# Proof of the Clustered Hadwiger Conjecture 

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#### Abstract

Hadwiger's Conjecture asserts that every $K_{h}$-minor-free graph is properly ( $h-1$ )-colourable. We prove the following improper analogue of Hadwiger's Conjecture: for fixed $h$, every $K_{h}$-minor-free graph is $(h-1)$-colourable with monochromatic components of bounded size. The number of colours is best possible regardless of the size of monochromatic components. It solves an open problem of Edwards, Kang, Kim, Oum and Seymour [SIAM J. Disc. Math. 2015], and concludes a line of research initiated in 2007. Similarly, for fixed $t \geqslant s$, we show that every $K_{s, t}$-minor-free graph is $(s+1)$-colourable with monochromatic components of bounded size. The number of colours is best possible, solving an open problem of van de Heuvel and Wood [J. London Math. Soc. 2018]. We actually prove a single theorem from which both of the above results are immediate corollaries. For an excluded apex minor, we strengthen the result as follows: for fixed $t \geqslant s \geqslant 3$, and for any fixed apex graph $X$, every $K_{s, t}$-subgraph-free $X$ -minor-free graph is $(s+1)$-colourable with monochromatic components of bounded size. The number of colours is again best possible.


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## 1 Introduction

### 1.1 Hadwiger's Conjecture

Our starting point is Hadwiger's Conjecture [31], which suggests a deep relationship between graph colourings and graph minors ${ }^{1}$. A colouring of a graph $G$ is a function that assigns one colour to each vertex of $G$. For an integer $k \geqslant 1$, a $k$-colouring is a colouring using at most $k$ colours. A colouring of a graph is proper if each pair of adjacent vertices receives distinct colours. The chromatic number $\chi(G)$ of a graph $G$ is the minimum integer $k$ such that $G$ has a proper $k$-colouring. A graph $H$ is a minor of a graph $G$ if $H$ is isomorphic to a graph that can be obtained from a subgraph of $G$ by contracting edges. A graph $G$ is $H$-minor-free if $H$ is not a minor of $G$. Let $K_{h}$ be the complete graph on $h$ vertices. Hadwiger [31] famously conjectured that every $K_{h}$-minor-free graph is properly $(h-1)$-colourable. This is widely considered to be one of the most important open problems in graph theory. By Wagner's characterization of $K_{5}$-minor-free graphs [67], the case $h=5$ is equivalent to the 4-Colour Theorem [1, 55]. The conjecture is true for $h \leqslant 6$ [59], and is open for $h \geqslant 7$. The best known upper bound on the chromatic number of $K_{h}$-minor-free graphs remained of order $O(h \sqrt{\log h})$ $[38,63]$ since the 1980 s, until a sequence of breakthrough results $[48,53]$ culminated in a $O(h \log \log h)$ bound due to Delcourt and Postle [10]. It is open whether every $K_{h^{-}}$ minor-free graph is $O(h)$-colourable. See Seymour's survey [61] for more on Hadwiger's Conjecture.

### 1.2 The Clustered Hadwiger Conjecture

As mentioned above, one of the main ways to approach Hadwiger's Conjecture has been to try to minimise the number of colours in a proper colouring of a $K_{h}$-minor-free graph. A second natural approach is to fix a number of colours close to Hadwiger's bound (at $h-1$ for instance), and try to obtain a colouring that is close to being proper. This leads to the notion of improper colourings of $K_{h}$-minor-free graphs; see [20, 22, 33-35, 40$45,47,49,50,52,65,68]$ and the references therein. A monochromatic component with respect to a colouring of a graph $G$ is a connected component of the subgraph of $G$ induced by all the vertices assigned a single colour. A colouring has clustering $c$ if every monochromatic component has at most $c$ vertices. Note that a colouring with clustering 1 is precisely a proper colouring. The clustered chromatic number $\chi_{\star}(\mathcal{G})$ of a graph class $\mathcal{G}$ is the minimum integer $k$ for which there exists an integer $c$ such that

[^1]every graph in $\mathcal{G}$ is $k$-colourable with clustering $c$. See [69] for an extensive survey on this topic, and see $[2,25]$ for connections between clustered colouring and asymptotic dimension in geometric group theory and site percolation in probability theory. One of the earliest works on clustered colouring was by Kleinberg, Motwani, Raghavan, and Venkatasubramanian [37], who used it as a tool to design algorithms for evolving databases.

Consider the clustered chromatic number of the class of $K_{h}$-minor-free graphs. So-called standard examples provide a lower bound of $h-1$ (regardless of the clustering function); see Proposition 9 below. A natural weakening of Hadwiger's Conjecture, sometimes called the Clustered Hadwiger Conjecture, asserts that every $K_{h}$-minor-free graph is $(h-1)$-colourable with clustering at most some function $f(h)$. This conjecture was first asked as an open problem by Edwards et al. [22]. This line of research was initiated in 2007 by Kawarabayashi and Mohar [35], who proved the first $O(h)$ upper bound on the number of colours. Their bound was $\left\lceil\frac{31}{2} h\right\rceil$, which was successively improved to $\left\lceil\frac{7 h-3}{2}\right\rceil$ by Wood [68] ${ }^{2}$, to $4 h-4$ by Edwards et al. [22], to $3 h-3$ by Liu and Oum [41], to $2 h-2$ independently by Norin [47], van den Heuvel and Wood [65] and Dvořák and Norin [20], and most recently to $h$ by Liu and Wood [44]. Note that Dvořák and Norin [20] showed that $h-1$ colours suffice for $h \leqslant 9$, and that Edwards et al. [22] proved that every $K_{h}$-minor-free graph has an $(h-1)$-colouring in which each monochromatic component has bounded maximum degree (which is significantly weaker than having bounded size).

