$$\log(2c^i n/\log n) \ge \log(2n) - 2\log(\log(n)) \in \Omega(\log(n)).$$

This gives the bound $f(n) + \cdots + f(c^{j}n) \in O(n(1+c+\cdots+c^{j})) \in O(n)$.

5.3 Edge connectivity certificate

We can easily use the spanning forest algorithm from last section to construct a sparse r-edge connectivity certificate by following the Nagamochi-Ibaraki approach of packing spanning forests. This would require O(nr) cut queries. In our edge connectivity algorithm, however, we will first do a star contraction on the input graph. This yields a contracted multigraph with significantly fewer vertices (say $q \ll n$), and we would like to construct an r-edge connectivity certificate with only O(rq) queries. This would easily follow from modifying the spanning forest algorithm from the last section to find a spanning forest of a multigraphs with q vertices using O(q) cut queries. However, it is not clear whether this is possible.¹³

In the following theorem we show that it is nevertheless possible to obtain a sparse r-edge connectivity certificate for a q-vertex contraction of an n-vertex simple graph efficiently, namely with $O(n + rq \log(n)/\log(q))$ cut queries. A key idea, as in the classic sequential algorithm of Nagamochi-Ibaraki, is to build the r spanning forests in parallel.

Theorem 5.9 (Formal version of Theorem 2.5). Let G = (V, E) be an n-vertex simple graph, and let G' = (V', E') be a contraction of G with q supervertices for $q \ge \log^{2+\varepsilon}(n)$ for some $\varepsilon > 0$. There is a zero-error randomized algorithm that makes $O(n + rq \log(n)/\log(q))$ cut queries in expectation and outputs a sparse r-edge connectivity certificate for G'.

Proof. We will make use of Theorem 3.5 and find F_1, \ldots, F_r such that F_i is a spanning forest of $(V', E' \setminus \bigcup_{j=1}^{i-1} F_j)$. We will follow the algorithm from Theorem 1.2 to find these r spanning forests in parallel.

The algorithm proceeds in rounds. We maintain the invariant that each F_i is a collection of trees $F_1^{(i)}, \ldots, F_{t_i}^{(i)}$ in the graph $G_i' = (V', E' \setminus \bigcup_{j=1}^{i-1} F_j)$. We let $S_1^{(i)}, \ldots, S_{t_i}^{(i)}$ be the partition of V' induced by the connected components of the trees in F_i . Each F_i is initialized to be empty, and thus corresponds to the trivial partition of V' by sets of size one. A key property that we maintain is that the partitions form a laminar family: the partition $S_1^{(i+1)}, \ldots, S_{t_i+1}^{(i+1)}$ is a refinement of the partition $S_1^{(i)}, \ldots, S_{t_i}^{(i)}$. This property means that if a (super-)vertex $U \in S_j^{(r)}$ has no edge leaving $S_j^{(r)}$, then U will not have an edge leaving any of $S_j^{(r-1)}, \ldots, S_j^{(1)}$ either.

The adjacency matrix of the contracted graph G' is no longer Boolean. To still take advantage of separating matrices as in Lemma 5.4, we will actually operate on the vertices of V instead of the (super-)vertices of V'. To aid in this we use the notation $T_j^{(i)} = \bigcup_{U \in S_j^{(i)}} U$, for all $i = 1, \ldots, r$ and i.

We initialize all $v \in V$ as Active. If at some point we discover that $v \in T_j^{(r)}$ satisfies $|E(v, \overline{T}_j^{(r)})| = 0$ then we change v to Inactive. Indeed, by the aforementioned laminar property we know that $|E(v, \overline{T}_j^{(i)})| = 0$ for all $i \leq r$ as well, and so it will not have an outgoing edge with respect to any of the forests.

We again proceed in rounds, until the number of active components in F_r has decreased by a factor of $\Omega(\log n)$. Let G_i' be the graph G' with supervertices contracted according to the edges in F_i , for $i=1,\ldots,r$ at this point of the algorithm. By the laminar property, the number of supervertices in each of these contracted graphs is at most $t_r \in O(q/\log n)$. We first complete finding a spanning forest of G_1' using the simple spanning forest algorithm Lemma 5.1 with O(q) queries. We then remove the edges found in completing the spanning forest of G_1' from G_2' and complete finding a spanning forest for G_2' via the simple spanning forest algorithm with O(q) queries. We continue in this way removing previous edges found and finding spanning forests for each G_i' for $i=3,\ldots,r$ to finish finding a sparse r-edge connectivity certificate with O(rq) more queries.

¹³In particular, the separating matrix machinery encounters additional logarithmic factors in working with the non-Boolean adjacency matrix of a weighted graph, which seem hard to avoid.

Let us now describe a generic round k of the algorithm, where we have partitions $S_1^{(i)}, \ldots, S_{t_i}^{(i)}$ for $i = 1, \ldots, r$. In round k we will simulate queries to the graph G where all edges already a part of F_1, \ldots, F_r are removed. We let E_k denote this set of edges.

Step 1. This step is very similar to step 1 of the algorithm in Theorem 1.2. For $j=1,\ldots,t_r$ we query $|E_k(v,\overline{T}_j^{(r)})|$ for $v\in T_j^{(r)}\cap \mathsf{Active}$. If $|E_k(v,\overline{T}_j^{(r)})|=0$ then v becomes Inactive; if $|E(v,\overline{T}_j^{(r)})|>0$ then v becomes the representative of $T_j^{(r)}$ and we move on to $T_{j+1}^{(r)}$ without any further queries in $T_j^{(r)}$. The number of queries in this step is $O(t_r+w)$ where w is the number of vertices that become Inactive in this round. Again, over all rounds the contribution to the number of queries from vertices becoming inactive is O(n).

Step 2. Let $t'_r \leq t_r$ be the number of representatives found in the previous step, and let us assume that these are representatives for the sets $T_1^{(r)}, \ldots, T_{t'_r}^{(r)}$. For $j=1,\ldots,t'_r$ we color each $T_j^{(r)}$ red or blue independently at random with equal probability, and give all vertices inside it the same color. For each red representative $v \in T_j^{(r)}$ we query its number of blue neighbors and let W be the set of all red representatives where this number is positive. Consider the submatrix M of the adjacency matrix of G whose rows are labeled by elements of W and columns are labeled by blue vertices. By Lemma 5.8 with k=10 for every element of W we can learn the name of a blue neighbor with $O(t'_r + t'_r \log(n)/\log(2t'_r/\log(n))) = O(t'_r \log(n)/\log(2t'_r/\log(n)))$ cut queries.

Via this process we learn |W| edges. We add each of these edges into the spanning forest F_i for the least value of i where it does not create a cycle. As this set of edges is necessarily cycle free with respect to F_r , all of the edges can be inserted somewhere, and so the total number of sets in the r forests goes down by at least |W|. The expected size of |W| is at least $t'_r/4$ as we expect half of the representatives to be red, and at least half of these to have a neighbor that is blue.

Total number of queries. The number of queries in a round depends on the number of components in the last spanning forest F_r . Apart from the queries made discovering inactive vertices, which we know is O(n) over the course of the entire algorithm, the number of queries made in a round is $O(t'_r \log(n)/\log(2t'_r/\log(n)))$. Thus we must analyze how t'_r decreases over the course of the algorithm.

Claim 5.10. The number of active components in F_r decreases by a factor of 1/2 after 16r rounds with probability at least 1/2.

Proof. Fix a round k, and suppose that at the start of round k the number of active components of F_r is α . Note that then the total number of components over all F_1, \ldots, F_r is at most $r\alpha$ by the laminar property of these components. We define two random variables at round k+i. Let Q_i be the random variable denoting the number of active components of F_r at the start of round k+i, and let W_i be the random variable denoting the number of edges found in round k+i. The key fact we need from the preceding discussion is that $\mathbb{E}[W_i] \geq \mathbb{E}[Q_i]/4$.

The total expected number of edges we find after ℓ rounds is

$$\mathbb{E}\left[\sum_{i=1}^{\ell} W_i\right] = \sum_{i=1}^{\ell} \mathbb{E}[W_i] \ge \sum_{i=1}^{\ell} \mathbb{E}[Q_i]/4.$$

As every found edge decreases the number of components over all F_1, \ldots, F_r by one, and there are at most $r\alpha$ components in total, this expectation is upper bounded by $r\alpha$. As $\mathbb{E}[Q_i]$ is a non-increasing function with i it must therefore be the case that $\mathbb{E}[Q_i] \leq \alpha/4$ for all $i \geq 16r$. Therefore, by Markov's inequality $\Pr[Q_i \geq \alpha/2] \leq 1/2$ for all $i \geq 16r$.

We can now bound the total number of queries from the $O(t_r' \log(n)/\log(2t_r'/\log(n)))$ terms in a similar way as we did in Theorem 1.2. Let T(s) be the cost of this term over the course of the algorithm starting from when $t_r' = s$. Let $f(s) = s \log(n)/\log(2s/\log(n))$ be the round cost. Then by Claim 5.10 we have $T(s) \leq 16rf(s) + T(s)/2 + T(s/2)/2$. This means $T(s) \leq 32rf(s) + T(s/2)$. With c = 1/2 and for any j, the quantity T(q) is hence of the order

$$32rq\log(n)\left(\frac{1}{\log(2q/\log(n))} + \frac{c}{\log(2cq/\log(n))} + \dots + \frac{c^{j}}{\log(2c^{j}q/\log(n))}\right) + T(c^{j+1}q) .$$

Once $c^{j+1} \leq 1/\log(n)$ we switch to the simpler algorithm, and so $T(c^{j+1}q) \in O(qr)$. Hence it remains to bound the preceding sum for $c^j > 1/\log(n)$, in which case we can bound $\log(2c^iq/\log(n)) \geq \log(2q/\log^2(n)) \in \Omega(\log(q))$ for all $i \leq j$ because by assumption $q \geq \log^{2+\varepsilon}(n)$ for some $\varepsilon > 0$. The sum then becomes $O(rq\log(n)/\log(q))$, finalizing the proof.

For the edge connectivity algorithm we will make use of a Monte Carlo version of Theorem 5.9, which we state here for reference.

Corollary 5.11. Let G = (V, E) be an n-vertex simple graph, and let G' = (V', E') be a contraction of G with q supervertices where $q \ge \log^{2+\varepsilon}(n)$ for some $\varepsilon > 0$. Let $r \le n$ be a positive integer. There is a randomized algorithm that makes $O(n + rq \log(n)/\log(q))$ cut queries and with probability 99/100 outputs a sparse r-edge connectivity certificate for G' and otherwise outputs FAIL.

