New Lower Bounds for Adaptive Tolerant Junta Testing

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Abstract

We prove a $k^{-\Omega(\log(\varepsilon_2-\varepsilon_1))}$ lower bound for adaptively testing whether a Boolean function is ε_1 -close to or ε_2 -far from k-juntas. Our results provide the first superpolynomial separation between tolerant and non-tolerant testing for a natural property of boolean functions under the adaptive setting. Furthermore, our techniques generalize to show that adaptively testing whether a function is ε_1 -close to a k-junta or ε_2 -far from (k + o(k))-juntas cannot be done with poly $(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries. This is in contrast to an algorithm by Iyer, Tal and Whitmeyer [CCC 2021] which uses poly $(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries to test whether a function is ε_1 -close to a k-junta or ε_2 -far from $O(k/(\varepsilon_2 - \varepsilon_1)^2)$ -juntas.

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1 Introduction

Junta Testing. We say a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is a k-junta if it only depends on k of its input variables. As one of the most fundamental classes of Boolean functions, juntas have received significant attention during the past few decades in extensive areas such as learning theory [Blu94, BL97, MOS03, Val15] (where juntas are used to model learning concepts in the presence of many irrelevant features) and analysis of Boolean functions [O'D14] (where juntas are used as good approximations of other classes of Boolean functions). Juntas have also been studied intensively in property testing. Under the standard model, the goal of the junta testing problem is to understand how many queries are needed by a randomized algorithm to decide whether a function $f : \{0,1\}^n \to \{0,1\}$ is a k-junta or ε -far from k-juntas, where we say f is ε -far from k-juntas if f and g disagree on at least $\varepsilon 2^n$ entries for every k-junta $g : \{0,1\}^n \to \{0,1\}$. Testing juntas, in particular, was initially of interest because of its connections to testing long-codes (which is related to PCPs) [BGS98, PRS02], but it is also a very natural question and highly motivated by feature selection in machine learning.

This problem has been settled now under both the adaptive and non-adaptive settings.¹ After a sequence of work [PRS02, CG04, FKR⁺04, Bla08, Bla09, BGSMdW13, STW15, CST⁺18, Sağ18], starting with [PRS02], it was shown that $\tilde{\Theta}(k/\varepsilon)$ queries are necessary [CG04, Sağ18] and sufficient [Bla09] for adaptive junta testing; $\tilde{\Theta}(k^{3/2}/\varepsilon)$ queries are necessary [CST⁺18] and sufficient [Bla08] for non-adaptive algorithms. Note that all the bounds are independent of n, the number of variables in the function, which can be much larger than k.

Tolerant Testing. A drawback of the standard testing model is that the algorithm is allowed to reject functions that are very close to having the property. Indeed all aforementioned junta testing algorithms would reject a function immediately once it has been found to have more than k relevant variables, no matter how big their influences are in the function. This makes algorithms under the standard model less applicable in more realistic scenarios that arise from experimental design and data analysis, where even a correct instance with the desired property may be subject to a small amount of noise.

To address this question, Parnas, Ron, and Rubinfeld [PRR06] introduced tolerant testing, a natural generalization of the standard testing model. For the tolerant testing of juntas, the goal of a $(k, \varepsilon_1, \varepsilon_2)$ -tester, for some $0 \le \varepsilon_1 < \varepsilon_2 < 1$, is to tell whether a given function $f : \{0, 1\}^n \to \{0, 1\}$ is ε_1 -close to a k-junta or is ε_2 -far from k-juntas, where we say f is ε_1 -close to a k-junta g if f and g disagree on no more than $\varepsilon_1 2^n$ entries. So the standard model corresponds to the case of $\varepsilon_1 = 0$. In general, tolerant testing of a property can be much more challenging than its standard testing, as intuitively seeing a single violation of the property is no longer sufficient to reject the input and the tester needs to estimate the distance of the input to the property.²

While it is believed that tolerant testing is much harder and current best upper bounds for well studied properties of Boolean functions such as juntas (which we review next) and monotonicity all remain exponential in their counterparts under the non-tolerant model [ITW21, FR10], there is no known superpolynomial separation between tolerant and non-tolerant testings for natural properties of Boolean functions.³ Our work proves the first such superpolynomial separation, using junta testing. We also believe our approach may be fruitful in proving tolerant testing lower bounds for other properties such as monotonicity.

¹A testing algorithm is adaptive if its queries can depend on results from previous queries.

²It was shown in [PRR06] that distance approximation and (fully) tolerant testing are equivalent up to a $\log(1/\varepsilon)$ factor in the query complexity.

 $^{^{3}}$ Using PCPs, [FF05] showed the existence of a class of Boolean functions that has a strong separation between the two models.

Previous Results on Tolerant Junta Testing. Indeed testing juntas under the tolerant model turns out to be challenging. Even after much effort, there remains an exponential gap in our understanding of its query complexity. After a sequence of work [CFGM12, BCE⁺19, DMN19, ITW21], the best algorithm up to date [ITW21] (which is highly adaptive) still needs $\exp(\tilde{O}(\sqrt{k/(\varepsilon_2 - \varepsilon_1)}))$ queries. And there has been no progress on the lower bound side: the current best lower bound for adaptive testers remains to be the $\tilde{\Omega}(k/\varepsilon)$ lower bound from the standard testing model.

There has been more success on lower bounds when the tester is non-adaptive. [LW19] showed that any non-adaptive algorithm requires $\tilde{\Omega}(k^2)$ queries, for some constants $0 < \varepsilon_1 < \varepsilon_2$. Later, for $\varepsilon_1 = O(1/k^{1-\eta})$ and $\varepsilon_2 = O(1/\sqrt{k})$, the bound was improved to $2^{k^{\eta}}$ for any $0 < \eta < 1/2$ [PRW22].

Relaxed Tolerant Junta Testing. Given the state of the art, the following relaxed, easier model has been considered in the literature [FKR⁺04, BBM11, RT13, DMN19, ITW21], with the hope of developing algorithms that are significantly more efficient than the best tolerant algorithm to date: For some $k \leq k'$ and $\varepsilon_1 < \varepsilon_2$, the goal of a $(k, k', \varepsilon_1, \varepsilon_2)$ -tester is to decide whether a given function is ε_1 -close to a k-junta or is ε_2 -far from k'-juntas.

The relaxed model was first posed by Fischer et al. in [FKR⁺04] under the non-tolerant setting (with $\varepsilon_1 = 0$), who asked whether a $(k, 2k, 0, \varepsilon)$ -tester still requires $\Omega(k)$ queries. Blais et al. [BBM11] proved that any $(k, k+O(\sqrt{k}), 0, \varepsilon)$ -tester must make $\Omega(k)$ queries for $\varepsilon = \Theta(1)$. This was improved by Ron and Tsur [RT13] to an $\tilde{\Omega}(k)$ lower bound for $(k, 2k, 0, \varepsilon)$ -testers for some constant ε .

As before, under the relaxed, tolerant setting $(\varepsilon_1 > 0)$, the best known lower bound remains to be the $\tilde{\Omega}(k)$ lower bound of [RT13] inherited from the non-tolerant setting. In contrast, much more efficient algorithms are known when k' is sufficiently larger than k. First Blais et al. [BCE⁺19] gave a $(k, 4k, \varepsilon/16, \varepsilon)$ -tester that makes $\mathsf{poly}(k, \varepsilon^{-1})$ queries. De et al. [DMN19] removed the restriction of $\varepsilon_2/\varepsilon_1 \ge 16$ and proved that $\mathsf{poly}(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries suffice for a $(k, O(k^2/(\varepsilon_2 - \varepsilon_1)^2), \varepsilon_1, \varepsilon_2)$ tester. More recently, Iyer et al. in [ITW21] obtained a $(k, O(k/(\varepsilon_2 - \varepsilon_1)^2), \varepsilon_1, \varepsilon_2)$ -tester that makes $\mathsf{poly}(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries.

Our Contribution. Obtaining stronger adaptive lower bounds for tolerant junta testing has been an important open problem in the property testing of Boolean functions (see e.g., the open problem posed in [ITW21]). In this paper, we make progress on this question by giving a superpolynomial separation between tolerant and non-tolerant adaptive junta testing:

Theorem 1. Let $\varepsilon_1, \varepsilon_2 : 0.01 \le \varepsilon_1 < \varepsilon_2 \le 0.49$ be two parameters such that $\varepsilon_2 - \varepsilon_1 \ge 2^{-O(k:99)}$.⁴ Then any $(k, \varepsilon_1, \varepsilon_2)$ -tolerant junta tester requires $k^{-\Omega(\log(\varepsilon_2 - \varepsilon_1))}$ queries.

Theorem 1 rules out the possibility of any tolerant junta tester that makes $poly(k) \cdot F(\varepsilon_1, \varepsilon_2)$ many queries, where F is an arbitrary function. In particular, this means that there is no tolerant junta tester that makes poly(k) queries for all constants $\varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1 < \varepsilon_2 < 1$. Moreover, when $\varepsilon_2 - \varepsilon_1$ is polynomially small (e.g., $\varepsilon_1 = 1/3$ and $\varepsilon_2 = 1/3 + 1/k^a$ for any constant a > 0), Theorem 1 gives a lower bound of $k^{\Omega(\log k)}$ for adaptive tolerant junta testing.

We remark that our proof naturally extends the linear lower bound of Chockler and Gutfreund [CG04]. Namely, we prove Theorem 1 by showing that determining whether a function on $(k + \ell)$ variables is ε_2 -far or ε_1 -close to a k-junta requires $k^{\Omega(\ell)}$ queries. Our main technical insight is that utilizing k-wise independence in the construction of hard instances essentially allows us to assume that the tester is non-adaptive in the lower bound proof.

Next, by making some slight modifications to our construction, we can extend our techniques to the relaxed tolerant junta testing setting. Namely, we prove that

 $^{^{4}}$ Both constants 0.01 and 0.49 are arbitrary; any constants that are positive and strictly less than 1/2 would work.

Theorem 2. Let $\varepsilon_1, \varepsilon_2, k$ and γ be parameters such that $0.01 \leq \varepsilon_1 < \varepsilon_2 \leq 0.49, \varepsilon_2 - \varepsilon_1 \geq 2^{-k^{0.1}}$ and $\gamma \geq 1/k$ but is sufficiently small. Then any $(k, (1+\gamma)k, \varepsilon_1, \varepsilon_2)$ -tester has query complexity

$$\left(\frac{1}{\gamma}\right)^{\Omega(-\log(\varepsilon_2 - \varepsilon_1))}$$

Setting $\varepsilon_2 - \varepsilon_1 = 1/k$, we get a superpolynomial lower bound (in k) whenever $\gamma = o(1)$. Hence Theorem 2 rules out the possibility of any $(k, k + o(k), \varepsilon_1, \varepsilon_2)$ -tester that makes $\mathsf{poly}(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries, which complements the need of a sufficiently large gap between k' and k to obtain testers with $\mathsf{poly}(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries in [BCE⁺19, DMN19, ITW21].