Our first contribution is to prove the Clustered Hadwiger Conjecture ${ }^{3}$, thereby solving the above-mentioned open problem of Edwards et al. [22].

Theorem 1. Every $K_{h}$-minor-free graph is $(h-1)$-colourable with clustering at most some function $f(h)$.

### 1.3 Excluding a Complete Bipartite Minor

Consider the clustered chromatic number of the class of $K_{s, t}$-minor-free graphs, where $K_{s, t}$ is the complete bipartite graph with parts of size $s \geqslant 1$ and $t \geqslant s$. This question is of particular interest since the answer turns out to not depend on $t$. Van den Heuvel and Wood [65] proved a lower bound of $s+1$ (for $t \geqslant \max \{s, 3\}$; see Proposition 9 below), and observed that results of Edwards et al. [22] and Ossona de Mendez et al. [52] imply an upper bound of $3 s$, which was improved to $2 s+2$ by Dvořák and Norin [20], and

[^2]further improved to $s+2$ by Liu and Wood [42]. Our second main result resolves the question.

Theorem 2. Every $K_{s, t}$-minor-free graph is $(s+1)$-colourable with clustering at most some function $f(s, t)$.

Combined with the above-mentioned lower bound, Theorem 2 shows that the clustered chromatic number of the class of $K_{s, t}$-minor-free graphs equals $s+1$. This resolves an open problem proposed by van den Heuvel and Wood [65].

### 1.4 Colin de Verdière Parameter

The Colin de Verdière parameter $\mu(G)$ is an important graph invariant introduced by Colin de Verdière $[7,8]$; see $[60,66]$ for surveys. It is known that $\mu(G) \leqslant 1$ if and only if $G$ is a disjoint union of paths, $\mu(G) \leqslant 2$ if and only if $G$ is outerplanar, $\mu(G) \leqslant 3$ if and only if $G$ is planar, and $\mu(G) \leqslant 4$ if and only if $G$ is linklessly embeddable. A famous conjecture of Colin de Verdière [7] states that $\chi(G) \leqslant \mu(G)+1$, which implies the 4 -colour theorem and is implied by Hadwiger's Conjecture. The following clustered analogue was conjectured by Wood [69].

Theorem 3. The clustered chromatic number of the class of graphs with Colin de Verdière parameter $\mu$ equals $\mu+1$.

The lower bound in Theorem 3 is proved in [69]. The upper bound follows immediately from either Theorem 1 or Theorem 2 since graphs with Colin de Verdière parameter $\mu$ are $K_{\mu+2}$-minor-free $[7,8]$ and are $K_{\mu, \max \{\mu, 3\}-m i n o r-f r e e ~[66] . ~}^{\text {. }}$.

### 1.5 A Common Generalization

We in fact prove a common generalization of Theorems 1 and 2 using the following definition of Campbell, Clinch, Distel, Gollin, Hendrey, Hickingbotham, Huynh, Illingworth, Tamitegama, Tan, and Wood [6]. Let $\mathcal{J}_{s, t}$ be the set of all graphs $K_{s} \oplus T$ where $T$ is a $t$-vertex tree. Here the complete join $G_{1} \oplus G_{2}$ is the graph obtained from the disjoint union of graphs $G_{1}$ and $G_{2}$ by adding all edges between $G_{1}$ and $G_{2}$. A graph is $\mathcal{J}_{s, t}$-minor-free if it contains no graph in $\mathcal{J}_{s, t}$ as a minor ${ }^{4}$.

[^3]Theorem 4. Every $\mathcal{J}_{s, t}$-minor-free graph is $(s+1)$-colourable with clustering at most some function $f(s, t)$.

Since $K_{s, t}$ is a subgraph of each graph in $\mathcal{J}_{s, t}$, Theorem 4 implies Theorem 2. Since $\mathcal{J}_{h-2,2}=\left\{K_{h}\right\}$, Theorem 4 with $s=h-2$ implies Theorem 1.

The proof of Theorem 4 is constructive and yields a polynomial-time colouring algorithm (using known polynomial-time algorithms $[20,30,36]$ for finding the decomposition in Theorem 11 as a starting point).

### 1.6 Excluding a Complete Bipartite Subgraph

As mentioned above, Liu and Wood [42] proved that $K_{s, t}$-minor-free graphs are $(s+2)$ colourable with bounded clustering. They actually proved a stronger result, where the number of colours is determined by an excluded complete bipartite subgraph, as expressed in the following results. For a graph $X$, a graph $G$ is $X$-subgraph-free if no subgraph of $G$ is isomorphic to $X$.

Theorem 5 ([42, Theorem 5]). For any integers $s, t, w \in \mathbb{N}$, every $K_{s, t}$-subgraph-free graph of treewidth at most $w$ is $(s+1)$-colourable with clustering at most some function $f(s, t, w)$.

We use an extension of Theorem 5 as one of the tools in the current paper (see Lemma 26).
Theorem 6 ([42, Theorem 2]). For any integers $t \geqslant s \geqslant 1$ and for any graph $X$, every $K_{s, t}$-subgraph-free $X$-minor-free graph is $(s+2)$-colourable with clustering at most some function $f(s, t, X)$.

In both these theorems, the number of colours is best possible [42]. Our Theorem 2 uses fewer colours than Theorem 6 but makes a stronger assumption of excluding $K_{s, t}$ as a minor. So the results are incomparable.

