# Tight Space Lower Bound for Pseudo-Deterministic Approximate Counting 

Ofer Grossman Meghal Gupta Mark Sellke


#### Abstract

We investigate one of the most basic problems in streaming algorithms: approximating the number of elements in the stream. Famously, [Mor78] gave a randomized algorithm achieving a constant-factor approximation error for streams of length at most $N$ in space $O(\log \log N)$. We investigate the pseudo-deterministic complexity of the problem and prove a tight $\Omega(\log N)$ lower bound, thus resolving a problem of [GGMW20].


## 1 Introduction

The study of streaming algorithms originated with the seminal paper of Morris [Mor78], which gave a low-memory randomized algorithm to approximately count a number of elements which arrive online in a stream. Roughly speaking, the idea is to have a counter which approximates $\log n$, where $n$ is the number of elements seen in the stream so far. Each time the algorithm encounters an element from the stream, it increases the counter with probability about $\frac{1}{n}$. As later proved by [Fla85], Morris's algorithm achieves a constant-factor approximation error for streams of length at most $N$ in space $O(\log \log N)$.

Morris's algorithm has the property that running it multiple times on the same stream may result in different approximations. That is, if Alice runs the algorithm on the same stream as Bob (but using different randomness), Alice may get some approximation (such as $2^{30}$ ), and Bob (running the same algorithm but with independent randomness) may get a different approximation (such as $2^{29}$ ). Is this behavior inherent? That is, could there exist a low-space algorithm which, while being randomized, for all streams with high probability both Alice and Bob will end up with the same approximation for the length? Such algorithms, namely those which output the same output with high probability when run multiple times on the same input, are called pseudo-deterministic.

This question of the pseudo-deterministic space complexity of approximate counting in a stream was first posed in [GGMW20]. Our main result fully resolves the problem by giving a tight $\Omega(\log N)$ lower bound. In fact, our lower bound applies for an easier threshold version of the problem asking to distinguish between streams of length at most $N$ versus at least $M$ for any integers $N<M$. Moreover, it depends only on $N$, i.e. the problem is hard even when $M$ is arbitrarily larger than $N$.

Theorem 1.1 (Informal). For any $N<M$, a pseudo-deterministic streaming algorithm to distinguish between streams of length at most $N$ and at least $M$ must use $\Omega(\log N)$ space.

Concurrently, [BKKS23] showed a non-trivial $\Omega(\sqrt{\log n / \log \log n})$ space lower bound for this problem using different techniques.

Motivation for Pseudo-Determinism. Randomization is a powerful tool in algorithm design. For example, almost all classic streaming algorithms (such as for heavy hitters, approximate counting, $\ell_{p}$ approximation, finding a nonzero entry in a vector for turnstile algorithms, counting the number of distinct elements in a stream) cannot be derandomized without incurring a dramatic blowup in space complexity. However randomized algorithms also have several downsides compared to deterministic algorithms:

- They have a nonzero probability of error or failure.
- Random bits must be sampled, which may be infeasible or computationally expensive.
- The output is not predictable. That is, multiple executions of the algorithm on the same input may result in different outputs.

The first two points above have both significantly informed research on randomized algorithms. For example, the first point motivates the distinction between Monte Carlo and Las Vegas algorithms as well as RP versus BPP, and the need for improvements over naive amplification [IZ89, GM20]. The second point has led to work on pseudorandom generators [HILL99], how to recycle random bits [IZ89], and efficient extractors [Tre01], to mention a few. The study of pseudo-determinism as introduced by Gat and Goldwasser [GG11] seeks to address the third problem. Our result shows that the third downside cannot be avoided by any algorithm that solves the approximate counting (or threshold) problem in low space.

### 1.1 Related Work

Prior Work on Approximate Counting in a Stream. The study of streaming algorithms began with the work Morris [Mor78] on approximate counting; a rigorous analysis was given later in [Fla85]. As explained earlier, the (randomized) Morris counter requires logarithmically fewer states than a (deterministic) exact counter. The approximate counter has been useful as a theoretical primitive for other streaming algorithms [AJKS02, GS09, KNW10, BDW18, JW19] and its performance has been evaluated extensively [Cve07, Csü10, XKNS21].

The optimal dependence on the error level and probability in approximate counting was determined recently in [NY22]. Moreover, [AAHNY22] studied the amortized complexity of maintaining several approximate counters rather than just one.

Prior Work on Pseudo-Determinism. Pseudo-deterministic algorithms were introduced by Gat and Goldwasser [GG11] who were motivated by applications to cryptography. Since then, such algorithms have since been studied in the context of standard (sequential algorithms) [Gro15, OS17], average case algorithms [Hol17], parallel algorithms [GG17, GG21], decision tree algorithms [GGR13, Gol19, DPV21, GIPS21], interactive proofs [GGH18, GGH19], learning algorithms [OS18], approximation algorithms [OS18, DPV18], low space algorithms [GL19], complexity
theory [LOS21, DPWV22] and more. In several of these settings, pseudo-deterministic algorithms were shown to outperform deterministic algorithms and sometimes even match the state of the art performance for randomized algorithms.

The paper [GGMW20] initiated the study of pseudo-deterministic streaming algorithms. They examined various problems (such as finding heavy hitters, $\ell_{2}$ approximation, and finding a nonzero entry in a vector for turnstile algorithms) and asked whether approximate counting admits a pseudodeterministic algorithm using $O(\log \log n)$ bits of space. We give an optimal $\Omega(\log n)$ lower bound.

As this work was being finalized, [BKKS23] independently showed a $\Omega(\sqrt{\log n / \log \log n})$ lower bound. Their proof proceeds via reduction to one-way communication complexity. Our proof, on the other hand, takes a completely different approach and analyzes the complexity directly by modeling the streaming algorithm as a Markov chain and showing it behaves as an ensemble of cyclic parts in a suitable sense.

### 1.2 Main Result

We consider the problem of pseudo-deterministic approximate counting in a stream. We first recall the definition of a pseudo-deterministic streaming algorithm:

Definition 1. We define a streaming algorithm $\mathcal{A}$ to be pseudo-deterministic if there exists some function $F$ such that for all valid input streams $x$,

$$
\mathbb{P}_{r}[\mathcal{A}(x, r)=F(x)] \geq 2 / 3
$$

where $r$ is the randomness sampled and used by the algorithm $\mathcal{A}$.
Our main result gives a tight bound of $\Omega(\log N)$ bits of memory. In fact, we prove that pseudo-deterministically distinguishing between streams of length $T \leq N$ and $T \geq f(N)$ requires $\Omega(\log N)$ bits of memory for any function $f: \mathbb{N} \rightarrow \mathbb{N}$ where $f(N)>N$.

Definition 2 (Pseudo-Deterministic Approximate Threshold Problem). Fix $N, M \in \mathbb{N}$. Suppose $\mathcal{A}$ is a pseudo-deterministic streaming algorithm such that with $T$ the stream length:

1. $F(T)=1$ for $T \leq N$,
2. $F(T)=0$ for $T \geq M$.

Then we say $\mathcal{A}$ solves the ( $N, M$ )-pseudo-deterministic approximate threshold problem.
Theorem 1.2. For any $f: \mathbb{N} \rightarrow \mathbb{N}$ where $f(N)>N$, any sequence of pseudo-deterministic algorithms $\mathcal{A}_{N}$ which solve the $(N, f(N))$-approximate threshold problem requires $\Omega(\log N)$ bits of space.

Since counting is harder than thresholding, the next corollary is an immediate consequence.
Corollary 1.3. Any pseudo-deterministic streaming algorithm which solves approximate counting up to a constant multiplicative factor for stream lengths at most $N$ requires $\Omega(\log N)$ bits of space.

