Memory-Query Tradeoffs for Randomized Convex Optimization

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Abstract

We show that any randomized first-order algorithm which minimizes a d-dimensional, 1-Lipschitz convex function over the unit ball must either use $\Omega(d^{2-\delta})$ bits of memory or make $\Omega(d^{1+\delta/6-o(1)})$ queries, for any constant $\delta \in (0, 1)$ and when the precision ϵ is quasipolynomially small in d. Our result implies that cutting plane methods, which use $\tilde{O}(d^2)$ bits of memory and $\tilde{O}(d)$ queries, are Pareto-optimal among randomized first-order algorithms, and quadratic memory is required to achieve optimal query complexity for convex optimization.

1 Introduction

A fundamental problem in optimization and mathematical programming is convex optimization given access to a first-order oracle. Consider one of the canonical settings where the input is a 1-Lipschitz, convex function $F : \mathbb{B}^d \to \mathbb{R}$ over the *d*-dimensional unit ball \mathbb{B}^d . An algorithm has access to a first-order oracle of F: In each round, it can send a query point $\mathbf{x} \in \mathbb{B}^d$ to the oracle and receive a pair $(F(\mathbf{x}), \mathbf{g}(\mathbf{x}))$, where $\mathbf{g}(\mathbf{x}) \in \partial F(\mathbf{x}) \subseteq \mathbb{R}^d$ is a subgradient of F at \mathbf{x} . The goal is to find an ϵ -optimal point $\mathbf{x}^* \in \mathbb{B}^d$ satisfying $F(\mathbf{x}^*) - \operatorname{argmin}_{\mathbf{x} \in \mathbb{B}^d} F(\mathbf{x}) \leq \epsilon$. Convex optimization has a wide range of applications in numerous fields. In particular, it has served as one of the most important primitives in machine learning [Bub15]

The worst-case query complexity of minimizing a 1-Lipschitz, convex function $F : \mathbb{B}^d \to \mathbb{R}$ has long been known to be $\Theta(\min\{1/\epsilon^2, d\log(1/\epsilon)\})$ [NY83]. The two upper bounds can be achieved by subgradient descent $(O(1/\epsilon^2)$ [Nes03]) and cutting-plane methods $(O(d\log(1/\epsilon))$ [Lev65, BV04, Vai96, AV95, LSW15, JLSW20]), respectively. However, even when $\epsilon \ll 1/\sqrt{d}$ (which is arguably the more interesting regime), gradient descent has been the dominant approach for convex optimization despite its suboptimal worst-case query complexity, and cutting-plane methods are less frequently used in practice. One noticeable drawback of cutting-plane methods is its memory requirement, which has become a more and more important resource in the era of massive datasets. While cutting-plane methods require quadratic $\Omega(d^2 \log(1/\epsilon))$ bits of memory (e.g., to either store all subgradients queried so far or at least an ellipsoid in \mathbb{R}^d), in contrast, linear $O(d\log(1/\epsilon))$ bits of memory suffice for gradient descent. It is a natural question to ask whether there is an algorithm that can achieve the better of the two worlds.

This motivates a COLT 2019 open problem posed by Woodworth and Srebro [WS19] to study memory-query tradeoffs of convex optimization. The first result along this direction was proved by Marsden, Sharan, Sidford and Valiant [MSSV22]. They showed that any randomized first-order algorithm that minimizes a 1-Lipschitz, convex function $F : \mathbb{B}^d \to \mathbb{R}$ with $\epsilon = 1/\text{poly}(d)$ accuracy must either use $d^{1.25-\delta}$ bits of memory or make $d^{1+(4/3)\delta}$ many queries, for any constant $\delta \in [0, 1/4]$. Recently, Blanchard, Zhang and Jaillet [BZJ23] showed that any deterministic algorithm must use either $d^{2-\delta}$ bits of memory or $d^{1+\delta/3}$ queries. Their tradeoffs imply that cutting-plane methods are Pareto-optimal among deterministic algorithms on the memory-query curve of convex optimization.

1.1 Our Contribution

The result of Blanchard, Zhang and Jaillet [BZJ23] left open the question about whether similar tradeoffs hold for randomized algorithms. We believe that this is an important question to address because numerous gradient descent methods (e.g. stochastic gradient descent [Bub15], randomized smoothing [DBW12], variance reduction [JZ13]) are inherently randomized. Many exponential separations are known between deterministic and randomized algorithms for optimization problems, including escaping saddle points [DJL⁺17, JNG⁺21] and volume estimations [Vem05]. The role of randomness becomes even more critical in connection to memory. Almost all known streaming algorithms and dimension reduction technique are randomized [Woo14].

In this paper, we show that cutting-plane methods are in fact Pareto-optimal among randomized algorithms and thus, *optimal query complexity for convex optimization requires quadratic memory*. This is a corollary of the following memory-query tradeoff:

Theorem 1.1. Let d be a sufficiently large integer, $\epsilon = \exp(-\log^5 d)$ and $\delta \in (0, 1)$. Any algorithm that finds an ϵ -optimal point of a d-dimensional 1-Lipschitz convex function requires either $\Omega(d^{2-\delta})$ bits of memory or makes at least $\Omega(d^{1+\delta/6-o(1)})$ queries.

After reviewing additional related work in Section 1.2, we give an overview of our techniques in Section 2. At a high level, our proof is based on a novel reduction to a two-player, three-round communication problem which we call the *correlated orthogonal vector game*. Crucially, it differs from the *orthogonal vector game* considered in both [MSSV22] and [BZJ23] in two aspects: (a) It suffices for Bob to output a single vector that is orthogonal to vectors of Alice, but (b) the vector cannot be an arbitrary one but needs to have a strong correlation with a random vector that Bob sends to Alice in the second round. See Section 2.1. The major challenge is to prove a lower bound for the correlated orthogonal vector game, from which Theorem 1.1 follows. For this purpose, we develop a recursive encoding scheme which we explain in more details in Section 2.2. We believe that our techniques may have applications in understanding memory-query tradeoffs in other settings.

1.2 Additional related work

Learning with limited memory The role of memory for learning has been extensively studied in the past few years, including memory-sample tradeoffs for parity learning [Raz17, Raz18, GRT18, GRT19, GKLR21, LRZ23] and linear regression [SSV19], memory-regret tradeoffs for online learning [PZ23, PR23, SWXZ22, WZZ23, ACNS23], memory bounds for continual learning [CPP22], communication/memory bounds for statistical estimation [BBFM12, GMN14, SD15, BGM⁺16] and many others [SVW16, DS18, DKS19, GLM20, BHSW20, Fel20, BBF⁺21, BBS22].

Literature of convex optimization There is a long line of literature on convex optimization, we refer readers to [Nes03, Bub15] for a general coverage of the area. In particular, there is a long line of work [NY83, Nes03, WS16, BGP17, SEAR18, BHSW20] on establishing query lower bounds for finding approximate minimizers of Lipschitz functions, with access to a subgradient oracle. A recent line of work [Nem94, BS18, BJL⁺19, DG19] studies lower bounds for parallel algorithms. Memory efficient optimization algorithms have also been proposed, including limited-memory-BFGS [LN89, Noc80], conjugate gradient descent [HS⁺52, FR64, HZ06], sketched/subsampled quasinewton method [PW17, RKM19] and recursive cutting-planes [BJJ23].

2 Technique overview

We provide a high-level overview of techniques behind the proof of Theorem 1.1. To establish the memory-query tradeoffs, we use the same family of hard instance as [MSSV22] and our contribution is a novel and improved analysis. Define

$$F(\mathbf{x}) = \frac{1}{\sqrt{dL}} \cdot \max\left\{ L \|\mathbf{A}\mathbf{x}\|_{\infty} - 1, \max_{i \in [N]} \left(\langle \mathbf{v}_i, \mathbf{x} \rangle - i\gamma \right) \right\}$$
(1)

where $\mathbf{A} \sim \{-1, 1\}^{(d/2) \times d}$, $\mathbf{v}_i \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$ independently and uniformly at random with $\mathcal{H}_d := \{-1, 1\}^d$. Letting $\delta \in (0, 1)$ be the constant in Theorem 1.1, we use the standard choice of

$$\gamma = \widetilde{O}(d^{-\delta/4})$$
 and $N = \widetilde{O}(d^{\delta/6})$

for Nemirovski function, and $L = \exp(\log^5 d)$ is a large scaling factor.

The function (1) consists of two parts: the projection term $\|\mathbf{A}\mathbf{x}\|_{\infty}$ and the Nemirovski function $\max_{i \in [N]} (\langle \mathbf{v}_i, \mathbf{x} \rangle - i\gamma)$. From a high level, the Nemirovski function enforces that any convex optimization algorithm, with high probability, has to obtain $\mathbf{v}_1, \ldots, \mathbf{v}_N$ in this order, and to obtain the subgradient \mathbf{v}_{i+1} , the algorithm must submit a query \mathbf{x} that has $\Omega(\gamma)$ -correlation with \mathbf{v}_i and at the same time, is almost orthogonal to \mathbf{A} , as enforced by the projection term.

2.1 Reduction to correlated orthogonal vector game

At the heart of our proof is the new correlated orthogonal vector game. It is an abstract model that captures the hardness of the Nemirovski function and the projection in the hard functions F.

Definition 2.1 $((s,\xi)$ -Correlated orthogonal vector game). Let $d \in \mathbb{N}$ be the dimension. Let $k, s, n \in [d]$ and $\xi \in (0,1)$ be input parameters. The (s,ξ) -correlated orthogonal vector game is a two-player three-round communication problem: Alice receives a matrix $\mathbf{A} \sim \{-1,1\}^{(d/2)\times d}$ and Bob receives a vector $\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$. A deterministic (k,n)-communication protocol proceeds in three rounds:

Round 1: Alice sends Bob a message \mathbf{M} of length dk.

Round 2: Bob sends Alice the vector \mathbf{v} .

Round 3: Alice sends Bob n rows $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathsf{row}(\mathbf{A}) \cup \{\mathsf{nil}\}.$

At the end, Bob is required to output a vector $\mathbf{x} \in \mathbb{B}^d$, such that, with probability at least 1/2 (over the randomness of \mathbf{A} and \mathbf{v}),

Orthogonality: x *is almost orthogonal to* **A**, *i.e.*, $\|\mathbf{A}\mathbf{x}\|_{\infty} \leq \xi$, and

Correlation: x has large correlations with **v**, *i.e.*, $|\langle \mathbf{v}, \mathbf{x} \rangle| \ge \sqrt{s/d}$.

A randomized (k, n)-communication protocol is a distribution of deterministic protocols.

Suppose that there is a randomized convex optimization algorithm that uses $S = d^{2-\delta}$ bits of memory, makes $T = d^{1+\delta/6-o(1)}$ queries, and finds an ϵ -optimal solution of a random F with probability at least 2/3, where ϵ is quasipolynomially small in d. We show in Section 4 that such an algorithm can be used to obtain a (k, n)-communication protocol with k = S/d and $n \approx T/N$ that solves the (s, ξ) -correlated orthogonal vector game with $s \approx d\gamma^2 \approx d^{1-\delta/2}$ and a quasipolynomially small ξ . The novelty is mainly in the definition of the correlated orthogonal vector game and that we can manage to prove a strong lower bound for it, as sketched in the next subsection; the reduction itself uses standard techniques from the literature.

2.2 Lower bound for the correlated orthogonal vector game

We next that prove any (k, n)-communication protocol for (s, ξ) -correlated orthogonal vector game, requires either sending $kd \geq \Omega(s^2)$ bits in the first round or sending $n \geq d^{1-o(1)}$ rows in the third round. By our choice of parameters, this implies the third round message must contain $T/N = d^{1-o(1)}$ rows, and yields a query lower bound of $T \geq d^{1+\delta/6-o(1)}$.

Remark 2.2 (Sharpness of parameters). We remark on the relation of k, s, d, n. A communication protocol that sends n = d/2 rows in the third round could of course resolve the game. Hence $n \ge d^{1-o(1)}$ is almost the best one can prove. Meanwhile, for the first round, Alice could send a k-dimensional subspace U (this costs roughly kd bits) that is orthogonal to **A**, and even without the third round message, Bob can extract a vector $\mathbf{x} \in U$ that satisfies $|\langle \mathbf{x}, \mathbf{v} \rangle| \ge O(\sqrt{k/d})$. This implies that $k \ge \Omega(s)$ is the best one can prove. Our lower bound does not exactly match this and we only manage to prove $kd \ge \Omega(s^2)$. Though the later already suffices for an improved (quadratic) lower bound for randomized algorithms.

The proof of this lower bound turns out to be challenging and it is our main technical contribution. We prove by contradiction and assume (1) $kd \ll s^2$ and (2) $\Delta := d/n \ge \exp(\log d/\log \log d)$. Our lower bound is established via an iterative encoding argument: We provide an encoding procedure, built upon the (too-good-to-be-true) communication protocol, that encodes (almost) all matrices $\mathbf{A} \in \{0,1\}^{(d/2) \times d}$ into a set of size no more than $2^{d^2/2}/d$. Warm up: Encoding in the bit model Our encoding argument is delicate and we first illustrate the basic idea in a simpler "bit model" of the correlated orthogonal vector game: Alice sends n bits (instead rows) in the third round. This is strictly weaker than the original model because n rows can transmit at least n bits of message (even when they are required to be row vectors of \mathbf{A}).