Theorem 6 implies that $K_{h}$-minor-free graphs are $(h+1)$-colourable with bounded clustering (since $K_{h}$ is a minor of $K_{h-1, h-1}$ ). Liu and Wood [44] pushed this proof further to reduce the number of colours to $h$, as mentioned above. These results are presented over a series of three articles [42-44]. The main tool introduced in the first article of the series [43] shows (via a technical 70-page proof) that $K_{s, t}$-subgraph-free graphs of bounded layered treewidth are $(s+2)$-colourable with bounded clustering. It is

[^4]open whether $s+1$ colours (which would be best possible) suffice in this setting [42, 43]. Bounded layered treewidth is not a minor-closed property. However, for minor-closed classes, having bounded layered treewidth is equivalent to excluding an apex graph as a minor [18]. Here a graph is apex if it can be made planar by deleting at most one vertex. Our next result shows that $s+1$ colours suffice for apex-minor-free $K_{s, t}$-subgraph-free graphs.

Theorem 7. For any integers $t \geqslant s \geqslant 3$ and for any apex graph $X$, every $K_{s, t}$-subgraphfree $X$-minor-free graph is $(s+1)$-colourable with clustering at most some function $f(s, t, X)$.

Several notes on Theorem 7 are in order:

- Theorem 5 is equivalent to saying that $K_{s, t}$-subgraph-free graphs excluding a fixed planar graph as a minor are $(s+1)$-colourable with bounded clustering. Theorem 7 strengthens this result to the setting of apex-minor-free graphs (for $s \geqslant 3$ ).
- The bound on the number of colours in Theorem 7 is tight, simply because $s+1$ colours is tight for bounded treewidth graphs [42, 43].
- Some non-trivial lower bound on $s$ is needed in Theorem 7, since the hexagonal grid graph is $K_{5}$-minor-free and $K_{1,7}$-subgraph-free, but every 2-colouring has unbounded clustering by the Hex Lemma [32]. Theorem 7 with $s=2$ is open, even for planar graphs [43].
- Our proof of Theorem 7 in the case $s \geqslant 4$ is reasonably short and simple, and is presented in Section 2.5. The case $s=3$ is more difficult, and requires tools for dealing with $K_{2, t}$-subgraphs in surfaces that are also required by the proofs of Theorems 1, 2 and 4.

For the sake of clarity, we now summarise how the present paper uses results of Liu and Wood [42, 43, 44]. Lemma 15 is a result of Liu and Wood [42] for colouring $K_{s, t^{-}}$ subgraph-free graphs of bounded treewidth. We use this to provide a simple proof of $(h+4)$-colourability for $K_{h}$-minor-free graphs as a way to introduce some of the key ideas used in our main proof. Lemma 26 extends this lemma, and we provide a full proof that uses one lemma by Liu and Wood [43] (our Lemma 25) which has a simple 1-paragraph proof. Lemma 27 is another result for colouring bounded treewidth graphs that extends a result of Liu and Wood [44]. Again, we provide a full proof. The present paper does not use layered treewidth or the 70-page proof mentioned above.

### 1.7 Clustered 4-Colouring Theorems

It is well-known that the clustered chromatic number of the class of planar graphs equals 4, where the upper bound follows from the 4 -Colour Theorem or a weaker result of Cowen, Cowen, and Woodall [9], and the lower bound follows from the $s=3$ case of Proposition 9 below. In fact, much more general results are known for 4-colouring with bounded clustering.

The Euler genus of a surface with $h$ handles and $c$ cross-caps is $2 h+c$. The Euler genus of a graph $G$ is the minimum integer $g \geqslant 0$ such that there is an embedding of $G$ in a surface of Euler genus $g$; see [46] for more about graph embeddings in surfaces. Dvořák and Norin [20] proved the following elegant generalization of the 4-Colour Theorem.

Theorem 8 ([20]). Every graph of Euler genus g has a 4-colouring with clustering at most some function $f(g)$.

It follows from Euler's formula that graphs of Euler genus $g$ are $K_{3,2 g+3}$-minor-free. Thus the $s=3$ case of Theorem 2 generalises Theorem 8 to the setting of $K_{3, t}$-minor-free graphs. This setting is substantially more general since disjoint unions of $K_{5}$ are $K_{3,3^{-}}$ minor-free, but have unbounded Euler genus. This highlights the utility of considering excluded complete bipartite minors.

Theorem 7 with $s=3$ provides a further generalization where the $K_{3, t}$-minor-free assumption is relaxed to apex-minor-free and $K_{3, t}$-subgraph-free. Again, this is a substantial generalization since, for example, the 1-subdivision of $K_{3, n}$ is $K_{5}$-minor-free and $K_{3,3}$-subgraph-free, but contains a $K_{3, n}$ minor.

Dvořák and Norin's proof of Theorem 8 uses the so-called 'island' method. Our proof uses and builds on this approach; see Lemma 29 below.

### 1.8 Proof Outline

This section highlights the main challenges in adapting existing techniques to prove our results and give a high-level sketch of the proof of Theorem 4 that gives a rough idea of how we overcome these challenges. The key to our proofs is the novel use of 'graph product structure theory' in partnership with the Graph Minor Structure Theorem of Robertson and Seymour [58]. Graph product structure theory is a recently developed field that describes graphs in a complicated graph class as subgraphs of a product of simpler graphs (along with some other operations).

The starting point for these recent developments is the Planar Graph Product Structure Theorem of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [17], which says that
every planar graph $G$ is a subgraph of the strong product of a graph $H$ of treewidth 8 and a path $P$, written as $G \subseteq H \boxtimes P$. This often allows a question about planar graphs $(G)$ to be reduced to an analogous question about graphs of bounded treewidth $(H)$, which is usually easier to solve. Extending the solution for $H$ to the product $H \boxtimes P$ is often straightforward. This approach has been used to solve several decades-old problems in mathematics and theoretical computer science [3, 4, 13, 14, 17], including bounds on the queue number of planar graphs and the construction of nearly optimal adjacency labelling schemes for planar graphs.