Remark 1.4. Our result also implies a space lower bound for solving a variant of the heavy hitters problem pseudo-deterministically. Consider a version of this problem where given a $\{0,1\}$-valued stream, we aim to output a bit that appeared at least $10 \%$ of the time. It follows from Corollary 1.3 that any pseudo-deterministic streaming algorithm solving this problem requires $\Omega(\log N)$ bits of space. Indeed, if it is public information that the first $9 N$ bits are 0 and the rest are 1 , then we are reduced to the $(N, 81 N)$-approximate threshold problem by counting the 1 's. Randomized algorithms again can solve the problem with $O(\log \log N)$ bits as it suffices to maintain a pair of Morris counters for the 0 's and 1's separately. (Note that while heavy hitters is often considered to be solvable deterministically in constant space, such solutions typically assume a model where exact counting uses constant space.)

### 1.3 Markov Chain Formulation

It will be helpful to reframe the approximate threshold problem and pseudo-determinism in the language of Markov chains. Note that any b-bit pseudo-deterministic streaming algorithm can be described as a Markov chain on $2^{b}$ states. Throughout the rest of the paper, we use this formulation instead, which we make precise below.

Let $\mathcal{M}_{n}$ be an arbitrary Markov chain on state space $V=\left\{v_{1}, \ldots, v_{n}\right\}$ with starting state $x_{0}=v_{1}$, and let $x_{1}, x_{2}, \ldots$ be the random states at each subsequent time. Moreover, let $U \subseteq V$ be a distinguished subset of $V$.

Definition 3 (Markov Chain Solution to Pseudo-Deterministic Approximate Thresholding). We say $\left(\mathcal{M}_{n}, U\right)$ is a $(N, f(N))$-solution to the approximate threshold problem if:
(i) $\mathbb{P}\left[x_{t} \in U\right] \geq 2 / 3$ for all $t \leq N$,
(ii) $\mathbb{P}\left[x_{t} \in U\right] \leq 1 / 3$ for all $t \geq f(N)$.

We say $\mathcal{M}_{n}$ is pseudo-deterministic if $\mathbb{P}\left[x_{t} \in U\right] \in[0,1 / 3] \cup[2 / 3,1]$ for all $t$.
We will show the following:
Theorem 1.5. For any pseudo-deterministic $\left(\mathcal{M}_{n}, U\right)$ that $(N, f(N))$-solves the approximate threshold problem for some $f(N)>N$, it holds that $n \geq N^{\Omega(1)}$.

It is easy to see that Theorem 1.5 implies Theorem 1.2 by viewing a streaming algorithm using $b$ bits of memory as a Markov chain on $2^{b}$ states. Here $U$ is the set of algorithm states on which the algorithm outputs 1 .

## 2 Technical Overview

### 2.1 Illustrative Examples

We begin by describing two examples of Markov chains that fail to solve the pseudo-deterministic approximate threshold problem, but end up being surprisingly illustrative of the general case.

Example 1: Threshold Morris Counter. We first describe how to solve approximate thresholding in the usual randomized (non-pseudo-deterministic) setting, where $M=10 \mathrm{~N}$ and for $t \in[N, 10 N]$, it is not required that $\mathbb{P}\left[x_{t} \in U\right] \in[0,1 / 3] \cup[2 / 3,1]$. We will use a version of a Morris counter with a simple two-state implementation (since we aim to threshold rather than count).

Our Markov chain will have states $\left(v_{1}, v_{2}\right)$ where $v_{2}$ is terminal, and the transition from $v_{1}$ to $v_{2}$ occurs with probability $\frac{1}{3 N}$. Formally, $\mathbb{P}\left[x_{t+1}=v_{2} \mid x_{t}=v_{2}\right]=1$, and $\mathbb{P}\left[x_{t+1}=v_{2} \mid x_{t}=\right.$ $\left.v_{1}\right]=\frac{1}{3 N}$. Setting $U=\left\{v_{1}\right\}$, this chain gives a randomized algorithm for $(N, 10 N)$ approximate thresholding. However, this example is not pseudo-deterministic because the function $t \mapsto \mathbb{P}\left[x_{t} \in\right.$ $U]$ decays gradually. In particular it is easy to see that $\mathbb{P}\left[x_{t+1} \in U\right] \geq 0.99 \cdot \mathbb{P}\left[x_{t} \in U\right]$, so by the discrete-time intermediate value theorem there exists $t$ such that $\mathbb{P}\left[x_{t} \in U\right] \notin[0,1 / 3] \cup[2 / 3,1]$.

Example 2: Prime Cycles. In our second example, we discuss a futile attempt to solve a simpler version of the approximate threshold problem where it is given that the stream's length is at most $10 M=100 \mathrm{~N}$. Informally speaking, on the first step the Markov chain picks a prime $p_{i}$. On all subsequent steps, it simply records the current time modulo $p_{i}$. A main idea in our proof, outlined in the next subsection, will be to show that any Markov chain $\mathcal{M}_{n}$ behaves similarly to this example.

Let $k=\left[n^{1 / 4}, 2 n^{1 / 4}\right]$ and choose $k$ distinct primes $p_{1}, \ldots, p_{k} \in\left[n^{1 / 3}, 2 n^{1 / 3}\right]$. Noting that $\sum_{i=1}^{k} p_{i} \leq n$, we can construct a Markov chain $\mathcal{M}_{n}$ which contains an initial state $v_{0}$ as well as a deterministic $p_{i}$-cycle for each $1 \leq i \leq k$. Here a $p$-cycle consists of vertices $\left(v_{0}^{(p)}, \ldots, v_{p-1}^{(p)}\right)$ with dynamics deterministically incrementing the subscript by 1 modulo $p$ each time. At the first timestep, $\mathcal{M}_{n}$ moves to $x_{1}=v_{1}^{\left(p_{i}\right)}$ for a uniformly random $p_{i}$. One could try to solve the pseudo-deterministic approximate threshold problem (with restricted stream length at most 100 N ) by choosing a special subset $U \subseteq \bigcup_{1 \leq i \leq k}\left\{v_{0}^{\left(p_{i}\right)}, \ldots, v_{p_{i}-1}^{\left(p_{i}\right)}\right\}$ of the state space of $\mathcal{M}_{n}$.

We argue below that this is not possible, which will serve as a useful warmup for the main proof. Note that by the discrete-time intermediate value theorem, for any such solution there must exist $0 \leq T \leq 9 N$ so that $\mathbb{P}\left[x_{t} \in U\right] \geq 2 / 3$ holds for roughly half of the values $t \in[T, T+1, \ldots, T+N]$. To contradict pseudo-determinism, it suffices to show an upper bound

$$
\begin{equation*}
\sum_{t \in[T, T+1, \ldots, T+N]}\left(\mathbb{P}\left[x_{t} \in U\right]-1 / 2\right)^{2} \leq o(N) . \tag{2.1}
\end{equation*}
$$

Since it involves squares of probabilities, the left-hand side can be rewritten as a sum over all pairs $p_{i}, p_{j}$ of cycle lengths. Moreover $p_{i} p_{j} \leq O\left(n^{2 / 3}\right) \ll N$, so the Chinese remainder theorem implies that every pair of distinct primes behaves almost independently on $[T, T+1, \ldots, T+N]$. Combining, it is not difficult to conclude (2.1), thus contradicting pseudo-determinism.

### 2.2 Proof Outline

Suppose for sake of contradiction that the Markov chain $\left(\mathcal{M}_{n}, U\right)$ is pseudo-deterministic and solves the approximate threshold problem. The main idea of our proof is to extract from $\mathcal{M}_{n}$ a subsystem behaving roughly like Example 2 above, in the sense that it is an ensemble of cycles with different period lengths. We leverage this behavior around a time $T$ where $\mathbb{P}\left[x_{s} \in U\right] \geq 2 / 3$ holds
for roughly half of the values $s \in[T, T+\operatorname{poly}(n)]$, which exists by the discrete-time intermediate value theorem. Finally, we use Fourier analysis to prove there exists some $s \in[T, T+\operatorname{poly}(n)]$ where $\mathbb{P}\left[x_{s} \in U\right]$ is not too biased, contradicting the pseudo-determinism of $\left(\mathcal{M}_{n}, U\right)$.