In this model, Bob's output vector \mathbf{x} , is a function of the first round message $\mathbf{M} \in \{0, 1\}^{kd}$, the third round message $\mathbf{b} \in \{0, 1\}^n$ and its input vector $\mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d$, and we denote as $\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{b}}$. Fix a matrix $\mathbf{A} \in \{-1, 1\}^{(d/2) \times d}$ and the first round message $\mathbf{M}_{\mathbf{A}}$, the choice of \mathbf{v} is still uniformly at random, and we observe that the collection $\{\mathbf{x}_{\mathbf{M}_{\mathbf{A}},\mathbf{v},\mathbf{b}}\}_{\mathbf{v}\in\frac{1}{\sqrt{d}}}\mathcal{H}_d,\mathbf{b}\in\{0,1\}^n$ should contain at least s linearly independent vectors that are orthogonal to \mathbf{A} , and this continues to hold w.h.p. when one restricts to a subset $V^* \subseteq \mathcal{H}_d$ of size O(s). That is to say, using a probabilistic argument, there exists a set $V^* \subseteq \frac{1}{\sqrt{d}}\mathcal{H}_d$ ($|V^*| = O(s)$) such that $\{\mathbf{x}_{\mathbf{M}_{\mathbf{A}},\mathbf{v},\mathbf{b}}\}_{\mathbf{v}\in V^*,\mathbf{b}\in\{0,1\}^n}$ contains at least s linearly independent vectors that are orthogonal to \mathbf{A} , for at least 1/2-fraction of matrices $\mathbf{A} \in \{0,1\}^{(d/2) \times d}$. We denote this set as $\mathcal{A}_{\mathsf{nice}}$

Consider the following (one-shot) encoding strategy: Fix a message $\mathbf{M} \in \{0,1\}^{kd}$, a s-tuple $\mathbf{x}_1, \ldots, \mathbf{x}_s \in \{\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{b}}\}_{\mathbf{v}\in V^*,\mathbf{b}\in\{0,1\}^n}$ that are linearly independent, include all matrices $\mathbf{A} \in \{-1,1\}^{(d/2)\times d}$ whose rows are orthogonal to $\mathbf{x}_1, \ldots, \mathbf{x}_s$. The above argument implies that all matrices $\mathbf{A} \in \mathcal{A}_{\mathsf{nice}}$ are included, but on the hand, the encoding scheme only encodes

$$2^{kd} \times (2^n s)^s \times 2^{(d/2) \times (d-s)} \le 2^{d^2/2 + kd + ns \log_2(s) - ds/2} \ll \frac{1}{2} \cdot 2^{d^2/2} = |\mathcal{A}_{\mathsf{nice}}|$$

different matrices. Here the first term of LHS is the number of message \mathbf{M} , the second term is the number of s-tuple $\mathbf{x}_1, \ldots, \mathbf{x}_s$ (note this is the place we need $|V^*| \leq O(s)$) and the third term is the number of matrices \mathbf{A} that are orthogonal to the s-tuple.

Encoding in the row model We next design an encoding strategy based on the (too-good-to-be-true) communication protocol that sends n rows instead of n bits, which turns out to be much more challenging and requires an iterative encoding. In the row model, the output \mathbf{x} of Bob depends on the message $\mathbf{M} \in \{0,1\}^{kd}$, the vector $\mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d$ as well as n rows $\mathbf{R} \subseteq \operatorname{row}(\mathbf{A}) \cup \{\operatorname{nil}\}$, and we denote it as $\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}$. The first step is similar and we select a subset $V^* \subseteq \frac{1}{\sqrt{d}} \mathcal{H}_d$ that is "representive". In particular, one can select a subset $V^* \subseteq \frac{1}{\sqrt{d}} \mathcal{H}_d$, of size at most 2^s , such that $\{\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}\}_{\mathbf{v}\in V^*,\mathbf{R}\in\operatorname{row}(\mathbf{A})\cup\{\operatorname{nil}\}}$ contains at least $N_0 = 2^{\Omega(s)}$ "well-spread" output vectors $\mathbf{x}_1,\ldots,\mathbf{x}_{N_0}$ that are orthogonal to \mathbf{A} , for 1/2-fraction of matrices $\mathbf{A} \in \{-1,1\}^{(d/2)\times d}$. Here "well-spread" means the pairwise distance are at least $\alpha_0 = 1/d^8$. Again, we denote this set as $\mathcal{A}_{\mathsf{nice}}$.

To apply a similar encoding strategy, we need to fix the message \mathbf{M} , and the *s*-tuple from $\mathsf{T}^{\mathbf{M}}(\mathbf{A}) := \{\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}\}_{\mathbf{v}\in V^*,\mathbf{R}\in\mathsf{row}(\mathbf{A})\cup\{\mathsf{nil}\}}$, this turns out to be circular because it already needs the full knowledge of \mathbf{A} . Hence, a natural idea is to select a subset of rows \mathbf{A}_1 of \mathbf{A} , and uses the *s*-tuple from $\mathsf{T}^{\mathbf{M}}(\mathbf{A}_1) = \{\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}\}_{\mathbf{v}\in V^*,\mathbf{R}\in\mathsf{row}(\mathbf{A}_1)\cup\{\mathsf{nil}\}}$. Let $\mathbf{A}_2 = \mathbf{A}\setminus\mathbf{A}_1$ be the removed sub-matrix and we need its size to be as large as possible – this is the part where we get compressions. Meanwhile, one can not hope to remove too many rows, because removing one row would decrease the size of $\mathsf{T}^{\mathbf{M}}(\mathbf{A})$ by a factor of roughly $(1 - 1/\Delta)$. There are $N_0 = 2^{\Omega(s)}$ well-spread vectors in $\mathsf{T}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A})$ (that are also orthogonal to \mathbf{A}), so one only affords to remove at most $s\Delta$ rows.

The encoding strategy would fix a message $\mathbf{M} \in \{0,1\}^{kd}$, a submatrix $\mathbf{A}_1 \in \{-1,1\}^{(d/2-s\Delta)\times d}$, a *s*-tuple $\mathbf{x}_1, \ldots, \mathbf{x}_s \in \{\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{b}}\}_{\mathbf{v}\in V^*,\mathbf{R}\in \mathsf{row}(\mathbf{A}_1)}$ that are (1) linearly independent, and (2) orthogonal to \mathbf{A}_1 , then it includes all matrices $\widetilde{\mathbf{A}}_2 \in \{-1,1\}^{s\Delta\times d}$ whose rows are orthogonal to $\mathbf{x}_1, \ldots, \mathbf{x}_s$. It encompass \mathcal{A}_{nice} and has size at most¹

$$2^{kd} \times 2^{(d/2-s\Delta) \times d} \times (d^n 2^s)^s \times 2^{s\Delta \times (d-s)} \approx 2^{d^2/2+kd+sn\log_2(d)-s^2\Delta}.$$

This is much larger than $2^{d^2/2}$ and not useful at all! We note in the above expression, the first term is the number of message \mathbf{M} , the second term is the total number of \mathbf{A}_1 , the third term is the number of *s*-tuple $\mathbf{x}_1, \ldots, \mathbf{x}_s$ and the last term is the number of orthogonal matrices $\widetilde{\mathbf{A}}_2$.

Iterative encoding Our key observation is that: If there are too many s-tuples that are linearly independent and orthogonal to \mathbf{A}_1 , then there must exists a large number of well-spread vectors in $\mathsf{T}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}_1) = \{\mathbf{x}_{\mathbf{M}_{\mathbf{A}},\mathbf{v},\mathbf{R}}\}_{\mathbf{v}\in V^*,\mathbf{R}\in\mathsf{row}(\mathbf{A}_1)\cup\{\mathsf{nil}\}}$ that are orthogonal to \mathbf{A}_1 . In particular, if there are $2^{\Omega(s^2\Delta)}$ different s-tuples, then there are $N_1 := 2^{\Omega(s\Delta)}$ well-spread vectors that are orthogonal to \mathbf{A}_1 . This guarantee is stronger than the original one: $N_0 = 2^{\Omega(s)}$ well-spread vectors that are orthogonal to \mathbf{A} . Therefore, if we fail to compress \mathbf{A} by removing $s\Delta$ rows, we can try to compress it by removing another $s\Delta^2$ rows. We can repeat it for $\log \log(d)$ iterations and it yields our final iterative encoding scheme.

There are a few subtle yet critical details that are (intentionally) omitted by us, e.g., the "well-spreadness" (the choice of pairwise distance α) are doubly exponentially decreasing at each iteration, and we need robustly linear independence (similar to [MSSV22]), we refer readers to Section 5 for these technical details.

2.3 Comparison with [MSSV22, BZJ23]

We compare our techniques with previous work of [MSSV22, BZJ23]. Both proofs are based on reductions to the following orthogonal vector game: Alice receives a random matrix $\mathbf{A} \in \{-1,1\}^{(d/2)\times d}$, sends Bob a message $\mathbf{M} \in \{0,1\}^{kd}$ and n rows \mathbf{R} from \mathbf{A} , and Bob is required to output $\Omega(k)$ linearly independent vectors that are orthogonal to \mathbf{A} . On the one hand, [MSSV22] proves a lower bound of $n \geq \Omega(d)$ for orthogonal vector game. On the other hand, they show that any S-bit, T-query randomized algorithm can be used to obtain a protocol for the orthogonal game with k = S/d and n = T/(N/k) and thus, using $n \geq \Omega(d)$, one has $T = \Omega(Nd^2/S)$. However, N can only be $O(d^{1/3})$ in the Nemirovski function, which limits their tradeoffs to apply only up to $S = d^{4/3}$ (for T to be superlinear). The tradeoffs of [BZJ23], on the other hand, are based on the idea that, when against deterministic algorithms, one can adaptively build the vector \mathbf{v}_i in the Nemirovski function (basing on the algorithm's queries so far) and make it orthogonal to previous queries. This increases the size N of Nemirovski function to be d and thus, the tradeoff holds even near $S = d^2$. However, the argument of [BZJ23] is dedicated to deterministic algorithms and can be easily sidestepped by randomized algorithms.

The correlated orthogonal vector game introduced in this paper is more fine-grained. It proceeds in three rounds (instead of one round as in the orthogonal vector game) and Bob is required to output only one orthogonal vector that has large correlations with the random vector \mathbf{v} . It is not surprising that our encoding argument gives an alternative proof for the orthogonal vector game: One simply enumerates the message \mathbf{M} , the *n* rows \mathbf{R} , and the rest rows of matrix \mathbf{A} , which are required to be orthogonal to the $\Omega(k)$ output vectors from (\mathbf{M}, \mathbf{R}) . Proving a strong lower bound for the correlated orthogonal vector game is technically the most challenging part of the paper.

¹We also need to fix the row indices of A_1 , but it is a lower order term that we omit here for simplicity.

3 Preliminaries

Notation Let $[n] = \{1, 2, ..., n\}$ and $[n_1 : n_2] = \{n_1, n_1 + 1, ..., n_2\}$. Let \mathbb{B}^d be the *d*-dimensional unit ball, \mathbb{S}^d be the *d*-dimensional unit sphere, and $\mathcal{H}_d = \{-1, 1\}^d$ be the *d*-dimensional Boolean hypercube. Given a point $\mathbf{x} \in \mathbb{R}^d$ and $\delta > 0$, we write $\mathbb{B}(\mathbf{x}, \delta) := \{\mathbf{y} : \|\mathbf{x} - \mathbf{y}\|_2 \le \delta\}$ to denote the δ -ball centered at \mathbf{x} . For a set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$, we use $\mathsf{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m) \subseteq \mathbb{R}^d$ to denote the subspace spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_m$. Given a subspace S and vector $\mathbf{x} \in \mathbb{R}^d$, we use $\mathsf{proj}_S(\mathbf{x}) \in \mathbb{R}^d$ to denote the projection of \mathbf{x} onto S. Given a matrix \mathbf{A} , we write $\mathsf{row}(\mathbf{A})$ to denote the set of its row vectors. We write $x \sim X$ if the random variable x is drawn uniformly at random from a set X.

We consider the following oracle model of convex optimization.

Definition 3.1 (Memory constrained convex optimization). Let $d \in \mathbb{N}$ be the dimension. An S-bit and T-query convex optimization deterministic algorithm with first-order oracle access runs as follows. Given access to a first-order oracle of a 1-Lipschitz convex function $F : \mathbb{B}^d \to \mathbb{R}$,

- 1. The algorithm starts by initializing the memory to be a string $\mathbf{M}_0 \in \{0, 1\}^S$;
- 2. During iteration $t \in [T]$, the algorithm picks a query $\mathbf{x}_t \in \mathbb{B}^d$ based on \mathbf{M}_{t-1} , obtains from the oracle both $F(\mathbf{x}_t) \in \mathbb{R}$ and a subgradient $\mathbf{g}(\mathbf{x}_t) \in \partial F(\mathbf{x}_t) \subseteq \mathbb{R}^d$, and updates the memory state to $\mathbf{M}_t \in \{0,1\}^S$ based on $\mathbf{M}_{t-1}, F(\mathbf{x}_t)$ and $\mathbf{g}(\mathbf{x}_t)$.
- 3. Finally, after T iterations, the algorithm outputs a point $\mathbf{x}_T \in \mathbb{B}^d$ based on \mathbf{M}_{T-1} .

An S-bit and T-query randomized algorithm is a distribution over deterministic algorithms. We say a randomized algorithm finds an ϵ -optimal point with probability at least $1 - \rho$ if for any f,

$$F(\mathbf{x}_T) - \operatorname{argmin}_{\mathbf{x} \in \mathbb{B}^d} F(\mathbf{x}) \le \epsilon.$$

with probability at least $1 - \rho$.

4 Reduction to correlated orthogonal vector game

Let $\delta \in (0, 1)$ be the constant in Theorem 1.1. We use the following hard instances from [MSSV22]:

$$F(\mathbf{x}) = \frac{1}{\sqrt{dL}} \cdot \max\left\{ L \|\mathbf{A}\mathbf{x}\|_{\infty} - 1, \max_{i \in [N]} \left(\langle \mathbf{v}_i, \mathbf{x} \rangle - i\gamma \right) \right\}$$

where $\mathbf{A} \sim \{-1, 1\}^{(d/2) \times d}$, $\mathbf{v}_i \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$ (so that each \mathbf{v}_i is a unit vector), and

$$\gamma = \frac{\log^2 d}{d^{\delta/4}}, \quad N = \frac{d^{\delta/6}}{\log^4 d} \quad \text{and} \quad L = \exp\left(\log^5 d\right).$$

It is easy to verify that F is 1-Lipschitz and convex.