6 Edge connectivity with $O(n \log \log n)$ cut queries

6.1 Sparse star contraction

In our edge connectivity algorithm with quantum cut or matrix-vector multiplication queries we used uniform star contraction—we randomly chose a set of center vertices R by taking each vertex with probability $p = \Theta(\log(n)/\delta(G))$, and considered the bipartite directed subgraph $H = (V, \overrightarrow{\operatorname{cut}}(V \setminus R))$. For every vertex in $V \setminus R$ we then independently at random chose an outgoing edge in H and contracted the set of selected edges. For these algorithms we could afford to learn the entire subgraph H when $\delta(G) \geq \sqrt{n}$ within the desired $\widetilde{O}(\sqrt{n})$ query bound.

In the randomized cut query model it is too expensive to learn H entirely. With our main tool for learning a bipartite graph, Lemma 5.4, we expect to spend $O(n\ell)$ queries to learn H, where ℓ is the maximum degree of a vertex in $V \setminus R$. To achieve our goal of an O(n) cut query algorithm, therefore, we would like to work with a directed subgraph H where the left hand side has *constant* degree.

To get an H where vertices on the left hand side have constant degree in expectation using star contraction we would have to take $p = \Theta(1/\delta(G))$. In this case, however, in expectation $\Omega(n)$ vertices would have no neighbor in R at all, thus doing 1-out contraction on H would not greatly reduce the number of vertices in the contracted graph.

The solution in this section is to perform sparse star contraction that uses two different sampling probabilities. First we randomly choose a set of center vertices R by taking each vertex with a slightly larger probability $p = \Theta(\log(\delta(G))/\delta(G))$ and letting $H = (V, \operatorname{cut}(V \setminus R))$. By a Chernoff bound, with constant probability now only $O(n/\delta(G))$ vertices in $V \setminus R$ have no outgoing edge in H. Let $S \subseteq V \setminus R$ be the set of vertices with positive outdegree in H. We then find a subgraph $H' = (S \cup R, A)$ of H where every vertex in S has an outgoing edge, but the maximum degree of a vertex in S is constant. In doing a random 1-out contraction on H', the resulting contracted graph G' will still only have $|R| + O(n/\delta(G)) \in O(n \log(\delta(G))/\delta(G))$ vertices, and intuitively we can hope to learn such an H' with only O(n) cut queries as the left hand side has constant degree.

The tricky part of doing this is to ensure that H' is still (α, β) -good for contracting with respect to a non-trivial minimum cut for some $\alpha < 1$ and constant β . We find the graph H' by using Recover-k-From-All (Algorithm 5.7), taking k to be a large constant, to learn k neighbors of every vertex in S. These learned edges define the graph H'. To review, what happens in Recover-k-From-All is that we first bucket the vertices in S into buckets with similar degrees in H. For a bucket with degree approximately r, we then run LearnBucket (Algorithm 5.5) which samples a subset of $R' \subseteq R$ by selecting each vertex of R with probability 2k/r. In expectation, each vertex in the bucket has a constant number of neighbors in R', as k is a constant. With cut queries we can easily check which vertices in the bucket were successfully "caught", where a vertex is caught if its number of neighbors in R' is within a constant factor of 2k, its expectation. For all the vertices caught we then learn all their neighbors in the subsample and add these edges to H'. We then repeat this routine until all vertices in the bucket are caught.

From the point of view of a single vertex $v \in S$ with degree r in H, its neighbors in H' will be its neighbors in a random subset R' of R, where each vertex of R is taken with probability p, conditioned on $d_{R'}(v)$ being close to its expectation. We need to show that this process does not select too high a fraction of edges from a non-trivial minimum cut even when p = 2k/r for a large constant k. Specifically, we want to upper bound the probability that $c_{R'}(v)/d_{R'}(v) \geq c(v)/d(v) + 1/10$. This requires a different proof than we used in Lemma 4.6 where $p = \Omega(\log(n)/\delta(G))$. With p this large we can argue by a Chernoff bound that with high probability $d_{R'}(v) = \Omega(\log n)$ and then again by a Chernoff bound that the probability that $c_{R'}(v)$ is both $\Omega(\log n)$ and greatly exceeds its expectation is negligible. When p = 2k/r with constant k, such an argument would only upper bound the probability that $c_{R'}(v)/d_{R'}(v) \geq c(v)/d(v) + 1/10$ by an absolute constant. This is not good enough for us because the number of vertices incident on a non-trivial minimum cut can be $\Omega(\delta(G))$, so this does not allow us to use a union bound.

The key to our proof is to show that for p=2k/r we can upper bound the probability that $c_{R'}(v)/d_{R'}(v) \geq c(v)/d(v)+1/10$ by a small constant times $\frac{c(v)}{kd(v)}$. By relating the failure probability to c(v)/d(v) and taking k to be a large enough constant, we can again use a union bound since we know that $\sum_{v \in N(C)} c(v)/d(v) \leq 2$. We prove this in Lemma 6.1. Then in Lemma 6.3 we formally verify that (a small modification of) Recover-k-From-All has the required properties needed to show that H' is indeed (α, β) -good for contracting. We prove these lemmas in the next subsection before giving a randomized algorithm for edge connectivity making $O(n \log \log n)$ cut queries in Section 6.3. To get down to O(n) cut queries one more trick is needed, which is postponed to Section 7.

6.2 Preparatory lemmas

Lemma 6.1. Let G = (V, E) be a simple n-vertex graph and let $C \subseteq E$. Let $v \in N(C)$ and $k \ge 10$. Choose a set R by putting each vertex of V into R independently at random with probability $p \ge 2k/d(v)$. Let $0 < f \le pd(v)/2$ and $g \ge 2pd(v)$. Then

$$\Pr_{R} \left[\frac{c_{R}(v)}{d_{R}(v)} \ge \frac{c(v)}{d(v)} + \frac{1}{10} \mid f \le d_{R}(v) \le g \right] \le \frac{200}{k} \frac{c(v)}{d(v)} .$$

The proof is deferred to Appendix A. At a high-level, the idea of the proof is the following. We already computed $\mathbb{E}_R[\frac{c_R(v)}{d_R(v)} \mid f \leq d_R(v) \leq g] = c(v)/d(v)$ in Proposition 4.5. To prove Lemma 6.1 we also compute $\mathbb{E}_R[\frac{c_R(v)^2}{d_R(v)^2} \mid f \leq d_R(v) \leq g]$. This allows us to upper bound the variance of $c_R(v)/d_R(v)$ by 2c(v)/(kd(v)) in Proposition A.5. We then obtain Lemma 6.1 by Chebyshev's inequality.

The next lemma summarizes the state of affairs after choosing the set R of centers using $p = \Theta(\log(\delta(G))/\delta(G))$.

Lemma 6.2. Let G = (V, E) be an n-vertex simple graph with minimum degree $d \ge 5 \cdot 10^6$ and let C be a non-trivial minimum cut of G. Let $p = \frac{10^5 \log(d)}{d}$ and choose a set R by putting each $v \in V$ into R independently at random with probability p. With probability at least 2/3 over the choice of R the following conditions will simultaneously hold

- 1. $|R| < \frac{3 \cdot 10^5 n \log(d)}{d}$.
- 2. $|\{v \in V : d_R(v) \le 5 \cdot 10^4 \log(d)\}| \le \frac{n}{10^3 d}$
- 3. The graphs $H = (V, \overrightarrow{cut}(V \setminus R))$ and G[R] are (3/5, 8)-good for contracting with respect to C^{14} .

Proof. We will upper bound the probability that each item *does not* happen. A union bound will then give the lemma.

Item 1 The expected size of R is $10^5 n \log(d)/d$. As the elements of R are chosen independently we can apply a Chernoff bound to see that the probability that $|R| \ge 3 \cdot 10^5 n \log(d)/d$ is at most $\exp(-10^5 n \log(d)/d) < 10^{-3}$.

Item 2 For $v \in V$ we have $\mathbb{E}_R[d_R(v)] \geq 10^5 \log(d)$. As the elements of R are chosen independently we can apply a Chernoff bound to see that the probability over R that $d_R(v) \leq 5 \cdot 10^4 \log(d)$ is at most $\exp(-10^4 \log(d))$. Therefore the expected number of v with $d_R(v) \leq 5 \cdot 10^4 \log(d)$ is at most $n/(10^6 d)$, and by Markov's inequality item (2) holds except with probability at most 10^{-3} .

Item 3 As C is a non-trivial minimum cut, we know that $c(v)/d(v) \leq 1/2$ for every $v \in V$. As we sample with probability $p = \frac{10^5 \log(d)}{d} \geq \frac{10^5 \log(d)}{d(v)}$ we can apply Lemma 6.1 with $k = 10^5 \log(d)/2$ to obtain $\Pr_R[c_R(v)/d_R(v) \geq 3/5 \mid d_R(v) > 0] \leq \frac{400}{10^5 \log(d)} \frac{c(v)}{d(v)}$ for any $v \in V$. Thus as $\sum_{c \in N(C)} c(v)/d(v) \leq 2$, by a union bound the probability that any v violates this is at most $800/(10^5 \log(d)) \leq 10^{-3}$.

Since $\mathbb{E}[c_R(v)/d_R(v) \mid d_R(v) > 0] = c(v)/d(v)$ by Proposition 4.5, the probability

$$\sum_{v:c_R(v)>0} c_R(v)/d_R(v) \ge 8$$

is at most 1/4 by Markov's inequality. This shows that both H and G[R] are (3/5,8)-good for contracting with respect to C except with probability at most $1/4 + 10^{-3}$.

Summing the three failure probabilities, overall the failure probability is at most $3 \cdot 10^{-3} + 1/4 < 1/3$, giving the lemma.

To learn neighbors in R of vertices in $V \setminus R$ we will use the next lemma. This lemma describes a worst-case version of the algorithm Recover-k-From-All (Algorithm 5.7) that was used in the spanning forest algorithm. However, we need to make some further observations about this algorithm, namely that neighbors are learned in a sufficiently random way that we are able to apply Lemma 6.1.