Recurrent Behavior on Moderate Time-Scales. The first step is to bypass the issue of permanent irreversible transitions as in Example 1 by considering a random time. We choose a random time $1 \leq t \leq \operatorname{poly}(n)$ and show that with high probability the chain behaves recurrently at vertex $x_{t}$. More precisely, let $t+r$ be the next time at which $x_{t+r}=x_{t}$ (where $r=\infty$ if the chain never returns). We show in Lemma 3.1 that with high probability,

$$
\begin{align*}
\mathbb{P}[r \leq 10 n] & \geq 1 / 2  \tag{2.2}\\
\mathbb{P}\left[r \leq n^{18}\right] & \geq 1-n^{-16} \tag{2.3}
\end{align*}
$$

Using this guarantee, we can view the Markov chain from the perspective of the (random) state $x_{t}$ by considering the sequence of cycle lengths. The bound (2.3) allows us to define this process up to large $\operatorname{poly}(n)$ number of time steps, with all cycle lengths at most $n^{18}$ with high probability. This circumvents the behavior in Example 1: although the chain might eventually transition to a terminal state and never return to $x_{t}$, it tends not to do so during a certain poly $(n)$ time window.

Decomposition into Periodic Parts. Next, using (2.2), it follows that there exists an integer $1 \leq m_{v} \leq 10 n$ such that conditioned on $x_{t}=v$, each cycle returning to $v$ has probability $\Omega(1 / n)$ to have length exactly $m_{v}$. We call cycles of length exactly $m_{v}$ special, and condition also on a masked cycle length process $\vec{\ell}$. The process $\vec{\ell}$ hides the special cycles, but reveals all other cycle lengths in order, until stopping once a moderately large fixed poly $(n)$ number of non-special cycles have been revealed. ${ }^{1}$

As an explicit example, suppose that at vertex $v$, there is a $1 / 4$ probability of arriving back at $v$ after 3 steps, $1 / 4$ probability of arriving back at $v$ after 5 steps, $1 / 3$ probability of arriving back at $v$ after 7 steps, and $1 / 6$ probability of arriving back at $v$ after 11 steps, and let $m_{v}=7$. Suppose that on a specific run of the Markov chain starting at $v$, the sequence of cycles taken are of lengths $(3,5,7,11,7,5,3,7)$. Then $\vec{\ell}=(3,5,11,5,3)$.

After this point, the number of hidden $m_{v}$ cycles is approximately Gaussian with poly $(n)$ standard deviation. In particular, its probability mass function is almost constant on short scales, which lets us approximate the function

$$
\mathbb{P}\left[x_{s} \in U \mid x_{t}=v\right]
$$

by an $m_{v}$-periodic function for large $s$. Averaging over the randomness of $x_{t}$, we have approximated $\mathbb{P}\left[x_{s} \in U\right]$ by an average of periodic functions with periods $m \leq 10 n .{ }^{2}$ This completes the structural phase of the proof. We now turn to finding a time $s$ such that $\mathbb{P}\left[x_{s} \in U\right] \in[1 / 3,2 / 3]$.

[^0]Analysis of Periodic Decomposition. The last step is to analyze this mixture-of-cyclic behavior at a time-region $T$ on which $\mathbb{P}\left[x_{s} \in U\right] \geq 2 / 3$ holds for roughly half of the values $s \in[T, T+\operatorname{poly}(n)]$; such $T$ exists by the discrete-time intermediate value theorem. If the different periods $m_{v}$ were relatively prime, intuitively each pair of cyclic behaviors would be approximately independent over the range $[T, T+\operatorname{poly}(n)]$ since $m_{v} m_{v}^{\prime} \leq 100 n^{2} \ll \operatorname{poly}(n)$. This pairwise almost independence would allow us to upper bound $\sum_{s \in[T, T+\mathrm{poly}(n)]}\left(\mathbb{P}\left[x_{s} \in U\right]-1 / 2\right)^{2}$ similarly to Example 2, thus contradicting pseudo-determinism.

To handle general period lengths, the key idea is to divide out common prime factors and restrict to a corresponding arithmetic progression. Precisely, for each prime $p$ we let $e_{p} \geq 0$ be the largest integer such that $m_{v}$ has at least some constant probability to be a multiple of $p^{e_{p}}$. Then it can be shown that $\beta=\prod_{p \text { prime }} p^{e_{p}} \leq \operatorname{poly}(n)$, so we restrict attention to a fixed arithmetic progression with difference $\beta$.

Restricting to this arithmetic progression causes the different cycles to behave almost as if they are relatively prime. Precisely, the "reduced period lengths" $M_{v}=m_{v} / \operatorname{gcd}\left(m_{v}, \beta\right)$ have the property that any fixed prime $p$ has low probability of dividing $M_{v}$. This property does not imply pairwise independence of different cycles: it could still be true that $\operatorname{gcd}\left(M_{v}, M_{v}^{\prime}\right)>1$ holds with high probability for independent $M_{v}, M_{v}^{\prime}$. However using Fourier analysis, we were still able to show that it implies a non-trivial upper bound on $\sum_{s \in[T, T+\operatorname{poly}(n)]}\left(\mathbb{P}\left[x_{s} \in U\right]-1 / 2\right)^{2}$. As mentioned above, such an upper bound contradicts pseudo-determinism which completes the proof.

## 3 Proof of Theorem 1.5

Assume $n \geq n_{0}$ is sufficiently large and let the threshold $N>n^{34}$. We will show such a Markov chain cannot $(N, M)$-solve the approximate counting problem for any $M$. We will replace the intervals $[0,1 / 3]$ and $[2 / 3,1]$ by $\left[0,10^{-4}\right]$ and $\left[1-10^{-4}, 1\right]$ in the statement of Theorem 1.5 , which is equivalent by amplification.

### 3.1 Recurrent Behavior on Moderate Time-Scales

Given any $v \in V$, let $\mu_{v}$ be the return-time distribution from $v$, i.e. the distribution for the smallest $r \geq 1$ such that $x_{r}=v$, starting from $x_{0}=v$. (We let $r=\infty$ if this never holds again.)

Lemma 3.1. There exists $1 \leq t \leq N / 100$ and a subset $S \subseteq V$ such that:
(i) $\mathbb{P}\left[x_{t} \in S\right] \geq 1 / 2$.
(ii) For all $v \in S$ :

$$
\begin{aligned}
& \mathbb{P}_{r \sim \mu_{v}}(r \in[0,10 n]) \geq 1 / 2, \\
& \mathbb{P}_{r \sim \mu_{v}}\left(r \in\left[0, n^{18}\right]\right) \geq 1-n^{-16} .
\end{aligned}
$$

Proof. We first show that at least 0.85 -fraction of the values of $1 \leq t \leq N / 100$ satisfy:

$$
\begin{equation*}
\mathbb{E}_{x_{t}} \mathbb{P}_{r \sim \mu_{x_{t}}}(r \in[0,10 n]) \geq 0.85 \tag{3.1}
\end{equation*}
$$

Partition the interval $[1, N / 100]$ into intervals $I_{j}=[1+(j-1) 10 n, 10 j n]$ of length $10 n$ for $1 \leq j \leq N / 1000 n$. Consider a fixed run of the Markov chain $x_{1} \ldots x_{N / 100}$. For each interval $I_{j}$, at most $n$ values of $z_{i}$ for $i \in I_{j}$ will not return to themselves at some later time in $I_{j}$. Thus, in a fixed run of the Markov chain, the return time back to $x_{t}$ starting from time $t$ is at most $10 n$ steps for at least 0.9 fraction of times $1 \leq t \leq N / 100$. Averaging over all runs of the Markov chain, we get that

$$
\mathbb{E}_{t \in[1, N / 100]} \mathbb{E}_{x_{t}} \mathbb{P}_{r \sim \mu_{x_{t}}}(r \in[1,10 n]) \geq 0.9
$$

and therefore at least 0.85 fraction of $0<t<N / 100$ satisfy (3.1) by Markov's inequality. By an identical argument, for at least 0.85 fraction of $1 \leq t \leq N / 100$,

$$
\mathbb{E}_{x_{t}} \mathbb{P}_{r \sim \mu_{x_{t}}}\left(r \in\left[0, n^{18}\right]\right) \geq 1-10 n^{-17}
$$

In particular, there exists a value of $t$ satisfying both conditions, which will be the value of $t$ in the lemma statement.