This approach is, of course, limited to graph families that admit this type of product structure. Dujmović et al. [17] proved that apex-minor-free graphs are the most general minor-closed classes contained in the product of a bounded treewidth graph and a path, so this reduction cannot be applied to the graph classes considered by Theorems 1, 2 and 4 (except for a few small values of $h$ or $s$ ).

Since not all minor-closed classes admit the type of product structure described above, we start by applying the Graph Minor Structure Theorem of Robertson and Seymour [58], which describes graphs excluding a fixed minor in terms of graphs called 'torsos' that are formed from a surface-embedded subgraph, by adding vortices in the surface, and apex vertices with unrestricted neighbourhoods. The torsos are then pasted together in a tree-like way (described by a tree-decomposition) using clique-sums.

A natural strategy for colouring a $\mathcal{J}_{s, t}$-minor-free graph is to colour each torso one-byone beginning with the root torso. With this strategy, when colouring a subsequent non-root torso, the vertices it shares with its parent torso are already precoloured with colours assigned while colouring the parent torso. These precoloured vertices are in one clique-sum whose size is bounded by a function of $s$ and $t$, but this bound may be much larger than $s$, which is a major obstacle since Theorem 4 promises a colouring with at most $s+1$ colours.

For the purposes of this high-level description, call a clique-sum small if it has at most $s$ vertices, and large otherwise (although still of size bounded by a function of $s$ and $t)$. Using the approach described above, the precoloured vertices of torsos are either part of a small or a large clique-sum. Since the natural strategy is not applicable to large clique-sums we conduct those clique-sums before any colouring. In particular, we call a maximal collection of torsos pasted together by only large clique-sums a curtain. (This choice of name will become clear later.) Now one can think of the Robertson-Seymour decomposition as a collection of curtains glued together using small clique-sums, which we call a tree of curtains. To colour the entire graph, it suffices to colour each curtain one-by-one beginning with the curtain that contains the root torso. The critical advantage of proceeding this way is that each non-root curtain comes with
at most $s$ precoloured vertices.
To colour each individual curtain, we use a variant of the Graph Minor Structure Theorem by Dvořák and Thomas [21], which classifies apex vertices as either major or non-major. Edges incident to major apex vertices are unrestricted, but non-major apex vertices are not adjacent to vertices in the part of the surface-embedded subgraph avoiding the vortices. A graph is $k$-apex if it has $k$ vertices whose removal leaves a planar graph. Dvorák and Thomas [21] showed that for $k$-apex-minor-free graphs, the number of major apex vertices is at most $k-1$. In our case, $\mathcal{J}_{s, t}$ includes an $(s-2)$-apex graph. So the number of major apex vertices is at most $s-3$. From now on we group the non-major apex vertices with the vortices (loosely speaking).

An important consequence of the Dvořák and Thomas [21] result is that (after some manipulation) any large clique-sum only involves (major and non-major) apex vertices and vertices from vortices of the two torsos being summed. Therefore, none of the clique-sums used to make a curtain touch the surface-embedded parts of the torsos, except those vertices on the boundary of a vortex.

The above material is presented in Section 2. To illustrate the utility of these preliminary ideas, Section 2 also describes a short proof that $K_{h}$-minor-free graphs have clustered colourings using $h+4$ colours, and we prove Theorem 7 for $s \geqslant 4$.

Our task now is to colour each curtain, given a set of at most $s$ precoloured vertices. Recall that each torso in the curtain is described by a set of at most $s-3$ major apex vertices, a surface-embedded subgraph, and a collection of vortices and non-major apex vertices. We now apply the so-called 'island' method of Esperet and Ochem [24] and Dvořák and Norin [20]. A $d$-island is a set of vertices, each of which has at most $d$ neighbours outside $I$. If the surface-embedded subgraph contains a 3 -island $I$ disjoint from the vortices and of bounded size, then it is an $s$-island in the overall graph (including the at most $s-3$ major apex vertices as possible neighbours). Delete $I$, apply induction, and greedily colour each vertex in $I$ by a colour not used on the at most $s$ neighbours outside $I$. Each new monochromatic component is contained within $I$, and thus has bounded size. Now we may assume the surface-embedded subgraph of each torso (within a curtain) has no 3-island of bounded size disjoint from the vortices.

The next step employs graph product structure theory. A result of Dujmović et al. [17] says that each torso (without the major apex vertices) is a subgraph of the strong product of a bounded treewidth graph and a path (generalising the result for planar graphs mentioned above). This product structure can be described in terms of partitions and layerings. For each torso with the major apex vertices removed, we obtain a partition of the vertex-set with connected parts, and a layering such that the intersection of each part and each layer has bounded size (thus, loosely speaking, each part in the partition
is 'long and skinny'). In addition, the first layer contains exactly the vortices and the non-major apex vertices. This means that each large clique-sum used to paste together a pair of torsos in the curtain $G$ is restricted to the major apex vertices and the vertices in the first layer of each torso. Dujmović et al. [17] proved that the minor obtained by contracting each part of the partition results in a quotient graph that has bounded treewidth.