Lemma 6.3. Let G = (V, E) be an n-vertex simple graph and $C \subseteq E$. Let $h \ge 10$ be an integer. Let $S, T \subseteq V$ be disjoint subsets such that $|S| \ge |T|^{1/3}$ and $d_T(v) \ge h$ for all $v \in S$. Suppose that $H = (S \cup T, \overrightarrow{E}(S,T))$ is (α,β) -good for contracting with respect to C. There is a randomized algorithm that makes O(h|S|) cut queries and with probability at most 1/100 outputs FAIL, and

¹⁴The fact that G[R] is good for contracting will only be used in the O(n) algorithm in the next section.

otherwise explicitly outputs a graph $H' = (S \cup T, A)$ with $A \subseteq \overrightarrow{E}(S, T)$ where every vertex in S has outdegree at least h in H' and that with probability at least $1/10 + 200\beta/h$ is $(\alpha + 1/10, 10\beta)$ -good for contracting with respect to C.

Algorithm 6.4 WC-Recover-k-From-All[G](S, T, k)

Input: Cut query access to a simple graph G = (V, E), two disjoint subsets $S, T \subseteq V$ with $d_T(v) \geq h$ for all $v \in S$, and a parameter k.

Output: The adjacency list Z of a directed graph $H' = (S \cup T, A)$ with $A \subseteq \overrightarrow{E}(S, T)$ such that all vertices in S have outdegree at least min $\{k, h\}$ in H'.

- 1: Let A be the adjacency matrix of G and M = A(S, T).
- 2: Run Recover-k-From-All[M](k) (Algorithm 5.7) and terminate with FAIL if it makes more than 100 times its expected number of queries. Otherwise output the adjacency list Z returned.

Proof. We run a worst-case query complexity version of Recover-k-From-All Algorithm 5.7, which we call WC-Recover-k-From-All[G](S,T,k) (Algorithm 6.4), with k=h. The original Recover-k-From-All is stated as a zero-error algorithm and a bound is given on its expected number of queries. Here we want a worst-case bound on the number of queries, so we set a clock on Recover-k-From-All and terminate, outputting FAIL, if it makes more than 100 times its expected number of queries. The probability that this happens is at most 1/100. As S,T are disjoint, we can compute x^TMy for any $x \in \{0,1\}^{|S|}, y \in \{0,1\}^{|T|}$ with 3 cut queries to G by Corollary 3.7. Thus by Lemma 5.8 the number of queries made is O(h|S|), using the assumption $|S| \ge |T|^{1/3}$.

Now suppose that Recover-k-From-All terminates within 100 times its expected number of queries and let Z be the adjacency list returned. This adjacency list defines the directed graph H'. By Lemma 5.8 every list Z[v] has at least k items, thus as we take k = h every vertex in S has outdegree at least h in H'. It remains to show that H' is $(\alpha + 1/10, 10\beta)$ -good for contracting with respect to C with probability at least $1/10 + 200\beta/h$.

By the "further" statement of Lemma 5.8 the elements in Z[v] are the neighbors of v in a set Q chosen by placing each vertex $u \in T$ into Q with probability at least $\min\{2k/d_T(v), 1\}$, conditioned on Q having at least f and at most g neighbors of v, for $0 < f \le qd_T(v)/2$ and $g \ge 2qd_T(v)$. By Proposition 4.5 we know that

$$\mathbb{E}_Q\left[\frac{c_Q(v)}{d_Q(v)} \mid f \le d_Q \le g\right] = \frac{c_T(v)}{d_T(v)} .$$

As H is (α, β) -good for contracting, by linearity of expectation and Markov's inequality we therefore have that except with probability at most 1/10 over the choice of Q

$$\sum_{v \in S} \left[\frac{c_Q(v)}{d_Q(v)} \right] \le 10\beta .$$

Further as $h \ge 10$ and Q satisfies the hypotheses of Lemma 6.1, we can invoke this lemma to obtain

$$\Pr_{Q} \left[\frac{c_{Q}(v)}{d_{Q}(v)} \ge \frac{c_{T}(v)}{d_{T}(v)} + \frac{1}{10} \mid f \le d_{Q}(v) \le g \right] \le \frac{200}{h} \frac{c_{T}(v)}{d_{T}(v)} .$$

This will hold for all $v \in S$ except with probability $200\beta/h$ by a union bound. This shows that H' is $(\alpha + 1/10, 10\beta)$ -good for contracting with respect to C except with probability $1/10 + 200\beta/h$. \square

6.3 Algorithm and correctness

We are now ready for the main result of this section.

Theorem 6.5. There is a randomized algorithm that computes the edge connectivity of a simple graph G with probability at least 2/3 after $O(n \log \log n)$ cut queries. If $\delta(G) > \log^{10}(n)$ then only O(n) cut queries are needed.

Algorithm 6.6 Randomized $O(n \log \log n)$ cut query edge connectivity algorithm

Input: Cut query access to a simple graph G = (V, E) with adjacency matrix A.

Output: With constant probability outputs the edge connectivity of G.

- 1: Compute the minimum degree d of G.
- 2: **if** $d < 5 \cdot 10^6$ **then**
- 3: Compute a sparse d-edge connectivity certificate via Corollary 5.11.
- 4: Return the edge connectivity of this certificate.
- 5: Choose a random set R by putting each vertex in R independently with probability $p = 10^5 \log(d)/d$.
- 6: For each $v \in V$ compute $d_R(v) = |E(v, R \setminus \{v\})|$.
- 7: if $|R| \ge 3 \cdot 10^5 n \log(d)/d$ or $|\{v \in V : d_R(v) \le 5 \cdot 10^4 \log(d)\}| > n/(10^3 d)$ then
- 8: Return FAIL.
- 9: $S \leftarrow (V \setminus R) \cap \{v : d_R(v) > 5 \cdot 10^4 \log(d)\}.$
- 10: Run WC-Recover-k-From-All $[A(S,R)](5\cdot10^3)$. If this returns FAIL then return FAIL, otherwise let H' be the output.
- 11: Do a random 1-out contraction on H' and let G' be the resulting multigraph.
- 12: **if** $d \le \log^{10} n$ **then**
- 13: Find a sparse d-edge connectivity certificate F of G' via Corollary 5.11.
- 14: Compute a bipartition (Y, \overline{Y}) of V' corresponding to a minimum cut of F.
- 15: **else**
- 16: Compute a bipartition (Y, \overline{Y}) of V' corresponding to a minimum cut of G' via the algorithm of [MN20] (Theorem 3.10).
- 17: $W \leftarrow \bigcup_{W_i \in Y} W_i$.
- 18: Return $\min\{d, |\operatorname{cut}(W)|\}$.

Proof. The claim follows from Algorithm 6.6. In the first step with n cut queries we compute the minimum degree d. In line 2 we then handle the small degree case. As $\lambda(G) \leq d$, the edge connectivity of a sparse d-edge connectivity certificate of G will equal $\lambda(G)$. This step succeeds with probability at least 99/100 by Corollary 5.11 and takes O(nd) = O(n) cut queries.

We now assume we are in the $d \geq 5 \cdot 10^6$ case. As can be seen on line 18, the output of the algorithm is the minimum of d and |cut(W)| for a subset $W \subseteq V$. Thus if $d = \lambda(G)$ the algorithm will always correctly return d.

Let us therefore focus on the case $\lambda(G) < d$, and let C be a fixed non-trivial minimum cut of G. We randomly choose a set R by putting each v into R with probability $p = 10^5 \log(d)/d$. Note that p < 1 as we have already handled the small d case. R will satisfy the conditions of Lemma 6.2 with respect to C with probability at least 2/3. We condition on this good event happening for the rest of the proof. In particular, items 1 and 2 of Lemma 6.2 mean that we will not fail in line 7.

Let $S = (V \setminus R) \cap \{v : d_{R}(v) > 5 \cdot 10^{4} \log(d)\}$. By item 3 of Lemma 6.2 and Proposition 4.3 we know that $H = (S \cup R, \overrightarrow{E}(S, R))$ is (3/5, 8)-good for contracting with respect to C. On line 10 we run the algorithm WC-Recover-k-From-All on the sets S and R with $k = 5 \cdot 10^{3}$. By

Lemma 6.3 this takes O(n) cut queries. Further, by the same lemma, with probability at least 99/100 this algorithm will not fail, in which case it outputs a directed graph $H' = (S \cup R, A)$ with $A \subseteq \overrightarrow{E}(S,R)$, where every vertex in S has outdegree at least 1. Further, as H is (3/5,8)-good for contracting, by Lemma 6.3 H' will be (7/10,80)-good for contracting except with probability at most $1/10 + 1600/(5 \cdot 10^3) \le 1/2$. Thus overall the algorithm has succeeded up to this point with probability at least $(2/3) \cdot (99/100) \cdot (1/2) \ge 3/10$.

On line 11 we do a random 1-out contraction on H'. As H' is (7/10, 80)-good for contracting with respect to C, by Corollary 4.2 we do not contract an edge of C with probability at least $(1/5)^{110}$. In this case we will have $\lambda(G') = \lambda(G)$.

Let us compute the number N of supervertices in G'. This is at most $|R| + n/(10^3 d) \le 4 \cdot 10^5 n \log(d)/d$ because for every vertex in S we have contracted an edge connecting it to a vertex in R, since every vertex in S has an outgoing edge in H'. Therefore if $d \ge \log^{10} n$, the number of vertices in G' is $O(n/\log^9 n)$ and we can run the minimum cut algorithm of [MN20] (Theorem 3.10) on G' on line 16 to compute $\lambda(G')$ with $O(N\log^8 N) = O(n)$ cut queries. This algorithm succeeds with high probability.

If $d \leq \log^{10} n$, then we find sparse d-edge connectivity certificate in line 13 using the algorithm from Corollary 5.11. This algorithm correctly outputs a sparse d-edge connectivity certificate with probability 99/100 and otherwise outputs FAIL. The number of cut queries is $O(n + n \log(n) \log(d)/\log(n \log(d)/d)) = O(n \log d) = O(n \log \log n)$.

Thus with probability at least $(3/10) \cdot (1/5)^{110} \cdot (99/100)$ we will have $\lambda(G) = |\operatorname{cut}(W)|$ for the set W defined on line 17. Therefore by repeating the whole algorithm a sufficiently large constant number of times and outputting the minimum of $|\operatorname{cut}(W)|$ over all sets W produced we can output the edge connectivity with probability at least 2/3. The cut query complexity is dominated by line 13 and is $O(n \log \log n)$. In the case $d > \log^{10}(n)$ we avoid doing this step and only make O(n) cut queries.