Notation. We start with some simple notation. Given a set $S \subseteq [n]$ and a binary string $x \in \{0,1\}^n$, we let $x|_S$ be the restriction of x to the coordinates in S. Similarly, given a string $x \in \{0,1\}^S$ and $y \in \{0,1\}^T$ for disjoint sets $S, T \subseteq [n]$ we let $x \sqcup y$ denote the string $z \in \{0,1\}^{S \cup T}$ with $z|_S = x$ and $z|_T = y$. For a set $S \subseteq [n]$ and $x \in \{0,1\}^n$, we take $x^{\oplus S}$ to denote the string $y \in \{0,1\}^n$ with $y|_{[n]\setminus S} = x|_{[n]\setminus S}$ and $y_s \neq x_s$ for all $s \in S$. Finally, we let $x_{-i} = x_1x_2...x_{i-1}x_{i+1}...x_n$.

 $B_n(r)$ will denote the hamming ball of radius r centered at 0^n and B(x, r) will be the hamming ball around a point $x \in \{0, 1\}^n$ of radius r.

We let \mathbb{I}_E denote the indicator variable for E. Additionally, MAJ denotes the majority function, breaking ties arbitrarily.

2 Lower Bounds for Tolerant Junta Testing

We will actually prove a stronger lower bound which allows for non-constant ε_1 and ε_2 .

Theorem 3. For any $0 < \varepsilon_1 < \varepsilon_2 \le 0.49$ and integer k with $\varepsilon_2 - \varepsilon_1 \ge 2^{-O(k^{.99})}$ and

$$(1 - \varepsilon_1/\varepsilon_2)^{-1} \ge \log^{O(1)}(1/\varepsilon_2),$$

any $(k, \varepsilon_1, \varepsilon_2)$ -tolerant junta tester must make at least $k^{-\Omega(\log(1-\varepsilon_1/\varepsilon_2))}$ queries.

We'll assume that $1 - \varepsilon_1/\varepsilon_2$ is sufficiently small. Note that we can always take the constant in the Ω notation to be sufficiently small, so that when $1 - \varepsilon_1/\varepsilon_2$ is large the theorem only gives an $\Omega(k)$ lower bound, which follows from [CG04].

The key insight for our lower bound is the following observation:

Lemma 1. Let n be an integer divisible by 4, \mathcal{D} be the uniform distribution over $\{0,1\}^n$, and \mathcal{D}' be the uniform distribution over $\{x \in \{0,1\}^n : \bigoplus_{i=1}^n x_i = 1\}$. Then

$$\mathbb{E}_{\mathbf{x}\sim\mathcal{D}'}\left[\operatorname{dist}(\mathbf{x},\{0^n,1^n\})\right] \leq \mathbb{E}_{\mathbf{x}\sim\mathcal{D}}\left[\operatorname{dist}(\mathbf{x},\{0^n,1^n\})\right] - \frac{1}{n},$$

where 0^n and 1^n are the all zeros and all ones strings respectively and dist is the hamming distance.

We remark that the $\frac{1}{n}$ term in the lemma is not tight, but will suffice for our purposes. The proof uses a coupling argument; we defer it to the end of the section.

Our construction will be parametrized by integers ℓ and r as well as a value $p \in [0, \frac{1}{2})$, which will be specified at the end (in the proof of Theorem 3) by k, ε_1 and ε_2 and in particular, we will make sure that r < k. It may help the reader to set r = 0 and p = 0, which will lead to a lower bound construction that works for some $\varepsilon_2 = \frac{1}{2} - o_\ell(1)$ and $\varepsilon_1 = \varepsilon_2 - 2^{-O(\ell)}$ (in which case \mathcal{D}_{NO} is simply the uniform distribution over all functions). Setting r and p appropriately (as we do later in the proof of Theorem 3) can help us shift ε_1 and ε_2 to where we want. We will proceed by Yao's principle and give two distributions \mathcal{D}_{YES} and \mathcal{D}_{NO} , both over boolean functions from $\{0,1\}^n$ to $\{0,1\}$ with $n := k + \ell$. We show that they are ε_1 -close and ε_2 -far from k-juntas, respectively, and show that it is difficult for any deterministic algorithm to distinguish them with few queries.

 \mathcal{D}_{NO} : A boolean function $\mathbf{f}: \{0,1\}^n \to \{0,1\}$ is drawn as follows:

- (a) For each $x \in \{0,1\}^n$ with $x|_{[r]} \neq 0^r$, independently draw $\mathbf{f}(x)$ from Bernoulli(p).
- (b) For each $x \in \{0,1\}^n$ with $x|_{[r]} = 0^r$, draw $\mathbf{f}(x)$ uniformly and independently at random.

 \mathcal{D}_{YES} : A boolean function $\mathbf{f}: \{0,1\}^n \to \{0,1\}$ is drawn as follows:

- (a) Randomly choose a set $\mathbf{J} \subseteq \{r+1, ..., n\}$ of size ℓ .
- (b) For each $x \in \{0,1\}^n$ with $x|_{[r]} \neq 0^r$, independently draw $\mathbf{f}(x)$ from Bernoulli(p).
- (c) For each $x \in \{0,1\}^n$ with $x|_{[r]} = 0^r$ and $x|_{\mathbf{J}} \neq 1^{\ell}$, draw $\mathbf{f}(x)$ uniformly and independently at random.
- (d) For each $x \in \{0,1\}^n$ with $x|_{[r]} = 0^r$ and $x|_{\mathbf{J}} = 1^{\ell}$, set $\mathbf{f}(x)$ to be such that

$$\bigoplus_{y \in \{0,1\}^{\mathbf{J}}} \mathbf{f}(x|_{[n] \setminus \mathbf{J}} \sqcup y) = 1.$$

The key property of the two distributions is that to distinguish \mathcal{D}_{YES} and \mathcal{D}_{NO} we must at least query a pair of points $x, x^{\oplus \mathbf{J}}$ for some x. (In fact, we must query, for some $x, x^{\oplus S}$ for all $S \subseteq \mathbf{J}$ to get any evidence, but this is more than we need for the lower bound.)

Lemma 2. Consider a set of points $x^{(1)}, ..., x^{(m)} \in \{0, 1\}^n$ and $y_1, ..., y_m \in \{0, 1\}$. Then

$$\Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}\left[\forall i \ \mathbf{f}(x^{(i)}) = y_i \mid \forall i, j \quad x^{(i)} \neq (x^{(j)})^{\oplus \mathbf{J}}\right] = \Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}\left[\forall i \ \mathbf{f}(x^{(i)}) = y_i\right]$$

Proof. Consider a $J \subseteq \{r+1, ..., n\}$ such that $x^{(i)} \neq (x^{(j)})^{\oplus J}$ for all $i, j \in [m]$. We will show that

$$\Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}\left[\forall i \ \mathbf{f}(x^{(i)}) = y_i \wedge \mathbf{J} = J\right] = \Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}\left[\forall i \ \mathbf{f}(x^{(i)}) = y_i\right] \cdot \Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}\left[\mathbf{J} = J\right].$$

Indeed, without loss of generality let $x^{(1)}, ..., x^{(a)}$ be such that $x^{(i)}|_{[r]} \neq 0^r$ and $x^{(a+1)}...x^{(m)}$ be such that $x^{(i)}|_{[r]} = 0^r$. Now note that for any $\rho \in \{0, 1\}^{[n] \setminus \mathbf{J}}$, the following 2^{ℓ} bits

$$\{\mathbf{f}(\rho \sqcup z) : z \in \{0,1\}^{\ell}\}\$$

are $(2^{\ell}-1)$ -wise independent. So it follows that the events $\mathbf{f}(x^{(i)}) = y_i$ are all independent and

$$\Pr_{\mathbf{f}, \mathbf{J} \sim \mathcal{D}_{YES}} \left[\forall i \ \mathbf{f}(x^{(i)}) = y_i \mid \mathbf{J} = J \right] = \left(\prod_{i=1}^a p^{y_i} (1-p)^{1-y_i} \right) \cdot \frac{1}{2^{m-a}} = \Pr_{\mathbf{f} \sim \mathcal{D}_{NO}} \left[\forall i \ \mathbf{f}(x^{(i)}) = y_i \right]$$

This finishes the proof of the lemma.

It now follows that \mathcal{D}_{YES} and \mathcal{D}_{NO} are hard to distinguish.

Lemma 3. Any deterministic algorithm ALG that distinguishes between \mathcal{D}_{YES} and \mathcal{D}_{NO} with probability at least 2/3, i.e., $\Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \ accepts] \geq 2/3$ and $\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \ rejects] \geq 2/3$, must make at least $\Omega\left(\sqrt{\binom{k-r}{\ell}}\right)$ queries.

Proof. Towards a contradiction, suppose ALG distinguishes the distributions and makes at most $Q = \frac{1}{10}\sqrt{\binom{k-r}{\ell}}$ queries. Since ALG is deterministic, it corresponds to a decision tree. But now observe that for a particular path p of the decision tree, we have that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p] = \Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p|p \text{ doesn't query a pair } x, x^{\oplus \mathbf{J}}].$$

by Lemma 2. We then conclude that

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p] \le \frac{\Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p]}{\Pr_{\mathbf{f}, \mathbf{J} \sim \mathcal{D}_{YES}}[p \text{ doesn't query a pair } x, x^{\oplus \mathbf{J}}]}$$

Now observe

$$\Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[p \text{ doesn't query a pair } x, x^{\oplus\mathbf{J}}] \ge 1 - \frac{Q^2}{\binom{k-r}{\ell}} \ge .99.$$

Thus

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p] \le 1.02 \cdot \Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p]$$

Summing over all rejecting paths we conclude that

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ rejects}] \le 1.02 \cdot \Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ rejects}] < 1/2,$$

a contradiction. So any tester must make $\Omega\left(\sqrt{\binom{k-r}{\ell}}\right)$ queries as claimed.