Next applying Markov's inequality to (3.1) shows for some $S_{1} \subseteq V$ with $\mathbb{P}\left[x_{t} \in S_{1}\right] \geq 0.6$,

$$
\mathbb{P}_{r \sim \mu_{v}}(r \in[1,10 n]) \geq 1 / 2
$$

Similarly for some $S_{2} \subseteq V$ such that $\mathbb{P}\left[x_{t} \in S_{2}\right]>0.9$,

$$
\mathbb{P}_{r \sim \mu_{v}}\left(r \in\left[0, n^{18}\right]\right) \geq 1-100 n^{-17} \geq 1-n^{-16} .
$$

Taking $S=S_{1} \cap S_{2}$ completes the proof.

### 3.2 Decomposition into Periodic Parts

From now on we fix $t, S$ as in the previous subsection. For each $v \in S$, define the period length

$$
m_{v} \equiv \underset{1 \leq m \leq 10 n}{\operatorname{argmax}} \mu_{v}(m)
$$

Starting from $x_{t}=v$, we define a sequence of steps as follows. Let $\left(C_{1}, C_{2}, \ldots\right)$ be the sequence of cycle lengths returning back to location $x_{t}$, i.e. $x_{t+C_{1}+\cdots+C_{j}}$ for $j \geq 0$ are exactly the times $s \geq t$ with $x_{s}=x_{t}$. When $C_{i}=m_{v}$, we assign $C_{i}$ to be special with probability

$$
\begin{equation*}
\mathbb{P}\left[C_{i} \text { special } \mid C_{i}=m_{v}\right]=\frac{n^{-2} \mu_{v}\left(\left\{1,2, \ldots, n^{18}\right\}\right)}{\mu_{v}\left(m_{v}\right)} . \tag{3.2}
\end{equation*}
$$

This choice of probability results in $\mathbb{P}\left[C_{i}\right.$ special $\left.\mid C_{i} \leq n^{18}\right]=n^{-2}$, which will be useful later. In all other cases where $C_{i} \leq n^{18}$, we assign $C_{i}$ to be typical. We define the special-steps sequence $\left(s_{1}, s_{2}, \ldots\right)=\left(m_{v}, m_{v}, \ldots\right)$ to be the subsequence of special cycle lengths, and the typical-steps sequence $\left(\ell_{1}, \ell_{2}, \ldots\right)$ to consist of the typical cycle lengths. Note that cycles of length greater than $n^{18}$ are neither special nor typical.

If $x_{t}=v$, let $\tau_{v}$ be the first time that $n^{14}$ typical steps have been completed. (We let $\tau_{v}$ be undefined or infinite for all other states $v$.) Let $Y$ be the number of special cycles until time $\tau_{x_{t}}$, and let $s_{\text {tot }}=Y m_{v}$ be the total duration of these special cycles.

For each $v \in S$, let $E_{v}$ be the event that $x_{t}=v$, and that all cycles until time $\tau_{v}$ are special or typical (in particular $\tau_{v}<\infty$ on this event).

Lemma 3.2. For any $v \in S$, we have $\mathbb{P}\left[E_{v} \mid x_{t}=v\right] \geq 0.99$.
Proof. Note that each cycle length $C_{i}$ is independent and has probability at least $1-n^{-16}$ of being at most $n^{18}$ by Lemma 3.1. We find that with high probability the first $n^{15}$ cycle lengths are special or typical:

$$
\begin{equation*}
\mathbb{P}\left[\max _{1 \leq i \leq n^{15}} C_{i} \leq n^{17}\right] \geq 1-n^{-1} . \tag{3.3}
\end{equation*}
$$

Conditioned on the event in (3.3), each $C_{i}$ is labeled typical independently with probability at least $1 / 2$. Hence the number of typical steps among the first $n^{15}$ cycles stochastically dominates a binomial $\operatorname{Bin}\left(n^{15}, 1 / 2\right)$ variable, and in particular is at least $n^{14}$ with probability at least $1-n^{-1}$. On this event, $\tau_{v}$ occurs during the first $n^{15}$ cycles and so $E_{v}$ holds with overall probability at least $\left(1-n^{-1}\right)\left(1-n^{-1}\right) \geq 0.99$.

Next, we use our concept of special and typical cycles to describe the distribution of $\tau_{v}$, which we recall is the first time that $n^{14}$ typical steps have been completed conditioned on $x_{t}=v$.

Definition 4. A probability distribution $\nu$ on $\mathbb{Z}$ is $m$-interval-periodic if it is supported on a discrete interval $\{I, I+1, \ldots, J\}$ and within this interval, $\nu(j)=\nu(j+m)$ depends only on $j$ $\bmod m$. The range of $\nu$ is $J-I$.

Similarly a sequence $\vec{\nu}=\left(\nu_{I}, \nu_{I+1}, \ldots, \nu_{J}\right)$ of probability distributions on $V$ is $m$-intervalperiodic if $\nu_{j}=\nu_{j+m}$ depends only on $j \bmod m$.

Definition 5. Let $\mu, \nu$ be probability distributions on a countable set $\mathcal{I}$ and $0<c<1$ be a constant. We say $\mu$ is a $c$-cover for $\nu$ if $\mu(i) \geq c \nu(i)$ for all $i \in \mathcal{I}$.

Let $\mu_{\text {tot }, v, \vec{\ell}}$ be the distribution of $\tau_{v}$ conditionally on the event $E_{v}$ and the sequence $\vec{\ell}$ of $n^{14}$ typical steps. In fact, it is easy to see from (3.2) that conditionally on $\left(v, E_{v}\right)$ the number $z_{i}$ of special steps between each adjacent pair $\left(\ell_{i-1}, \ell_{i}\right)$ of typical steps is exactly the geometric distribution Geom $\left(1-n^{-2}\right)$. Thus $\mathbb{P}\left[z_{i}=j\right]=\left(1-n^{-2}\right) n^{-2 j}$ for each $j \in \mathbb{Z}_{\geq 0}$, and the mean and variance are known to be

$$
\begin{aligned}
\mathbb{E}\left[z_{i}\right] & =\frac{1}{n^{2}-1} \\
\operatorname{Var}\left[z_{i}\right] & =\frac{n^{2}}{\left(n^{2}-1\right)^{2}}
\end{aligned}
$$

Lemma 3.3. For each $v \in S$ and $\vec{\ell}$, there exists an $m_{v}$-interval-periodic distribution $\nu_{v, \vec{\ell}}$ with range $n^{6}<w_{v}<n^{8}$ such that the distribution $\mu_{\text {tot }, v, \vec{\ell}}$ is a 0.06 -cover of $\nu_{v, \vec{\ell}}$.
Proof. Let $Z=\sum_{i=1}^{n^{14}} z_{i}$ be the total number of special cycles until time $\tau_{v}$. We first show that $Z$ obeys a central limit theorem as $n \rightarrow \infty$. Note that $w_{i}=z_{i}-\mathbb{E}\left[z_{i}\right]$ satisfies $\mathbb{E}\left[w_{i}\right]=0$, $\mathbb{E}\left[w_{i}^{2}\right]=\Theta\left(n^{-2}\right)$ and $\mathbb{E}\left[\left|w_{i}\right|^{3}\right]=\Theta\left(n^{-2}\right)$. Hence it follows from the Berry-Esseen theorem that with $\Phi(A)=\mathbb{P}^{y \sim \mathcal{N}(0,1)}[y \leq A]$ the cumulative distribution function of the standard Gaussian, and $\mu_{Z}, \sigma_{Z}$ the mean and standard deviation of $Z$ :

$$
\sup _{A \in \mathbb{R}}\left|\mathbb{P}\left[\frac{Z-\mu_{Z}}{\sigma_{Z}} \leq A\right]-\Phi(A)\right| \leq \frac{C n^{-2}}{n^{-3} \sqrt{n^{14}}} \leq O\left(1 / n^{2}\right)
$$