First-order oracle Given \mathbf{A} , \mathbf{v}_i and the function F they define, we will use \mathbf{g} below as the subgradient part of the first-order oracle of F: on any query point $\mathbf{x} \in \mathbb{B}^d$, if the maximum is achieved at either $\frac{L \cdot \langle \mathbf{A}_j, \mathbf{x} \rangle - 1}{\sqrt{dL}}$ or $\frac{-L \cdot \langle \mathbf{A}_j, \mathbf{x} \rangle - 1}{\sqrt{dL}}$, for some row \mathbf{A}_j of \mathbf{A} $(j \in [d/2])$, \mathbf{g} returns either \mathbf{A}_j/\sqrt{d} or $-\mathbf{A}_j/\sqrt{d}$ accordingly with the smallest such $j \in [d/2]$; otherwise, \mathbf{g} returns $\mathbf{v}_i/(\sqrt{dL})$ with the smallest $i \in [N]$ that achieves the maximum. In the rest of the section, when F is the input function, the algorithm has access to F and \mathbf{g} for subgradients of F as the first-order oracle.

Our main result in this section is the following lemma, which shows that any query-efficient randomized convex optimization algorithm would imply an efficient deterministic communication protocol for the correlated orthogonal vector game. **Lemma 4.1** (Reduction). Let $\epsilon = 1/(d^2L)$. If there is a randomized convex optimization algorithm that uses $S = d^{2-\delta}$ bits of memory, makes $T = d^{1+\delta/6-o(1)}$ queries and finds an ϵ -optimal point with probability at least 2/3, then there is a deterministic (k, n)-communication protocol that succeeds with probability at least 1/2 for the (s, ξ) -correlated orthogonal vector game, with parameters

$$k = \frac{S}{d} = d^{1-\delta-o(1)}, \quad n = \frac{40T}{N} = d^{1-o(1)}, \quad s = d^{1-\delta/2}\log^2 d \quad and \quad \xi = \frac{2}{L}$$

Assume such a randomized convex optimization algorithm exists. Since randomized algorithms are distributions of deterministic algorithms, there must be a deterministic S-bit-memory, T-query algorithm that outputs an ϵ -optimal point of F with probability at least 2/3 (over the randomness of **A** and **v**_i). Let ALG denote such an algorithm. For convenience we assume that the last point that ALG queries is the same point it outputs at the end; it is without loss of generality since such a requirement can only increase the number of queries by one. We use it to obtain a deterministic communication protocol for the correlated orthogonal vector game in the rest of the section.

Given a function F, we first use the execution of ALG on F to define the *correlation time*:

Definition 4.2 (Correlation time). For each $i \in [N]$, let $t_i \in [T] \cup \{\infty\}$ be the first time that ALG submits a query $\mathbf{x}_{t_i} \in \mathbb{B}^d$ such that

- 1. \mathbf{x}_{t_i} has a large correlation with \mathbf{v}_i : $|\langle \mathbf{x}_{t_i}, \mathbf{v}_i \rangle| \geq \gamma/4$; and
- 2. \mathbf{x}_{t_i} is almost orthogonal to \mathbf{A} : $\|\mathbf{A}\mathbf{x}_{t_i}\|_{\infty} \leq \xi$.

If no such query exists during the execution of ALG, then we set $t_i = \infty$.

First we show that, with high probability, the correlation time are in order $t_1 \leq t_2 \leq \cdots \leq t_N$:

Lemma 4.3. Fix any $\mathbf{A} \in \{-1, 1\}^{(d/2) \times d}$. With probability at least $1 - d^{-\omega(1)}$ over the randomness of $\mathbf{v}_1, \ldots, \mathbf{v}_N$, we have $t_1 \leq t_2 \leq \cdots \leq t_N$.

Proof. Consider the process of first initializing $t_1 = \cdots = t_N = \infty$ and then in each round $t \in [T]$, setting $t_i = t$ if $t_i = \infty$ at the end of round t-1 but the point \mathbf{x}_t queried by ALG in round t satisfies both conditions $|\langle \mathbf{x}_t, \mathbf{v}_i \rangle| \ge \gamma/4$ and $||\mathbf{A}\mathbf{x}_t||_{\infty} \le \xi$. Note that if the event of the lemma is violated, then there must exist $t \in [T]$ and $i \in [N-1]$ such that the following two events hold:

- 1. Event E_1 : At the end of round t-1, we have $t_i = t_{i+1} = \infty$; and
- 2. Event E_2 : At the end of round t, we have $t_i = \infty$ but $t_{i+1} = t$.

Fixing $t \in [N]$ and $i \in [N-1]$, we show in the rest of the proof that the probability of $E_1 \wedge E_2$ is at most $d^{-\omega(1)}$. The lemma follows from a union bound on t and i.

We start the following simple claim.

Claim 4.4. Assuming E_1 , ALG never receives $\mathbf{v}_{i+1}/(\sqrt{dL})$ as the subgradient before round t.

Proof. Given that $t_i = t_{i+1} = \infty$ at the end of round t - 1, every point **x** queried by ALG before round t satisfies one of the following conditions: either

1. $\|\mathbf{A}\mathbf{x}\|_{\infty} > \xi$, in which case the maximum in $F(\mathbf{x})$ must be achieved by $L\|\mathbf{A}\mathbf{x}\|_{\infty} - 1$; or

2. $|\langle \mathbf{x}, \mathbf{v}_i \rangle| < \gamma/4$ and $|\langle \mathbf{x}, \mathbf{v}_{i+1} \rangle| < \gamma/4$, in which case we have

$$\langle \mathbf{x}, \mathbf{v}_{i+1} \rangle - (i+1)\gamma < (\gamma/4) - (i+1)\gamma < -(\gamma/4) - i\gamma < \langle \mathbf{x}, \mathbf{v}_i \rangle - (i+1)\gamma.$$

So in both cases, $\mathbf{v}_{i+1}/(\sqrt{dL})$ cannot be a subgradient of F returned at **x**.

We finish the proof by showing that the probability of E_2 conditioning on E_1 is at most $d^{-\omega(1)}$. To this end, consider running ALG on F that is defined using the fixed **A** and random $\mathbf{v}_1, \ldots, \mathbf{v}_N$, and E_1 holds at the end of round t-1. Let H denote the query-answer history of the t-1 rounds so far. Let $\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}$ be the t-1 points queried in H, and let \mathbf{x}_t be the point to be queried by ALG next in round t given H. By Claim 4.4, $\mathbf{v}_{i+1}/\sqrt{dL}$ is never returned as a subgradient in H.

Before proceeding to round t, we reveal all vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$ except \mathbf{v}_{i+1} . Let \mathbf{v}_{-i} denote the tuple that contains these N-1 vectors revealed. We claim that given \mathbf{v}_{-i} , the following set of vectors captures exactly candidates for \mathbf{v}_{i+1} to (i) be consistent with H and (ii) satisfy E_1 :

$$V := \left\{ \mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d : |\langle \mathbf{v}, \mathbf{x}_\tau \rangle| \le \frac{\gamma}{4} \text{ for all } \tau \in [t-1] \text{ with } \|\mathbf{A}\mathbf{x}_\tau\|_{\infty} \le \xi \right\}.$$
 (2)

To see this is the case, it is trivial that \mathbf{v}_{i+1} needs to be in V for both (i) and (ii) to hold. On the other hand, $\mathbf{v}_{i+1} \in V$ is sufficient for (i) and (ii) since no subgradient in H is relevant to \mathbf{v}_{i+1} .

It then suffices to show that the probability of $\mathbf{v} \sim V$ satisfying $|\langle \mathbf{v}, \mathbf{x}_t \rangle| \geq \gamma/4$ is at most $d^{-\omega(1)}$. First we have by Khintchine inequality (Lemma A.1) and using $\delta < 1$ that

$$\Pr_{\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d} \left[\mathbf{v} \in V \right] \ge 1 - d^{-\omega(1)}.$$

On the other hand, we have by Chernoff bound that

$$\Pr_{\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d} \left[|\langle \mathbf{v}, \mathbf{x}_t \rangle| \ge \gamma/4 \right] \le d^{-\omega(1)}$$

It follows that the probability of $\mathbf{v} \sim V$ satisfying $|\langle \mathbf{v}, \mathbf{x}_t \rangle| \geq \gamma/4$ is at most $d^{-\omega(1)}$.

Next, we observe the optimal value of F is small with high probability:

Lemma 4.5. Fix $\mathbf{A} \in \{-1, 1\}^{(d/2) \times d}$. With probability at least $1 - d^{-\omega(1)}$ over $\mathbf{v}_1, \ldots, \mathbf{v}_N$ we have

$$\operatorname{argmin}_{\mathbf{x}\in\mathbb{B}^d}F(\mathbf{x}) \leq -\frac{1}{\sqrt{dL}}\cdot\frac{1}{\sqrt{N}\log^2 d}$$

Proof. Let $\mathbf{U}_{\mathbf{A}}$ be an orthonormal matrix that spans the row space of \mathbf{A} . Let $\hat{\mathbf{v}}_i = (\mathbf{I} - \mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^{\top})\mathbf{v}_i$ be projection \mathbf{v}_i onto the orthogonal space of $\mathbf{U}_{\mathbf{A}}$. It is clear that $\hat{\mathbf{v}}_i \perp \mathsf{row}(\mathbf{A})$ for any $i \in [N]$.

Consider

$$\mathbf{x} = -\frac{1}{\sqrt{N}\log d} \cdot \sum_{i=1}^{N} \hat{\mathbf{v}}_i.$$

It suffices to prove that, with probability at least $1 - d^{-\omega(1)}$, one has

- $F(\mathbf{x}) \leq -\frac{1}{\sqrt{dL}} \cdot \frac{1}{\sqrt{N}\log^2 d}$, and
- $\|\mathbf{x}\|_2 \leq 1.$

For the first claim, it is clear that $\|\mathbf{A}\mathbf{x}\|_{\infty} = 0$ since $\hat{\mathbf{v}}_i$ $(i \in [N])$ is orthogonal to the row space of **A**. Furthermore, for any $i \in [N]$, one has

$$\langle \mathbf{v}_{i}, \mathbf{x} \rangle = \mathbf{v}_{i} (\mathbf{I} - \mathbf{U}_{\mathbf{A}} \mathbf{U}_{\mathbf{A}}^{\top}) \mathbf{x} = \langle \hat{\mathbf{v}}_{i}, \mathbf{x} \rangle$$

$$= \left\langle \hat{\mathbf{v}}_{i}, -\frac{1}{\sqrt{N} \log(d)} \sum_{j=1}^{N} \hat{\mathbf{v}}_{i} \right\rangle$$

$$\leq -\frac{1}{\sqrt{N} \log d} \| \hat{\mathbf{v}}_{i} \|_{2}^{2} + \frac{1}{\sqrt{N} \log d} \left| \sum_{j \neq i} \langle \hat{\mathbf{v}}_{i}, \hat{\mathbf{v}}_{j} \rangle \right|.$$

$$(3)$$

The first step holds since $(\mathbf{I} - \mathbf{U}_{\mathbf{A}} \mathbf{U}_{\mathbf{A}}^{\top})\mathbf{x} = \mathbf{x}$, the second and third step follow from the definition of $\hat{\mathbf{v}}_i$ and \mathbf{x} .

We bound the two terms separately. For the first term, since $\hat{\mathbf{v}}_i$ is the projection of \mathbf{v}_i onto the orthogonal space of $\mathbf{U}_{\mathbf{A}}$ (which has rank at least d/2), by Lemma A.3, with probability at least $1 - d^{-\omega(1)}$, one has

$$\|\hat{\mathbf{v}}_{i}\|_{2}^{2} \ge \frac{1}{2} - \frac{\log d}{\sqrt{d}}.$$
(4)

For the second term, with probability at least $1 - d^{-\omega(1)}$, one has

$$\left\langle \hat{\mathbf{v}}_{i}, \sum_{j \neq i} \hat{\mathbf{v}}_{j} \right\rangle = \mathbf{v}_{i}^{\top} (\mathbf{I} - \mathbf{U}_{\mathbf{A}} \mathbf{U}_{\mathbf{A}}^{\top}) (\sum_{j \neq i} \mathbf{v}_{j})$$

$$\leq \frac{\log d}{\sqrt{d}} \cdot \| (\mathbf{I} - \mathbf{U}_{\mathbf{A}} \mathbf{U}_{\mathbf{A}}^{\top}) (\sum_{j \neq i} \mathbf{v}_{j}) \|_{2}$$

$$\leq \frac{\log d}{\sqrt{d}} \cdot \| \sum_{j \neq i} \mathbf{v}_{j} \|_{2}$$

$$\leq \frac{\log d}{\sqrt{d}} \cdot \sqrt{N} \log d \leq \frac{1}{4}.$$

$$(5)$$

Here the second step holds due to Khintchine inequality (Lemma A.1), the fourth step holds due to Chernoff bound, and the last step holds due to the choice of parameters.

Combining Eq. (3)(4)(5), we conclude that

$$\langle \mathbf{v}_i, \mathbf{x} \rangle \le -\frac{1}{\sqrt{N}\log^2(d)} \quad \forall i \in [N]$$

and therefore, complete the proof of the first claim.

For the second claim, with probability at least $1 - d^{-\omega(1)}$

$$\|\sum_{i=1}^{N} \hat{\mathbf{v}}_{i}\|_{2} = \|(\mathbf{I} - \mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^{\top})\sum_{i=1}^{N} \mathbf{v}_{i}\|_{2} \le \|\sum_{i=1}^{N} \mathbf{v}_{i}\|_{2} \le \sqrt{N}\log(d)$$

We finish the proof here.