7 Edge connectivity with O(n) cut queries

The bottleneck in the algorithm from the previous section is that the contracted graph G' had $\Omega(n \log(\delta(G))/\delta(G))$ vertices. We would like to get it down to $O(n/\delta(G))$ so that we can compute a sparse $\delta(G)$ -edge connectivity certificate of G' with O(n) queries by Corollary 5.11.

The bound on the number of vertices in G' resulted because we had to choose a set R of $\Theta(n \log(\delta(G))/\delta(G))$ centers in order to ensure that a sufficient number of vertices had a neighbor in R. Note, however, that we did not contract any edges *inside* the induced subgraph G[R]. As R was chosen randomly, however, each vertex in R has $\Theta(\log(\delta(G)))$ neighbors in R in expectation. As all but $O(n/\delta(G))$ vertices are connected to a vertex in R, we could further reduce the number of vertices in G' by contracting edges in G[R].

We could potentially do this via another round of star contraction inside G[R]: as each vertex in R has $\Theta(\log(\delta(G)))$ neighbors in R in expectation, we could randomly sample $R' \subseteq R$ by taking each vertex of R to be in R' with probability $p' = \log\log(\delta(G))/\log\delta(G)$. The expected size of R' is $n\log\log(\delta(G))/\delta(G)$ and with constant probability all but $O(n/\delta(G))$ many $v \in R$ have a neighbor in R'. Following this idea through can give an $O(n\log\log\log n)$ cut query algorithm for edge connectivity. To actually get the number of vertices down to $O(n/\delta(G))$, we follow a different approach based on 2-out contraction rather than star contraction.

¹⁵In fact, repeatedly applying the same argument yields a query complexity $O(n \log \log \cdots \log n)$ for any constant number of log's.

7.1 2-out contraction on the centers

We will take advantage of the following lemma shown by Ghaffari, Nowicki, and Thorup about the number of vertices in a graph after a random 2-out contraction.

Lemma 7.1 ([GNT20, Lemma 2.5]). Let G = (V, E) be a simple n-vertex graph with minimum degree ℓ . Independently for each $v \in V$ choose two outgoing edges $\{v, u_1\}, \{v, u_2\}$ uniformly at random and add them to a set X. Then with high probability the graph (V, X) has $O(n/\ell)$ connected components.

A nice quality of this lemma is that it can also be applied to a *subgraph* of G. In other words, if we learn a subgraph H = (V, E') of G = (V, E) such that all vertices in V have degree at least h in H then by doing 2-out contraction restricted to edges of H, we can still reduce the number of vertices in the corresponding contraction of G to O(n/h).

This will be our approach with the induced subgraph G[R]. If we could learn a subgraph H of G[R] where every vertex has degree $\Omega(\log \delta(G))$, then by doing 2-out contraction on H we could reduce the number of vertices in the contraction of G[R] by a $\log \delta(G)$ factor, i.e. down to $O(n/\delta(G))$. Furthermore, using the algorithm WC-Recover-k-From-All with $k = \log \delta(G)$ we can hope to learn such a subgraph with $O(n \log^2(\delta(G))/\delta(G)) = O(n)$ cut queries.

A direct obstacle to this plan is that G[R] can have $\Omega(n/\delta(G))$ many vertices with $o(\log \delta(G))$ neighbors in R. Luckily, we can deal with this by slightly generalizing Lemma 7.1. We show that if there is a degree threshold h such that only O(n/h) vertices have degree less than h, then after 2-out contraction the contracted graph still has only O(n/h) supervertices.

While this appropriately reduces the size of the contracted graph, a second obstacle remains: we have to ensure that we do not contract any edge of a non-trivial minimum cut. This is again where the randomness properties of WC-Recover-k-From-All shown in Lemma 6.3 come in handy. Using this lemma we will show that we can explicitly learn a directed subgraph H = (R, A) of G[R] such that (i) all but $O(n/\delta(G))$ vertices have degree at least h in H, and (ii) H is (α, β) -good for contracting with respect to a non-trivial minimum cut for some $\alpha < 1$ and constant β . We show how to do this in Lemma 7.3 below.

As we view H as a directed graph, we also need to generalize Lemma 7.1 to this case, where we sample only from outgoing edges, not incoming ones. Both generalizations are captured in the following lemma. The proof follows the original proof of [GNT20] with minor modifications, and we defer it to Appendix B. Similar to Lemma 7.1, this lemma also applies when H = (V, A) is a subgraph of a larger graph G = (V, E).

Lemma 7.2 (cf. [GNT20, Lemma 2.5]). Let H = (V, A) be an n-vertex directed graph such that all but τ vertices have out-degree at least $\ell \geq 4$. Independently for each $v \in V$ choose two outgoing edges $(v, u_1), (v, u_2)$ uniformly at random and add them to a set X. Then with high probability the graph (V, X) has at most $\tau + 2n/\ell$ weakly connected components.

Next we give the algorithm based on WC-Recover-k-From-All that we will use to build the directed subgraph H of G[R] on which we will perform the 2-out contraction.

Lemma 7.3. Let G = (V, E) be an n-vertex simple graph and $C \subseteq E$. Suppose that G is (α, β) -good for contracting with respect to C. Let $h \ge \max\{1500\beta, 35\}$ be such that for all but τ vertices in $v \in V$ it holds that $d(v) \ge 4h$. There is a randomized algorithm that makes O(hn) cut queries and with probability at most 3/100 outputs FAIL, and otherwise outputs a directed subgraph H of G where all but $\tau+n/h$ vertices in V have outdegree at least h in H. Further, H is $(\alpha+1/5, 100\beta)$ -good for contracting with respect to C with probability at least 1/2.

Algorithm 7.4 LearnSubgraph[G](h)

Input: Cut query access to a simple graph G = (V, E) with adjacency matrix A and a parameter h such that for all but τ vertices in $v \in V$ it holds that $d(v) \geq 8h$.

Output: A directed subgraph H of G where all but $\tau + n/h$ vertices have outdegree $\geq h$.

- 1: Randomly partition V into two sets V_1 and $V_2 = V \setminus V_1$ by putting each vertex independently at random into V_1 with probability 1/2 and otherwise into V_2 .
- 2: $V_1' \leftarrow \{v \in V_1 : d_{V_2}(v) \ge h\}, V_2' \leftarrow \{v \in V_2 : d_{V_1}(v) \ge h\}.$
- 3: if $|V \setminus (V_1' \cup V_2')| > \tau + n/h$ then return FAIL.
- 4: Run WC-Recover-k-From-All $[A(V'_1, V_2)](h)$ and WC-Recover-k-From-All $[A(V'_2, V_1)](h)$. If either call returns FAIL then return FAIL. Otherwise, let Z_1 and Z_2 be the outputs.
- 5: Return the directed graph H defined by the concatenation of Z_1 and Z_2 .

Proof. The algorithm is given in Algorithm 7.4. Let us first check the probability that we fail on line 3. We first randomly partition V into two sets V_1 and $V_2 = V \setminus V_1$. Let $V_1' = \{v \in V_1 : d_{V_2}(v) \ge h\}$ and $V_2' = \{v \in V_2 : d_{V_1}(v) \ge h\}$. If $v \in V_1$ has $d(v) \ge 4h$ then the probability that v is not in V_1' is at most $\exp(-h/4)$ by a Chernoff bound (Eq. (5)). The same is true for any $v \in V_2$, therefore the expected number of vertices with degree at least 4h that are not in $V_1' \cup V_2'$ is at most $n \exp(-h/4)$. By Markov's inequality therefore we have $|V \setminus (V_1' \cup V_2')| \le \tau + n/h$ except with probability $h \cdot \exp(-h/4) \le 1/100$ as $h \ge 35$. Thus the probability that we fail on line 3 is at most 1/100. Checking this condition can be done with O(n) cut queries as we can compute $d_{V_2}(v)$ with a constant number of cut queries, and likewise for $d_{V_1}(v)$.

By Proposition 4.5 we have $\mathbb{E}[c_{V_2}(v)/d_{V_2}(v) \mid d_{V_2}(v) > 0] = c(v)/d(v)$ for every $v \in V_1$ and $\mathbb{E}[c_{V_1}(v)/d_{V_1}(v) \mid d_{V_1}(v) > 0] = c(v)/d(v)$ for every $v \in V_2$. Thus by Markov's inequality we have

$$\sum_{v \in N(C) \cap V_1'} \frac{c_{V_2}(v)}{d_{V_2}(v)} + \sum_{v \in N(C) \cap V_2'} \frac{c_{V_1}(v)}{d_{V_1}(v)} \le 10\beta \tag{9}$$

except with probability at most 1/10.

Further, by Lemma 6.1 as we sample with probability p = 1/2 we can apply Lemma 6.1 with k = h together with a union bound to obtain that except with probability at most $200\beta/h$ we have

$$\frac{c_{V_2}(v)}{d_{V_2}(v)} \le \frac{c(v)}{d(v)} + \frac{1}{10} \text{ for all } v \in V_1',
\frac{c_{V_1}(v)}{d_{V_1}(v)} \le \frac{c(v)}{d(v)} + \frac{1}{10} \text{ for all } v \in V_2'.$$
(10)

To summarize, Eq. (9) and Eq. (10) show that the graph $F = (V, \overrightarrow{E}(V_1', V_2) \cup \overrightarrow{E}(V_2', V_1))$ is $(\alpha + 1/10, 10\beta)$ -good for contracting with respect to C except with probability at most $1/10 + 200\beta/h$. We now condition on this good event that F is $(\alpha + 1/10, 10\beta)$ -good for contracting.

The goal now is to learn h neighbors in V_2 of every vertex in V_1' , and vice versa, which we do by running WC-Recover-k-From-All $[A(V_1', V_2)](h)$ and WC-Recover-k-From-All $[A(V_2', V_1)](h)$. The total number of queries is O(hn) by Lemma 6.3. If either call outputs FAIL, then we abort and output FAIL, which happens with probability at most 2/100. We now condition on both of these calls being successful and let Z_1 and Z_2 be the adjacency lists returned. The directed subgraph H is defined by the concatenation of Z_1 and Z_2 . When these calls do not fail, every vertex of H with positive outdegree has outdegree at least h, as each list in Z_1, Z_2 has at least h neighbors by Lemma 6.3. Thus the number of vertices with zero outdegree in H is at most $|V \setminus (V_1' \cup V_2')| \le \tau + n/h$ assuming we did not FAIL in line 3. In summary, with probability at most 3/100 the algorithm

outputs FAIL, and otherwise it always returns a directed subgraph H where all but at most $\tau + n/h$ vertices have outdegree at least h. Further, item 3 of Lemma 6.3 together with the fact that F is $(\alpha + 1/10, 10\beta)$ -good for contracting tells us that H is $(\alpha + 1/5, 100\beta)$ -good for contracting with respect to C except with probability at most $1/10+200\beta/h$. Thus overall, H will be $(\alpha+1/5, 100\beta)$ -good for contracting with respect to C except with probability at most $2/10+400\beta/h \le 1/2$. \square

7.2 Algorithm and correctness

We are now ready to give a randomized O(n) cut query algorithm for edge connectivity.