In particular, for $n$ large enough,

$$
\begin{aligned}
& \mathbb{P}\left[Z \in\left[\mu_{Z}-2 \sigma_{Z}, \mu_{Z}-\sigma_{Z}\right]\right] \geq \Phi(-1)-\Phi(-2)-0.001 \geq 0.13 \\
& \mathbb{P}\left[Z \in\left[\mu_{Z}+\sigma_{Z}, \mu_{Z}+2 \sigma_{Z}\right]\right] \geq \Phi(2)-\Phi(1)-0.001 \geq 0.13
\end{aligned}
$$

By averaging, there exist integers $a_{1} \in\left[\mu_{Z}-2 \sigma_{Z}, \mu_{Z}-\sigma_{Z}\right]$ and $a_{2} \in\left[\mu_{Z}+\sigma_{Z}, \mu_{Z}+2 \sigma_{Z}\right]$ with

$$
\mathbb{P}\left[Z=a_{1}\right], \mathbb{P}\left[Z=a_{2}\right] \geq \frac{1}{8 \sigma_{Z}}
$$

Next, we claim the probability mass function of $Z$ is log-concave; this implies that

$$
\begin{equation*}
\mathbb{P}[Z=a] \geq \frac{1}{8 \sigma_{Z}}, \quad \forall a \in\left[\mu_{Z}-\sigma_{Z}, \mu_{Z}+\sigma_{Z}\right] \tag{3.4}
\end{equation*}
$$

Since $\mu_{\text {tot }, v, \vec{\ell}}$ is exactly the law of $m_{v} Z$ and $\sigma_{Z}=n^{7} \sqrt{\operatorname{Var}\left[z_{1}\right]}=n^{6}(1 \pm O(1 / n))$, the statement (3.4) readily implies the desired result. It remains only to prove that $Z$ has log-concave probability mass function.

For this, first note that $Z$ is a negative binomial distribution and its probability mass function takes the exact form

$$
\mathbb{P}[Z=k]=\binom{k+n^{14}-1}{n^{14}-1} n^{-14 k}\left(1-n^{-2}\right)^{n^{14}}
$$

(This description is well-known and follows from an elementary stars and bars argument.) Hence it suffices to show that $k \mapsto\binom{k+n^{14}-1}{n^{14}-1}$ is log-concave. For this we note that $k \mapsto k+c$ is log-concave for each $c \geq 0$ on $k \in \mathbb{Z}_{\geq 0}$ and

$$
\binom{k+n^{14}-1}{n^{14}-1}=\frac{\left(k+n^{14}-1\right)\left(k+n^{14}-2\right) \cdots(k+1)}{\left(n^{14}-1\right)!}
$$

is proportional to a product of such sequences.
We remark that the conditioning on $\vec{\ell}$ played no role above as it just shifts $\mu_{t o t, v, \vec{\ell}}$. However, in the next subsection it will be important to apply Lemma 3.3 after conditioning on $\vec{\ell}$.

For each $v \in S$, typical-steps sequence $\vec{\ell}$, and time $s>t$, define the probability distribution

$$
\widetilde{\mu}_{s}=\widetilde{\mu}_{s}(v, \vec{\ell})
$$

on $V$ to be the conditional law of $x_{s}$ given $\left(E_{v}, \vec{\ell}\right)$.
Lemma 3.4. For each $v \in S$ and $\vec{\ell}$ and $T^{\prime}>n^{33}$, there exists an $m_{v}$-interval-periodic sequence $\vec{\omega}=\vec{\omega}_{v}=\left(\omega_{T^{\prime}}, \ldots, \omega_{T^{\prime}+n^{5}}\right)$ of distributions that is 0.03 -covered by the sequence $\widetilde{\mu}_{T^{\prime}} \ldots \widetilde{\mu}_{T^{\prime}+n^{5}}$.

Proof. For any vertex $u$ and time $r$, define $P^{r}(u)$ to be the distribution of the Markov chain state after $r$ steps starting from vertex $u$. Let

$$
\operatorname{supp}\left(\nu_{v, \vec{\ell}}\right)=\{I, I+1, \ldots, J\}=\left\{I\left(\nu_{v, \vec{\ell}}\right), I\left(\nu_{v, \vec{\ell}}\right)+1, \ldots, J\left(\nu_{v, \vec{\ell}}\right)\right\}
$$

be the support of $\nu_{v, \vec{\ell}}$. Moreover let $\bar{\nu}_{v, \bar{\ell}}(0), \ldots, \bar{\nu}_{v, \bar{\ell}}\left(m_{v}-1\right)$ be such that $\nu_{v, \bar{\ell}}(j)=\bar{\nu}_{v, \vec{\ell}}(j \bmod m)$ for $I \leq j \leq J$.

We explicitly compute $\widetilde{\mu}_{s}$ for $T^{\prime} \leq s \leq T^{\prime}+n^{5}$. With inequalities of positive measures indicating that the difference is a positive measure, we have

$$
\begin{aligned}
\widetilde{\mu}_{s}(v, \vec{\ell}) & \geq \sum_{d=1}^{s} \mathbb{P}\left[\tau_{v}=d \mid\left(x_{d}=v, \vec{\ell}\right)\right] \cdot P^{s-d}(v) \\
& \geq 0.06 \sum_{d=1}^{s} \nu_{v, \vec{\ell}}(d) \cdot P^{s-d}(v) \\
& =0.06 \sum_{u=I-T^{\prime}}^{J-T^{\prime}-n^{5}} \nu_{v, \vec{\ell}}(s-u) \cdot P^{u}(v) \\
& \geq 0.06\left(\frac{J-I-n^{5}-2 m}{m}\right) \cdot \sum_{i=1}^{m} \bar{\nu}_{v, \vec{\ell}}(s-i \bmod m) \sum_{\substack{I-T^{\prime} \leqq u \leq J-T^{\prime}-n^{5} \\
u \equiv i \\
\bmod m}} P^{u}(v) .
\end{aligned}
$$

Note that this lower bound is exactly $m$-periodic for $s$ in the stated range. We now show it has a significant total mass. Note that

$$
\begin{aligned}
& \left(\frac{J-I+n^{5}+2 m}{m}\right) \cdot \sum_{i=1}^{m} \bar{\nu}_{v, \bar{\ell}}(s-i \bmod m) \sum_{\substack{I-T^{\prime} \leq u \leq J-T^{\prime}-n^{5} \\
u \equiv i}} P^{u}(v) \\
& \geq \sum_{t=1}^{s} \nu_{v, \bar{\ell}}(t) \cdot P^{s-t}(v)
\end{aligned}
$$

has total mass at least 0.99 since clearly $\mathbb{P}\left[\tau_{v}<s \mid E_{v}\right] \geq 0.99$ (by Markov's inequality on $s_{\text {tot }}$ ).
Since $J-I \geq n^{6}$ we have

$$
\frac{J-I-n^{5}-2 m}{J-I+n^{5}+2 m} \geq 0.99
$$

so we conclude that the lower bound above has a total mass of at least 0.03 .

### 3.3 Analysis of Periodic Decomposition

We fix the values $m_{v}$ and distributions $\nu_{v, \vec{\ell}}$ from Lemma 3.3. Next, we construct a "global period length" $\beta$ as follows. For each prime $p$, let $e_{p}$ be the largest value such that

$$
\mathbb{P}\left[p^{e_{p}} \text { divides } m_{v} \mid v \in S\right] \geq 0.55
$$

and define $\beta=\prod_{p} p^{e_{p}}$.
Lemma 3.5. It holds that $\beta \leq n^{2}$.

Proof. It suffices to observe that

$$
\log (10 n) \geq \mathbb{E}\left[\log \left(m_{v}\right) \mid v \in S\right] \geq \log \left(\beta^{0.55}\right)
$$

The former holds since $m_{v} \leq 10 n$ while the latter holds by Markov's inequality. Hence $\beta \leq$ $(10 n)^{20 / 11} \leq n^{2}$.