It follows from Lemma 4.5 that $t_N \leq T$ with probability at least $(2/3) - d^{-\omega(1)}$: Lemma 4.6. With probability at least $(2/3) - d^{-\omega(1)}$, we have $t_N \leq T$. *Proof.* We claim that $t_N \leq T$ whenever (1) ALG finds an ϵ -optimal point of F and (2) the event of Lemma 4.5 holds. The lemma follows from Lemma 4.5 and the assumption that ALG succeeds with probability at least 2/3. To prove the claim, we note that by (2),

$$\operatorname{argmin}_{\mathbf{x}\in\mathbb{B}^d} F(\mathbf{x}) \leq -\frac{1}{\sqrt{dL}} \cdot \frac{1}{\sqrt{N}\log^2 d}.$$

On the other hand, if $t_N = \infty$, then every point \mathbf{x}_t queried by ALG satisfies

$$F(\mathbf{x}_t) \ge \frac{1}{\sqrt{dL}} \cdot \max\left\{ L \| \mathbf{A}\mathbf{x}_t \|_{\infty} - 1, \langle \mathbf{v}_N, \mathbf{x}_t \rangle - N\gamma \right\} \ge \frac{1}{\sqrt{dL}} \cdot \left(-N - \frac{1}{4} \right) \gamma \ge \operatorname{argmin}_{\mathbf{x} \in \mathbb{B}^d} F(\mathbf{x}) + \epsilon.$$

Here the second step follows from $t_N = \infty$ and the last step follows from our choice of parameters. Given that we assume ALG outputs \mathbf{x}_N , this contradicts with the assumption that ALG succeeds in finding an ϵ -optimal point of F. This finishes the proof of the lemma.

By a simple averaging argument, there exists an $i \in [N-1]$ with the following property:

Lemma 4.7. There exists an $i \in [N-1]$ such that both $t_i \leq t_{i+1} \leq T$ and

$$t_{i+1} - t_i \le \frac{20T}{N} = \frac{n}{2}$$

hold with probability at least 3/5.

Proof. We always condition on the high probability event of Lemma 4.3. Let \mathcal{E} denote the event that $t_N \leq T$ and by Lemma 4.6, we have $\Pr[\mathcal{E}] \geq \frac{2}{3} - d^{-\omega(1)}$. We prove by contradiction and suppose there is no such index i, then we have $\Pr[t_{i+1} - t_i \geq \frac{20T}{N}] > \frac{3}{5}$ for all $i \in [N-1]$, and therefore, $\Pr[t_{i+1} - t_i \geq \frac{20T}{N} |\mathcal{E}] > \frac{1}{15}$. This means $\mathbb{E}[t_{i+1} - t_i |\mathcal{E}] > \frac{1}{15} \cdot \frac{20T}{N} \geq \frac{4T}{3N}$, and

$$\mathbb{E}\left[t_N - t_1|\mathcal{E}\right] = \mathbb{E}\left[\sum_{i=1}^{N-1} t_{i+1} - t_i|\mathcal{E}\right] > (N-1) \cdot \frac{4T}{3N} > T$$

This contradicts the fact that $t_N \leq T$ when \mathcal{E} happens.

We remark that all lemmas proved so far stand regardless of the memory constraints.

Reduction. Now we are ready to use ALG to design a randomized (k, n)-communication protocol for the (s, ξ) -correlated orthogonal vector game that succeeds with probability at least 1/2. Given that a randomized protocol is a distribution of deterministic protocols, Lemma 4.1 follows.

Proof of Lemma 4.1. Let $i \in [N-1]$ be an integer that satisfies the condition of Lemma 4.7. The communication protocol is described in Figure 1. Recall that in the game Alice receives a matrix $\mathbf{A} \sim \{-1, 1\}^{(d/2) \times d}$ and Bob receives a vector $\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$. In the protocol, Alice and Bob sample independently $\mathbf{v}_1, \ldots, \mathbf{v}_N \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$ using public randomness. Let

$$V = (\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \dots, \mathbf{v}_N)$$
 and $V' = (\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}, \mathbf{v}_{i+2}, \dots, \mathbf{v}_N),$

that is, V' replaces \mathbf{v}_{i+1} with \mathbf{v} . Let $F_{\mathbf{A},V}$ (or $F_{\mathbf{A},V'}$) denote the function defined using \mathbf{A} and those vectors in V (or V', respectively). Roughly speaking, Alice would first optimize the function $F_{\mathbf{A},V}$ to time t_i , then send its memory state \mathbf{M} to Bob. It continues to optimize $F_{\mathbf{A},V'}$ for n iterations and sends all subgradients to Bob. Bob would simulate ALG using \mathbf{M} and these subgradients, output any query \mathbf{x} that is orthogonal to \mathbf{A} and has large correlation with \mathbf{v} .

Communication protocol for the correlated orthogonal vector game

- Alice and Bob use public randomness to sample $\mathbf{v}_1, \ldots, \mathbf{v}_N \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$.
- Round 1: Alice runs ALG on the function $F_{\mathbf{A},V}$, from which Alice obtains t_i and sends the memory state $\mathbf{M} \in \{0,1\}^{kd}$ of ALG after $t_i 1$ rounds to Bob.
- Round 2: Bob sends **v** to Alice.
- Round 3: Alice runs ALG on the function $F_{\mathbf{A},V'}$. For each $j \in [n]$, Alice sets \mathbf{a}_j to be \mathbf{A}_i if the subgradient returned by $F_{\mathbf{A},V'}$ in round $t_i + j 1$ is either \mathbf{A}_i/\sqrt{d} or $-\mathbf{A}_i/\sqrt{d}$, and set $\mathbf{a}_j = \mathsf{nil}$ otherwise. Alice sends $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathsf{row}(\mathbf{A}) \cup \{\mathsf{nil}\}$ to Bob.
- Output: Bob runs ALG for n rounds, starting with memory state **M**, the first message from Alice. For each round j = 1, ..., n, let \mathbf{x}_j be the query point of ALG:
 - 1. If $\mathbf{a}_i \neq \mathsf{nil}$, Bob sets

$$\left(\frac{L|\langle \mathbf{a}_j, \mathbf{x}_j \rangle| - 1}{\sqrt{d}L}, \pm \frac{\mathbf{a}_j}{\sqrt{d}}\right)$$

be the pair returned by the oracle, where the sign of the subgradient can be determined by the sign of $\langle \mathbf{a}_j, \mathbf{x}_j \rangle$, and continue the execution of ALG;

2. If $\mathbf{a}_j = \mathsf{nil}$, Bob finds the vector $\mathbf{v}_\ell \in V'$ with the smallest index that maximizes $\langle \mathbf{v}_\ell, \mathbf{x}_j \rangle - \ell \gamma$, sets

$$\left(\frac{\langle \mathbf{v}_{\ell}, \mathbf{x}_{j} \rangle - \ell \gamma}{\sqrt{d}L}, \frac{\mathbf{v}_{\ell}}{\sqrt{d}L}\right)$$

be the pair returned by the oracle and continue the execution of ALG.

Bob outputs any query \mathbf{x}_j $(j \in [n])$ that satisfies $|\langle \mathbf{v}, \mathbf{x}_j \rangle| \ge \gamma/4$ and $F(\mathbf{x}_j) \le \frac{1}{\sqrt{dL}}$. If no such query exists, Bob fails and outputs an arbitrary vector, say **0**.

Figure 1: Reduction from convex optimization to the correlated orthogonal vector game

It is clear from the description of the protocol that it is a (k, n)-communication protocol. We finish the proof by showing that it succeeds with probability at least 1/2. We first condition on the high probability event of Lemma 4.3, for V and V'. Note it happens with probability at least $1 - d^{-\omega(1)}$. The convex optimization algorithm has the same transcripts for $F_{\mathbf{A},V}$ and $F_{\mathbf{A},V'}$, during time $t \in [t_i - 1]$. This is because it only receives subgradients from $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\} \cup \mathsf{row}(\mathbf{A})$ before time t_i by Lemma 4.3 and Claim 4.4. Hence, we can assume the memory state \mathbf{M} comes from optimizing the function $F_{\mathbf{A},V'}$.

Next, we condition on the event of Lemma 4.7, which asserts $t_{i+1} - t_i \leq n/2$ and it happens with probability at least 3/5. It is easy to see that Bob, who knows \mathbf{M} , V' and $\mathbf{a}_1, \ldots, \mathbf{a}_n$, can simulate ALG for function $F_{\mathbf{A},V'}$ up to time $t_{i+1} \leq t_i + n$. Therefore, it must submit a query \mathbf{x} that satisfies $\|\mathbf{A}\mathbf{x}\|_{\infty} \leq \frac{2}{L} = \xi$ and $|\langle \mathbf{x}, \mathbf{v} \rangle| \geq \gamma/4 \geq \sqrt{s/d}$. We finish the proof of reduction here.

5 Lower bound for the correlated orthogonal vector game

In this section, We prove the following lower bound for correlated orthogonal vector game. Theorem 1.1 follows directly by combining the lower bound with Lemma 4.1.

Theorem 5.1 (Lower bound of correlated orthogonal vector game). Let d be a sufficiently large integer. Let $\delta \in (0,1)$, $k = d^{1-\delta-o(1)}$, $s = d^{1-\delta/2}\log^2(d)$, $\xi = 2\exp(-\log^5(d))$ Then any deterministic (k, n)-communication protocol that solves the (s, ξ) -correlated orthogonal vector game with probability at least 1/2 requires $n \ge d \cdot \exp(-\log d/\log \log d)$.

We prove by contradiction and assume $n \leq d \cdot \exp(-\log d/\log \log d)$ from now on.

Communication protocol. A deterministic (k, n)-communication protocol works as follows:

- (Round 1) Alice receives a matrix $\mathbf{A} \in \{-1, 1\}^{(d/2) \times d}$, and let $\mathbf{M}_{\mathbf{A}} \in \{0, 1\}^{kd}$ be the message sent in the first round.
- (Round 3) Recall Alice obtains both A and v after the second round. Let

$$\mathbf{r}_{i,\mathbf{A},\mathbf{v}} \in \{-1,1\}^d \cup \{\mathsf{nil}\}$$

be the *i*-th row sent to Bob $(i \in [n])$ and let $\mathbf{R}_{\mathbf{A},\mathbf{v}} = (\mathbf{r}_{1,\mathbf{A},\mathbf{v}},\ldots,\mathbf{r}_{n,\mathbf{A},\mathbf{v}})$ be the collection of rows sent by Alice.

• (Output) At the end, Bob outputs a vector $\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}$ based on the message \mathbf{M} , the vector \mathbf{v} and the collection of rows \mathbf{R} . We write $\mathbf{x}_{\mathbf{A},\mathbf{v}} = \mathbf{x}_{\mathbf{M}\mathbf{A},\mathbf{v},\mathbf{R}\mathbf{A},\mathbf{v}}$ to denote Bob's output under the input pair (\mathbf{A},\mathbf{v}) .

Note the output vector $\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}$ is well-defined for any message $\mathbf{M} \in \{0,1\}^d$, vector $\mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d$ and any collection of rows $\mathbf{R} \in (\{-1,1\}^d \cup \{\mathsf{nil}\})^n$.

The following lemma shows that, without loss of generality, one can assume that the output $\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}$ has unit norm.

Lemma 5.2 (Unit norm). If there is a (k, n) communication protocol for (s, ξ) -correlated orthogonal vector game, then there is a (k, n) communication protocol that always output unit vectors and solves (s, ξ') -correlated orthogonal vector game, where $\xi' = \sqrt{d\xi}$

Proof. For any message $\mathbf{M} \in \{0,1\}^{kd}$, vector $\mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d$ and collection of rows $\mathbf{R} \in (\{-1,1\}^d \cup \{\mathsf{nil}\})^n$, consider the output vector $\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}$:

- If $\|\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}\|_2 < \sqrt{s/d}$, one can change it to an arbitrary unit vector and it does not reduce the success probability of the protocol.
- If $\sqrt{s/d} \leq \|\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}\|_2 \leq 1$, then we can scale the output to $\frac{\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}}{\|\mathbf{x}_{\mathbf{M},\mathbf{v},\mathbf{R}}\|_2}$ and it has unit norm. The correlation with any vector can only increase, and by a factor of at most $\sqrt{d/s} \leq \sqrt{d}$, hence by relaxing the orthogonal condition by a factor of \sqrt{d} , the success probability does not decrease.

We complete the proof here.

For any matrix $\mathbf{B} \in \{-1, 1\}^{m \times d}$ and any collection of rows $\mathbf{R} \in (\{-1, 1\}^d \cup \{\mathsf{nil}\})^n$, we write $\mathbf{R} \subseteq \mathsf{row}(\mathbf{B}) \cup \{\mathsf{nil}\}$ if each row of \mathbf{R} either is nil or belongs to $\mathsf{row}(\mathbf{B})$.

Definition 5.3 (Table). Let $m \in [d/2]$. Given any message $\mathbf{M} \in \{0,1\}^{kd}$, any matrix $\mathbf{B} \in$ $\{-1,1\}^{m\times d}$ and any set of vectors $V\subseteq \frac{1}{\sqrt{d}}\mathcal{H}_d$

$$\mathsf{T}^{\mathbf{M}}(\mathbf{B}, V) := \{\mathbf{x}_{\mathbf{M}, \mathbf{v}, \mathbf{R}} : \mathbf{v} \in V, \mathbf{R} \subseteq \mathsf{row}(\mathbf{B}) \cup \{\mathsf{nil}\}\} \subseteq \mathbb{S}^{d}$$

That is to say, $\mathsf{T}^{\mathbf{M}}(\mathbf{B}, V)$ contains all possible outputs by Bob, when the message is \mathbf{M} , the vector \mathbf{v} is from V and the n rows are taken from \mathbf{B} .

Note the definition of table is flexible that we do not need **B** to have exact d/2 rows. Given a table $\mathsf{T}^{\mathbf{M}}(\mathbf{B}, V)$, we mostly care about entries that are (almost) orthogonal to **B**. Recall $\xi' = \sqrt{d\xi}$.

Definition 5.4 (Orthogonal entry). Given a table $\mathsf{T}^{\mathbf{M}}(\mathbf{B}, V)$, let $\mathsf{O}^{\mathbf{M}}(\mathbf{B}, V)$ contain all entries in $\mathsf{T}^{\mathbf{M}}(\mathbf{B}, V)$ that are (almost) orthogonal to \mathbf{B} , i.e.,

$$\mathsf{O}^{\mathbf{M}}(\mathbf{B}, V) := \left\{ \mathbf{x} : \mathbf{x} \in \mathsf{T}^{\mathbf{M}}(\mathbf{B}, V), \|\mathbf{B}\mathbf{x}\|_{\infty} \le \xi' \right\}.$$

5.1The existence of a succinct table

The goal of this subsection is to select a small subset $V^* \subseteq \frac{1}{\sqrt{d}} \mathcal{H}_d$, of size at most 2^s , that is representative.