Next, we need to understand exactly how far functions in \mathcal{D}_{NO} are from being k-juntas. To do so, we'll need the following helper lemma

Lemma 4. Let $\mathbf{x}_1, ..., \mathbf{x}_n$ be independent boolean valued random variables such that $\mathbf{x}_i \sim \text{Bernoulli}(p_i)$ for constants $p_i \in [0, 1/2]$. If $\frac{1}{2} - \frac{1}{n} \sum_{i=1}^n p_i \ge \delta$ then

$$\left| \mathbb{E}_{\mathbf{x}_1,\dots,\mathbf{x}_n} \left[\operatorname{dist}(\mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_n, \{0^n, 1^n\}) \right] - \sum_{i=1}^n p_i \right| \le n e^{-n\delta^2/3}$$

Proof. We start by proving that it is unlikely that $\mathbf{x}_1, ..., \mathbf{x}_n$ is closer 1^n than 0^n . Indeed, the probability of this occurring is

$$\Pr_{\mathbf{x}_1,\dots,\mathbf{x}_n}\left[\sum_{i=1}^n \mathbf{x}_i > n/2\right] \le e^{-n\delta^2/3}$$

by a Hoeffding bound. So we conclude that

$$\left| \mathbb{E}_{\mathbf{x}_1,\dots,\mathbf{x}_n} \left[\operatorname{dist}(\mathbf{x}_1\dots\mathbf{x}_n, \{0^n, 1^n\}) \right] - \mathbb{E}_{\mathbf{x}_1,\dots,\mathbf{x}_n} \left[\operatorname{dist}(\mathbf{x}_1\dots\mathbf{x}_n, 0^n) \right] \right| \le n e^{-n\delta^2/3}.$$

The result now follows as

$$\mathbb{E}[\operatorname{dist}(\mathbf{x}_1, \dots, \mathbf{x}_n, 0^n)] = \sum_{i=1}^n p_i$$

This finishes the proof of the lemma.

Let Δ_t be defined as follows:

$$\Delta_t := \frac{1}{t} \cdot \mathbb{E}_{\mathbf{x}_1 \dots \mathbf{x}_t \sim \text{Bernoulli}(\frac{1}{2})} \left[\text{dist}(\mathbf{x}_1, \dots, \mathbf{x}_t, \{0^t, 1^t\}) \right].$$

Lemma 5. Suppose that $\ell \leq k$, then

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}} \left[\operatorname{dist}(\mathbf{f}, \mathcal{J}_k) \le p(1 - 2^{-r}) + \Delta_{2^{\ell}} \cdot 2^{-r} - e^{-2^{\ell} \cdot (1/2 - p)^2/12} - 2^{-k/3} \right] = o_k(1)$$

where \mathcal{J}_k denotes the class of k-juntas.

Proof. Fix a set $S \subseteq [n]$ of size k. Let \mathcal{J}_S denote the set of k-juntas on S. Now take $I = S \cap [r]$ and note that

$$\operatorname{dist}(\mathbf{f}, \mathcal{J}_S) = \frac{1}{2^k} \sum_{\rho \in \{0,1\}^S} \sum_{y \in \{0,1\}^{[n] \setminus S}} \frac{1}{2^\ell} \left| \mathbf{f}(\rho \sqcup y) - \operatorname{MAJ}(\{\mathbf{f}(\rho \sqcup y) : y \in \{0,1\}^{[n] \setminus S}\}) \right|$$

We will lower bound $\mathbb{E}_{\mathbf{f}}[\operatorname{dist}(\mathbf{f}, \mathcal{J}_S)]$. Consider some fixed $\rho \in \{0, 1\}^S$. If $\rho|_I \neq 0$ then

$$\frac{1}{2^\ell} \cdot \mathbb{E}_{\mathbf{f}} \left[\sum_{y \in \{0,1\}^{[n] \setminus S}} \mathbf{f}(\rho \sqcup y) \right] = p.$$

So by Lemma 4 it follows that

$$\mathbb{E}_{\mathbf{f}}\left[\frac{1}{2^{\ell}}\sum_{y\in\{0,1\}^{[n]\setminus S}}\left|\mathbf{f}(\rho\sqcup y) - \mathrm{MAJ}(\{\mathbf{f}(\rho\sqcup y): y\in\{0,1\}^{[n]\setminus S}\})\right|\right] \ge p - e^{-2^{\ell} \cdot (1/2-p)^2/3}.$$

So now suppose that $\rho_I = 0^{|I|}$. We take cases on |I|. First suppose that |I| < r. Then

$$\frac{1}{2^{\ell}} \cdot \mathbb{E}_{\mathbf{f}}\left[\sum_{y \in \{0,1\}^{[n] \setminus S}} \mathbf{f}(\rho \sqcup y)\right] = p(1 - 2^{|I| - r}) + \frac{1}{2} \cdot 2^{|I| - r}$$

as the $2^{|I|-r}$ fraction of x values with $x|_{[r]} = 0^r$ are distributed according to Bernoulli $(\frac{1}{2})$ and the rest are distributed according to Bernoulli(p). Using Lemma 4 along with the fact that $r - |I| \ge 1$, we get

$$\mathbb{E}_{\mathbf{f}}\left[\frac{1}{2^{\ell}}\sum_{y\in\{0,1\}^{[n]\setminus S}}\left|\mathbf{f}(\rho\sqcup y) - \mathrm{MAJ}(\{\mathbf{f}(\rho\sqcup y): y\in\{0,1\}^{[n]\setminus S}\})\right|\right] \ge p(1-2^{|I|-r}) + \frac{1}{2} \cdot 2^{|I|-r} - e^{-2^{\ell} \cdot (1/2-p)^2/12}.$$

All together we see that in this case

$$\mathbb{E}_{\mathbf{f}}[\operatorname{dist}(\mathbf{f},\mathcal{J}_S)] \ge (1-2^{-|I|})p + 2^{-|I|} \left(p(1-2^{|I|-r}) + \frac{1}{2} \cdot 2^{|I|-r} \right) - e^{-2^{\ell} \cdot (1/2-p))^2/12}$$
$$= p(1-2^{-r}) + \frac{1}{2^{r+1}} - e^{-2^{\ell} \cdot (1/2-p))^2/12}$$

Now suppose that |I| = r. It then follows that all the entries are Bernoulli $(\frac{1}{2})$ and

$$\frac{1}{2^{\ell}} \cdot \mathbb{E}_{\mathbf{f}} \left[\sum_{y \in \{0,1\}^{[n] \setminus S}} \left| \mathbf{f}(\rho \sqcup y) - \mathrm{MAJ}(\{\mathbf{f}(\rho \sqcup y) : y \in \{0,1\}^{[n] \setminus S}\}) \right| \right] = \Delta_{2^{\ell}}$$

So we conclude that in this case

$$\mathbb{E}_{\mathbf{f}}\left[\operatorname{dist}(\mathbf{f}, \mathcal{J}_S)\right] \ge p(1 - 2^{-r}) + \Delta_{2^{\ell}} \cdot 2^{-r} - e^{-2^{\ell} \cdot (1/2 - p)^2/12}$$

Since $\Delta_{2^{\ell}} < \frac{1}{2}$, it follows that for any set $S \subseteq [n]$

$$\mathbb{E}_{\mathbf{f}}\left[\operatorname{dist}(\mathbf{f}, \mathcal{J}_S)\right] \ge p(1 - 2^{-r}) + \Delta_{2^{\ell}} \cdot 2^{-r} - e^{-2^{\ell} \cdot (1/2 - p)^2/12}.$$

We now note that $dist(\mathbf{f}, \mathcal{J}_S)$ is the average of independent random variables in [0, 1]. Namely, it is the average the random variables

$$\mathbf{z}_{\rho} := \sum_{y \in \{0,1\}^{[n] \setminus S}} \frac{1}{2^{\ell}} \left| \mathbf{f}(\rho \sqcup y) - \mathrm{MAJ}(\{\mathbf{f}(\rho \sqcup y) : y \in \{0,1\}^{[n] \setminus S}\}) \right|$$

Thus, by Hoeffding's inequality we have that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}\left[\operatorname{dist}(\mathbf{f},\mathcal{J}_S)\leq\mathbb{E}[\operatorname{dist}(\mathbf{f},\mathcal{J}_S)]-2^{-k}t\right]\leq e^{-2t^2/2^k}$$

Taking t to be $2^{2k/3}$, it follows that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}\left[\operatorname{dist}(\mathbf{f},\mathcal{J}_S) \leq \mathbb{E}[\operatorname{dist}(\mathbf{f},\mathcal{J}_S)] - 2^{-k/3}\right] \leq e^{-2\cdot 2^{k/3}}$$

We can now take a union bound over all subsets to conclude that

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}} \left[\operatorname{dist}(\mathbf{f}, \mathcal{J}_k) \le p(1 - 2^{-r}) + \Delta_{2^{\ell}} \cdot 2^{-r} - e^{-2^{\ell} \cdot (1/2 - p)^2/12} - 2^{-k/3} \right] \le e^{-2 \cdot 2^{k/3}} (k + \ell)^k = o_k(1)$$

We will also need an analogous result for functions drawn from \mathcal{D}_{YES} .

Lemma 6. Suppose that $\ell \geq 2$, then

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{YES}} \left[\operatorname{dist}(\mathbf{f}, \mathcal{J}_k) \ge p(1 - 2^{-r}) + (\Delta_{2^{\ell}} - 2^{-2^{\ell}}) \cdot 2^{-r} + e^{-2^{\ell} \cdot (1/2 - p)^2/12} + 2^{-k/3} \right] = o_k(1)$$

where \mathcal{J}_k denotes the class of k-juntas.

Proof. The proof follows very similarly to Lemma 5.

$$\begin{split} \mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[\operatorname{dist}(\mathbf{f},\mathcal{J}_{[n]\setminus\mathbf{J}})] \\ &= \frac{1}{2^{k}} \mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}} \left[\sum_{\rho\in\{0,1\}^{[n]\setminus\mathbf{J}}} \sum_{y\in\{0,1\}^{\mathbf{J}}} \frac{1}{2^{\ell}} \left| \mathbf{f}(\rho\sqcup y) - \operatorname{MAJ}(\{\mathbf{f}(\rho\sqcup y): y\in\{0,1\}^{\mathbf{J}}\}) \right| \right] \\ &\leq p(1-2^{-r}) + e^{-2^{\ell} \cdot (1/2-p)^{2}/12} + \left(\Delta_{2^{\ell}} - 2^{-2^{\ell}}\right) \cdot 2^{-r} \end{split}$$

where $p(1-2^{-r}) + e^{-2^{\ell} \cdot (1/2-p)^2/12}$ corresponds to the contribution from terms with $\rho|_{[r]} \neq 0^r$ and $(\Delta_{2^{\ell}} - 2^{-2^{\ell}}) \cdot 2^{-r}$ bounds the expectation for terms with $\rho|_{[r]} = 0^r$ by Lemma 1. Applying a Hoeffding bound now gives us the desired result.