Theorem 7.5. There is a randomized algorithm that computes the edge connectivity of a simple graph with probability at least 2/3 after O(n) cut queries.

Algorithm 7.6 Randomized O(n) cut query edge connectivity algorithm

Input: Cut query access to a simple graph G = (V, E) with minimum degree $d < \log^{10}(n)$. **Output:** With constant probability output the edge connectivity of G.

- 1: Compute the minimum degree d of G.
- 2: **if** $d < 5 \cdot 10^6$ **then**
- 3: Compute a sparse d-edge connectivity certificate via Corollary 5.11.
- 4: Return the edge connectivity of this certificate.
- 5: Choose a random set R by putting each vertex into R independently at random with probability $p = 10^5 \log(d)/d$.
- 6: For each $v \in V$ compute $d_R(v) = |E(v, R \setminus \{v\})|$.
- 7: if $|R| \ge 3 \cdot 10^5 n \log(d)/d$ OR $|\{v \in V : d_R(v) \le 5 \cdot 10^4 \log(d)\}| > n/(10^3 d)$ then
- 8: Return FAIL.
- 9: Let $S = (V \setminus R) \cap \{v : d_R(v) > 5 \cdot 10^4 \log(d)\}$ and A be the adjacency matrix of G.
- 10: Run WC-Recover-k-From-All $[A(S,R)](5\cdot 10^3)$. If this returns FAIL then return FAIL, otherwise let H' be the output.
- 11: Run LearnSubgraph[G[R]](h) (Algorithm 7.4) with $h = 5 \cdot 10^4 \log(d)$. If this returns FAIL then return FAIL, otherwise let H be the output.
- 12: Take a random 1-out sample of H' and a random 2-out sample of H and contract all selected edges in G. Let G' be the resulting multigraph.
- 13: Return FAIL if G' has more than $n/(10^3d) + 3|R|/h$ vertices.
- 14: Find a sparse d-edge connectivity certificate F of G' via Corollary 5.11.
- 15: Compute a bipartition (Y, \overline{Y}) of V' corresponding to a minimum cut of F.
- 16: $W \leftarrow \bigcup_{W_i \in Y} W_i$.
- 17: Return $\min\{d, |\operatorname{cut}(W)|\}$.

Proof. We can restrict to the case that the minimum degree $\delta(G) \leq \log^{10} n$ because an O(n) cut query algorithm for the case of larger degree is already handled by Theorem 6.5. The algorithm is given in Algorithm 7.6. As argued in the proof of Algorithm 6.6, the algorithm will always return correctly when the edge connectivity is achieved by a trivial cut. Let us therefore analyze the case that the edge connectivity is achieved by a non-trivial cut C.

The algorithm is identical to Algorithm 6.6 until line 11. From the proof of Theorem 6.5, at this point of the algorithm with probability at least 3/10 we will be in the state where

• The set R satisfies the three conditions of Lemma 6.2.

• The call to WC-Recover-k-From-All did not fail, and the returned graph H' is (7/10, 80)-good for contracting with respect to C.

Next, on line 11 we run Algorithm 7.4 on G[R] with $h = 5 \cdot 10^3 \log(d)$. This takes $O(n \log^2(d)/d) = O(n)$ cut queries by Lemma 7.3. Note that by item 2 of Lemma 6.2, at most $n/(10^3d)$ vertices in R have $d_R(v) < 8h < 5 \cdot 10^4 \log(d)$, thus we can take $\tau = n/(10^3d)$ in Lemma 7.3. Further, by item 3 of Lemma 6.2 G[R] is (3/5, 8)-good for contracting with respect to C. With $\beta = 8$ our choice of h satisfies $h \ge 1500\beta$ as we are in the case $d \ge 5 \cdot 10^6$. Thus we are in a position to apply Lemma 7.3, which tells us that with probability at least 1/2 the directed subgraph H of G[R] returned by the algorithm will be (7/10, 800)-good for contracting with respect to C. Let us assume this is the case, and the probability the algorithm reaches this good state is at least (3/10)(1/2) = 3/20. Further the number of vertices in R with outdegree less than h in H is at most $n/(10^3d) + |R|/h$.

As argued in the proof of Algorithm 6.6, the probability we do not select an edge of C in taking a 1-out sample of H' is at least $(1/5)^{110}$. As H is (7/10,800)-good for contracting with respect to C, by Proposition 4.1 the probability that we do not select an edge of C in taking a random 2-out sample of H is at least $(3/10)^{2286}$. We apply Lemma 7.2 with degree threshold h to see that with high probability (for concreteness say 99/100) the number of vertices in the contraction of G[R] by the edges in the 2-out sample will be the number of vertices with outdegree < h, which is at most $n/(10^3d) + |R|/h$, plus 2|R|/h. In particular, we do not fail in line 13 with probability at least 99/100. Overall, we are now in the good case that all steps of the algorithm have been successful with probability at least $(3/20) \cdot (1/5)^{110} \cdot (3/10)^{2286} \cdot (99/100)$, and G' has at most 7n/d vertices.

Finally, we find a sparse d-edge connectivity certificate F of G'. By Corollary 5.11 this succeeds with probability at least 99/100 and takes $O(n + n \log(n)/\log(n/d))$ queries, which is O(n) overall as $d < \log^{10} n$. Hence with probability at least $(5/24) \cdot (1/5)^{110} \cdot (3/10)^{2286} \cdot (99/100)^2$ we will correctly output the edge connectivity on line 17. As we never output a value that is less than the edge connectivity, we can repeat the whole algorithm a sufficiently large but constant number of times and output the minimum of the values returned to boost the success probability to 2/3. \square

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A Proof of Lemma 6.1

Throughout this appendix we will use the following notation. Let 0 < c < d be positive integers. Let $X_1, \ldots, X_c, Z_1, \ldots, Z_{d-c}$ be independent and identically distributed Bernoulli random variables that are 1 with probability p. Let $X = \sum_{i=1}^{c} X_i$ and $Y = \sum_{i=1}^{c} X_i + \sum_{i=1}^{d-c} Z_i$. Note that $X \sim B(c, p), Y \sim B(d, p)$ are both binomial random variables.

Although the most of the statements in this appendix will be purely probabilistic, one can keep in mind the following scenario. We have a graph G = (V, E) and a subset of edges C. Say that a vertex $v \in V$ had degree d and has c edges of C incident to it. We then sample a subset of edges incident to v by independently taking each edge with probability p. $X_i = 1$ represents the event that the ith edge of C incident to v is selected, and $Z_i = 1$ the event that the ith non-edge of C incident to v is selected. Then $X = \sum_{i=1}^{c} X_i$ is the random variable for the total number of edges of C incident to v selected and $V = X + \sum_{i=1}^{d-c} Z_i$ is the random variable for the total number of edges incident to v selected.

Proposition A.1.

$$\mathbb{E}[X^2 \mid Y = b] = \frac{cb}{d} + \frac{c(c-1)b(b-1)}{d(d-1)} .$$

Proof. By linearity of conditional expectation, $\mathbb{E}[X^2 \mid Y=b] = \sum_{i,j} \mathbb{E}[X_i X_j \mid Y=b]$. In the proof of Proposition 4.5 we have already computed that $\mathbb{E}[X_i^2 \mid Y=b] = \mathbb{E}[X_i \mid Y=b] = b/d$. Recall that X_i and Z_j are identically distributed, so also $\mathbb{E}[Z_i^2 \mid Y=b] = b/d$. For the same reason, the following expected values are all equal (i) $\mathbb{E}[X_i X_j \mid Y=b]$ for $i \neq j$, (ii) $\mathbb{E}[Z_i Z_j \mid Y=b]$ for any i, j. We then obtain the following for any $i \neq j$:

$$\mathbb{E}[Y^{2} \mid Y = b] = b^{2}$$

$$\mathbb{E}[(X_{1} + \dots + X_{c} + Z_{1} + \dots + Z_{d-c})^{2} \mid Y = b] = b^{2}$$

$$\mathbb{E}[d(d-1)X_{i}X_{j} + dX_{i}^{2} \mid Y = b] = b^{2}$$

$$\implies \mathbb{E}[X_{i}X_{j} \mid Y = b] = \frac{b(b-1)}{d(d-1)}.$$

There are c terms of the form $\mathbb{E}[X_i^2 \mid Y = b]$ and c(c-1) terms of the form $\mathbb{E}[X_i X_j \mid Y = b]$, giving the proposition.

Definition A.2 (Conditional first inverse moment). Let d be a positive integer and f, g be integers with $0 < f \le g \le d$. Let $p \in (0,1]$. Let $Y \sim B(d,p)$ be a binomial random variable. Define $Q(d,p,f,g) = \mathbb{E}[1/Y \mid f \le Y \le g]$.

Proposition A.3. Let $0 < f \le g \le d$. Then

$$\mathbb{E}[X^2/Y^2 \mid f \le Y \le g] = \frac{c(c-1)}{d(d-1)} + \left(\frac{c}{d} - \frac{c(c-1)}{d(d-1)}\right) Q(d, p, f, g) .$$

Proof. Let $\gamma = \Pr[f \leq Y \leq g]$. Then we have

$$\mathbb{E}[X^{2}/Y^{2} \mid f \leq Y \leq g] = \frac{1}{\gamma} \sum_{b=f}^{g} \frac{1}{b^{2}} \sum_{a=0}^{c} a^{2} \Pr[X = a, Y = b]$$

$$= \frac{1}{\gamma} \sum_{b=f}^{g} \frac{\Pr[Y = b]}{b^{2}} \mathbb{E}[X^{2} \mid Y = b]$$

$$= \frac{1}{\gamma} \sum_{b=f}^{g} \Pr[Y = b] \left(\frac{c}{bd} + \frac{c(c-1)(b-1)}{d(d-1)b} \right)$$

$$= \frac{c(c-1)}{d(d-1)} + \left(\frac{c}{d} - \frac{c(c-1)}{d(d-1)} \right) \mathbb{E}[1/Y \mid f \leq Y \leq g] . \square$$

In order to apply Proposition A.3 we will need to upper bound Q(d, p, f, g). Calculating the inverse moments of a truncated binomial distribution is a well-studied problem and precise asymptotic estimates are known, see e.g. [MW99]. For our purposes a looser estimate suffices and we opt for a simple self-contained proof adapted from [CS72].