Let $\mathcal{A}_{suc} \subseteq \{-1,1\}^{\frac{d}{2} \times d}$ contain all matrices **A** such that the protocol succeeds with probability at least 1/4 (over the randomness of **v**) when Alice receives **A** as input. We have $|\mathcal{A}_{suc}| \geq \frac{1}{4} \cdot 2^{d^2/2}$ because the protocol succeeds with probability at least 1/2 over a random pair of A and v. First, we prove the outputs $\{\mathbf{x}_{\mathbf{A},\mathbf{v}}\}_{\mathbf{v}\in\frac{1}{\sqrt{d}}\mathcal{H}_d}$ are spread out for $\mathbf{A}\in\mathcal{A}_{\mathsf{suc}}$.

Lemma 5.5. Let $c_1 > 0$ be some sufficiently small constant and $K = \exp(c_1 s)$. For any fixed matrix $A \in \mathcal{A}_{suc}$ and fixed set of vectors $\mathbf{y}_1, \ldots, \mathbf{y}_K \in \mathbb{S}^d$, one has

$$\Pr_{\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d} \left[\|\mathbf{A}\mathbf{x}_{\mathbf{A},\mathbf{v}}\|_{\infty} \le \xi' \wedge \|\mathbf{x}_{\mathbf{A},\mathbf{v}} - \mathbf{y}_i\|_2 > \frac{1}{d^8} \,\forall i \in [K] \right] \ge 1/8.$$
(6)

Proof. Let $V_{\text{close}} \subseteq \frac{1}{\sqrt{d}} \mathcal{H}_d$ be the set of vectors **v** that has large correlations with some \mathbf{y}_i , i.e.,

$$V_{\mathsf{close}} = \bigcup_{i \in [K]} \left\{ \mathbf{v} : |\langle \mathbf{v}, \mathbf{y}_i \rangle| \ge \sqrt{s/2d}, \mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d \right\}.$$

We first prove the size of V_{close} is not large. For any fixed \mathbf{y}_i $(i \in [K])$, by Lemma A.3, we have that

$$\Pr_{\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d} \left[|\langle \mathbf{v}, \mathbf{y}_i \rangle| \ge \sqrt{s/2d} \right] \le \exp(-c_2 s)$$

for some constant $c_2 > c_1 > 0$. Taking a union bound over $i \in [K]$, we have

$$\Pr_{\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d} \left[\exists i \in [K] : |\langle \mathbf{v}, \mathbf{y}_i \rangle| \ge \sqrt{s/2d} \right] \le K \cdot \exp(-c_2 s) \le \exp(-(c_2 - c_1)s) \le \frac{1}{8}$$

Hence, we have $|V_{\text{close}}| \leq \frac{1}{8} \cdot 2^d$. Let $V' \subseteq \frac{1}{\sqrt{d}} \mathcal{H}_d$ be the set of vectors such that Eq. (6) holds. It suffices to prove that the protocol succeeds on (\mathbf{A}, \mathbf{v}) only if $\mathbf{v} \in V_{\mathsf{close}} \cup V'$, as this would imply $|V'| \ge \frac{1}{8} \cdot 2^d$. For any vector $\mathbf{v} \in \frac{1}{\sqrt{d}} \mathcal{H}_d \setminus (V_{\mathsf{close}} \cup V')$, if $\|\mathbf{A}\mathbf{x}_{\mathbf{A},\mathbf{v}}\|_{\infty} \leq \xi'$, then $\mathbf{x}_{\mathbf{A},\mathbf{v}}$ must be close to some \mathbf{y}_i (since $\mathbf{v} \notin V'$). Then we have

$$|\langle \mathbf{x}_{\mathbf{A},\mathbf{v}},\mathbf{v}\rangle| \le |\langle \mathbf{y}_i,\mathbf{v}\rangle| + \frac{1}{d^8} \le \sqrt{\frac{s}{2d}} + \frac{1}{d^8} < \sqrt{\frac{s}{d}}$$

where the first step follows from $\|\mathbf{x}_{\mathbf{A},\mathbf{v}} - \mathbf{y}_i\|_2 \leq \frac{1}{d^8}$, the second step follows from $\mathbf{v} \notin V_{\text{close}}$. We finish the proof here.

Definition 5.6 (α -cover). Let $\alpha > 0$ and $X \subseteq \mathbb{R}^d$ be a set of points. Let $\mathcal{N}(X, \alpha) \subseteq X$ be the α -covering of set X, which is defined as the largest set $X' \subseteq X$ such that for any $\mathbf{x}_1, \mathbf{x}_2 \in X'$, $\|\mathbf{x}_1 - \mathbf{x}_2\|_2 \ge \alpha$.

Now, we prove that there exists a "small" set of vectors $V^* \subseteq \frac{1}{\sqrt{d}}H_d$, such that the $(1/d^8)$ -cover of $O^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V^*)$ is large, for most $\mathbf{A} \in \mathcal{A}_{\mathsf{suc}}$. We derive the existence using the probabilistic method.

Lemma 5.7. Let $c_1 > 0$ be a sufficiently small constant and $K = \exp(c_1 s)$. For any fixed matrix $\mathbf{A} \in \mathcal{A}_{suc}$, with probability least $1 - \exp(-s/2)$ over the random draw of $V = \{\mathbf{v}_1, \dots, \mathbf{v}_{16sK}\},\$

$$\left| \mathcal{N} \left(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V), 1/d^8 \right) \right| \geq K.$$

Proof. We partition the set V into K groups, where the *i*-th group

$$V_i = \{\mathbf{v}_{16s(i-1)+1}, \dots, \mathbf{v}_{16si}\} \quad \forall i \in [K]$$

We would construct a large set \mathcal{N} from $O^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V)$ such that the pairwise distance is large. Initially, $\mathcal{N}_0 = \emptyset$. At the *i*-th step, suppose $\mathcal{N}_{i-1} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\} \subseteq O^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V_1 \cup \cdots \cup V_{i-1})$ be the set constructed thus far, we would add an entry from $O^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V_i)$ that is apart from existing vectors in \mathcal{N}_{i-1} .

Note for each $j \in [16s]$, consider the entry $\mathbf{x}_{\mathbf{A},\mathbf{v}_{16s(i-1)+j}} \in \mathsf{T}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V_i)$, by Lemma 5.5, we have

$$\Pr_{\mathbf{v}_{16s(i-1)+j} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d} \left[\|\mathbf{A}\mathbf{x}_{\mathbf{A},\mathbf{v}_{16s(i-1)+j}}\|_{\infty} \le \xi' \land \|\mathbf{x}_{\mathbf{A},\mathbf{v}_{16s(i-1)+j}} - \mathbf{x}_{\tau}\|_2 > \frac{1}{d^8} \,\forall \tau \in [i-1] \right] \ge 1/8.$$

Since $\{\mathbf{v}_{16s(i-1)+j}\}_{j\in[16s]}$ are sampled independently from $\frac{1}{\sqrt{d}}\mathcal{H}_d$, with probability at least $1 - \exp(-s)$, there exists $j \in [16s]$ such that

- $\|\mathbf{x}_{\mathbf{A},\mathbf{v}_{16s(i-1)+j}} \mathbf{x}_{\tau}\|_{2} \geq \frac{1}{d^{8}}$ for any $\tau \in [i-1]$, and
- $\|\mathbf{A}\mathbf{x}_{\mathbf{A},\mathbf{v}_{16s(i-1)+j}}\|_{\infty} \leq \xi'.$

Hence we can add $\mathcal{N}_i = \mathcal{N}_{i-1} \cup \{\mathbf{x}_{\mathbf{A}, 16s(i-1), j}\}$. Taking a union bound over $i \in [K]$, we have with probability at least $1 - \exp(-s/2)$, one has $|\mathcal{N}(\mathbf{O}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V), 1/d^8)| \geq K$.

By Lemma 5.7, we conclude that

Lemma 5.8. There exists a set $V^* \subseteq \frac{1}{\sqrt{d}} \mathcal{H}_d$ with size at most $16sK \leq 2^s$, such that the set

$$\mathcal{A}_{\mathsf{nice}} := \left\{ \mathbf{A} \in \mathcal{A}_{\mathsf{suc}} : |\mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V^*), 1/d^8)| \ge 2^{s/\log(d)} \right\}.$$

has size at least $\frac{1}{8} \cdot 2^{d^2/2}$.

We would fix the set of V^* from now on, and we only consider matrix $\mathbf{A} \in \mathcal{A}_{\mathsf{nice}}$. In the rest of the proof, we would omit V^* when there is no confusion. That is to say, we write the table $\mathsf{T}^{\mathbf{M}}(\mathbf{A}) := \mathsf{T}^{\mathbf{M}}(\mathbf{A}, V^*)$ and its orthogonal entries $\mathsf{O}^{\mathbf{M}}(\mathbf{A}) := \mathsf{O}^{\mathbf{M}}(\mathbf{A}, V^*)$.

5.2 Encoding

Up to this point, we have proved that there exists a large set of matrices $\mathcal{A}_{nice} \subseteq \{-1, 1\}^{(d/2) \times d}$, such that, for any $\mathbf{A} \in \mathcal{A}_{nice}$, the table $\mathsf{T}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A})$ contains many different orthogonal entries: The $(1/d^8)$ -cover of $\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A})$ is of size at least $2^{s/\log(d)}$. We establish the contradiction by proving this is simply impossible! Our strategy is to find an encoding strategy such that (1) it encodes every matrix in \mathcal{A}_{nice} , but at the same time (2) the number of matrices encoded are at most $2^{d^2/2}/d$.

We first introduce the notion of robustly linearly independent sequence.²

Definition 5.9 (γ -robustly linearly independent sequence). Let $\gamma \in (0,1), L \in [d]$. A sequence of unit vectors $\mathbf{y}_1, \ldots, \mathbf{y}_L$ is γ -robustly linearly independent, if for any $j \in [L-1]$,

$$\|\mathbf{y}_j - \operatorname{proj}_{\mathsf{span}}(\mathbf{y}_1, \dots, \mathbf{y}_{j-1})(\mathbf{y}_j)\|_2 \ge \gamma$$

That is to say, \mathbf{y}_i has a non-trivial component that is orthogonal to the linear span of $\mathbf{y}_1, \ldots, \mathbf{y}_{i-1}$.

The definition of γ -RLI sequence allows for arbitrary length, but in the rest of section, we would consider sequence of length

$$L := \frac{s}{4\log^8 d}$$

Parameters. We introduce a few parameters. Let

$$\Delta := \frac{d}{n} \ge \exp(\log d / \log \log d).$$

Let H be the smallest integer such that

$$\frac{s}{\log^2(d)} \cdot \left(\frac{\Delta}{\log^5(d)}\right)^H \ge \frac{d}{10},$$

we know that $H \leq \log \log d$ by our choice of parameters.

The encoding strategy proceeds in H levels, and the step size

$$s_h = \frac{s}{\log^2(d)} \cdot \left(\frac{\Delta}{\log^5(d)}\right)^h$$
, $\forall h \in [0: H-1]$ and $s_H = \frac{d}{10}$.

Denote the partial sum as $s_{\leq h} = \sum_{i=1}^{h} s_i$ and $s_{\leq 0} = 0$ (note they exclude s_0).

We consider doubly exponentially decreasing radius of cover $\{\alpha_h\}_{h\in[H]}$,

$$\alpha_0 = \frac{1}{d^8}, \quad \alpha_h = \alpha_{h-1}^8 = d^{-8^{h+1}} \,\forall h \in [0:H]$$

Note we still guarantee $\alpha_H \ge \exp(-8\log^4(d)) \gg \xi' = \sqrt{d} \exp(-\log^5(d)).$

Definition 5.10 (Partition). At the h-th level $(h \in [H])$, we consider partitions of rows [d/2] that satisfy

$$P_{h,1} \cup P_{h,2} \cup P_{h,3} = [d/2], \quad |P_{h,1}| = d/2 - s_{\leq h}, \ |P_{h,2}| = s_h, \ |P_{h,3}| = s_{\leq h-1}$$

and let P_h be the collection of all such partitions.

Given a partition $P_h \in \mathsf{P}_h$ and three sub-matrices $\mathbf{A}_{h,1} \in \{-1,1\}^{(d/2-s_{\leq h})\times d}$, $\mathbf{A}_{h,2} \in \{-1,1\}^{s_h \times d}$ and $\mathbf{A}_{h,3} \in \{-1,1\}^{s_{\leq h-1} \times d}$, Let $[\mathbf{A}_{h,1}, \mathbf{A}_{h,2}, \mathbf{A}_{h,3}]_{P_h} \in \{-1,1\}^{(d/2)\times d}$ be the matrix induced naturally by the partition. That is, the *j*-th row of $\mathbf{A}_{h,\tau}$ is located at the $P_{h,\tau}(j)$ -th row, where $P_{h,\tau}(j)$ is the *j*-th element in $P_{h,\tau}$ ($\tau = 1, 2, 3, j \in [|P_{h,\tau}|]$).

Finally, we set $\Gamma_h := 2^{s_h d - 2kd}$ to be the threshold for level h.

²Our definition of γ -robustly linearly independent sequence is slightly different from [MSSV22], roughly speaking, γ -RLI in our work implies ($\gamma^2/2$)-RLI of [MSSV22] and vice versa.