Let $M_{v}=m_{v} / \operatorname{gcd}\left(m_{v}, \beta\right)$. By definition, for any prime $p$ we have

$$
\begin{equation*}
\mathbb{P}\left[p \text { divides } M_{x_{t}} \mid x_{t} \in S\right] \leq 0.55 \tag{3.5}
\end{equation*}
$$

Recalling Subsection 1.3, for each time $s$ let $F(s)=0$ if $\mathbb{P}\left[x_{s} \in U\right] \in[0,1 / 3]$ and $F(s)=1$ otherwise. Moreover define

$$
\bar{F}_{a}=\frac{1}{\left\lfloor n^{5} / \beta\right\rfloor} \sum_{i=1}^{\left\lfloor n^{5} / \beta\right\rfloor} F((a+i) \beta) .
$$

Lemma 3.6. There exists $a \in \mathbb{N}$ such that $\left|\bar{F}_{a}-1 / 2\right| \leq 0.01$.
Proof. This follows by the discrete-time intermediate value theorem. Indeed since $N \geq n^{5}$ we have $\bar{F}_{0}=1$, while $\bar{F}_{a}=0$ for $a \geq f(N)$. Clearly $\left|\bar{F}_{a+1}-\bar{F}_{a}\right| \leq 0.01$ for all $a$.

Throughout the rest of the proof, let $T=a \beta$ for $a$ as in Lemma 3.6. We note that $T>n^{33}$ and therefore satisfies the requirements for $T^{\prime}$ in Lemma 3.4. For each time $T \leq s \leq T+n^{5}$, let

$$
\bar{\omega}_{s}=\mathbb{E}^{x_{t}, \vec{\ell}}\left[\mathbb{P}^{v \sim \omega_{x_{t}, \ell, s}}[v \in U] \mid x_{t} \in S \text { and } E_{x_{t}}\right] .
$$

(Note that $\left\{x_{t} \in S\right.$ and $\left.E_{x_{t}}\right\}$ is just an event.)
Lemma 3.7. $\bar{\omega}_{s} \geq 0.99$ if $F(s)=1$ and $\bar{\omega}_{s} \leq 0.01$ if $F(s)=0$.
Proof. We focus on the latter statement since they are symmetric.
Given $x_{t}$, generate $w \sim P^{s-t}\left(x_{t}\right)$ to be the state of the Markov chain at time $s$ drawn from the original trajectory distribution. By Bayes' rule for expectations, since the event $\left\{x_{t} \in S\right.$ and $\left.E_{x_{t}}\right\}$ has probability at least $1 / 3$, conditioning on it increases the expectation of any non-negative random variable by a factor at most 3 . In particular:

$$
\begin{aligned}
\mathbb{E}^{x_{t}, \vec{\ell}}\left[\mathbb{P}[w \in U] \mid x_{t} \in S \text { and } E_{x_{t}}\right] & \leq \frac{\mathbb{E}^{x_{t}, \vec{\ell}}[\mathbb{P}[w \in U]]}{\mathbb{P}\left[x_{t} \in S \text { and } E_{x_{t}}\right]} \\
& =\frac{\mathbb{P}\left[x_{s} \in U\right]}{\mathbb{P}\left[x_{t} \in S \text { and } E_{x_{t}}\right]} \\
& \leq 3 \cdot 10^{-4} .
\end{aligned}
$$

Given $x_{t}$ and $\vec{\ell}$, we also let $\widetilde{w} \sim \omega_{x_{t}, \vec{\ell}, s}$ for $\omega_{x_{t}, \vec{\ell}, s}$ as in Lemma 3.4. Then the 0.03 -covering guarantee in Lemma 3.4 directly implies

$$
\begin{aligned}
& \mathbb{E}^{x_{t}, \vec{\ell}}\left[\mathbb{P}[w \in U] \mid x_{t} \in S \text { and } E_{x_{t}}\right] \\
& \geq 0.03 \cdot \mathbb{E}^{x_{t}, \vec{\ell}}\left[\mathbb{P}[\widetilde{w} \in U] \mid x_{t} \in S \text { and } E_{x_{t}}\right] \\
& =0.03 \cdot \bar{\omega}_{s} .
\end{aligned}
$$

Combining completes the proof.

Lemma 3.8. It holds that

$$
\left|\frac{1}{\left\lfloor n^{5} / \beta\right\rfloor} \sum_{i=1}^{\left\lfloor n^{5} / \beta\right\rfloor} \bar{\omega}_{(a+i) \beta}-\frac{1}{2}\right| \leq 0.02 .
$$

Proof. By Lemma 3.7, $\left|\bar{\omega}_{s}-F(s)\right| \leq 0.01$ for each $s$. Hence

$$
\left|\frac{1}{\left\lfloor n^{5} / \beta\right\rfloor} \sum_{i=1}^{\left\lfloor n^{5} / \beta\right\rfloor} \bar{\omega}_{(a+i) \beta}-\frac{1}{2}\right| \leq\left|\bar{F}_{a}-1 / 2\right|+0.01 \leq 0.02
$$

We will use the following lemma which is proved in Section 4.
Lemma 3.9. The following holds with $L=n^{3}$. Fix $n$ and let $S=\left(S_{1}, S_{2}, \ldots, S_{L}\right) \in[-1,1]^{L}$ be a random sequence which is $m$-periodic for some random $m=m(S) \leq 10 n$. Suppose that

$$
\begin{equation*}
-L / 10 \leq \mathbb{E} \sum_{d=1}^{L} S_{d} \leq L / 10 \tag{3.6}
\end{equation*}
$$

and that for each prime $p$,

$$
\mathbb{P}[p \text { divides } m] \leq 0.55
$$

Then there exists an index $1 \leq d \leq L$ such that

$$
\left|\mathbb{E}\left[S_{d}\right]\right| \leq 0.9
$$

Combining the above, we can finally prove Theorem 1.5.
Proof of Theorem 1.5: Given a putative $\left(\mathcal{M}_{n}, U\right)$, we apply Lemma 3.9 with

$$
S=S_{x_{d}, \vec{\ell}}=\mathbb{P}^{v \sim \omega_{x_{d}}, \ell, s}[v \in U] .
$$

where we map $[0,1] \rightarrow[-1,1]$ via $a \mapsto 2 a-1$. The first condition holds by Lemma 3.8 since $\mathbb{E}\left[S_{d}\right]=\bar{\omega}_{(a+d) \beta}$. The second condition holds by (3.5). Hence Lemma 3.9 implies that there exists $s$ such that $\bar{\omega}_{s} \in[0.02,0.98]$. This contradicts Lemma 3.7 above, giving a contradiction and completing the proof.

## 4 Proof of Lemma 3.9

Here we prove Lemma 3.9 using Fourier analysis. For each $m \geq 1$, let

$$
Z_{m}=\{0,1 / m, \ldots,(m-1) / m\}
$$

and set $Z_{m}^{*}=Z_{m} \backslash\{0\}$. Given a m-periodic sequence $S=\left(S_{1}, S_{2}, \ldots\right)$ let $\widehat{S}:[0,1) \rightarrow \mathbb{C}$ be its Fourier transform

$$
\widehat{S}(\alpha)= \begin{cases}\frac{1}{m} \sum_{j=1}^{m} e^{2 \pi i j \alpha} S_{j}, & \alpha \in Z_{m}  \tag{4.1}\\ 0, & \alpha \notin Z_{m}\end{cases}
$$

The definition (4.1) agrees with the Fourier transform of $S$ viewed as a periodic function on $\mathbb{Z}$ and hence is independent of the period $m$ - we could view $S$ as being $k m$ periodic for any $k \geq 1$ and $\widehat{S}$ would remain consistent. This allows us to use Fourier analysis on pairs $S, S^{\prime}$ of sequences with different period lengths $m, m^{\prime}$. Indeed with $M=m m^{\prime}$ we can define:

$$
\begin{align*}
\left\langle S, S^{\prime}\right\rangle & =\frac{1}{M} \sum_{i=1}^{M} S_{i} S_{i}^{\prime}  \tag{4.2}\\
\|S\|_{L^{2}}^{2} & =\frac{1}{M} \sum_{i=1}^{M} S_{i}^{2}  \tag{4.3}\\
\left\langle\widehat{S}, \widehat{S}^{\prime}\right\rangle & =\sum_{\alpha \in Z_{M}} \widehat{S}(\alpha) \widehat{S}^{\prime}(\alpha)  \tag{4.4}\\
\|\widehat{S}\|_{\ell^{2}}^{2} & =\sum_{\alpha \in Z_{m}}|\widehat{S}(\alpha)|^{2} \tag{4.5}
\end{align*}
$$