However, we do not perform the contractions that would create this bounded treewidth quotient graph, since doing so could interfere with a clique-sum and could introduce complete bipartite subgraphs that we wish to avoid for reasons that will become clear shortly. Instead, we reduce to a bounded treewidth graph by 'raising the curtain' as follows. Within each torso of a curtain $G$, we contract the parts of the corresponding partition while avoiding the first five layers. This produces a minor $G_{\uparrow}$ of $G$ on at most six layers and having bounded treewidth. Since these contractions avoid the first five layers of each torso, they avoid the vertices used in the clique-sums between torsos. Therefore, these clique-sums remain in $G_{\uparrow}$, and $G_{\uparrow}$ is also a curtain with a surface-embedded subgraph within each torso. Unlike $G, G_{\uparrow}$ has bounded treewidth.
However, it is not enough that $G_{\uparrow}$ has bounded treewidth. Indeed, the extremal examples have bounded treewidth (see Proposition 9). It is also critical that $G_{\uparrow}$ is a minor of $G$. However, because of the possibility of edges added to each torso, it is not immediate that $G_{\uparrow}$ is obtained by contracting connected subgraphs in $G$. To overcome this issue we show that our tree of curtains is 'lower-minor-closed'. We expect this property and in particular Theorem 22 may be of independent interest.

In Section 3 we prove two lemmas (that are technical extensions of results by Liu and Wood $[42,44]$ ) for colouring bounded treewidth graphs containing no $K_{s, t}$-subgraph. Our goal is to apply these results to $G_{\uparrow}$. However, without further work $K_{s, t}$-subgraphs may be present in $G_{\uparrow}$.

Any such problematic subgraph contains a large $K_{2, p}$-subgraph in the deeper layers of some surface-embedded subgraph of $G_{\uparrow}$. We eliminate these problematic $K_{2, p^{-}}$ surface-embedded subgraphs in two steps: (i) We do further contractions of connected surface-embedded subgraphs of $G_{\uparrow}$ that are separated from the rest of $G_{\uparrow}$ by short surfaceseparating cycles and major apex vertices. (ii) We remove certain redundant surface degree- 2 vertices. The result of this process is a minor $G_{\uparrow \bullet}$ of $G_{\uparrow}$ that shares a crucial property with $G_{\uparrow}$ : Each vertex of $G_{\uparrow}$ • corresponds to a long and skinny connected subgraph of $G$. (Establishing this property relies on the fact that 3 -islands have been previously removed from each surface-embedded subgraph.) These techniques for eliminating large $K_{2, t}$ subgraphs in surface-embedded graphs are presented in Section 4.1. We augment $G_{\uparrow}$. with a set of 'special' vertices, each of which dominates a connected
subgraph in the embedded part of some torso of $G_{\uparrow}$. The addition of these special vertices only increases the treewidth by a small amount. On this augmented graph we apply an enhanced version of the divide-and-conquer strategy for $K_{s, t}$-subgraph-free bounded treewidth graphs of Liu and Wood [42, 44] to find an $(s+1)$-colouring with bounded clustering where, for each special vertex $\alpha$, the subgraph dominated by $\alpha$ completely avoids the colour used by $\alpha$.

We now lower the curtain that was previously raised, and assign each vertex $w$ of $G$ the colour given to the vertex of $G_{\uparrow} \bullet$ that $w$ was contracted into. The resulting colouring of $G$ does not have bounded clustering, but its monochromatic components are all long and skinny (with respect to the layering and partition). This allows us to use the colours assigned to the special vertices as blocking colours to break any long and skinny monochromatic components into (short and skinny) pieces of bounded size.

In this way, we obtain an $(s+1)$-colouring of the curtain $G$ with bounded clustering. Then we use the top-down colouring strategy mentioned earlier to extend this colouring to an $(s+1)$-colouring of the tree of curtains, which is the original $\mathcal{J}_{s, t}$-minor-free graph. This completes the high-level description of the proof of Theorem 4.

The full proof of Theorem 4 is completed in Section 4.2. In Section 4.3 we complete the proof of Theorem 7 by explaining the changes needed for the case $s=3$. Section 5 concludes the paper by outlining polynomial-time algorithms for computing the colourings in Theorems 1, 2, 4 and 7.

Theorems 1 and 2 give the first optimal bounds on the clustered chromatic number of $K_{h}$-minor-free and $K_{s, t}$-minor-free graphs. At least as important as these results are the definitions and tools that we develop, including curtain decompositions, raised curtains, and skinnyness-preserving contractions to control $K_{2, t}$ subgraphs. Curtain decompositions have already found applications to other decomposition and colouring problems [16] and we expect that they will soon find more. Since the extremal examples for many problems include large $K_{2, t}$ subgraphs, we expect the tools for controlling $K_{2, t}$-subgraphs will also find further applications in different contexts. Indeed, the Planar Graph Product Structure Theorem mentioned above was initially developed to bound the queue-number of planar graphs [17], but has since been used to resolve a number of longstanding open problems on planar and other graph classes [3, 4, 12-15].

### 1.9 Lower Bounds

The number of colours in Theorems 1, 2 and 7 is best possible because of the following well-known 'standard' example [22, 39, 42, 65, 69], which we include for completeness.

Proposition 9. For each integer $s \geqslant 1$, there is a graph class $\mathcal{G}_{s}$, such that every graph in $\mathcal{G}_{s}$ has treewidth $s$ and is $K_{s, s+2}$-subgraph-free, and for any integer $c \geqslant 1$ there exists $G \in \mathcal{G}_{s}$ such that every s-colouring of $G$ has a monochromatic component on more than c vertices.

Proof. We proceed by induction on $s$, where $\mathcal{G}_{1}$ is the class of all paths on at least two vertices, which trivially satisfies the desired properties. Now assume the claim for some $s \geqslant 1$. As illustrated in Figure 1, let $\mathcal{G}_{s+1}$ be the class of all graphs obtained by taking arbitrarily many disjoint copies of graphs in $\mathcal{G}_{s}$ and adding one dominant vertex.