We can now put everything together and prove the lower bound.

Proof of Theorem 3. It only remains to set the parameters so any $(k, \varepsilon_1, \varepsilon_2)$ tolerant tester is forced to distinguish the two distributions. We let $\ell = \lfloor \frac{-1}{10} \log(1 - \varepsilon_1 \varepsilon_2^{-1}) \rfloor$. We then take r to be the smallest integer such that

$$\Delta_{2\ell} 2^{-r} \le \varepsilon_2 + 2^{-k/3} + e^{-2^{\ell} \cdot (0.009)^2/12}$$

and set p to satisfy

$$\varepsilon_2 + 2^{-k/3} + e^{-2^{\ell} \cdot (0.009)^2/12} = p(1 - 2^{-r}) + \Delta_{2^{\ell}} \cdot 2^{-r}$$

We assume that ℓ and k are sufficiently large such that $2^{-k/3} \leq .005$, $e^{-2^{\ell} \cdot (0.009)^2/12} \leq .005$, and $\Delta_{2^{\ell}} \geq .491$. Note that these together imply that $r \geq 0$ and $0 \leq p \leq .491$. Moreover, note that by the minimality of r, we have that

$$\Delta_{2^{\ell}} 2^{-r+1} \ge \varepsilon_2.$$

which implies that $2^{-r} \ge \varepsilon_2$ and thus assuming $\varepsilon_2 - \varepsilon_1 \ge 2^{-k/6}$, $r \le k/6$ as promised.

To see that a $(k, \varepsilon_1, \varepsilon_2)$ tolerant tester must distinguish \mathcal{D}_{YES} and \mathcal{D}_{NO} , first observe that by Lemma 5 and our choice of parameters, we have that functions in \mathcal{D}_{NO} are at least ε_2 -far from the set of k-juntas with high probability. On the other hand, by Lemma 6, functions from \mathcal{D}_{YES} are with high probability

$$p(1-2^{-r}) + (\Delta_{2^{\ell}} - 2^{-2\ell}) \cdot 2^{-r} + e^{-2^{\ell} \cdot (.009)^2/12} + 2^{-k/3}$$

close to some k-junta. We'll show this is at most ε_1 under our choice of parameters. To see this first note that

$$e^{-2^{\ell}(.009)^2/12} \le e^{-(1-\varepsilon_1\varepsilon_2^{-1})^{-1/10}(.009)^2/24} \le e^{-(1-\varepsilon_1\varepsilon_2^{-1})^{-1/11}}$$

where the second inequality assumes $(1 - \varepsilon_1 \varepsilon_2^{-1})^{-1}$ is sufficiently large. Now we assume that $(1 - \varepsilon_1 \varepsilon_2^{-1})^{-1} \ge \log^{12}(\frac{1}{\varepsilon_2})$, which we took as a hypothesis in the theorem, and observe

$$-2\log(\varepsilon_2 - \varepsilon_1) = -2\log(\varepsilon_2) - 2\log(1 - \varepsilon_1\varepsilon_2^{-1}) \le 2(1 - \varepsilon_1\varepsilon_2^{-1})^{-1/12} + 2(1 - \varepsilon_1\varepsilon_2^{-1})^{-1/12} \le (1 - \varepsilon_1\varepsilon_2^{-1})^{-1/11} \le (1 - \varepsilon_1\varepsilon_2^{-1})^{-1/11}$$

for $(1 - \varepsilon_1 \varepsilon_2^{-1})^{-1}$ sufficiently large. Thus,

$$e^{-(1-\varepsilon_1\varepsilon_2^{-1})^{-1/11}} \le e^{2\log(\varepsilon_2-\varepsilon_1)} = (\varepsilon_2-\varepsilon_1)^2.$$

Plugging these into our expression for the distance of functions in \mathcal{D}_{YES} to the set of k-juntas yields that they are

$$\varepsilon_2 + 4(\varepsilon_2 - \varepsilon_1)^2 - 2^{\frac{1}{5}\log(1 - \varepsilon_1\varepsilon_2^{-1})}\varepsilon_2 \le \varepsilon_2 + 4(\varepsilon_2 - \varepsilon_1)^2 - (1 - \varepsilon_1\varepsilon_2^{-1})^{\frac{1}{5}}\varepsilon_2$$
$$\le \varepsilon_2 + 4(\varepsilon_2 - \varepsilon_1)^2 - 10(1 - \varepsilon_1\varepsilon_2^{-1})\varepsilon_2 \le \varepsilon_1$$

close to being a k-junta with high probability, again assuming $(1 - \varepsilon_1 \varepsilon_2^{-1})^{-1}$ is sufficiently large. Thus any tolerant tester must distinguish the two distributions and the result follows from Lemma 3.

Finally, we conclude by proving Lemma 1.

Proof of Lemma 1. We'll prove the statement by a coupling argument. Note that we can sample from \mathcal{D} taking $\mathbf{x}_1, ..., \mathbf{x}_n$ to be i.i.d. Bernoulli $(\frac{1}{2})$ random variables. Similarly, we can sample from \mathcal{D}' by taking $\mathbf{x}_1, ..., \mathbf{x}_{n-1}$ to be i.i.d. Bernoulli $(\frac{1}{2})$ and then setting $\mathbf{x}_n = 1 \oplus \bigoplus_{i=1}^{n-1} x_i$. We now compute

$$\begin{split} \mathbb{E}_{\mathbf{x}\sim\mathcal{D}'}\left[\operatorname{dist}(\mathbf{x},\{0^{n},1^{n}\})\right] &= \mathbb{E}_{\mathbf{x}\sim\mathcal{D}'}\left[\operatorname{dist}(\mathbf{x}_{1}\mathbf{x}_{2}...\mathbf{x}_{n-1},\{0^{n-1},1^{n-1}\}) + \mathbb{I}_{\mathbf{x}_{n}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})}\right] \\ &= \mathbb{E}_{\mathbf{x}\sim\mathcal{D}',\mathbf{y}\sim\mathrm{Bernoulli}(\frac{1}{2})}\left[\operatorname{dist}(\mathbf{x}_{1}\mathbf{x}_{2}...\mathbf{x}_{n-1},\{0^{n-1},1^{n-1}\}) + \mathbb{I}_{\mathbf{x}_{n}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})} \right. \\ &+ \mathbb{I}_{\mathbf{y}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})} - \mathbb{I}_{\mathbf{y}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})}\right] \\ &= \mathbb{E}_{\mathbf{x}\sim\mathcal{D}}\left[\operatorname{dist}(\mathbf{x},\{0^{n},1^{n}\})\right] + \mathbb{E}_{\mathbf{x}\sim\mathcal{D}',\mathbf{y}\sim\mathrm{Bernoulli}(\frac{1}{2})}\left[\mathbb{I}_{\mathbf{x}_{n}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})} - \mathbb{I}_{\mathbf{y}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})}\right] \\ &= \mathbb{E}_{\mathbf{x}\sim\mathcal{D}}\left[\operatorname{dist}(\mathbf{x},\{0^{n},1^{n}\})\right] + \mathbb{E}_{\mathbf{x}\sim\mathcal{D}'}\left[\mathbb{I}_{\mathbf{x}_{n}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})\right] - \frac{1}{2} \end{split}$$

where the first equality uses the fact that $MAJ(\mathbf{x}_1, ..., \mathbf{x}_n) = MAJ(\mathbf{x}_1, ..., \mathbf{x}_{n-1})$ since n is a multiple of 4 and x has odd parity. Now note that

$$\Pr_{\mathbf{x}\sim\mathcal{D}'}\left[\mathbf{x}_{n}\neq\mathrm{MAJ}(\mathbf{x}_{1},...,\mathbf{x}_{n-1})\right] = \mathbb{E}_{\mathbf{i}\sim[n]}\left[\Pr_{\mathbf{x}\sim\mathcal{D}'}\left[\mathbf{x}_{\mathbf{i}}\neq\mathrm{MAJ}(\mathbf{x}_{-\mathbf{i}})\right]\right] = \mathbb{E}_{\mathbf{x}\sim\mathcal{D}'}\left[\Pr_{\mathbf{i}\sim[n]}\left[\mathbf{x}_{\mathbf{i}}\neq\mathrm{MAJ}(\mathbf{x}_{-\mathbf{i}})\right]\right].$$

Since 4|n we have that any string x of odd parity satisfies $MAJ(x_{-j}) = MAJ(x)$ for all j. Moreover, at most n/2 - 1 bits of x differ from the majority. Thus,

$$\mathbb{E}_{\mathbf{x}\sim\mathcal{D}'}\left[\Pr_{\mathbf{i}\sim[n]}[\mathbf{x}_{\mathbf{i}}\neq\mathrm{MAJ}(\mathbf{x}_{-\mathbf{i}})]\right] \leq \left(\frac{n}{2}-1\right)\frac{1}{n} = \frac{1}{2} - \frac{1}{n}$$

as desired.

3 Lower Bounds for Relaxed Junta Testing

Recall that for relaxed tolerant junta testing, a $(k, k', \varepsilon_1, \varepsilon_2)$ tester must accept functions that are ε_1 -close to some k-junta and reject those that are ε_2 -far from all k' > k juntas. To prove Theorem 2, it will suffice to show:

Theorem 4. For any $0.01 \le \varepsilon_1 < \varepsilon_2 \le 0.49$ and integer k with $\varepsilon_2 - \varepsilon_1 \ge 2^{-O(k)}$ and any $(k, k + \lfloor \frac{-\log(\varepsilon_2 - \varepsilon_1)}{20} \rfloor, \varepsilon_1, \varepsilon_2)$ -tolerant junta tester must make at least $\left(\frac{k}{-\log(\varepsilon_2 - \varepsilon_1)}\right)^{-\Omega(\log(\varepsilon_2 - \varepsilon_1))}$ queries.