Proposition A.4. Let $Y \sim B(d, p)$ and let $0 < f \le g$ be such that $\Pr[f \le Y \le g] \ge 1/2$. Then $Q(d, p, f, g) \le \frac{4}{pd}$.

Proof. We have

$$Q(d, p, f, g) = \frac{1}{\Pr[f \le Y \le g]} \sum_{b=f}^{g} \frac{\Pr[Y = b]}{b}$$

$$\le \frac{1}{\Pr[f \le Y \le g]} \sum_{b=1}^{d} \frac{1}{b} p^{b} (1 - p)^{d - b} \binom{d}{b}$$

$$\le \frac{2}{\Pr[f \le Y \le g]} \sum_{b=1}^{d} \frac{1}{b+1} p^{b} (1 - p)^{d - b} \binom{d}{b}$$

$$= \frac{2}{\Pr[f \le Y \le g]} \frac{1}{p(d+1)} \sum_{b=1}^{d} p^{b+1} (1 - p)^{d - b} \binom{d+1}{b+1}$$

$$\le \frac{2}{\Pr[f \le Y \le g]} \frac{1}{p(d+1)}$$

$$\le \frac{4}{pd}.$$

Proposition A.5. Let X, Y be the random variables defined in Proposition A.1. Then

$$\operatorname{Var}[X/Y \mid f \leq Y \leq g] \leq Q(d, p, f, g) \frac{c}{d}$$
.

Proof. For convenience let Q = Q(d, p, f, g). We have

$$\begin{aligned} \operatorname{Var}[X/Y \mid f \leq Y \leq g] &= \mathbb{E}[X^2/Y^2 \mid f \leq Y \leq g] - \mathbb{E}[X/Y \mid f \leq Y \leq g]^2 \\ &= \frac{c(c-1)}{d(d-1)} + \left(\frac{c}{d} - \frac{c(c-1)}{d(d-1)}\right)Q - \frac{c^2}{d^2} \\ &= \frac{c}{d}(d-c)\left(\frac{Q}{d-1} - \frac{1}{d(d-1)}\right) \\ &\leq Q\frac{c}{d} \ . \end{aligned}$$

We can now derive a more general version of Lemma 6.1.

Lemma A.6. Let G = (V, E) be a simple n-vertex graph and let $C \subseteq E$. Let $v \in N(C)$ and $k \ge 10$. Choose a set R by putting each vertex of V into R independently at random with probability $p \ge 2k/d(v)$. Let $0 < f \le pd(v)/2$ and $g \ge 2pd(v)$. Then for any $\alpha > 0$

$$\Pr_{R} \left[\frac{c_R(v)}{d_R(v)} \ge \frac{c(v)}{d(v)} + \alpha \sqrt{\frac{2}{k}} \mid f \le d_R(v) \le g \right] \le \frac{1}{\alpha^2} \frac{c(v)}{d(v)}.$$

Proof. Let c = c(v) and d = d(v). Let us first upper bound Q(d, p, f, g) with $p \ge 2k/d$. As $k \ge 10$ by a Chernoff bound (the "in particular" of Lemma 3.3) the probability that R contains between

f and g neighbors of v is at least 1/2. Thus we can apply Proposition A.4 to see that $Q \leq 2/k$. Therefore by Proposition A.5 we have $\operatorname{Var}[c_R(v)/d_R(v) \mid f \leq d_R(v) \leq g] \leq 2c/(kd)$. We can therefore apply Chebyshev's inequality to find

$$\Pr_{R} \left[\frac{c_R(v)}{d_R(v)} \ge \frac{c}{d} + t\sqrt{\frac{2c}{kd}} \mid f \le d_R(v) \le g \right] \le \frac{1}{t^2} .$$

Taking $t = \alpha \sqrt{d(v)/c(v)}$ gives the lemma.

Lemma 6.1 follows as an easy corollary by taking $\alpha = (1/10)\sqrt{k/2}$.

Lemma 6.1. Let G = (V, E) be a simple n-vertex graph and let $C \subseteq E$. Let $v \in N(C)$ and $k \ge 10$. Choose a set R by putting each vertex of V into R independently at random with probability $p \ge 2k/d(v)$. Let $0 < f \le pd(v)/2$ and $g \ge 2pd(v)$. Then

$$\Pr_{R} \left[\frac{c_{R}(v)}{d_{R}(v)} \ge \frac{c(v)}{d(v)} + \frac{1}{10} \mid f \le d_{R}(v) \le g \right] \le \frac{200}{k} \frac{c(v)}{d(v)} .$$

B Contraction lemma

To prove Lemma 7.2 we will closely follow the proof of the bound on the number of vertices after 2-out contraction given in [GNT20, Lemma 2.5]. To this end, we need the following definition and lemma.

Definition B.1 (Stochastic domination). Let X and Y be two random variables not necessarily defined on the same probability space. We say that Y stochastically dominates X, written $X \leq Y$, if for all $\lambda \in \mathbb{R}$ we have $\Pr[X \leq \lambda] \geq \Pr[Y \leq \lambda]$.

Lemma B.2 ([Doe20, Lemma 1.8.7]). Let X_1, \ldots, X_n be arbitrary binary random variables, and let Y_1, \ldots, Y_n be independent binary random variables. If $\Pr[X_i = 1 | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \le \Pr[Y_i = 1]$ for all $i = 1, \ldots, n$ and all $x_1, \ldots, x_{i-1} \in \{0, 1\}$ with $\Pr[X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] > 0$ then

$$\sum_{i=1}^{n} X_i \preceq \sum_{i=1}^{n} Y_i .$$

We are now ready to prove the following.

Lemma 7.2 (cf. [GNT20, Lemma 2.5]). Let H = (V, A) be an n-vertex directed graph such that all but τ vertices have out-degree at least $\ell \geq 4$. Independently for each $v \in V$ choose two outgoing edges $(v, u_1), (v, u_2)$ uniformly at random and add them to a set X. Then with high probability the graph (V, X) has at most $\tau + 2n/\ell$ weakly connected components.

Proof. We will prove the theorem by considering adding edges to E' in a specific order given by Algorithm B.3. In this algorithm we maintain a set \mathcal{P} of processed vertices, a set \mathcal{A} of active vertices, and a set \mathcal{S} of sampled vertices. We use the notation $A(v) = \{u \in V : (v, u) \in E\}$ and $u \in_R A(v)$ to denote choosing an element of A(v) uniformly at random.

Algorithm B.3 Procedure to add sampled edges

```
1: E' = \emptyset
 2: \mathcal{P} = \{ v \in V : \text{outdeg}(v) < \ell \}
 3: while \overline{\mathcal{P}} \neq \emptyset do
              Select v \in \overline{\mathcal{P}}
              \mathcal{A} \leftarrow \{v\}, \mathcal{S} \leftarrow \emptyset
 5:
              pflag \leftarrow 0
 6:
              while A \setminus S \neq \emptyset do
 7:
                                                                                                                 A run of this while loop is called a phase
                     Select v \in \mathcal{A} \setminus \mathcal{S}
                     \mathcal{S} \leftarrow \mathcal{S} \cup \{v\}
 9:
                     Sample u_1, u_2 \in_R A(v)
10:
                     Update \mathcal{A} \leftarrow \mathcal{A} \cup \{u_1, u_2\} and E' \leftarrow E' \cup \{\{v, u_1\}, \{v, u_2\}\}
11:
12:
                     if u_1 \in \mathcal{P} \vee u_2 \in \mathcal{P} then
                            pflag \leftarrow 1
13:
              \mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{A}
14:
```

Note that \mathcal{A} is always a connected via edges in E'. We are interested in the properties of \mathcal{A} when it is added to the set of processed vertices in Line 14. We wish to upper bound the number of times κ that \mathcal{A} is added to \mathcal{P} and the following two conditions hold

- 1. $|\mathcal{A}| < \ell$
- 2. pflag = 0

We can upper bound the number of connected components of G' by $\tau + n/\ell + \kappa$. This is because

- 1. The initial size of \mathcal{P} is τ ,
- 2. If pflag = 1 when \mathcal{A} is added to \mathcal{P} this means that \mathcal{A} is connected to a set of vertices that has already been processed and hence already counted,
- 3. The number of sets added where $A \ge \ell$ is at most n/ℓ .

The remaining case is where pflag = 0 and $A < \ell$, which is counted by κ .

Thus our task is to show that with high probability $\kappa \leq n/\ell$. To this end, define a random variable X_i to be 1 if at the end of the i^{th} phase $|\mathcal{A}| < \ell$ and pflag = 0, and 0 otherwise. In other words, $X_i = 1$ if and only if the i^{th} phase contributes to increasing κ .