Figure 1: Members of the families $\mathcal{G}_{1}, \mathcal{G}_{2}$, and $\mathcal{G}_{3}$.
Every graph in $\mathcal{G}_{s+1}$ has treewidth $s+1$ and is $K_{s+1, s+3}$-subgraph-free. For any $c$, there is a graph $H$ in $\mathcal{G}_{s}$ such that any $s$-colouring of $G$ has a monochromatic component on more than $c$ vertices. Let $G$ be obtained from $c$ disjoint copies of $H$ by adding a dominant vertex $v$. So $G \in \mathcal{G}_{s+1}$. In any colouring of $G$ with clustering $c$, some copy of $H$ avoids the colour assigned to $v$, as otherwise the monochromatic component containing $v$ would have at least $c+1$ vertices. By assumption, $H$ has at least $s+1$ colours, implying $G$ has at least $s+2$ colours. This proves the claim.

Proposition 9 shows that $h-1$ colours in Theorem 1 is best possible, since every graph with treewidth $s$ is $K_{s+2}$-minor-free. Similarly, Proposition 9 shows that $s+1$ colours in Theorem 2 is best possible, since the graph in Proposition 9 is $K_{s, \max \{s, 3\}}$-minor-free [65]. To see that the number of colours in Theorem 7 is best possible, note that every graph in $\mathcal{G}_{s}$ contains no $(s+1) \times(s+1)$ grid as a minor (since the $(s+1) \times(s+1)$ grid has treewidth $s+1$ ). Thus $\mathcal{G}_{s}$ excludes a planar (and thus apex) graph as a minor. If every graph in $\mathcal{G}_{s}$ is $k$-colourable with clustering at most some function $f(s)$, then $k \geqslant s+1$ by Proposition 9.

### 1.10 Notation

We use the following notation for a graph $G$. For $v \in V(G)$. let $N_{G}(v):=\{w \in V(G)$ : $v w \in E(G)\}$ and $N_{G}[v]:=N_{G}(v) \cup\{v\}$. For $S \subseteq V(G)$, let $N_{G}[S]:=\bigcup_{v \in S} N_{G}[v]$ and $N_{G}(S):=N_{G}[S] \backslash S$.

For two vertices $v$ and $w$ in the same component of a graph $G$, let $\operatorname{dist}_{G}(v, w)$ be the number of edges in a shortest path from $v$ to $w$ in $G$. If $v$ and $w$ are in distinct components of $G$ then $\operatorname{dist}_{G}(v, w):=\infty$. For a subset $S \subseteq V(G)$, we write $\operatorname{dist}_{G}(v, S):=$ $\min \left\{\operatorname{dist}_{G}(v, w): w \in S\right\}$. For $d \in \mathbb{N}$, let $N_{G}^{d}[S]:=\left\{v \in V(G): \operatorname{dist}_{G}(v, S) \leqslant d\right\}$.

## 2 Structure of $k$-Apex-Minor-Free Graphs

### 2.1 Graph Minor Structure Theorem

The Graph Minor Structure Theorem of Robertson and Seymour [58] describes the structure of graphs excluding a fixed minor using four ingredients: tree-decompositions, graphs on surfaces, vortices, and apex vertices. To describe this formally, we need the following definitions.

A tree-decomposition of a graph $G$ is a collection $\left(B_{x}: x \in V(T)\right)$ of subsets of $V(G)$ (called bags) indexed by the vertices of a tree $T$, such that (a) for every edge $u v \in E(G)$, some bag $B_{x}$ contains both $u$ and $v$, and (b) for every vertex $v \in V(G)$, the set $\left\{x \in V(T): v \in B_{x}\right\}$ induces a non-empty (connected) subtree of $T$. For each edge $x y \in E(T)$ the set $B_{x} \cap B_{y}$ is called an adhesion set. The adhesion of $\left(B_{x}: x \in V(T)\right)$ is $\max \left\{\left|B_{x} \cap B_{y}\right|: x y \in E(T)\right\}$. The width of $\left(B_{x}: x \in V(T)\right)$ is $\max \left\{\left|B_{x}\right|: x \in V(T)\right\}-1$. A path-decomposition is a tree-decomposition in which the underlying tree is a path, simply denoted by the corresponding sequence of bags $\left(B_{1}, \ldots, B_{n}\right)$. The treewidth of a graph $G$, denoted by $\operatorname{tw}(G)$, is the minimum width of a tree-decomposition of $G$.

If $\left(B_{x}: x \in V(T)\right)$ is a tree-decomposition of a graph $G$, then the torso $G\left\langle B_{x}\right\rangle$ of a bag $B_{x}$ is the graph obtained from the induced subgraph $G\left[B_{x}\right]$ by adding edges so that $B_{x} \cap B_{y}$ is a clique for each edge $x y \in E(T)$.

A rooted tree consists of a tree $T$ and a distinguished vertex of $T$ called the root. The depth of a vertex $v$ in a tree $T$ rooted at $r \in V(T)$ is $\operatorname{dist}_{T}(r, v)$. A tree-decomposition ( $B_{x}: x \in V(T)$ ) of a graph $G$ is rooted if $T$ is rooted. If $T$ is rooted at $r \in V(T)$, then $G\left\langle B_{r}\right\rangle$ is called the root torso.