We can then combine this result with the following observation:

Lemma 7. Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function, b be an integer, and take $F : \{0,1\}^{nb} \to \{0,1\}$ as

$$F(x) = f\left(\bigoplus_{i=1,\dots,b} x_i, \bigoplus_{i=b+1,\dots,2b} x_i, \dots, \bigoplus_{i=nb-b+1,\dots,nb} x_i\right).$$

Then for any $k \leq n$ we have that

$$\operatorname{dist}(F, \mathcal{J}_{kb}) = \operatorname{dist}(F, \mathcal{J}_{kb+b-1}) = \operatorname{dist}(f, \mathcal{J}_k)$$

We include a proof in the appendix, but intuitively any junta must use all the coordinates from a set of xor'd variables or no variables from it. This then gives the following corollary:

Corollary 1. Let k, ℓ, b be integers and $0 \le \varepsilon_1 \le \varepsilon_2 \le \frac{1}{2}$. If any $(k, k + \ell, \varepsilon_1, \varepsilon_2)$ tester must make $Q(k, \ell, \varepsilon_1, \varepsilon_2)$ queries, then any $(kb, (k + \ell)b + b - 1, \varepsilon_1, \varepsilon_2)$ tester must make $Q(k, \ell, \varepsilon_1, \varepsilon_2)$ queries.

Proof. We'll construct a $(k, k+\ell, \varepsilon_1, \varepsilon_2)$ tester using a $(kb, (k+\ell)b+b-1, \varepsilon_1, \varepsilon_2)$ tester ALG. Indeed, given a function $f : \{0, 1\}^n \to \{0, 1\}$, we construct F as in Lemma 7. We then run ALG on F and accept if ALG accepts and reject otherwise.

Note that $\operatorname{dist}(F, \mathcal{J}_{kb}) = \operatorname{dist}(f, \mathcal{J}_k)$ and $\operatorname{dist}(F, \mathcal{J}_{(k+\ell)b+b-1}) = \operatorname{dist}(f, \mathcal{J}_{k+\ell})$, so this indeed constitutes a $(k, k+\ell, \varepsilon_1, \varepsilon_2)$ tester. Since we can answer each query to F with at most one query to f, the corollary follows.

With this, we can prove Theorem 2.

Proof of Theorem 2. Note that it suffices to handle the case when $\gamma \geq \frac{1}{k \cdot 01}$: If $\frac{1}{k} \leq \gamma \leq \frac{1}{k \cdot 01}$, a $(k, (1+\gamma)k, \varepsilon_1, \varepsilon_2)$ tester is also a $(k, (1+\frac{1}{k \cdot 01})k, \varepsilon_1, \varepsilon_2)$ tester. Thus, the $k^{-\Omega(\log(\varepsilon_2-\varepsilon_1))}$ query lower bounds for a $(k, (1+\frac{1}{k \cdot 01})k, \varepsilon_1, \varepsilon_2)$ applies to the $(k, (1+\gamma)k, \varepsilon_1, \varepsilon_2)$ tester, which proves the desired result.

Let $\ell = \lfloor -\log(\varepsilon_2 - \varepsilon_1) \rfloor$ and set $k' = \lfloor \ell/(100\gamma) \rfloor$. Now observe that if γ is smaller than some appropriate absolute constant, we can apply Theorem 4 to get a $\left(\frac{1}{\gamma}\right)^{-\Omega(\log(\varepsilon_2 - \varepsilon_1))}$ lower bound for a $(k', k' + \lfloor \ell/20 \rfloor, \varepsilon_1, \varepsilon_2)$ tester. Now let $b = \lfloor k/k' \rfloor$. Applying Corollary 1 then extends this lower bound to $(k'b, k'b + b\lfloor \ell/20 \rfloor + b - 1, \varepsilon_1, \varepsilon_2)$ testers.

Note note that $k' \leq k^{.12}$. Thus

$$k'b + b|\ell/20| + b - 1 \ge k - k' + 5k\gamma - 5k'\gamma - 1 \ge (1 + 4\gamma)k > (1 + \gamma)k$$

assuming k is sufficiently large.

3.1 Weak Gap Lower Bound

It now remains to prove Theorem 4. At a high level, we follow the same proof as with Theorem 3 but with some changes the \mathcal{D}_{NO} distribution. Since most of the proofs are simple or identical to their counterparts in the original lower bound we banish them to the appendix.

Lemma 8. Let $d \le n$ be integers. There exists a coloring of the boolean cube $\chi : \{0,1\}^n \to [|B_n(d)|]$ such that for all $x, y \in \{0,1\}^n$ with $\operatorname{dist}(x,y) \le d$, $\chi(x) \ne \chi(y)$.

We'll also need the following fact.

Lemma 9. Let $\lambda_1, ..., \lambda_n$ be non-negative real numbers with $\sum_i \lambda_i = 1$. Let $\mathbf{X}_1, ..., \mathbf{X}_n$ be drawn uniformly and independently at random from $\{-1, 1\}$. Then

$$\mathbb{E}\left[\left|\sum_{i}\lambda_{i}\mathbf{X}_{i}\right|\right] \geq \mathbb{E}\left[\left|\sum_{i}\frac{1}{n}\mathbf{X}_{i}\right|\right].$$

Finally, standard results about random walks give us that

Lemma 10 (Folklore). Δ_t satisfies the following properties

(i) Δ_t is an increasing function in t

(ii) For t sufficiently large, $\frac{1}{2} - \frac{10}{\sqrt{t}} \le \Delta_t \le \frac{1}{2} - \frac{1}{10\sqrt{t}}$

With this we have everything we need to construct our new distributions. We again proceed by Yao's lemma and construct a \mathcal{D}_{YES} and \mathcal{D}_{NO} distributions, which are again boolean functions over $n := k + \ell$ variables. We take ℓ, r and d to be parameters, which we'll set later. We again will ensure that r < k and take $d = \Theta(\ell)$.

 \mathcal{D}_{NO} : A boolean function $\mathbf{f}: \{0,1\}^n \to \{0,1\}$ is drawn as follows:

- (a) For each $x \in \{0,1\}^n$ with $x|_{[r]} \neq 0^r$, independently draw $\mathbf{f}(x)$ from Bernoulli(p).
- (b) For each $x \in \{0,1\}^n$ with $x|_{[r]} = 0^r$, draw $\mathbf{f}(x)$ uniformly and independently at random.

 \mathcal{D}_{YES} : A boolean function $\mathbf{f}: \{0,1\}^n \to \{0,1\}$ is drawn as follows:

- (a) Randomly choose a set $\mathbf{J} \subseteq \{r+1, ..., n\}$ of size ℓ .
- (b) Let χ be a coloring of $\{0,1\}^{\mathbf{J}}$ from Lemma 8 such that points within distance d have different colors.
- (c) For each $\rho \in \{0,1\}^{[n] \setminus \mathbf{J}}$, sample $\mathbf{y}_1^{\rho}, ..., \mathbf{y}_{|B_{\ell}(d)|}^{\rho} \sim \operatorname{Bernoulli}(\frac{1}{2})$.
- (d) For each $x \in \{0,1\}^n$ with $x|_{[r]} \neq 0^r$, independently draw $\mathbf{f}(x)$ from Bernoulli(p).
- (e) For each $x \in \{0,1\}^n$ with $x|_{[r]} = 0^r$ set $\mathbf{f}(x) = \mathbf{y}_{\chi(x|_{\mathbf{J}})}^{x|_{[n]\setminus\mathbf{J}}}$

We now follow the previous proof. Note that by construction we have that

Lemma 11. Consider a set of points $x^{(1)}, ..., x^{(m)} \in \{0, 1\}^n$ and $y_1, ..., y_m \in \{0, 1\}$. Then

$$\Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}} \left[\forall i \ \mathbf{f}(x^{(i)}) = y_i \mid \forall i,j \quad x^{(i)}|_{[n]\backslash\mathbf{J}} \neq x^{(j)}|_{[n]\backslash\mathbf{J}} \lor \operatorname{dist}(x,y) \leq d \right] = \Pr_{\mathbf{f}\sim\mathcal{D}_{NO}} \left[\forall i \ \mathbf{f}(x^{(i)}) = y_i \right]$$

Lemma 12. Any deterministic algorithm ALG that distinguishes between \mathcal{D}_{YES} and \mathcal{D}_{NO} with probability at least 2/3, i.e., $\Pr_{\mathbf{f}\sim\mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \ accepts] \geq 2/3$ and $\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \ rejects] \geq 2/3$, must make at least $\Omega((k/\ell)^{d/2})$ queries.

Applying Lemma 5 with our new parameters then gives

Lemma 13. Suppose that $\ell \leq k$, then

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}\left[\operatorname{dist}(\mathbf{f},\mathcal{J}_{k+\ell/10}) \le p(1-2^{-r}) + \Delta_{2^{0.9\ell}} \cdot 2^{-r} - e^{-2^{0.9\ell} \cdot (1/2-p)^2/12} - 2^{-(k+\ell/10)/3}\right] = o_k(1)$$

where $\mathcal{J}_{k+\ell/10}$ denotes the class of $(k+\ell/10)$ -juntas.

Finally, we have that

Lemma 14. Suppose that $\ell \geq 2$, then

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{YES}} \left[\operatorname{dist}(\mathbf{f}, \mathcal{J}_k) \ge p(1 - 2^{-r}) + \Delta_{|B_\ell(d)|} \cdot 2^{-r} + e^{-2^\ell \cdot (1/2 - p)^2/12} + 2^{-k/3} \right] = o_k(1)$$

where \mathcal{J}_k denotes the class of k-juntas.

Combining these all together and setting parameters now gives us the theorem:

Proof of Theorem 4. It only remains to set the parameters so any $(k, \varepsilon_1, \varepsilon_2)$ tolerant tester is forced to distinguish the two distributions. We let $\ell = 10 \lfloor -\log(1 - \varepsilon_1 \varepsilon_2^{-1}) \rfloor$. We then take r to be the smallest integer such that

$$\Delta_{20.9\ell} 2^{-r} < \varepsilon_2 + 2^{-(k+0.1\ell)/3} + e^{-2^{0.9\ell} \cdot (0.009)^2/12}$$

and set p to satisfy

$$\varepsilon_2 + 2^{-(k+0.1\ell)/3} + e^{-2^{0.9\ell} \cdot (0.009)^2/12} = p(1-2^{-r}) + \Delta_{2^{0.9\ell}} \cdot 2^{-r}.$$

We assume that ℓ and k are sufficiently large such that $2^{-(k+0.1\ell)/3} \leq .005$, $e^{-2^{0.9\ell} \cdot (0.009)^2/12} \leq .005$, and $\Delta_{2^{0.9\ell}} \geq .491$. Note that these together imply that $r \geq 0$ and $0 \leq p \leq .491$. Moreover, note that by the minimality of r, we have that

$$\Delta_{2^{0.9\ell}} 2^{-r+1} \ge \varepsilon_2.$$

which implies that $2^{-r} \ge \varepsilon_2$, yielding r = O(1) and thus that r < k as promised.