Algorithm 1 Encoding

1: $\mathcal{A}_h \leftarrow \emptyset \ (h \in [H])$ 2: for h = 1, 2, ..., H do for each message $\mathbf{M} \in \{0,1\}^{kd}$, partition $P_h \in \mathsf{P}_h$, matrix $\mathbf{A}_{h,1} \in \{-1,1\}^{(d/2-s_{\leq h}) \times d}$ do 3: $\mathcal{S}_{\mathbf{M},P_h,\mathbf{A}_{h,1}} \leftarrow \emptyset$ 4: for each $(\alpha_{h-1}/4)$ -RLI sequence $\mathbf{x}_{h,1}, \ldots, \mathbf{x}_{h,L} \in O^{\mathbf{M}}(\mathbf{A}_{h,1})$ do $\mathcal{I}_{h,\mathbf{x}_{h,1},\ldots,\mathbf{x}_{h,L}} \leftarrow \{\mathbf{A}_{h,2} \in \{-1,1\}^{s_h \times d}, \|\mathbf{A}_{h,2}\mathbf{x}_{h,i}\|_{\infty} \leq \xi' \, \forall i \in [L]\}$ 5:6: $\mathcal{S}_{\mathbf{M},P_{h},\mathbf{A}_{h,1}} \leftarrow \mathcal{S}_{\mathbf{M},P_{h},\mathbf{A}_{h,1}} \cup \mathcal{I}_{h,\mathbf{x}_{h,1},\dots,\mathbf{x}_{h,L}}$ 7: end for 8: $\mathcal{J}_{\mathbf{M},P_h,\mathbf{A}_{h,1}} \leftarrow \{ [\mathbf{A}_{h,1},\mathbf{A}_{h,2},\mathbf{A}_{h,3}]_{P_h} : \mathbf{A}_{h,2} \in \mathcal{S}_{\mathbf{M},P_h,\mathbf{A}_{h,1}}, \mathbf{A}_{h,3} \in \{-1,1\}^{s \le h-1 \times d} \}$ 9: if $|\mathcal{S}_{\mathbf{M},P_h,\mathbf{A}_{h,1}}| \leq \Gamma_h$ then 10: $\mathcal{A}_h \leftarrow \mathcal{A}_h \cup \mathcal{J}_{\mathbf{M}, P_h, \mathbf{A}_{h, 1}}$ 11: 12:end if end for 13:14: end for 15: $\mathcal{A} \leftarrow \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_H$

Encoding algorithm The encoding strategy is formally stated at Algorithm 1. It divides into H levels. In the *h*-th level, it enumerates all possible messages $\mathbf{M} \in \{0,1\}^{kd}$, partitions $P_h \in \mathsf{P}_h$ and sub-matrices $\mathbf{A}_{h,1} \in \{-1,1\}^{(d/2-s_{\leq h})\times d}$. It remains to determine $\mathbf{A}_{h,2} \in \{-1,1\}^{s_h\times d}$ ($\mathbf{A}_{h,3}$ could take any value in $\{-1,1\}^{s_{\leq h-1}\times d}$). To this end, it checks the orthogonal entries $\mathbf{O}^{\mathbf{M}}(\mathbf{A}_{h,1})$ and enumerates all $(\alpha_{h-1}/4)$ -RLI sequence (of length L) in it. It includes matrices $\mathbf{A}_{h,2}$ that is orthogonal to one of the sequence and and takes a union over all of them (Line 5-8). Finally, the encoding algorithm includes this set of matrices (i.e., $S_{\mathbf{M},P_h,\mathbf{A}_{h,1}}$) only if its size is no more than $\Gamma_h = 2^{s_h d-2kd}$.

5.3 Analysis

Our goal is to prove the size of \mathcal{A} is at most $2^{d^2/2}/d$, but at the same time, it contains \mathcal{A}_{nice} , this would reach a contradiction. In particular, we prove

Lemma 5.11 (Upper bound on \mathcal{A}). The size of \mathcal{A} is at most $2^{d^2/2}/d$.

Lemma 5.12 (Lower bound on \mathcal{A}). $\mathcal{A}_{\mathsf{nice}} \subseteq \mathcal{A}$.

The proof of Lemma 5.11 is straightforward and follows simply from counting.

Lemma 5.13. For each level $h \in [H]$, the size of \mathcal{A}_h is at most $2^{d^2/2}/d^2$.

Proof. For any fixed $h \in [H]$, the total number of different messages $\mathbf{M} \in \{0, 1\}^{kd}$, partitions $P_h = (P_{h,1}, P_{h,2}, P_{h,3})$ and sub-matrices $\mathbf{A}_{h,1} \in \{-1, 1\}^{(d/2 - s \leq h) \times d}$ are at most

$$2^{kd} \times 2^{3d} \times 2^{d(d/2 - s_{\leq h})} \tag{7}$$

It remains to bound the size of $\mathcal{J}_{\mathbf{M},P_h,\mathbf{A}_{h,1}}$. Note it would not be counted if $|\mathcal{S}_{\mathbf{M},P_h,\mathbf{A}_{h,1}}| > \Gamma_h$. On the other side, when $|\mathcal{S}_{\mathbf{M},P_h,\mathbf{A}_{h,1}}| \leq \Gamma_h$, the total different choice of $\mathbf{A}_{h,2}$ is at most $\Gamma_h = 2^{ds_h - 2kd}$. We make no restrictions on $\mathbf{A}_{h,3}$ and its size is at most $2^{ds_{\leq h-1}}$. Multiply these numbers, we have

$$\left|\mathcal{J}_{\mathbf{M},P_{h},\mathbf{A}_{h,1}}\right| \le 2^{ds_{h}-2kd} \times 2^{ds_{\le h-1}}.$$
(8)

Combining Eq. (7)(8), we have

$$|\mathcal{A}_h| \le (2^{kd} \times 2^{3d} \times 2^{d(d/2 - s \le h)}) \times (2^{ds_h - 2kd} \times 2^{ds \le h - 1}) = 2^{d^2/2 + 3d - kd} \le 2^{d^2/2}/d^2.$$

We complete the proof here.

We set to prove Lemma 5.12, which is the major technical Lemma. We set

$$N_h = 2^{s_h/\log(d)} \quad \forall h \in [0:H]$$

Proof of Lemma 5.12. For any fixed $\mathbf{A} \in \mathcal{A}_{nice}$, we wish to prove $\mathbf{A} \in \mathcal{A}$. We prove by induction on h, and our inductive hypothesis is

• Inductive hypothesis. If $\mathbf{A} \notin \mathcal{A}_{\ell}$ for any $\ell = 0, 1, ..., h$, then there is a submatrix $\mathbf{A}_h \subseteq \mathbf{A}_{\ell}$ $(\widetilde{\mathbf{A}}_h \in \{-1, 1\}^{(d/2 - s \leq h) \times d})$, such that $|\mathcal{N}(\mathbf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_h), \alpha_h)| \geq N_h$.

In another word, our inductive hypothesis asserts that if **A** has not yet been added to \mathcal{A} till level h, then there is a submatrix $\widetilde{\mathbf{A}}_h$ that takes $(d/2 - s_{\leq h})$ rows from **A** and the α_h -cover of $O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_h)$ has size at least N_h .

The base case of h = 0 holds directly from the definition of $\mathcal{A}_{\text{nice}}$. In particular, it is clear that $\mathbf{A} \notin \mathcal{A}_0 = \emptyset$ and we would take $\widetilde{\mathbf{A}}_0 = \mathbf{A}$. By Lemma 5.15, we have that $|\mathcal{N}(\mathbf{O}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}), \alpha_0)| \ge N_0$.

Suppose the claim holds up to h, and the orthogonal entries $O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h})$ has a large α_{h} -cover. Our first step is to remove $s_{h+1} = s_{h} \cdot (\frac{\Delta}{\log^{5}(d)})$ different rows, $\mathbf{R}_{h} \in \{-1, 1\}^{s_{h+1} \times d}$, from $\widetilde{\mathbf{A}}_{h}$, and prove that the remaining sub-matrix $\widetilde{\mathbf{A}}_{h+1} = \widetilde{\mathbf{A}}_{h} \backslash \mathbf{R}_{h}$ still contains a large number of entries from the α_{h} cover. In particular,

Lemma 5.14 (Deleting rows). Suppose

$$\left| \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h}) \right| \geq N_{h} = 2^{s_{h}/\log(d)},$$

then there exists s_{h+1} rows of \mathbf{A} , denoted as \mathbf{R}_h , such that, after deleting \mathbf{R}_h from \mathbf{A}_h , the remaining matrix $\mathbf{\widetilde{A}}_{h+1} = \mathbf{\widetilde{A}}_h \backslash \mathbf{R}_h$ satisfies,

$$\left|\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h+1}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h})\right| \geq 2^{s_{h}/2\log(d)}.$$

Proof. It is clear if an entry $\mathbf{x} \in \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_h), \alpha_h)$, then it satisfies $\|\widetilde{\mathbf{A}}_{h+1}\mathbf{x}\|_{\infty} \leq \|\widetilde{\mathbf{A}}_h\mathbf{x}\|_{\infty} \leq \xi'$. Hence, all we need to prove is that a non-trivial amount of entries in $\mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_h), \alpha_h)$ survive after the deletion. Indeed, for any submatrix $\widetilde{\mathbf{A}} \subseteq \widetilde{\mathbf{A}}_h$ of at least $\frac{d}{10}$ rows, we wish to prove, there exists a row $\mathbf{r} \in \widetilde{\mathbf{A}}$, such that

$$\left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}} \setminus \{\mathbf{r}\}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h}) \right| \geq \left(1 - \frac{10}{\Delta} \right) \cdot \left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h}) \right|.$$
(9)

Assuming this is true, one can take a sequence of rows $\mathbf{r}_1,\ldots,\mathbf{r}_{s_{h+1}}$ and

$$\begin{aligned} \left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h} \setminus \{\mathbf{r}_{1}, \dots, \mathbf{r}_{s_{h+1}}\}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h}) \right| \\ \geq \left(1 - \frac{10}{\Delta} \right) \cdot \left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h} \setminus \{\mathbf{r}_{1}, \dots, \mathbf{r}_{s_{h+1}-1}\}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h}) \right| \\ \vdots \\ \geq \left(1 - \frac{10}{\Delta} \right)^{s_{h+1}} \cdot \left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}), \alpha_{h}) \right| \\ > 2^{-s_{h}/\log^{4}(d)} \cdot 2^{s_{h}/\log(d)} > 2^{s_{h}/2\log(d)} \end{aligned}$$

The first and second step follows from Eq. (9) and $\widetilde{\mathbf{A}}_h$ contains $d/2 - s_{\leq h} \geq d/2 - d/5 \geq \frac{d}{5}$ rows. We plug in the value of $s_{h+1} = s_h \cdot \left(\frac{\Delta}{\log^5(d)}\right)$ and $N_h = 2^{s_h/\log(d)}$ in the third step.

It remains to prove Eq. (9), it follows from a counting argument. In particular, we claim

$$\begin{split} & \sum_{\mathbf{r}\in\mathsf{row}(\widetilde{\mathbf{A}})} \left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}\backslash\{\mathbf{r}\}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}),\alpha_{h}) \right| \\ & \geq \left(|\mathsf{row}(\widetilde{\mathbf{A}})| - (d/\Delta) \right) \cdot \left| \mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{h}),\alpha_{h}) \right|. \end{split}$$

This is because for any entry \mathbf{x} in the RHS (i.e. $\mathbf{x} \in O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}) \cap \mathcal{N}(O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_h), \alpha_h)$), suppose it equals $\mathbf{x}_{\mathbf{M}_{\mathbf{A}},\{\mathbf{q}_1,\ldots,\mathbf{q}_n\},\mathbf{v}}$ for some rows $\mathbf{q}_1,\ldots,\mathbf{q}_n \in \mathsf{row}(\widetilde{\mathbf{A}}) \cup \{\mathsf{nil}\}$ and $\mathbf{v} \in V^*$, then it is also contained in $O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}\setminus\{\mathbf{r}\}) \cap \mathcal{N}(O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_h), \alpha_h)$ as long as $\mathbf{r} \notin \{\mathbf{q}_1,\ldots,\mathbf{q}_n\}$. This means \mathbf{x} has been counted for $|\mathsf{row}(\widetilde{\mathbf{A}}) - n| = |\mathsf{row}(\widetilde{\mathbf{A}}) - (d/\Delta)|$ times in the LHS.

Combining with the assumption that \mathbf{A} contains at least d/10 rows, we conclude the proof of Eq. (9), and complete the proof of Lemma.

We continue the proof of induction. In particular, let $\mathbf{R}_h, \widetilde{\mathbf{A}}_{h+1}$ be defined as in Lemma 5.14, we would consider the following enumeration

$$\mathbf{M} = \mathbf{M}_{\mathbf{A}}, \quad \mathbf{A}_{h+1,1} = \widetilde{\mathbf{A}}_{h+1} \in \{-1,1\}^{(d/2 - s_{\leq h+1}) \times d}, \quad \mathbf{A}_{h+1,3} = \mathbf{A} \setminus \widetilde{\mathbf{A}}_h \in \{-1,1\}^{s_{\leq h} \times d}$$
(10)

and the partition $P_{h+1} \in \mathsf{P}_{h+1}$ is the one, such that, $\mathbf{A} = [\widetilde{\mathbf{A}}_{h+1}, \mathbf{R}_h, \mathbf{A} \setminus \widetilde{\mathbf{A}}_h]_{P_{h+1}}$. It remains to fill in $\mathbf{A}_{h+1,2}$ with \mathbf{R}_h , to this end, we prove

Lemma 5.15. Suppose the message \mathbf{M} , the matrix $\mathbf{A}_{h+1,1}$ and the partition P_{h+1} are defined as Eq. (10), then $\mathbf{R}_h \in \mathcal{S}_{\mathbf{M}, P_{h+1}, \mathbf{A}_{h+1,1}}$

Proof. Consider the set

$$X := \mathsf{O}^{\mathbf{M}}(\mathbf{A}_{h+1,1}) \cap \mathcal{N}(\mathsf{O}^{\mathbf{M}}(\widetilde{\mathbf{A}}_h), \alpha_h)$$
(11)

By Lemma 5.14, the size $|X| \ge 2^{s_h/2 \log(d)}$ and for any $\mathbf{x} \in X$, $\|\mathbf{R}_h \mathbf{x}\|_{\infty} \le \|\mathbf{\widetilde{A}}_h \mathbf{x}\|_{\infty} \le \xi'$. Therefore, it suffices to prove that there exists an $(\alpha_h/4)$ -RLI sequence $\mathbf{x}_1, \ldots, \mathbf{x}_L \in X$, note it would imply $\mathbf{R}_h \in \mathcal{I}_{h+1,\mathbf{x}_1,\ldots,\mathbf{x}_L}$.