Let $G_{0}$ be a graph embedded in a surface $\Sigma$. A closed disc $D$ in $\Sigma$ is $G_{0}$-clean if the interior of $D$ is disjoint from $G_{0}$, and the boundary of $D$ only intersects $G_{0}$ in vertices of


Figure 2: A $(0, \ell)$-almost-embedded graph and a $(3, \ell)$-almost-embedded graph. The $G_{0}$-clean discs used to define vortices are yellow. Edges and vertices that participate in vortices are red. Non-major apex vertices and their incident edges are dark cyan. Major apex vertices and their incident edges are purple. Non-top edges and vertices are black.
$V\left(G_{0}\right)$. Let $x_{1}, \ldots, x_{n}$ be the vertices of $G_{0}$ on the boundary of $D$ in the order around $D$. A $D$-vortex consists of a graph $H$ and a path-decomposition $\left(B_{1}, \ldots, B_{n}\right)$ of $H$ such that $x_{i} \in B_{i}$ for each $i \in\{1,2, \ldots, n\}$, and $V\left(G_{0} \cap H\right)=\left\{x_{1}, \ldots, x_{n}\right\}$.

As illustrated in Figure 2, for integers $a, \hat{a}, g, r \geqslant 0$ and $w \geqslant 1$, an ( $a, \hat{a}, g, r, w$ )-almostembedding of a graph $G$ is a tuple $\mathcal{E}:=\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{r}\right)$ such that:

- $A \subseteq \hat{A} \subseteq V(G)$ with $|A| \leqslant a$ and $|\hat{A}| \leqslant \hat{a} ;$
- $G_{0}, G_{1}, \ldots, G_{r}$ are subgraphs of $G$ such that $G-\hat{A}=G_{0} \cup G_{1} \cup \cdots \cup G_{r}$;
- $N_{G}(v) \subseteq \hat{A} \cup V\left(G_{1}\right) \cup \cdots \cup V\left(G_{r}\right)$ for each $v \in \hat{A} \backslash A$;
- $G_{1}, \ldots, G_{r}$ are pairwise vertex-disjoint;
- $G_{0}$ is embedded in a surface $\Sigma$ of Euler genus at most $g$;
- there are pairwise disjoint $G_{0}$-clean discs $D_{1}, \ldots, D_{r}$ in $\Sigma$; and
- $G_{i}$ is a $D_{i}$-vortex with a path-decomposition $\left(B_{1}, \ldots, B_{n_{i}}\right)$ of width at most $w$, for each $i \in\{1,2, \ldots, r\}$.
The top of $G$ (with respect to $\mathcal{E}$ ) is $\hat{A} \cup V\left(G_{1} \cup \cdots \cup G_{r}\right)$. The near-top of $G$ is the union of the top of $G$ and the neighbourhood of the top of $G$ in $G_{0}$.

The graph $G_{0}$ is called the embedded part of $\mathcal{E}$. The vertices in $\hat{A}$ are called apex vertices; those in $A$ are called major apex vertices and those in $\hat{A} \backslash A$ are called non-major apex
vertices. Major apex vertices can be adjacent to any vertex of $G$ but the neighbourhood of non-major apex vertices is restricted to the top of $G$. The precise number, $a$, of major apex vertices in an almost-embedding is critical in this work, so we say that a graph is $(a, \ell)$-almost-embeddable if it has an $(a, \hat{a}, g, r, w)$-almost-embedding for some $\hat{a}, g, r, w \leqslant \ell$. A graph $G$ equipped with an $(a, \ell)$-almost-embedding of $G$ is said to be ( $a, \ell$ )-almost-embedded.

We need the following result, which is probably well known to experts in the field, but for which we have not been able to find any reference. A closed curve in a surface $\Sigma$ is contractible if it bounds a region of $\Sigma$ homeomorphic to a disc. For any terms related to graphs on surfaces that are not defined here, the reader is referred to the monograph by Mohar and Thomassen [46].

Lemma 10. Let $G$ be a 2-connected graph embedded in a surface $\Sigma$ so that every cycle in $G$ is contractible. Then there is a region $R \subseteq \Sigma$ bounded by a cycle of $G$ such that $R$ is homeomorphic to a disc and $G \subseteq R$, and in particular $G$ is planar.

Proof. First suppose that $\Sigma$ is the sphere. Then the embedding of $G$ in $\Sigma$ is planar and since $G$ is 2-connected each face $f$ bounds a cycle, implying $\Sigma \backslash f$ is homeomorphic to a disc and contains $G$.

Now assume that $\Sigma$ has positive Euler genus, which implies that the embedding of $G$ in $\Sigma$ is not cellular (since any cellular embedding in a surface of positive Euler genus contains a non-separating cycle, which is non-contractible). It follows that there is a face $f$ of $G$ that is not homeomorphic to a disc.

Consider a boundary walk $W$ of $f$. Note that a priori, $W$ might not be a cycle, but since cycles form a basis for closed walks in $G$ and all cycles of $G$ are contractible, $W$ is also contractible. Consider an arbitrary small tubular neighbourhood $N$ of $W$ on $\Sigma$, and let $C$ be the closed curve bounding $f \backslash N$ on $\Sigma$. Then $C$ is contractible, and the connected region $R$ of $\Sigma-C$ contained in $f$ is not homeomorphic to a disc. Thus $\Sigma-R$ is homeomorphic to a disc, and moreover, $\Sigma-R$ contains $G$. It follows that $G$ has a planar embedding in $\Sigma-R$ and since $G$ is 2 -connected, all its faces in the embedding are bounded by cycles. In particular, $W$ is indeed a cycle. We conclude that $G$ has an embedding in a region of $\Sigma$ that is homeomorphic to a disc, and is bounded by a cycle of $G$.