Finally, we set d to be the largest integer such that

$$|B_{\ell}(d)| \le 2^{0.1\ell}$$

This clearly implies that $d = \Theta(\ell)$ as promised.

To see that a $(k, \varepsilon_1, \varepsilon_2)$ tolerant tester must distinguish \mathcal{D}_{YES} and \mathcal{D}_{NO} , first observe that by Lemma 13 and our choice of parameters, we have that functions in \mathcal{D}_{NO} are at least ε_2 -far from the set of $(k + \ell/10)$ -juntas with high probability. On the other hand, by Lemma 14 and Lemma 10, functions from \mathcal{D}_{YES} are with high probability at most

$$p(1-2^{-r}) + \Delta_{2^{0.1\ell}} \cdot 2^{-r} + e^{-2^{\ell} \cdot (.009)^2/12} + 2^{-k/3}$$

close to some k-junta. We'll show this is at most ε_1 under our choice of parameters. Note this is at equal to

$$\varepsilon_2 + \Delta_{2^{0.1\ell}} \cdot 2^{-r} - \Delta_{2^{0.9\ell}} \cdot 2^{-r} + e^{-2^{\ell} \cdot (.009)^2/12} + e^{-2^{0.9\ell} \cdot (0.009)^2/12} + 2^{-k/3} + 2^{-(k+0.1\ell)/3}$$

Again by Lemma 10, we have that if $\varepsilon_2 - \varepsilon_1$ is sufficiently small

$$\Delta_{2^{0.1\ell}} \cdot 2^{-r} - \Delta_{2^{0.9\ell}} \cdot 2^{-r} \le 10\sqrt{2^{-0.9\ell}} - \frac{1}{10}\sqrt{2^{-0.1\ell}} \le -\frac{1}{20}\sqrt{2^{-0.1\ell}}$$

Combining this with our hypothesis that $\varepsilon_2 - \varepsilon_1 \ge 2^{k/6}$ and the fact that $2^{-r} \ge \varepsilon_2$, we get that the distance of functions from \mathcal{D}_{YES} is at most

$$\varepsilon_2 - \frac{1}{20}\sqrt{2^{-0.1\ell}} \cdot \varepsilon_2 + 4(\varepsilon_1 - \varepsilon_2)^2 \le \varepsilon_2 - \frac{1}{20}(1 - \varepsilon_1 \varepsilon_2^{-1})^{\frac{1}{2}} \varepsilon_2 + (\varepsilon_2 - \varepsilon_1)^2 \le \varepsilon_2 - 2(1 - \varepsilon_1 \varepsilon_2^{-1})\varepsilon_2 + (\varepsilon_2 - \varepsilon_1)^2 < \varepsilon_1 - \varepsilon_2 + \varepsilon_2 - \varepsilon_$$

again assuming $\varepsilon_2 - \varepsilon_1$ is sufficiently small. The proof is now complete since

$$\ell/10 \ge -\log(\varepsilon_2 - \varepsilon_1) + \log(\varepsilon_2) - 1 \ge -\log(\varepsilon_2 - \varepsilon_1)/20.$$

since $\varepsilon_2 \geq 0.01$.

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4 Barriers to Stronger Lower Bounds

There are several interesting open questions raised by this work. Can we further improve the lower bound for tolerant junta testing? For the relaxed model, can we rule out $(k, 2k, \varepsilon_1, \varepsilon_2)$ -testers that make poly $(k, (\varepsilon_2 - \varepsilon_1)^{-1})$ queries? Additionally, can we use our techniques to prove lower bounds for testing monotonicity (tolerantly)?

We conclude with some limitations of our approach to getting stronger bounds. Optimistically, one may hope that asking for smaller than $(2^{\ell} - 1)$ -wise independence may give better bounds, but this will not work naively even if we change \mathcal{D}_{NO} . Indeed, suppose we have a distribution \mathcal{D}_{NO} over boolean functions on $\{0,1\}^n$ with $n := k + \ell$. Let \mathcal{D}_{YES}^J for a subset $J \subseteq [n]$ of size ℓ also be a distribution over boolean functions on $\{0,1\}^n$. Moreover, assume that when $\mathbf{f} \sim \mathcal{D}_{YES}^J$ and $\mathbf{g} \sim \mathcal{D}_{NO}$ we have that for any $\rho \in \{0,1\}^{[n]\setminus J}$ and any m points $y^{(1)}, ..., y^{(m)} \in \{0,1\}^J$, $\mathbf{f}(\rho \sqcup y^{(1)}), ..., \mathbf{f}(\rho \sqcup y^{(1)})$ and $\mathbf{g}(\rho \sqcup y^{(1)}), ..., \mathbf{g}(\rho \sqcup y^{(m)})$ are identically distributed.

It turns out that under these assumptions, if we follow our proof strategy naively we cannot prove better lower bounds. To see this, we will need the following lemma.

Lemma 15. Let \mathcal{D}_1 and \mathcal{D}_2 be distributions over boolean functions $f : \{0,1\}^{\ell} \to \{0,1\}$. Moreover, suppose that for any $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)} \in \{0,1\}^{\ell}$ and $y_1, ..., y_m \in \{0,1\}$ we have that

$$\Pr_{\mathbf{f} \sim \mathcal{D}_1} \begin{bmatrix} \forall i \quad \mathbf{f}(x_i) = y_i \end{bmatrix} = \Pr_{\mathbf{g} \sim \mathcal{D}_2} \begin{bmatrix} \forall i \quad \mathbf{g}(x_i) = y_i \end{bmatrix}$$

Then

$$\left| \mathbb{E}_{\mathbf{f} \sim \mathcal{D}_1} \left[\operatorname{dist}(\mathbf{f}, \{0, 1\}) \right] - \mathbb{E}_{\mathbf{g} \sim \mathcal{D}_2} \left[\operatorname{dist}(\mathbf{g}, \{0, 1\}) \right] \right| \le O\left(\sqrt{\frac{\log(m)}{m}}\right).$$

 $where \ {\rm dist}(f,\{0,1\}):=\min\{{\rm Pr}_{\mathbf{x}\sim\{0,1\}^\ell}[f(x)=0],{\rm Pr}_{\mathbf{x}\sim\{0,1\}^\ell}[f(x)=1]\}.$

We leave the proof for the appendix. That said, intuitively such a pair of distributions would give a lower bound against estimating the distance of a function $f : \{0,1\}^{\ell} \to \{0,1\}$ to constant, and we know that this can be done without many samples.

Now note that an algorithm that makes n^r queries can query every point in a ball B(x,r) for some $x \in \{0,1\}^n$. Thus, in order to avoid giving any evidence that could distinguish \mathcal{D}_{YES}^J and \mathcal{D}_{NO} , we must take $m \geq \binom{\ell}{r}$. But Lemma 15, implies that if **f** and **g** are ϵ_1 -close and ϵ_2 -far from being a junta on $[n] \setminus J$ with high probability then $m \leq O((\epsilon_2 - \epsilon_1)^{-3})$. Simplifying, we see that we at best get a $n^{-\Omega(\log(\epsilon_2 - \epsilon_1))}$ lower bound.

Moreover, we can also observe that a better lower bound cannot use a uniformly random functions as \mathcal{D}_{NO} : Namely, we can distinguish a uniformly random distribution from a distribution on functions closer than random to k-juntas by sampling a random $x \in \{0, 1\}^n$ and querying all points within hamming distance r. We then check if there is a set $S \subseteq [n]$ of size k such that $B(x,r) \cap \{y \in \{0,1\}^n : y_S = x_S\}$ is biased. For r suitably large, we expect few biased balls under the random distribution, but for functions that are closer than random to juntas we expect to see a biased ball with reasonable probability. Formally,

Lemma 16. Let \mathcal{D}_{YES} a distribution over boolean functions $f: \{0,1\}^n \to \{0,1\}^n$ such that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{YES}}[\operatorname{dist}(\mathbf{f},\mathcal{J}_k)\geq\frac{1}{2}-\varepsilon]=o(1)$$

where $\varepsilon \geq \Omega(2^{-(n-k)/10})$ and $\varepsilon \geq 2^{-n/128}$. Take \mathcal{D}_{NO} to be the uniform distribution over all boolean functions. Then there exists an algorithm ALG that makes at most $n^{O(\log(1/\varepsilon))}$ queries and distinguishes \mathcal{D}_{YES} and \mathcal{D}_{NO} with probability 2/3.

We again leave the proof for the appendix. Note that this rules out the possibility of a better lower bound with n = poly(k). When $n = k^{\omega(1)}$, we remark that the techniques of [ITW21] are likely able to remove our testers dependence on n and give a $(k/\varepsilon^2)^{O(\log(1/\varepsilon))}$ query bound.

Thus, further improvements must move beyond *m*-wise independence and random functions. A promising way of circumventing both these barriers would be to consider using distributions \mathcal{D}_1 and \mathcal{D}_2 over functions $f: \{0,1\}^{\ell} \to \{0,1\}$ that are only identically distributed on balls B(x,r) for $x \in \{0,1\}^{\ell}$ rather than distributed identically for any *m* points. Indeed, [PRW22] shows that there are distributions \mathcal{D}_1 and \mathcal{D}_2 over functions $f: \{0,1\}^n \to \{0,1\}$ with functions in \mathcal{D}_2 being $\Omega(1)$ closer (in expectation) to constant than those from \mathcal{D}_1 and such that \mathcal{D}_1 and \mathcal{D}_2 are identically distributed along any ball B(x,r) of radius $O(\sqrt{n})$.

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A Missing Proofs from Section 3

Proof of Lemma 7. For simplicity of notation, we'll assume $f : \{\pm 1\}^n \to \{\pm 1\}$. We'll show that $\operatorname{dist}(F, \mathcal{J}_{kb}) \leq \operatorname{dist}(f, \mathcal{J}_k)$ and $\operatorname{dist}(F, \mathcal{J}_{kb+b-1}) \geq \operatorname{dist}(f, \mathcal{J}_k)$. Since $\operatorname{dist}(F, \mathcal{J}_{kb}) \geq \operatorname{dist}(F, \mathcal{J}_{kb+b-1})$ the result follows.

We start by showing $dist(F, \mathcal{J}_{kb}) \leq dist(f, \mathcal{J}_k)$. Indeed, let $S \subseteq \{\pm 1\}^n$ be a minimum set of changes needed to make f into a k-junta. Let

$$m(x) = \left(\bigoplus_{i=1\dots b} x_i, \bigoplus_{i=b+1,\dots,2b} x_i, \dots, \bigoplus_{i=nb-b+1,\dots,nb} x_i\right)$$

To get a kb-junta, it clearly suffices to change the values of F under $m^{-1}(S)$. Since $m^{-1}(S) = |S| 2^{nb}/2^n$, we conclude dist $(F, \mathcal{J}_{kb}) \leq \text{dist}(f, \mathcal{J}_k)$.