We prove by contradiction and assume the maximum length of $(\alpha_h/4)$ -RLI sequence is at most L' < L, then there exists an L'-dimensional subspace (formed by the maximum sequence), represented by the orthonormal $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_{L'}] \in \mathbb{R}^{d \times L'}$, such that

$$\|(\mathbf{I} - \mathbf{U}\mathbf{U}^{\top})\mathbf{x}\|_2 < \frac{\alpha_h}{4}, \quad \forall \mathbf{x} \in X.$$
 (12)

Consider the grid

$$G_{L'} = \left\{ \sum_{i=1}^{L'} \lambda_i \mathbf{u}_i : \lambda_i = 0 \pm \xi, \pm 2\xi, \dots, \pm 1, \forall i \in [L'] \right\}$$

Then the size of the grid is most

$$|G_{L'}| \le (4/\xi)^{L'} \le \exp(L\log(4/\xi)) < \exp(s/2\log^3(d)) \le \exp(s_h/2\log(d)).$$

where the third step follows from the choice of parameters $L = \frac{s}{4 \log^8(d)}$, $\xi = 2 \exp(-\log^5(d))$, the last step follows from $s_h \ge s_0 = s/\log^2(d)$.

Therefore, there exists $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that the projections $\mathbf{U}\mathbf{U}^{\top}\mathbf{x}_1, \mathbf{U}\mathbf{U}^{\top}\mathbf{x}_2$ are in the same grid, then we have

$$\|\mathbf{x}_{1} - \mathbf{x}_{2}\|_{2} \le \|(\mathbf{U}\mathbf{U}^{\top})(\mathbf{x}_{1} - \mathbf{x}_{2})\|_{2} + \|(\mathbf{I} - \mathbf{U}\mathbf{U}^{\top})(\mathbf{x}_{1} - \mathbf{x}_{2})\|_{2} \le 2d\xi + 2 \cdot \frac{\alpha_{h}}{4} < \alpha_{h}$$

Here the first step follows from the triangle inequality, the second step follows from Eq. (12) and $\mathbf{U}\mathbf{U}^{\top}\mathbf{x}_1, \mathbf{U}\mathbf{U}^{\top}\mathbf{x}_2$ are at the same grid. This contradicts with the fact that points in X have pairwise distance at least α_h (see the definition at Eq. (11)). We complete the proof here.

Due to Lemma 5.15, we can conclude that, if $|\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}}| \leq \Gamma_{h+1} = 2^{s_{h+1}d-2kd}$, then the encoding algorithm would add \mathbf{A} to \mathcal{A}_{h+1} , and we can finish the induction. Hence, we focus on the case of $|\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}}| > \Gamma_{h+1} = 2^{s_{h+1}d-2kd}$. In this case, let \hat{N}_{h+1} be the size of α_{h+1} -cover in $O^{\mathbf{M}}(\mathbf{A}_{h+1,1})$, that is

$$\hat{N}_{h+1} := |\mathcal{N}(\mathsf{O}^{\mathbf{M}}(\mathbf{A}_{h+1,1}), \alpha_{h+1})|.$$

We give a lower bound on \hat{N}_{h+1} in terms of $|\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1},1}|$, in particular, we prove

Lemma 5.16. Suppose the message \mathbf{M} , the matrix $\mathbf{A}_{h+1,1}$ and the partition P_{h+1} are defined as Eq. (10), then $|\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}}| \leq (\hat{N}_{h+1})^L \cdot 2^{s_{h+1} \cdot (d-\Omega(L))}$.

We need the following bound on the number of orthogonal vectors to a RLI sequence.

Lemma 5.17 (Number of orthogonal vectors). Given any $(\alpha_h/4)$ -RLI sequence $\mathbf{x}_1, \ldots, \mathbf{x}_L$, define

$$\widetilde{\mathcal{I}}_{h+1,\mathbf{x}_1,\ldots,\mathbf{x}_L} := \bigcup_{\widetilde{\mathbf{x}}_i \in \mathbb{B}(\mathbf{x}_i, 2\alpha_{h+1}), \forall i \in [L]} \mathcal{I}_{h+1,\widetilde{\mathbf{x}}_1,\ldots,\widetilde{\mathbf{x}}_L}.$$

Then we have

$$\left| \widetilde{\mathcal{I}}_{h+1,\mathbf{x}_1,\ldots,\mathbf{x}_L} \right| \le 2^{s_{h+1}(d-cL)}$$

for some absolute constant c > 0.

Proof. We apply Lemma A.4 for the sequence $\mathbf{x}_1, \ldots, \mathbf{x}_L$, with $\delta = (\alpha_h/4)^2/2$, $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_L] \in \mathbb{R}^{d \times L}$, then there exists an orthonormal matrix $\mathbf{U} \in \mathbb{R}^{d \times (L/2)}$, such that

$$\|\mathbf{U}^{\top}\mathbf{a}\|_{\infty} \leq \frac{d}{\delta} \|\mathbf{X}^{\top}\mathbf{a}\|_{\infty} = \frac{32d}{\alpha_h^2} \|\mathbf{X}^{\top}\mathbf{a}\|_{\infty} \quad \forall \mathbf{a} \in \mathbb{R}^d.$$
(13)

Now for any matrix $\mathbf{R} \in \widetilde{\mathcal{I}}_{h+1,\mathbf{x}_1,\ldots,\mathbf{x}_L}$, there exists a sequence $\widetilde{\mathbf{x}}_1,\ldots,\widetilde{\mathbf{x}}_L$, such that $\widetilde{\mathbf{x}}_i \in \mathbb{B}(\mathbf{x}_i, 2\alpha_{h+1})$ for $i \in [L]$, and $\mathbf{R} \in \mathcal{I}_{h+1,\widetilde{\mathbf{x}}_1,\ldots,\widetilde{\mathbf{x}}_L}$. Let $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1,\ldots,\widetilde{\mathbf{x}}_L] \in \mathbb{R}^{d \times L}$, for each row \mathbf{r} of \mathbf{R} , one has

$$\|\mathbf{U}^{\top}\mathbf{r}\|_{\infty} \leq \frac{32d}{\alpha_h^2} \|\mathbf{X}^{\top}\mathbf{r}\|_{\infty} \leq \frac{32d}{\alpha_h^2} (\|\mathbf{\widetilde{X}}^{\top}\mathbf{r}\|_{\infty} + \sqrt{d} \cdot 2\alpha_{h+1}) \leq \frac{32d\xi'}{\alpha_h^2} + \frac{64d\sqrt{d}\alpha_{h+1}}{\alpha_h^2} \leq 1/d^5.$$
(14)

Here the first step follows from Eq. (13), the second step follows from $\widetilde{\mathbf{x}}_i \in \mathbb{B}(\mathbf{x}_i, 2\alpha_{h+1})$ for every $i \in [L]$ and $\|\mathbf{r}\|_2 = \sqrt{d}$. The last step follows from the choice of parameters.

Let $\mathsf{R}_{\mathsf{or}} \subseteq \{-1,1\}^d$ contain all possible **r** that satisfies Eq. (14). We wish to bound the size of R_{or} , by Lemma 5.7, we have that

$$\Pr_{\mathbf{r} \sim \{-1,1\}^d} \left[\|\mathbf{U}^\top \mathbf{r}\|_2^2 \le 1/d^8 \right] \le 2^{-cL}$$

for some constant c > 0. Hence, we have $|\mathsf{R}_{\mathsf{or}}| \leq 2^d \cdot 2^{-cL} = 2^{d-cL}$. Since we have proved $\mathbf{r} \in \mathsf{R}_{\mathsf{or}}$ for every row of $\mathbf{R} \in \widetilde{\mathcal{I}}_{h+1,\mathbf{x}_1,\ldots,\mathbf{x}_L}$, and \mathbf{R} has s_{h+1} rows, we conclude the proof of Lemma.

Now we can go back to the proof of Lemma 5.16

Proof of Lemma 5.16. Let X_h contain all $(\alpha_h/4)$ -RLI sequence of length L in $O^{\mathbf{M}}(\mathbf{A}_{h+1,1})$, then ones has

$$\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}} = \bigcup_{(\mathbf{x}_1,\dots,\mathbf{x}_L)\in X_h} \mathcal{I}_{h+1,\mathbf{x}_1,\dots,\mathbf{x}_L}$$

Construct Y_h as follows. For any $\mathbf{x}_1, \ldots, \mathbf{x}_L$ in the α_{h+1} -cover $\mathcal{N}(\mathsf{O}^{\mathbf{M}}(\mathbf{A}_{h+1,1}), \alpha_{h+1})$, if there exists a sequence $\mathbf{y}_1, \ldots, \mathbf{y}_L$, such that

- $\mathbf{y}_i \in \mathbb{B}(\mathbf{x}_i, \alpha_{h+1}),$
- $\mathbf{y}_i \in \mathsf{O}^{\mathbf{M}}(\mathbf{A}_{h+1,1})$, and
- $\mathbf{y}_1, \ldots, \mathbf{y}_L$ forms an $(\alpha_h/4)$ -RLI sequence,

then we add $(\mathbf{y}_1, \ldots, \mathbf{y}_L)$ to Y_h . If there are multiple sequences, we only add one of them.

First, we have $|Y_h| \leq (\hat{N}_{h+1})^L$ because we look at each *L*-tuple in $\mathcal{N}(\mathsf{O}^{\mathbf{M}}(\mathbf{A}_{h+1,1}), \alpha_{h+1})$ once. Next, we claim that for any sequence $\mathbf{x}_1, \ldots, \mathbf{x}_L \in X_h$, there exists $(\mathbf{y}_1, \ldots, \mathbf{y}_L) \in Y_h$ such that $\|\mathbf{x}_i - \mathbf{y}_i\|_2 \leq 2\alpha_{h+1}$. To see this, let $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_L$ from the α_{h+1} -cover $\mathcal{N}(\mathsf{O}^{\mathbf{M}}(\mathbf{A}_{h+1,1}), \alpha_{h+1})$, such that $\|\tilde{\mathbf{x}}_i - \mathbf{x}_i\|_2 \leq \alpha_{h+1}$. Then there must exists $(\mathbf{y}_1, \ldots, \mathbf{y}_L) \in Y_h$ such that $\|\tilde{\mathbf{x}}_i - \mathbf{y}_i\|_2 \leq \alpha_{h+1}$. This is sufficient for our purpose.

Hence, we can conclude that

$$\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}} = \bigcup_{(\mathbf{x}_1,\dots,\mathbf{x}_L)\in X_h} \mathcal{I}_{h+1,\mathbf{x}_1,\dots,\mathbf{x}_L} \subseteq \bigcup_{(\mathbf{y}_1,\dots,\mathbf{y}_L)\in Y_h} \widetilde{\mathcal{I}}_{h+1,\mathbf{y}_1,\dots,\mathbf{y}_L}$$

By Lemma 5.17, we conclude

$$|\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}}| \le |Y_h| \cdot 2^{s_{h+1}(d-cL)} \le (\hat{N}_{h+1})^L \cdot 2^{s_{h+1}(d-\Omega(L))}.$$

This completes the proof of Lemma.

By Lemma 5.16 and our assumption that $|\mathcal{S}_{\mathbf{M},P_{h+1},\mathbf{A}_{h+1,1}}| \geq \Gamma_{h+1}$, we have that

$$(\hat{N}_{h+1})^L \cdot 2^{s_{h+1} \cdot (d-\Omega(L))} \ge |\mathcal{S}_{\mathbf{M}, P_{h+1}, \mathbf{A}_{h+1, 1}}| \ge \Gamma_{h+1} = 2^{s_{h+1}d-2kd}$$

and therefore,

$$\hat{N}_{h+1} \ge 2^{s_{h+1}/\log(d)} = N_{h+1}$$

this is because $kd = d^{2-\delta-o(1)}$ and $s_{h+1}L \ge s^2/\operatorname{poly}\log(d) \ge d^{2-\delta}/\operatorname{poly}\log(d)$.

This wraps up the induction.

For any matrix $\mathcal{A} \in \mathcal{A}_{\mathsf{nice}}$, if $\mathcal{A} \notin \mathcal{A}_h$ for any h = 0, 1, 2, ..., H, then the above induction implies that there exists a sub-matrix $\widetilde{\mathbf{A}}_H \subseteq \mathbf{A}$ ($\widetilde{\mathbf{A}}_H \in \{-1, 1\}^{(d/2-s \leq H) \times d}$), such that

$$|\mathcal{N}(\mathsf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{H}), \alpha_{H})| \ge N_{H} = 2^{s_{H}/\log d} \ge 2^{d/10\log(d)}.$$

This is simply impossible, because the total number of entries in $\mathcal{N}(O^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{H}), \alpha_{H})$ is at most

$$\left| \mathcal{N}(\mathbf{O}^{\mathbf{M}_{\mathbf{A}}}(\widetilde{\mathbf{A}}_{H}), \alpha_{H}) \right| \leq |\mathsf{T}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A})| = |\mathsf{T}^{\mathbf{M}_{\mathbf{A}}}(\mathbf{A}, V^{*})|$$
$$\leq |V^{*}| \cdot (d/2 + 1)^{n} \leq 2^{s+n \log_{2}(d)} \leq 2^{d/10 \log(d)}.$$

Here the third follows from the definition of a table (see Definition 5.3), and there are at most $(d/2+1)^n$ combinations of the third round message given **A**, the fourth step follows from $|V^*| \leq 2^s$ (see Lemma 5.8) – this is the only place that we use the size of V^* is not large.

Combining the above inequalities, we have proved that $\mathcal{A}_{nice} \subseteq \mathcal{A}$ and finish the proof of Lemma 5.12.