Let us consider the probability that at the end of a phase on line $14 |\mathcal{A}| = x$ and pflag = 0. In this case we chose 2x many samples, and exactly x+1 of these were already in \mathcal{A} . Following Ghaffari, Nowicki, and Thorup [GNT20] we say a sample is caught if it is already in \mathcal{A} . The only fact needed to make the [GNT20] proof go through is that the probability a sample is caught is at most $\frac{x-1}{\ell}$ throughout the course of the phase. This holds in our case as \mathcal{P} is initialized to have all vertices of outdegree at most ℓ . Thus if pflag = 0 at the end of a phase then all vertices added to \mathcal{A} during the phase have outdegree at least ℓ and the probability that a sample on line 10 in already in \mathcal{A} is at most $\frac{x-1}{\ell}$.

in \mathcal{A} is at most $\frac{x-1}{\ell}$. There are $\binom{2x}{x+1}$ many sequences for the placement of the caught samples. Thus overall we can upper bound the probability that $|\mathcal{A}| = x$ by

$$P_x = {2x \choose x+1} \left(\frac{x-1}{\ell}\right)^{x+1} .$$

Following the calculation in [GNT20] (displayed equation, bottom of page 7) it follows that

$$\Pr[|\mathcal{A}| \le \frac{\ell}{8e^3}] \le \frac{8}{\ell^3} .$$

This means that $\Pr[X_i = 1 | X_1 = x_1 \dots X_{i-1} = x_{i-1}] \leq 8/\ell^3$. Now define independent random variables Y_i that take value 1 with probability $8/\ell^3$ and 0 otherwise. By Lemma B.2

$$\Pr[\sum_{i} X_{i} > \gamma] \le \Pr[\sum_{i} Y_{i} > \gamma] .$$

As the Y_i are independent we can upper bound the probability they exceed their expectation by a Chernoff bound. We have $\mathbb{E}[\sum_i Y_i] \leq 8n/\ell^3$. Thus for any $\gamma \geq 8n/\ell^3$ and $0 < \varepsilon$ we have by a Chernoff bound that $\Pr[\sum_i Y_i \geq (1+\varepsilon)\gamma)] \leq \exp(-\frac{\gamma\varepsilon^2}{2+\varepsilon})$.

If $n/(2\ell) \ge \log^2(n)$ then taking $\gamma = n/(2\ell)$ (which is at least $8n/\ell^3$ as $\ell \ge 4$) and $\varepsilon = 1$ tells us that $\kappa \le n/\ell$ except with probability $\exp(-\log^2(n)/3)$. If $n/(2\ell) < \log^2(n)$ then $8n/\ell^3 \le 64\log^8(n)/n^2$ and so we can take $\gamma = 8n/\ell^3$ and $\varepsilon = 1/(2\gamma)$ to see that $\kappa \le 1/2$ except with exponentially small probability.

C Proofs in streaming model

C.1 Space lower bound in explicit vertex arrival setting

In this section, we sketch a proof of the following observation.

Observation 4.13. [Follows from [Zel11]] Any one-pass streaming algorithm computing the edge connectivity of a simple graph in the explicit vertex arrival setting requires $\Omega(n^2)$ memory.

For completeness, we describe the reduction of [Zel11] from the Index function problem¹⁶ in the 2-party communication complexity setting to the problem of designing a one-pass streaming algorithm with $o(n^2)$ memory in the edge arrival setting that computes the minimum cut of a given graph. We then note that the exact same reduction can be implemented even if one considers the explicit vertex arrival setting, thus proving Observation 4.13.

Formally, in the Index function problem, Alice is given a binary string x of length ℓ , and Bob receives an index $i \in [\ell]$. Bob's goal is to learn the value of x_i . A well known result [KNR99] states that any one-way¹⁷ communication protocol that solves the Index function problem requires $\Omega(\ell)$ bits of communication.

The reduction: Let A be a one-pass streaming algorithm in the edge arrival setting using $o(n^2)$ bits of memory that computes the minimum cut in any given graph G = (V, E) on n vertices. Now, let x, i be an instance of the Index function problem such that x is of length $\frac{n^2-n}{2}$, and $i \in [\frac{n^2-n}{2}]$. Consider the following communication protocol in which Alice interprets x as the description of a simple graph G = (V, E) on n vertices, Alice feeds the edges of G to A in an arbitrary order, and then sends A's working memory to Bob, along with the degrees of all vertices in G. Bob interprets i as a pair of vertices a, b for which he wants to know whether $(a, b) \in E$. Bob continues the execution of A by extending G into a graph $G^* = (V^*, E^*), V \subseteq V^*, E \subseteq E^*$ as follows. Bob adds two cliques S, T, each on 3n vertices to the graph G, connects all vertices in T to all the vertices in the set $V \setminus \{a, b\}$, and connects all vertices in S to both S and S. Finally, Bob adds a final vertex S to the graph and connects it to S to both S and S vertices in S to both S and S to both S and S to both S to be adds a final vertex S to the graph and connects it to S to both S to both S to both S to be adds a final vertex S to the graph and connects it to S to both S to both S to be adds a final vertex S to the graph and connects it to S to be adds a final vertex S to the graph and connects it to S to be adds a final vertex S to the graph and connects it to S to be adds a final vertex S to the graph and connects it to S to be adds a final vertex S to the graph and connects it to S to be adds a final vertex S to the graph and connects it to S to be adds a final vertex S to the graph and connects it to S to S to be adds a final vertex S to the graph and connects it to S to S to S the final vertex S to S t

¹⁶[Zel11] refers to the problem as the bit vector probing problem

¹⁷A protocol consisting of a single message sent from Alice to Bob.

Zelke [Zel11] proves that computing the minimum cut in the resulting graph allows Bob to infer whether $(a, b) \in E$, thus proving the reduction.

Observation C.1. The above reduction can be implemented in the explicit vertex arrival setting.

Proof. We go over all insertions of edges to A and show that they can implemented in the vertex arrival setting. First, Alice inserts G = (V, E) into A in an arbitrary order, thus if $v_1, ..., v_n$ is any arbitrary order on the vertices of G, we can insert the edges of G into A by inserting the vertices in increasing order along with all incident edges to previously seen vertices. The same trick can be applied to the insertion of S and T into A with the addition that every $s \in S$ inserted is connected to not only previously seen vertices of S, but also to $\{a,b\}$, which are also previously seen vertices, and similarly for every vertex $t \in T$ and the edges connecting t to $V \setminus \{a,b\}$. Lastly, c can clearly be added in the vertex arrival setting as it is the last vertex, so in particular all of its edges are incident to previously seen vertices.

Observation C.1 combined with the soundness of the reduction proved in [Zel11] concludes the proof of Observation 4.13.

C.2 Proof of parallel sampling lemma

Lemma 4.15. There is a sampling procedure that operates within $\widetilde{O}(n)$ space and, given a stream S of n vertices, outputs $Y_1, \ldots, Y_r \subseteq [n]$ after reading the first $|\bigcup_{i=1}^r Y_i|$ vertices. The distribution D on (Y_1, \ldots, Y_r) defined by the procedure admits the following property. For every $R_1, \ldots, R_r \subseteq [n]$:

$$\Pr_{S \sim S_n, (Y_1, \dots, Y_r) \sim D}[Y_1 = R_1, \dots, Y_r = R_r] = \prod_{i=1}^r \Pr_{X_i \sim B([n], p)}[X_i = R_i] .$$

Proof. We begin with designing a procedure that outputs independent samples $Y_1, \ldots, Y_r \sim B([n], p)$. The procedure does not operate on a stream of vertices yet, and instead just samples vertices uniformly at random when necessary.

Algorithm C.2 Sampling independent subsets

Input: A probability parameter p, number of sets to sample r, and size of the universe n. **Output:** Sets Y_1, \ldots, Y_r that are independent samples from B([n], p).

- 1: Independently sample $X_1, \ldots, X_r \sim B([n], p)$.
- 2: For i = 1, ..., r let $k_i = |X_i|$.
- 3: Let $f_1 = k_1$ and for i = 2, ..., r let $f_i = |X_i \setminus \bigcup_{j=1}^{i-1} X_j|$.
- 4: $Z_0 \leftarrow \emptyset$.
- 5: **for** i = 1, ..., r **do**
- 6: Sample a uniformly random set S_1 of size $k_i f_i$ from Z_{i-1} .
- 7: Sample a uniformly random set S_2 of size f_i from $[n] \setminus Z_{i-1}$.
- 8: $Y_i \leftarrow S_1 \cup S_2 \text{ and } Z_i \leftarrow Z_{i-1} \cup Y_i$.
- 9: Output Y_1, \ldots, Y_r .

Proposition C.3. Given input parameters n, p, and r, Algorithm C.2 samples independent and identically distributed sets $Y_1, \ldots, Y_r \sim B([n], p)$.

Proof. Let $(Y_1, \ldots, Y_r) \sim \mathcal{A}$ indicate a sample from Algorithm C.2. For any sets $R_1, \ldots, R_r \subseteq [n]$ we show that the probability the algorithm outputs R_1, \ldots, R_r is equal to $\Pr_{X_1, \ldots, X_r \sim B([n], p)}[X_1 = R_1 \wedge \cdots \wedge X_r = R_r]$. We will decompose the latter probability as a product of conditional probabilities.

The first conditional event we consider is that X_1, \ldots, X_r satisfy some basic size and intersection requirements to be equal to R_1, \ldots, R_r . Specifically, let \mathcal{S} be the event that $|X_i| = |R_i|$ for $i = 1, \ldots, r$ and that $|X_i \setminus \bigcup_{i=1}^{i-1} X_j| = |R_i \setminus \bigcup_{i=1}^{i-1} R_j|$ for $i = 2, \ldots, r$. Then

$$\Pr_{X_1,\dots,X_r \sim B([n],p)} [X_1 = R_1 \wedge \dots \wedge X_r = R_r]$$

$$= \Pr_{X_1,\dots,X_r \sim B([n],p)} [\mathcal{S}] \cdot \Pr_{X_1,\dots,X_r \sim B([n],p)} [X_1 = R_1 \wedge \dots \wedge X_r = R_r \mid \mathcal{S}] .$$

Let S' be the analogous event that $Y_1, \ldots, Y_r \sim A$ satisfy the same size and intersection requirements of R_1, \ldots, R_r . Note that by the first 3 lines of the algorithm we have $\Pr_{X_1, \ldots, X_r \sim B([n], p)}[S] = \Pr_{(Y_1, \ldots, Y_r) \sim A}[S']$. To prove the lemma it thus suffices to show

$$\Pr_{X_1,\dots,X_r \sim B([n],p)}[X_1 = R_1 \wedge \dots \wedge X_r = R_r \mid \mathcal{S}] = \Pr_{(Y_1,\dots,Y_r) \sim \mathcal{A}}[Y_1 = R_1 \wedge \dots \wedge Y_r = R_r \mid \mathcal{S}'] .$$

This will follow from showing

1.
$$\Pr_{X_1,...,X_r \sim B([n],p)}[X_1 = R_1 \mid \mathcal{S}] = \Pr_{(Y_1,...,Y_r) \sim \mathcal{A}}[Y_1 = R_1 \mid \mathcal{S}']$$
, and

2. for
$$i = 2, ..., r$$

$$\begin{aligned} \Pr_{X_1, \dots, X_r \sim B([n], p)}[X_i &= R_i \mid \mathcal{S}, X_j = R_j, 1 \leq j < i] \\ &= \Pr_{(Y_1, \dots, Y_r) \sim \mathcal{A}}[Y_i = R_i \mid \mathcal{S}, Y_j = R_j, 1 \leq j < i] \end{aligned}.$$

Item 1 follows directly as, conditioned on S, X_1 is a uniformly random subset of [n] of size $|R_1|$, as is Y_1 conditioned on S'. Consider now the second item for an arbitrary i. Let $W = \bigcup_{j=1}^{i-1} R_j$. Let $k_i = |R_i|$ and $f_i = |R_i \setminus W|$. Conditioned on S, we know that $|X_i| = k_i$ and $|X_i| \in W| = f_i$. Thus subject to the conditional, X_i is the union of a uniformly random chosen set of size f_i from $[n] \setminus W$ and a uniformly random chosen set from W of size $k_i - f_i$. Similarly, conditioned on S and that $Y_j = R_j$ for $j = 1, \ldots, i-1$ so that $\bigcup_{j=1}^{i-1} Y_j = W$, in the algorithm Y_i is defined to be the union of a uniformly chosen set from $[n] \setminus W$ of size f_i and a uniformly random chosen set from W of size $k_i - f_i$. Thus the two sides of the equation in item 2 are equal.