We now claim that $\operatorname{dist}(F, \mathcal{J}_{kb+b-1}) \geq \operatorname{dist}(f, \mathcal{J}_k)$. Indeed, fix a set $J \subseteq [nb]$ of size kb + b - 1. Without loss of generality, we let ℓ be the largest integer such that $[0, \ell b] \subseteq J$ and assume every subsequent block is missing at least one element. Note that fixing a prefix $\rho \in \{\pm 1\}^J$ will then fix the first ℓ bits of m(x) to $z^{\rho} \in \{\pm 1\}^{\ell}$. We can then compute

$$dist(g, \mathcal{J}_J) = \frac{1}{2^{kb+b-1}} \sum_{\rho \in \{\pm 1\}^J} \frac{1}{2} - \frac{1}{2^{nb-kb-b+2}} \left| \sum_{y \in \{\pm 1\}^{[nb] \setminus J}} g(\rho \sqcup y) \right|$$
$$= \frac{1}{2^{kb+b-1}} \sum_{\rho \in \{\pm 1\}^J} \frac{1}{2} - \frac{1}{2^{nb-kb-b+2}} \cdot \frac{2^{nb-kb-b+\ell+1}}{2^n} \left| \sum_{b \in \{\pm 1\}^{[n] \setminus [\ell]}} f(z^{\rho} \sqcup b) \right|$$
$$= \frac{1}{2^{kb+b-1}} \frac{2^{kb+b-1}}{2^{\ell}} \sum_{a \in \{\pm 1\}^{\ell}} \frac{1}{2} - \frac{2^{\ell}}{2^{n+1}} \left| \sum_{b \in \{\pm 1\}^{[n] \setminus [\ell]}} f(a \sqcup b) \right|$$
$$= \frac{1}{2^{\ell}} \sum_{a \in \{\pm 1\}^{\ell}} \frac{1}{2} - \frac{1}{2^{n-\ell+1}} \left| \sum_{b \in \{\pm 1\}^{[n] \setminus [\ell]}} f(a \sqcup b) \right| = dist(f, \mathcal{J}_{[\ell]})$$

The result now follows since ℓ is at most k.

Proof of Lemma 8. If we make a graph G with vertices $\{0,1\}^n$ and an edge between x, y with $dist(x, y) \leq d$. Note every vertex in G has degree at most $B_n(d) - 1$. It then follows by a greedy coloring that we need at most $|B_n(d)|$ colors.

Proof of Lemma 9. Without loss of generality suppose that $\lambda_1 > \lambda_2$. We claim that

$$\mathbb{E}\left[\left|\sum_{i}\lambda_{i}\mathbf{X}_{i}\right|\right] \geq \mathbb{E}\left[\left|\frac{\lambda_{1}+\lambda_{2}}{2}(\mathbf{X}_{1}+\mathbf{X}_{2})+\sum_{i=3}^{n}\mathbf{X}_{i}\right|\right].$$

Let $\overline{\lambda} = \frac{\lambda_1 + \lambda_2}{2}$. Now note

$$\mathbb{E}\left[\left|\sum_{i=1}^{n}\lambda_{i}\mathbf{X}_{i}\right|\right] - \mathbb{E}\left[\left|\overline{\lambda}(\mathbf{X}_{1} + \mathbf{X}_{2}) + \sum_{i=3}^{n}\lambda_{i}\mathbf{X}_{i}\right|\right]$$
$$= \mathbb{E}_{\mathbf{X}_{3},...\mathbf{X}_{n}}\left[\mathbb{E}_{\mathbf{X}_{1},\mathbf{X}_{2}}\left[\left|\sum_{i=1}^{n}\lambda_{i}\mathbf{X}_{i}\right| - \left|\overline{\lambda}(\mathbf{X}_{1} + \mathbf{X}_{2}) + \sum_{i=3}^{n}\lambda_{i}\mathbf{X}_{i}\right|\right|\mathbf{X}_{3},...,\mathbf{X}_{n}\right]\right]$$
$$= \frac{1}{4}\mathbb{E}_{\mathbf{X}_{3},...\mathbf{X}_{n}}\left[\left|\lambda_{1} - \lambda_{2} + \sum_{i=3}^{n}\lambda_{i}\mathbf{X}_{i}\right| + \left|\lambda_{2} - \lambda_{1} + \sum_{i=3}^{n}\lambda_{i}\mathbf{X}_{i}\right| - 2\left|\sum_{i=3}^{n}\lambda_{i}\mathbf{X}_{i}\right|\right]$$

which is non-negative by Jensen's inequality. The lemma now follows by a limiting argument. \Box

Proof of Lemma 12. Towards a contradiction, suppose ALG distinguishes the distributions and makes at most $Q = \frac{1}{10} (k/\ell)^{d/2}$ queries. Call a pair of points $x, y \in \{0, 1\}^n$ bad if $x|_{[n]\setminus \mathbf{J}} = y|_{[n]\setminus \mathbf{J}}$ and dist $(x, y) \ge d$. Since ALG is deterministic, it corresponds to a decision tree. But now observe that for a particular path p of the decision tree, we have that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p] = \Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p|p \text{ doesn't query a bad pair}].$$

by Lemma 11. We then conclude that

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p] \le \frac{\Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p]}{\Pr_{\mathbf{f}, \mathbf{J} \sim \mathcal{D}_{YES}}[p \text{ doesn't query a bad pair]}$$

Now observe

$$\Pr_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[p \text{ doesn't query a bad pair}] \ge 1 - Q^2 \cdot (\ell/k)^d \ge .99.$$

Thus

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p] \leq 1.02 \cdot \Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ follows } p]$$

Summing over all rejecting paths we conclude that

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}}[\mathsf{ALG}(\mathbf{f}) \text{ rejects}] \leq 1.02 \cdot \Pr_{\mathbf{f} \sim \mathcal{D}_{YES}}[\mathsf{ALG}(\mathbf{f}) \text{ rejects}] < 1/2,$$

a contradiction. So any tester must make $\Omega((k/\ell)^{d/2})$ queries as claimed.

Proof of Lemma 14. We claim that **f** is close to a junta on $[n] \setminus \mathbf{J}$:

$$\begin{split} & \mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[\operatorname{dist}(\mathbf{f},\mathcal{J}_{[n]\setminus\mathbf{J}})] \\ &= \frac{1}{2^{k}} \mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}} \left[\sum_{\rho\in\{0,1\}^{[n]\setminus\mathbf{J}}} \sum_{y\in\{0,1\}^{\mathbf{J}}} \frac{1}{2^{\ell}} \left| \mathbf{f}(\rho\sqcup y) - \operatorname{MAJ}(\{\mathbf{f}(\rho\sqcup y): y\in\{0,1\}^{\mathbf{J}}\}) \right| \right] \\ &\leq p(1-2^{-r}) + e^{-2^{\ell} \cdot (1/2-p)^{2}/12} + \frac{1}{2^{k}} \mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}} \left[\sum_{\substack{\rho\in\{0,1\}^{[n]\setminus\mathbf{J}}} \sum_{y\in\{0,1\}^{\mathbf{J}}} \frac{1}{2^{\ell}} \left| \mathbf{f}(\rho\sqcup y) - \operatorname{MAJ}(\{\mathbf{f}(\rho\sqcup y): y\in\{0,1\}^{\mathbf{J}}\}) \right| \right] \end{split}$$

by Lemma 4. To bound the second term, let $\lambda_i = |\chi^{-1}(i)|/2^{\ell}$ and observe that for a fixed ρ with $\rho|_{[r]} = 0^r$ we have that

$$\begin{split} \mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}} \left[\sum_{y\in\{0,1\}^{\mathbf{J}}} \frac{1}{2^{\ell}} \left| \mathbf{f}(\rho\sqcup y) - \mathrm{MAJ}(\{\mathbf{f}(\rho\sqcup y): y\in\{0,1\}^{\mathbf{J}}\}) \right| \right] \\ &= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\mathbf{X}_{i}\sim\{-1,1\}} \left[\left| \sum_{i=1}^{|B_{\ell}(d)|} \lambda_{i}\mathbf{X}_{i} \right| \right] \leq \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\mathbf{X}_{i}\sim\{-1,1\}} \left[\left| \sum_{i=1}^{|B_{\ell}(d)|} \mathbf{X}_{i} \right| \right] = \Delta_{|B_{\ell}(d)|} \end{split}$$

by Lemma 9. Thus,

$$\mathbb{E}_{\mathbf{f},\mathbf{J}\sim\mathcal{D}_{YES}}[\text{dist}(\mathbf{f},\mathcal{J}_{[n]\setminus\mathbf{J}})] \le p(1-2^{-r}) + e^{-2^{\ell} \cdot (1/2-p)^2/12} + \Delta_{|B_{\ell}(d)|} 2^{-r}$$

as desired.