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References

- [ACNS23] Anders Aamand, Justin Chen, Huy Le Nguyen, and Sandeep Silwal. Improved space bounds for learning with experts. *https://arxiv.org/abs/2303.01453*, 2023.
- [AV95] David S Atkinson and Pravin M Vaidya. A cutting plane algorithm for convex programming that uses analytic centers. *Mathematical programming*, 69(1-3):1–43, 1995.
- [BBF⁺21] Gavin Brown, Mark Bun, Vitaly Feldman, Adam Smith, and Kunal Talwar. When is memorization of irrelevant training data necessary for high-accuracy learning? In Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, pages 123–132, 2021.
- [BBFM12] Maria Florina Balcan, Avrim Blum, Shai Fine, and Yishay Mansour. Distributed learning, communication complexity and privacy. In *Conference on Learning Theory*, pages 26–1. JMLR Workshop and Conference Proceedings, 2012.
- [BBS22] Gavin Brown, Mark Bun, and Adam Smith. Strong memory lower bounds for learning natural models. *Conference on Learning Theory (COLT)*, 2022.
- [BGM⁺16] Mark Braverman, Ankit Garg, Tengyu Ma, Huy L Nguyen, and David P Woodruff. Communication lower bounds for statistical estimation problems via a distributed data processing inequality. In Proceedings of the forty-eighth annual ACM symposium on Theory of Computing, pages 1011–1020, 2016.
- [BGP17] Gábor Braun, Cristóbal Guzmán, and Sebastian Pokutta. Lower bounds on the oracle complexity of nonsmooth convex optimization via information theory. *IEEE Transactions on Information Theory*, 63(7):4709–4724, 2017.
- [BHSW20] Mark Braverman, Elad Hazan, Max Simchowitz, and Blake Woodworth. The gradient complexity of linear regression. In *Conference on Learning Theory*, pages 627–647. PMLR, 2020.
- [BJJ23] Moïse Blanchard, Zhang Junhui, and Patrick Jaillet. Memory-constrained algorithms for convex optimization via recursive cutting-planes. arXiv preprint arXiv:2306.10096, 2023.
- [BJL⁺19] Sébastien Bubeck, Qijia Jiang, Yin-Tat Lee, Yuanzhi Li, and Aaron Sidford. Complexity of highly parallel non-smooth convex optimization. Advances in neural information processing systems, 32, 2019.
- [BS18] Eric Balkanski and Yaron Singer. Parallelization does not accelerate convex optimization: Adaptivity lower bounds for non-smooth convex minimization. *arXiv preprint arXiv:1808.03880*, 2018.
- [Bub15] Sébastien Bubeck. Convex optimization: Algorithms and complexity. Foundations and Trends® in Machine Learning, 8(3-4):231–357, 2015.

- [BV04] Dimitris Bertsimas and Santosh Vempala. Solving convex programs by random walks. Journal of the ACM (JACM), 51(4):540–556, 2004.
- [BZJ23] Moïse Blanchard, Junhui Zhang, and Patrick Jaillet. Quadratic memory is necessary for optimal query complexity in convex optimization: Center-of-mass is pareto-optimal. *arXiv preprint arXiv:2302.04963*, 2023.
- [CPP22] Xi Chen, Christos Papadimitriou, and Binghui Peng. Memory bounds for continual learning. In 2022 IEEE 63th Annual Symposium on Foundations of Computer Science (FOCS), 2022.
- [DBW12] John C Duchi, Peter L Bartlett, and Martin J Wainwright. Randomized smoothing for stochastic optimization. *SIAM Journal on Optimization*, 22(2):674–701, 2012.
- [DG19] Jelena Diakonikolas and Cristóbal Guzmán. Lower bounds for parallel and randomized convex optimization. In *Conference on Learning Theory*, pages 1132–1157. PMLR, 2019.
- [DJL⁺17] Simon S Du, Chi Jin, Jason D Lee, Michael I Jordan, Aarti Singh, and Barnabas Poczos. Gradient descent can take exponential time to escape saddle points. Advances in neural information processing systems, 30, 2017.
- [DKS19] Yuval Dagan, Gil Kur, and Ohad Shamir. Space lower bounds for linear prediction in the streaming model. In *Conference on Learning Theory*, pages 929–954, 2019.
- [DS18] Yuval Dagan and Ohad Shamir. Detecting correlations with little memory and communication. In *Conference On Learning Theory (COLT)*, 2018.
- [Fel20] Vitaly Feldman. Does learning require memorization? a short tale about a long tail. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, pages 954–959, 2020.
- [FR64] Reeves Fletcher and Colin M Reeves. Function minimization by conjugate gradients. The computer journal, 7(2):149–154, 1964.
- [GKLR21] Sumegha Garg, Pravesh Kumar Kothari, Pengda Liu, and Ran Raz. Memory-sample lower bounds for learning parity with noise. In 24th International Conference on Approximation Algorithms for Combinatorial Optimization Problems (APPROX 2021) and 25th International Conference on Randomization and Computation (RANDOM 2021), 2021.
- [GLM20] Alon Gonen, Shachar Lovett, and Michal Moshkovitz. Towards a combinatorial characterization of bounded-memory learning. Advances in Neural Information Processing Systems (NeurIPS), 2020.
- [GMN14] Ankit Garg, Tengyu Ma, and Huy Nguyen. On communication cost of distributed statistical estimation and dimensionality. *Advances in Neural Information Processing* Systems, 27, 2014.
- [GRT18] Sumegha Garg, Ran Raz, and Avishay Tal. Extractor-based time-space lower bounds for learning. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory* of Computing (STOC), 2018.

- [GRT19] Sumegha Garg, Ran Raz, and Avishay Tal. Time-space lower bounds for two-pass learning. In 34th Computational Complexity Conference (CCC), 2019.
- [HS⁺52] Magnus R Hestenes, Eduard Stiefel, et al. Methods of conjugate gradients for solving linear systems. Journal of research of the National Bureau of Standards, 49(6):409–436, 1952.
- [HZ06] William W Hager and Hongchao Zhang. A survey of nonlinear conjugate gradient methods. *Pacific journal of Optimization*, 2(1):35–58, 2006.
- [JLSW20] Haotian Jiang, Yin Tat Lee, Zhao Song, and Sam Chiu-wai Wong. An improved cutting plane method for convex optimization, convex-concave games, and its applications. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, pages 944–953, 2020.
- [JNG⁺21] Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M Kakade, and Michael I Jordan. On nonconvex optimization for machine learning: Gradients, stochasticity, and saddle points. *Journal of the ACM (JACM)*, 68(2):1–29, 2021.
- [JZ13] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. Advances in neural information processing systems, 26, 2013.
- [Lev65] Anatoly Yur'evich Levin. An algorithm for minimizing convex functions. In *Doklady Akademii Nauk*, volume 160, pages 1244–1247. Russian Academy of Sciences, 1965.
- [LN89] Dong C Liu and Jorge Nocedal. On the limited memory bfgs method for large scale optimization. *Mathematical programming*, 45(1-3):503–528, 1989.
- [LRZ23] Qipeng Liu, Ran Raz, and Wei Zhan. Memory-sample lower bounds for learning with classical-quantum hybrid memory. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, 2023.
- [LSW15] Yin Tat Lee, Aaron Sidford, and Sam Chiu-wai Wong. A faster cutting plane method and its implications for combinatorial and convex optimization. In 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 1049–1065. IEEE, 2015.
- [MSSV22] Annie Marsden, Vatsal Sharan, Aaron Sidford, and Gregory Valiant. Efficient convex optimization requires superlinear memory. In *Conference on Learning Theory*, pages 2390–2430. PMLR, 2022.
- [Nem94] Arkadi Nemirovski. On parallel complexity of nonsmooth convex optimization. *Journal* of Complexity, 10(4):451–463, 1994.
- [Nes03] Yurii Nesterov. Introductory lectures on convex optimization: A basic course, volume 87. Springer Science & Business Media, 2003.
- [Noc80] Jorge Nocedal. Updating quasi-newton matrices with limited storage. *Mathematics of computation*, 35(151):773–782, 1980.
- [NY83] Arkadij Semenovič Nemirovskij and David Borisovich Yudin. Problem complexity and method efficiency in optimization. 1983.
- [PR23] Binghui Peng and Aviad Rubinstein. Near optimal memory-regret tradeoff for online learning. arXiv preprint arXiv:2303.01673, 2023.

- [PW17] Mert Pilanci and Martin J Wainwright. Newton sketch: A near linear-time optimization algorithm with linear-quadratic convergence. *SIAM Journal on Optimization*, 27(1):205–245, 2017.
- [PZ23] Binghui Peng and Fred Zhang. Online prediction in sub-linear space. In Proceedings of the Thirty Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, 2023.
- [Raz17] Ran Raz. A time-space lower bound for a large class of learning problems. In 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS), 2017.
- [Raz18] Ran Raz. Fast learning requires good memory: A time-space lower bound for parity learning. *Journal of the ACM (JACM)*, 66(1):1–18, 2018.
- [RKM19] Farbod Roosta-Khorasani and Michael W Mahoney. Sub-sampled newton methods. Mathematical Programming, 174:293–326, 2019.
- [SD15] Jacob Steinhardt and John Duchi. Minimax rates for memory-bounded sparse linear regression. In Conference on Learning Theory, pages 1564–1587. PMLR, 2015.
- [SEAR18] Max Simchowitz, Ahmed El Alaoui, and Benjamin Recht. Tight query complexity lower bounds for pca via finite sample deformed wigner law. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, pages 1249–1259, 2018.
- [SSV19] Vatsal Sharan, Aaron Sidford, and Gregory Valiant. Memory-sample tradeoffs for linear regression with small error. In *Proceedings of the 51st Annual ACM SIGACT Symposium* on Theory of Computing (STOC), 2019.
- [SVW16] Jacob Steinhardt, Gregory Valiant, and Stefan Wager. Memory, communication, and statistical queries. In *Conference on Learning Theory (COLT)*, 2016.
- [SWXZ22] Vaidehi Srinivas, David P. Woodruff, Ziyu Xu, and Samson Zhou. Memory bounds for the experts problem. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, 2022.
- [Vai96] Pravin M Vaidya. A new algorithm for minimizing convex functions over convex sets. Mathematical programming, 73(3):291–341, 1996.
- [Vem05] Santosh Vempala. Geometric random walks: a survey. Combinatorial and computational geometry, 52(573-612):2, 2005.
- [Woo14] David Woodruff. Sketching as a tool for numerical linear algebra. Foundations and $Trends(\widehat{R})$ in Theoretical Computer Science, 10(1–2):1–157, 2014.
- [WS16] Blake E Woodworth and Nati Srebro. Tight complexity bounds for optimizing composite objectives. Advances in neural information processing systems, 29, 2016.
- [WS19] Blake Woodworth and Nathan Srebro. Open problem: The oracle complexity of convex optimization with limited memory. In *Conference on Learning Theory*, pages 3202–3210, 2019.
- [WZZ23] David P Woodruff, Fred Zhang, and Samson Zhou. Streaming algorithms for learning with experts: Deterministic versus robust. *arXiv preprint arXiv:2303.01709*, 2023.

A Useful Lemma

Lemma A.1 (Khintchine's Inequality). Let $\sigma_1, \ldots, \sigma_n$ be i.i.d. Rademacher variables (i.e., $\sigma_i \sim \{-1,1\}$), and let x_1, \ldots, x_n be real numbers. Then there are constants $c_1, c_2 > 0$ so that

$$\Pr\left[\left|\sum_{i=1}^{n} \sigma_i x_i\right| \ge c_1 t \cdot \|x\|_2\right] \le \exp(-c_2 t^2).$$

Define the sub-Gaussian norm $||x||_{\psi_2}$ of a sub-Gaussian random variable x as

$$||x||_{\psi_2} := \inf\{K > 0 \text{ such that } \mathbb{E}[\exp(x^2/K^2)] \le 2\}.$$

We have

Lemma A.2 (Projection of sub-gaussian random variables, Lemma 40 of [MSSV22]). Let $\mathbf{x} \in \mathbb{R}^d$ be a random vector with *i.i.d* sub-Gaussian components which satisfy $\mathbb{E}[\mathbf{x}_i] = 0$, $\|\mathbf{x}_i\|_{\psi_2} \leq K$, and $\mathbb{E}[\mathbf{x}\mathbf{x}^{\top}] = s^2 I_d$. Let $\mathbf{U} \in \mathbb{R}^{d \times r}$ be an orthonormal matrix, then there is a constant c > 0, such that, for any $t \geq 0$

$$\Pr\left[|\|\mathbf{U}^{\top}\mathbf{x}\|_{2}^{2} - rs^{2}| \ge t\right] \le \exp\left(-c\min\left\{\frac{t^{2}}{rK^{4}}, \frac{t}{K^{2}}\right\}\right).$$

In particular, if the vector $\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$, then we have $\mathbb{E}[\mathbf{v}\mathbf{v}^\top] = \frac{1}{d}I$ and $\|\mathbf{v}_i\|_{\psi_2} \leq \frac{2}{\sqrt{d}}$, and there exists a constant c > 0 such that for any $t \geq 0$,

Lemma A.3 (Projection of random vectors in \mathcal{H}_d). Let $\mathbf{v} \sim \frac{1}{\sqrt{d}} \mathcal{H}_d$ and $\mathbf{U} \in \mathbb{R}^{d \times r}$ be an orthonormal matrix, then

$$\Pr\left[\left|\|\mathbf{U}^{\top}\mathbf{v}\|_{2}^{2} - \frac{r}{d}\right| \ge t\right] \le \exp\left(-c\min\left\{\frac{d^{2}t^{2}}{16r}, \frac{dt}{4}\right\}\right).$$

Lemma A.4 (Lemma 34 from [MSSV22]). Let $L \in [d]$, $\delta \in (0, 1]$. Suppose a sequence of unit norm vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in \mathbb{R}^d$ satisfies

$$\|\operatorname{proj}_{\mathsf{span}(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1})}(\mathbf{x}_i)\|_2 \le 1 - \delta.$$

Let $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_L] \in \mathbb{R}^{d \times L}$. There exists an orthonormal matrix $\mathbf{U} \in \mathbb{R}^{d \times (L/2)}$ such that for any vector $\mathbf{a} \in \mathbb{R}^d$,

$$\|\mathbf{U}^{\top}\mathbf{a}\|_{\infty} \leq \frac{d}{\delta}\|\mathbf{X}^{\top}\mathbf{a}\|_{\infty}$$