Next, we show how to apply Algorithm C.2 on a stream of vertices. To this end, line 7 is implemented by repeatedly reading subsequent vertices from the given stream S (instead of repeatedly sampling a vertex uniformly at random from the remaining vertices). As each read vertex is included in Y_i , after reading the first $\bigcup_{i=1}^r Y_i$ vertices we can output the generated subsets Y_1, \ldots, Y_r as required. It remains to argue that the probability of generating $Y_1 = R, \ldots, Y_r = R$, over a random stream S and the random choices made by the algorithm, is equal to $\prod_{i=1}^r \Pr_{X_i \sim B([n],p)} [X_i = R_i]$. By Proposition C.3, this is the case for the subsets Y_1, \ldots, Y_r generated by Algorithm C.2. Next, we argue that the probability of the original Algorithm C.2 generating $Y_1 = R, \ldots, Y_r = R$ (over the random choices of the procedure) is the same as the probability of the modified Algorithm C.2 generating $Y_1 = R, \ldots, Y_r = R$ (over a random stream and the random choices made by the algorithm). This is the case because while the latter samples the next vertex uniformly at random from the remaining vertices, while the latter read the next vertex from the stream, which for a random stream is chosen uniformly at random from the remaining vertices.

C.3 Proof of algorithm in complete vertex arrival setting

Theorem 4.16. There is a one-pass streaming algorithm, using $\widetilde{O}(n)$ memory, that given a simple graph G = (V, E) in the complete vertex arrival setting, computes the edge connectivity of G with high probability.

Proof. Similarly to the case of random vertex arrivals, we run in parallel $\log(n)$ independent instances of an algorithm, each of which uses a different estimate $d=2^{\ell}$ for the minimum degree $\delta(G)$, with $\ell=0,1,2,\ldots,\lceil\log(n)\rceil-1$. Each algorithm aborts if it uses more than $\widetilde{O}(n)$ memory, and we will show that if ℓ is such that $d \leq \delta(G) < 2d$ then with high probability the corresponding algorithm will not abort and have correct outcome $\lambda(G)$. Since we know $\delta(G)$ exactly by the end of the stream (we can keep track of all degrees with $\widetilde{O}(n)$ memory), we can filter out the correct outcome at the end of the algorithm.

In the remainder we describe the algorithm for an estimate d on the minimum degree. As in the case of random vertex arrival, the algorithm attempts uniform star contraction on G with $p = \frac{1200 \ln n}{d}$, and in parallel constructs a sparse 2d-edge connectivity certificate on the contracted graph, so that at the end of the stream we can compute the edge connectivity of this certificate. We will run $r \in \Theta(\log n)$ parallel repetitions of this. In the i^{th} repetition, we first sample the set R_i by choosing each vertex with probability $p = \frac{1200 \ln n}{d}$. Then, we read the vertices from the input stream. As each vertex v arrives with all of its incident edges, when $v \notin R_i$ we are able to choose a uniformly random edge incident on R_i and contract it in G'_i . In parallel, we build a 2d-connectivity certificate $F_1^i \cup \cdots \cup F_{2d}^i$ of G'_i of G'_i . We summarize the i^{th} parallel repetition in full detail. Recall that each repetition is aborted as soon as its memory usage exceed $\widetilde{O}(n)$.

- 1. Construct R_i by choosing each vertex with probability $p = \frac{1200 \ln n}{d}$. Initialize F_1^i, \ldots, F_{2d}^i as empty forests and set $R_i = \emptyset$. Initialize a mapping $r_i : V \to V$ to be the identity (through the stream this will keep track of the contracted vertices).
- 2. For the j^{th} vertex arrival v with edges $e_1, ..., e_\ell$ between v to all other vertices, the following is done:
 - (a) **Uniform star contraction:** If $v \in R_i$, do nothing. If $v \notin R_i$ we consider the set $N_{R_i}(v)$, i.e., the set of neighbors of v in R_i . If this set is empty, we about the ith instance. Otherwise, we pick a uniformly random center w from the center neighborhood $N_{R_i}(v)$ (if it exists) and set $r_i(v) = w$. This amounts to contracting the edge $\{v, w\}$. For each e_t among e_1, \ldots, e_ℓ , except for the contracted edge which is discarded, change the endpoints of $e_t = \{v, u\}$ to be $\{r_i(v), r_i(u)\}$, discard any self loops. At the end of the stream, the vertices with the same $r_i(\cdot)$ values constitute a vertex in G'_i .
 - (b) Maintaining of 2d-edge connectivity certificate: For each (relabelled) incident edge e_t among e_1, \ldots, e_ℓ , add e_t to F_k^i where k is the minimal index for which $F_k^i \cup \{e_t\}$ contains no cycles. If there is no such k, discard the edge.
- 3. If the repetition did not abort by the end of the stream, we compute the edge connectivity of the connectivity certificate $\lambda_i = \lambda(G'_i) \geq \lambda(G)$.

Finally, we combine the r parallel repetitions by outputting $\min\{\delta(G), \lambda_1, \dots, \lambda_r\}$.

Analysis. The analysis is very similar to that of Theorem 4.14, we include it here for completeness. It is enough to prove the correctness and a $\widetilde{O}(n)$ memory bound only for the algorithm that has an estimate d such that $d \leq \delta(G) < 2d$. To this end, we only need to argue that, with constant probability, in a single repetition we have $|G'| = \widetilde{O}(n/d)$ and if $\lambda(G) < \delta(G)$ then $\lambda(G) = \lambda(G')$ (it is easy to see that we never have $\lambda(G') < \lambda(G)$).

A single repetition implements uniform star contraction with $p = \frac{1200 \ln n}{d}$. First, we want to analyse |G'|. By item 1 of Proposition 4.4, $|R| \leq 2pn$ except with probability n^{-400} . Next, by item 2 of Proposition 4.4 and $d \leq d(v)$, the probability that a vertex in $[n] \setminus R$ has no neighbor in R is at most n^{-6} . By a union bound, $|G'| = |R| \leq 2pn = \widetilde{O}(n/d)$ except with probability at

most $n^{-400} + n^{-5}$. Second, we want to lower bound the probability that $\lambda(G) = \lambda(G')$, assuming that $\lambda(G') < \delta(G)$. Let C be a non-trivial minimum cut of G. By Lemma 4.6, $H = (V, \operatorname{cut}(V \setminus R))$ is (2/3, 8)-good for contracting with respect to C with probability at least 2/3 over the choice of C. By Corollary 4.2 performing a random 1-out contraction on C does not contract any edge of C with probability at least 3^{-12} by Corollary 4.2. Thus $\lambda(G') = \lambda(G)$ with probability at least $2/3 \cdot 3^{-12}$.

D Reduction from minimum degree to edge connectivity

Here we mention a simple reduction from computing the minimum degree of a simple graph G = (V, E) to computing the edge connectivity of a graph G' = (V', E') with |V'| = 2|V|.

Lemma D.1. Given a simple graph G = (V, E) on n vertices for which we need to find the edge connectivity, we can construct another simple graph G' = (V', E') such that:

- The size of the vertex set |V'| = 2n, and
- the edge connectivity $\lambda(G') = \delta(G) + n$.

Proof. The construction of G' is simple: The vertex set of G' is $V' = V \cup K$ where K is a set of size n. The edge set $E' = E_1 \cup E_2 \cup E_3$ constitutes the following three types of edges:

- 1. **Original edges:** $E_1 = E$ consists of all the original edges of G.
- 2. Cross edges: $E_2 = \{\{u, v\} \mid u \in V, v \in K\}$ consists of all edges between V and K.
- 3. Clique edges: $E_3 = \{\{u, v\} \mid u \neq v, u, v \in K\}$ creates a clique on K.

In simpler words, G' consists of G and a clique on vertices K with all cross edges present between V and K.

We can immediately note that, for any $v \in V$, the degree $d_{G'}(v)$ in G' is $d_G(V) + n$. For all vertices $v \in K$, the degree is 2n - 1. Hence a vertex v that has minimum degree in G also has minimum degree in G', which is at most 2n - 1.

We now argue that the value of any non-trivial cut in G' is at least 2n-1. Let $X \subset V'$ denote a side of the cut and let $a = |X \cap V|$ and $b = |X \cap K|$. For a non-trivial cut we have a + b > 1 and we can assume w.l.o.g. that $b \le n/2$ (otherwise consider X^c). Now note that

$$|\operatorname{cut}_{G'}(X)| = |\operatorname{cut}_{G}(X \cap V)| + |\operatorname{cut}_{K}(X \cap K)| + |E(X \cap V, K \setminus X)| + |E(V \setminus X, X \cap K)|$$

$$\geq |\operatorname{cut}_{K}(X \cap K)| + |E(X \cap V, K \setminus X)| + |E(V \setminus X, X \cap K)|$$

$$= b(n - b) + a(n - b) + (n - a)b.$$

Now if b=0 then a>1 and the right hand side is an>2n-1. If b=1 then $a\geq 1$ and the right hand side is $2n+a(n-2)-1\geq 2n-1$ assuming $n\geq 2$. Finally, if $n/2\geq b>1$ then we can lower bound the right hand side by $a(n-b)+(n-a)b=nb+a(n-2b)\geq nb\geq 2n$.