Parseval identity's modulo $M$ implies that $\left\langle S, S^{\prime}\right\rangle=\left\langle\widehat{S}, \widehat{S^{\prime}}\right\rangle$ and $\|S\|_{L^{2}}=\|\widehat{S}\|_{L^{2}}$.
The next proposition approximates averages on $1,2, \ldots, L$ by Fourier averages. Here and below we let $\mu$ be the law of the random sequence $S$, and recall that $S \sim \mu$ is $m(S)$ periodic for $m \leq 10 n$ almost surely.
Proposition 4.1. Then

$$
\begin{array}{r}
\left|\frac{1}{L} \sum_{d=1}^{L} \mathbb{P}^{S \sim \mu}\left[S_{d}=1\right]^{2}-\mathbb{E}^{S, S^{\prime} \sim \mu}\left\langle\widehat{S}, \widehat{S^{\prime}}\right\rangle\right| \leq O\left(n^{2} / L\right), \\
\left|\left(\frac{1}{L} \mathbb{E}^{S \sim \mu} \sum_{d=1}^{L} S_{d}\right)^{2}-\mathbb{E}^{S, S^{\prime} \sim \mu}\left[\widehat{S}(0) \widehat{S^{\prime}}(0)\right]\right| \leq O\left(n^{2} / L\right) \tag{4.7}
\end{array}
$$

Proof. Notice that by definition,

$$
\frac{1}{L} \sum_{d=1}^{L} \mathbb{P}^{S \sim \mu}\left[S_{d}=1\right]^{2}=\frac{1}{L} \mathbb{E}^{S, S^{\prime} \sim \mu}\left\langle\left(S_{1} \ldots, S_{L}\right),\left(S_{1}^{\prime}, \ldots, S_{T}^{\prime}\right)\right\rangle
$$

where $\left\langle\left(S_{1} \ldots, S_{L}\right),\left(S_{1}^{\prime}, \ldots, S_{T}^{\prime}\right)\right\rangle=\frac{1}{L} \sum_{d=1}^{L} S_{d} S_{d}^{\prime}$. Since $M=m m^{\prime} \leq 100 n^{2}$,

$$
\left|\frac{1}{L} \mathbb{E}^{S, S^{\prime} \sim \mu}\left\langle\left(S_{1} \ldots, S_{L}\right),\left(S_{1}^{\prime}, \ldots, S_{T}^{\prime}\right)\right\rangle-\left\langle S, S^{\prime}\right\rangle\right| \leq O\left(n^{2} / L\right)
$$

Combining and using $\left\langle S, S^{\prime}\right\rangle=\left\langle\widehat{S}, \widehat{S}^{\prime}\right\rangle$ by Parseval completes the proof of (4.6). The proof of (4.7) is similar since

$$
\left|\frac{1}{L} \mathbb{E}^{S \sim \mu} \sum_{d=1}^{L} S_{d}-\mathbb{E}^{S \sim \mu}[\hat{S}(0)]\right|=\left|\frac{1}{L} \mathbb{E}^{S \sim \mu} \sum_{d=1}^{L} S_{d}-\frac{1}{m(S)} \mathbb{E}^{S \sim \mu} \sum_{i=1}^{m(S)} S_{i}\right| \leq O(n / L) .
$$

Indeed $\left(\mathbb{E}^{S \sim \mu}[\widehat{S}(0)]\right)^{2}=\mathbb{E}^{S, S^{\prime} \sim \mu}\left[\widehat{S}(0) \widehat{S}^{\prime}(0)\right]$, and the function $x \mapsto x^{2}$ is Lipschitz on $x \in[-1,1]$.

The main idea to prove Lemma 3.9 is that $S, S^{\prime}$ are unlikely to have similar periods. In carrying out this argument we have to handle the bias $\widehat{S}(0)$ separately from the $\alpha \in(0,2 \pi)$ contributions. The next proposition gives an estimate to handle the latter contributions; note it is important that the last term below mixes $S, S^{\prime}$.

Proposition 4.2. For $m$ and $m^{\prime}$-periodic functions $S, S^{\prime}: \mathbb{Z} \rightarrow[-1,1]$,

$$
\left\langle\widehat{S}, \widehat{S}^{\prime}\right\rangle \leq \widehat{S}(0) \widehat{S}^{\prime}(0)+\left\|\left.\widehat{S}\right|_{Z_{m^{\prime}}^{*}}\right\|_{\ell^{2}}
$$

where $\left.\widehat{S}\right|_{Z_{m^{\prime}}^{*}}$ denotes the restriction of $\widehat{S}$ to $Z_{m^{\prime}}^{*}$.
Proof. The summand $\widehat{S}(\alpha) \widehat{S}^{\prime}(\alpha)$ in (4.4) is non-zero only when $\alpha \in Z_{g}$ for $g=\operatorname{gcd}\left(m, m^{\prime}\right)$. This yields an upper bound of

$$
\widehat{S}(0) \widehat{S}^{\prime}(0)+\left\|\left.\widehat{S}\right|_{Z_{g}^{*}}\right\|_{\ell^{2}} \cdot\left\|\left.\widehat{S}^{\prime}\right|_{Z_{g}^{*}}\right\|_{\ell^{2}}
$$

This immediately implies the claim since

$$
\left\|\left.\widehat{S}\right|_{Z_{g}^{*}}\right\|_{\ell^{2}} \leq\left\|\left.\widehat{S}\right|_{Z_{m^{\prime}}^{*}}\right\|_{\ell^{2}}
$$

(in fact, equality holds) while

$$
\left\|\left.\widehat{S}^{\prime}\right|_{Z_{g}^{*}}\right\|_{\ell^{2}} \leq\left\|\left.\widehat{S}^{\prime}\right|_{Z_{M}}\right\|_{\ell^{2}}=\left\|S^{\prime}\right\|_{L^{2}} \leq 1
$$

Lemma 4.3. Suppose $S, S^{\prime} \stackrel{\text { i.i.d. }}{\sim} \mu$ and $\mathbb{P}^{S \sim \mu}[p$ divides $m(S)] \leq \varepsilon$ for any prime $p$. Then

$$
\mathbb{E}^{S \sim \mu}\left[\left\|\left.\widehat{S}\right|_{Z_{m^{\prime}}^{*}}\right\|_{\ell^{2}}\right] \leq \sqrt{\varepsilon}
$$

Proof. Directly by the assumption, for any fixed $\alpha \neq 0$, we have $\mathbb{P}\left[\alpha \in Z_{m^{\prime}}^{*}\right] \leq \varepsilon$. Thus linearity of expectation implies

$$
\mathbb{E}^{S \sim \mu}\left[\left\|\left.\widehat{S}\right|_{Z_{m^{\prime}}^{*}}\right\|_{\ell^{2}}^{2}\right] \leq \varepsilon
$$

Applying Jensen's inequality completes the proof.
Proof of Lemma 3.9: By applying Markov's inequality to (4.6) and noting $n^{2} / L \leq 1 / n$ it suffices to show that

$$
\begin{equation*}
\mathbb{E}^{S, S^{\prime} \sim \mu}\left\langle\widehat{S}, \widehat{S}^{\prime}\right\rangle \stackrel{?}{\leq} 0.8<0.9^{2} \tag{4.8}
\end{equation*}
$$

We first deal with the bias term of the left-hand side. Combining (4.7) and (3.6) yields

$$
\mathbb{E}^{S, S^{\prime} \sim \mu}\left[\widehat{S}(0) \widehat{S}^{\prime}(0)\right] \leq 0.02
$$

For the remaining terms $\alpha \neq 0$, combining Proposition 4.2 with Lemma 4.3 (which holds with $\varepsilon=0.55$ by assumption) yields

$$
\mathbb{E}^{S, S^{\prime} \sim \mu} \sum_{\alpha \in Z_{M}^{*}} \widehat{S}(\alpha) \widehat{S}^{\prime}(\alpha) \leq \sqrt{0.55} \leq 0.75
$$

Summing the contributions establishes (4.8) and thus finishes the proof.