B Missing Proofs from Section 4

Proof of Lemma 15. Fix some function $h : \{0,1\}^{\ell} \to \{\pm 1\}$ and choose $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)} \in \{0,1\}^{\ell}$ uniformly and independently at random. Note that

$$\mathbb{E}_{\mathbf{x}^{(i)}}[h(\mathbf{x}^{(i)})] = \frac{\sum_{x \in \{0,1\}^{\ell}} h(x)}{2^{\ell}}$$

Moreover, since $h(\mathbf{x}^{(1)}), ..., h(\mathbf{x}^{(m)})$ are independent Chernoff-Hoeffding bounds give

$$\Pr_{\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}}\left[\left|\frac{1}{m}\sum_{i=1}^{m}h(\mathbf{x}^{(i)}) - \frac{\sum_{x\in\{0,1\}^{\ell}}h(x)}{2^{\ell}}\right| \ge \varepsilon\right] \le 2e^{-\Omega(\varepsilon^2 m)}$$

So we conclude that

$$\left| \mathbb{E}_{\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}} \left[\left| \frac{1}{m} \sum_{i=1}^{m} h(\mathbf{x}^{(i)}) \right| - \frac{\left| \sum_{x \in \{0,1\}^{\ell}} h(x) \right|}{2^{\ell}} \right] \right| \le \varepsilon + 4e^{-\Omega(\varepsilon^2 m)}.$$

It now follows that

$$\mathbb{E}_{\mathbf{f}\sim\mathcal{D}_1}\left[\left|\mathbb{E}_{\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}}\left[\left|\frac{1}{m}\sum_{i=1}^m (2\mathbf{f}(\mathbf{x}^{(i)})-1)\right| - \frac{\left|\sum_{x\in\{0,1\}^\ell} (2\mathbf{f}(x)-1)\right|}{2^\ell}\right]\right|\right] \le \varepsilon + 4e^{-\Omega(\varepsilon^2 m)}.$$

By Jensen's inequality,

$$\left| \mathbb{E}_{\mathbf{f} \sim \mathcal{D}_1, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}} \left[\left| \frac{1}{m} \sum_{i=1}^m (2\mathbf{f}(\mathbf{x}^{(i)}) - 1) \right| \right] - \mathbb{E}_{\mathbf{f} \sim \mathcal{D}_1} \left[\frac{\left| \sum_{x \in \{0,1\}^\ell} (2\mathbf{f}(x) - 1) \right|}{2^\ell} \right] \right| \le \varepsilon + 4e^{-\Omega(\varepsilon^2 m)}.$$

Analogously, we have that

$$\left| \mathbb{E}_{\mathbf{g} \sim \mathcal{D}_{2}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}} \left[\left| \frac{1}{m} \sum_{i=1}^{m} (2\mathbf{g}(\mathbf{x}^{(i)}) - 1) \right| \right] - \mathbb{E}_{\mathbf{g} \sim \mathcal{D}_{2}} \left[\frac{\left| \sum_{x \in \{0,1\}^{\ell}} (2\mathbf{g}(x) - 1) \right|}{2^{\ell}} \right] \right| \le \varepsilon + 4e^{-\Omega(\varepsilon^{2}m)}$$

But now as

$$\mathbb{E}_{\mathbf{f}\sim\mathcal{D}_{1},\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)}}\left[\left|\frac{1}{m}\sum_{i}(2\mathbf{f}(\mathbf{x}^{(i)})-1)\right|\right] = \mathbb{E}_{\mathbf{g}\sim\mathcal{D}_{2},\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)}}\left[\left|\frac{1}{m}\sum_{i}(2\mathbf{g}(\mathbf{x}^{(i)})-1)\right|\right]$$

we conclude that

$$\left| \mathbb{E}_{\mathbf{f} \sim \mathcal{D}_1} \left[\frac{\left| \sum_{x \in \{0,1\}^{\ell}} (2\mathbf{f}(x) - 1) \right|}{2^{\ell}} \right] - \mathbb{E}_{\mathbf{g} \sim \mathcal{D}_2} \left[\frac{\left| \sum_{x \in \{0,1\}^{\ell}} (2\mathbf{g}(x) - 1) \right|}{2^{\ell}} \right] \right| \le 2\varepsilon + 8e^{-\Omega(\varepsilon^2 m)}.$$

Finally, note that for a function $f:\{0,1\}^\ell \to \{0,1\}$

dist
$$(f, \{0, 1\}) = \frac{1}{2} - \frac{1}{2} \cdot \frac{\left|\sum_{x \in \{0, 1\}^{\ell}} (2f(x) - 1)\right|}{2^{\ell}}$$

Thus

$$\left| \mathbb{E}_{\mathbf{f} \sim \mathcal{D}_1} \left[\operatorname{dist}(\mathbf{f}, \{0, 1\}) \right] - \mathbb{E}_{\mathbf{g} \sim \mathcal{D}_2} \left[\operatorname{dist}(\mathbf{g}, \{0, 1\}) \right] \right| \leq \varepsilon + 4e^{-\Omega(\varepsilon^2 m)}.$$

Taking $\varepsilon = \Theta\left(\sqrt{\frac{\log(m)}{m}}\right)$ then gives the result.

Proof of Lemma 16. We consider the following algorithm, ALG: Set $r = \log(8/\varepsilon^2) + \log \log(32/\varepsilon)$. We'll assume that $\varepsilon \ge 1000 \cdot 2^{-(n-k)/10}$, which implies $r \le n-k$. Sample $m := 1024n^2/\varepsilon^2$ random points $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}$ and query every point in a ball of radius r around each $\mathbf{x}^{(i)}$. For each set $S \subseteq [n]$ of size k, compute

$$e_{S}(\mathbf{x}^{(i)}) := \frac{1}{|B_{n-k}(r)|} \left| \sum_{z \in B(\mathbf{x}^{(i)}, r) \cap \{y \in \{0,1\}^{n} : y_{S} = \mathbf{x}_{S}^{(i)}\}} (2\mathbf{f}(z) - 1) \right|$$

If for some set S,

$$\frac{1}{m}\sum_{i=1}^{m}\mathbb{I}(e_S(\mathbf{x}^{(i)}) > \varepsilon/2) \ge \varepsilon^2/8$$

then accept. Otherwise, reject.

Claim 1.

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{YES}}[ALG(\mathbf{f}) \ accepts] = 1 - o(1)$$

Proof. Let f be a function that is $(\frac{1}{2} - \varepsilon)$ -close to some junta on a set of relevant variables S of size k. Note

$$\operatorname{dist}(f, \mathcal{J}_S) = \frac{1}{2^k} \sum_{\rho \in \{0,1\}^S} \frac{1}{2} \left(1 - \frac{1}{2^{n-k}} \left| \sum_{y \in \{0,1\}^{[n] \setminus S}} 2f(\rho \sqcup y) - 1 \right| \right) \le \frac{1}{2} - \varepsilon.$$

Rearranging, we have that

$$\frac{1}{2^k} \sum_{\rho \in \{0,1\}^S} \frac{1}{2^{n-k}} \left| \sum_{y \in \{0,1\}^{[n] \setminus S}} 2f(\rho \sqcup y) - 1 \right| \ge 2\varepsilon.$$

By an averaging argument, it now follows that for at least $\varepsilon 2^k$ values of ρ we have that

$$\frac{1}{2^{n-k}} \left| \sum_{y \in \{0,1\}^{[n] \setminus S}} 2f(\rho \sqcup y) - 1 \right| \ge \varepsilon.$$

Fix a particular $\rho \in \{0,1\}^S$ such that the above holds and assume that

$$\frac{1}{2^{n-k}}\sum_{y\in\{0,1\}^{[n]\setminus S}}(2f(\rho\sqcup y)-1)\geq\varepsilon.$$

Now observe

$$\frac{1}{2^{n-k}} \sum_{y \in \{0,1\}^{[n] \setminus S}} \frac{1}{|B_{n-k}(r)|} \sum_{x \in B(y,r)} (2f(\rho \sqcup x) - 1) = \frac{1}{2^{n-k}} \sum_{y \in \{0,1\}^{[n] \setminus S}} (2f(\rho \sqcup y) - 1) \ge \varepsilon.$$

So by another averaging argument, we get that for at least $\varepsilon 2^{n-k-1}$ values of $y \in \{0,1\}^{[n]\setminus S}$

.

$$\frac{1}{|B_{n-k}(r)|} \left| \sum_{x \in B(y,r)} 2f(\rho \sqcup x) - 1 \right| \ge \varepsilon/2$$

An analogous argument gives the same result when

$$\frac{1}{2^{n-k}}\sum_{y\in\{0,1\}^{[n]\setminus S}}(2f(\rho\sqcup y)-1)\leq -\varepsilon.$$

We can now conclude that

$$\Pr_{\mathbf{x}_i}[e_S(\mathbf{x}^{(i)}) \ge \varepsilon/2] \ge \varepsilon^2/2$$

So by a Chernoff bound we conclude that

$$\Pr_{\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}}\left[\frac{1}{m}\sum_{i=1}^{m}\mathbb{I}(e_{S}(\mathbf{x}^{(i)}) > \varepsilon/2) \le \varepsilon^{2}/8\right] \le e^{-m\varepsilon^{2}/8}.$$

Since $m \ge 1024n^2/\varepsilon^2$, we conclude that we accept such a function f with high probability. Finally, as a function $\mathbf{f} \sim \mathcal{D}_{YES}$ is $(\frac{1}{2} - \varepsilon)$ close to a k-junta with high probability the result follows.

Claim 2.

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}[ALG(\mathbf{f}) \ accepts] = o(1)$$

Proof. Fix a set $S \subseteq [n]$ of size k. By a Hoeffding bound, we have that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO},\mathbf{x}^{(i)}}\left[e_{S}(\mathbf{x}^{(i)})\geq\varepsilon/2\right]\leq 2e^{-\varepsilon^{2}|B_{n-k}(r)|/8}\leq 2e^{-\varepsilon^{2}2^{r}/8}\leq 2e^{-\log(32/\varepsilon)}=\varepsilon/16$$

Since $r \leq n/4$, conditioned on dist $(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) \geq n/4$ for all $i \neq j$, we have that the events $\mathbb{I}(e_S(\mathbf{x}^{(i)}) \geq \varepsilon/2)$ are independent. (Note that without this assumption the events $\mathbb{I}(e_S(\mathbf{x}^{(i)}) \geq \varepsilon/2)$ are only independent given f.) Applying Chernoff bounds then gives us that

$$\Pr_{\mathbf{f}\sim\mathcal{D}_{NO}}\left[\frac{1}{m}\sum_{i=1}^{m}\mathbb{I}(e_{S}(\mathbf{x}_{i})>\varepsilon/2)\geq\varepsilon^{2}/8\middle|\operatorname{dist}(\mathbf{x}^{(i)},\mathbf{x}^{(j)})\geq n/4\quad\forall i\neq j\right]\leq e^{-\varepsilon m/64}.$$

Taking a union bound over all sets S,

$$\Pr_{\mathbf{f} \sim \mathcal{D}_{NO}} \left[\mathsf{ALG}(\mathbf{f}) \text{ accepts} \middle| \operatorname{dist}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) \ge n/4 \quad \forall i \neq j \right] \le n^k e^{-\varepsilon^2 m/64} = o(1).$$

It now remains to compute the probability that the $\mathbf{x}^{(i)}$'s are far from one another. By a Chernoff bound

$$\Pr_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left[\operatorname{dist}(\mathbf{x}^{(1)},\mathbf{x}^{(2)}) \le n/4 \right] \le e^{-n/16}.$$

A union bound then yields

$$\Pr_{\mathbf{x}^{(1)},\mathbf{x}^{(2)},\dots,\mathbf{x}^{(m)}} \left[\exists i \neq j : \quad \operatorname{dist}(\mathbf{x}^{(i)},\mathbf{x}^{(j)}) \le n/4 \right] \le m^2 e^{-n/16} \le 2^{20} \cdot n^4 \cdot 2^{n/32} \cdot e^{-n/16} = o(1).$$