## Acknowledgements

Thanks to Yang Liu for helpful discussions, and to Rachel Zhang and Naren Manoj for reviewing the manuscript. Ofer Grossman was supported by the NSF GRFP as a graduate student at MIT, and was visiting the Simons Institute for the Theory of Computing while part of this work was completed. Meghal Gupta was supported by a Chancellor's Fellowship as a graduate student at UC Berkeley.

## References

[AAHNY22] Ishaq Aden-Ali, Yanjun Han, Jelani Nelson, and Huacheng Yu. On the amortized complexity of approximate counting. arXiv preprint arXiv:2211.03917, 2022.
[AJKS02] Miklós Ajtai, TS Jayram, Ravi Kumar, and D Sivakumar. Approximate counting of inversions in a data stream. In Proceedings of the thiry-fourth annual ACM symposium on Theory of Computing, pages 370-379, 2002.
[BDW18] Arnab Bhattacharyya, Palash Dey, and David P Woodruff. An optimal algorithm for $\ell^{1}$-heavy hitters in insertion streams and related problems. ACM Transactions on Algorithms (TALG), 15(1):1-27, 2018.
[BKKS23] Vladimir Braverman, Robert Krauthgamer, Aditya Krishnan, and Shay Sapir. Lower Bounds for Pseudo-Deterministic Counting in a Stream. In 50th International Colloquium on Automata, Languages, and Programming (ICALP 2023), volume 261 of Leibniz International Proceedings in Informatics (LIPIcs), pages 30:1-30:14, Dagstuhl, Germany, 2023. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.
[Csü10] Miklós Csürös. Approximate counting with a floating-point counter. In COCOON, volume 6196, pages 358-367. Springer, 2010.
[Cve07] Andrej Cvetkovski. An algorithm for approximate counting using limited memory resources. ACM SIGMETRICS Performance Evaluation Review, 35(1):181-190, 2007.
[DPV18] Peter Dixon, A Pavan, and NV Vinodchandran. On Pseudodeterministic Approximation Algorithms. In 43rd International Symposium on Mathematical Foundations of Computer Science (MFCS 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.
[DPV21] Peter Dixon, A Pavan, and NV Vinodchandran. Complete problems for multipseudodeterministic computations. Innovations in Theoretical Computer Science, 2021.
[DPWV22] Peter Dixon, A Pavan, Jason Vander Woude, and NV Vinodchandran. Pseudodeterminism: Promises and Lowerbounds. In Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, pages 1552-1565, 2022.
[Fla85] Philippe Flajolet. Approximate counting: a detailed analysis. BIT Numerical Mathematics, 25(1):113-134, 1985.
[GG11] Eran Gat and Shafi Goldwasser. Probabilistic search algorithms with unique answers and their cryptographic applications. In Electronic Colloquium on Computational Complexity (ECCC), volume 18, page 136, 2011.
[GG17] Shafi Goldwasser and Ofer Grossman. Bipartite Perfect Matching in Pseudo-Deterministic NC. In 44th International Colloquium on Automata, Languages, and Programming (ICALP 2017). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.
[GG21] Sumanta Ghosh and Rohit Gurjar. Matroid intersection: A pseudo-deterministic parallel reduction from search to weighted-decision. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021). Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.
[GGH18] Shafi Goldwasser, Ofer Grossman, and Dhiraj Holden. Pseudo-deterministic proofs. In 9th Innovations in Theoretical Computer Science Conference (ITCS 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.
[GGH19] Michel Goemans, Shafi Goldwasser, and Dhiraj Holden. Doubly-efficient pseudodeterministic proofs. arXiv preprint arXiv:1910.00994, 2019.
[GGMW20] Shafi Goldwasser, Ofer Grossman, Sidhanth Mohanty, and David P Woodruff. Pseudodeterministic streaming. In 11th Innovations in Theoretical Computer Science Conference (ITCS 2020). Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.
[GGR13] Oded Goldreich, Shafi Goldwasser, and Dana Ron. On the possibilities and limitations of pseudodeterministic algorithms. In Proceedings of the 4th conference on Innovations in Theoretical Computer Science, pages 127-138. ACM, 2013.
[GIPS21] Shafi Goldwasser, Russell Impagliazzo, Toniann Pitassi, and Rahul Santhanam. On the pseudo-deterministic query complexity of NP search problems. In Proceedings of the 36th Computational Complexity Conference, page 1, 2021.
[GL19] Ofer Grossman and Yang P Liu. Reproducibility and pseudo-determinism in log-space. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 606-620. SIAM, 2019.
[GM20] Ofer Grossman and Dana Moshkovitz. Amplification and Derandomization without Slowdown. SIAM Journal on Computing, 49(5):959-998, 2020.
[Gol19] Oded Goldreich. Multi-pseudodeterministic algorithms. In Electronic Colloquium on Computational Complexity (ECCC), 2019.
[Gro15] Ofer Grossman. Finding primitive roots pseudo-deterministically. In Electronic Colloquium on Computational Complexity (ECCC), volume 22, page 207, 2015.
[GS09] André Gronemeier and Martin Sauerhoff. Applying approximate counting for computing the frequency moments of long data streams. Theory of Computing Systems, 44:332-348, 2009.
[HILL99] Johan Håstad, Russell Impagliazzo, Leonid A Levin, and Michael Luby. A pseudorandom generator from any one-way function. SIAM Journal on Computing, 28(4):1364-1396, 1999.
[Hol17] Dhiraj Holden. A note on unconditional subexponential-time pseudo-deterministic algorithms for BPP search problems. arXiv preprint arXiv:1707.05808, 2017.
[IZ89] Russell Impagliazzo and David Zuckerman. How to recycle random bits. In FOCS, volume 30, pages 248-253, 1989.
[JW19] Rajesh Jayaram and David P Woodruff. Towards optimal moment estimation in streaming and distributed models. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2019.
[KNW10] Daniel M Kane, Jelani Nelson, and David P Woodruff. On the exact space complexity of sketching and streaming small norms. In Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms, pages 1161-1178. SIAM, 2010.
[LOS21] Zhenjian Lu, Igor C Oliveira, and Rahul Santhanam. Pseudodeterministic algorithms and the structure of probabilistic time. In Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, pages 303-316, 2021.
[Mor78] Robert Morris. Counting large numbers of events in small registers. Communications of the ACM, 21(10):840-842, 1978.
[NY22] Jelani Nelson and Huacheng Yu. Optimal Bounds for Approximate Counting. In Proceedings of the 41st ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, 2022.
[OS17] Igor C Oliveira and Rahul Santhanam. Pseudodeterministic constructions in subexponential time. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 665-677, 2017.
[OS18] Igor C Oliveira and Rahul Santhanam. Pseudo-derandomizing learning and approximation. In LIPIcs-Leibniz International Proceedings in Informatics, volume 116. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.
[Tre01] Luca Trevisan. Extractors and pseudorandom generators. Journal of the ACM, 48(4):860879, 2001.
[XKNS21] Jingyi Xu, Sehoon Kim, Borivoje Nikolic, and Yakun Sophia Shao. Memory-efficient hardware performance counters with approximate-counting algorithms. In 2021 IEEE International Symposium on Performance Analysis of Systems and Software (ISPASS), pages 226228. IEEE, 2021.


[^0]:    ${ }^{1}$ Technically, we will assign length $m_{v}$ cycles to be non-special with some positive probability. This is necessary if the cycle length always equals $m_{v}$, for example.
    ${ }^{2}$ We formalize this approximation using a notion of comparison we call $c$-covering, see Definition 5.