A graph $X$ is $k$-apex if $X-A$ is planar for some set $A \subseteq V(X)$ of at most $k$ vertices. We use the following version of the Graph Minor Structure Theorem, due to Dvořák and Thomas [21]. (A version of Theorem 11 also appears in Dvorák and Kawarabayashi [19, Theorem 16].)

Theorem 11 ([21, Theorem 12]). For every integer $k \geqslant 1$ and every $k$-apex graph $X$ there exists an integer $\ell_{0}$ such that every $X$-minor-free graph $G$ has a tree-decomposition ( $\left.B_{x}: x \in V(T)\right)$ such that for each node $x \in V(T)$ :
(i) the torso $G\left\langle B_{x}\right\rangle$ is a $\left(k-1, \ell_{0}\right)$-almost-embedded graph; and
(ii) every 3-cycle in the embedded part of $G\left\langle B_{x}\right\rangle$ is the boundary of a 2-cell face (that is, a face homeomorphic to an open disc).

For graphs $H$ and $G$, an $H$-model in $G$ is a set $\left\{G_{x}: x \in V(H)\right\}$ of pairwise vertexdisjoint connected subgraphs of $G$, such that for each $x y \in E(H)$ there is an edge of $G$ between $G_{x}$ and $G_{y}$. Each subgraph $G_{x}$ is called a branch set of the model. Observe that $H$ is a minor of $G$ if and only if there is an $H$-model in $G$. An $H$-model $\left\{G_{x}: x \in V(H)\right\}$ in $G$ is faithful if each vertex $x \in V(H)$ is in $G_{x}$. That is, in a faithful model, each branch set is indexed by some vertex of $G$ that is in the branch set it indexes. If there is a faithful $H$-model in $G$, then $H$ is a faithful minor of $G$. This terminology is due to Grohe [29].

Let $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ be a tree-decomposition of a graph $G$ in which each torso $G\left\langle B_{x}\right\rangle$ is $(a, \ell)$-almost-embedded. For each $x \in V(T)$, the lower torso $G\left\{B_{x}\right\}$ is the supergraph of $G\left[B_{x}\right]$ obtained by adding each edge $u v \in E\left(G\left\langle B_{x}\right\rangle\right) \backslash E\left(G\left[B_{x}\right]\right)$ if neither $u$ nor $v$ is in the top of $G\left\langle B_{x}\right\rangle$. Equivalently, $G\left\{B_{x}\right\}$ is the subgraph of $G\left\langle B_{x}\right\rangle$ in which edge $u v \in E\left(G\left\langle B_{x}\right\rangle\right)$ is removed if and only if $u v \notin E\left(G\left[B_{x}\right]\right)$ and at least one of $u$ or $v$ is in the top of $G\left\langle B_{x}\right\rangle$. We say that $\mathcal{T}$ is lower-minor-closed if, for each $x \in V(T)$, $G$ contains a faithful $G\left\{B_{x}\right\}$-model $\left\{G_{v}: v \in B_{x}\right\}$ such that, for each vertex $v$ in the top of $G\left\langle B_{x}\right\rangle, G_{v}$ consists only of the vertex $v$. Note that this implies that $G\left\{B_{x}\right\}$ is a minor of $G$.

The definitions of torso and lower-minor-closed are motivated by the fact that we will eventually contract subgraphs of $G\left[B_{x}\right]$ that are connected in $G\left\langle B_{x}\right\rangle$ (but not necessarily connected in $G\left[B_{x}\right]$ ) in order to obtain a bounded treewidth graph $G_{\uparrow}^{x}$. Each connected subgraph of $G\left\langle B_{x}\right\rangle$ that we contract will avoid all vertices in the top of $G\left\langle B_{x}\right\rangle$, so it is also a connected subgraph of $G\left\{B_{x}\right\}$. This ensures that, even though we contracted sets of vertices that are not connected in $G\left[B_{x}\right]$, the contracted graph $G_{\uparrow}^{x}$ is a minor of $G\left\{B_{x}\right\}$, so it is a minor of $G$. This ensure that $G_{\uparrow}^{x}$ is $X$-minor-free if $G$ is $X$-minor-free.

The tree-decomposition of Theorem 11 can be modified to obtain the decomposition described in Lemma 12 below. The proof of Lemma 12, which involves some careful restructuring of the $\left(k-1, \ell_{0}\right)$-almost-embedded torsos in the tree-decomposition of Theorem 11, is presented in Appendix A. It is also possible to prove Theorems 4 and 7 with a version of this lemma that does not guarantee Property (3) (see Lemma 44 in Appendix A), by working alternately with torsos $G\left\langle B_{x}\right\rangle$ and induced subgraphs $G\left[B_{x}\right]$. We prefer to work with Lemma 12 because it makes our proof conceptually simpler and
is useful, even critical, in other applications [16].
Lemma 12. For every integer $k \geqslant 1$ and every $k$-apex graph $X$ there exists an integer $\ell$ such that every $X$-minor-free graph $G$ has a rooted tree-decomposition $\mathcal{T}:=\left(B_{x}: x \in\right.$ $V(T))$ such that:
(1) for each $x \in V(T)$, the torso $G\left\langle B_{x}\right\rangle$ is a $(k-1, \ell)$-almost-embedded graph;
(2) for each edge $x y$ of $T$ where $y$ is the parent of $x$;
(a) $B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$,
(b) $B_{x} \cap B_{y}$ is contained in the near-top of $G\left\langle B_{y}\right\rangle$ or $\left|B_{x} \cap B_{y}\right| \leqslant k+2$, and
(c) $B_{x} \cap B_{y}$ contains at most three vertices not in the top of $G\left\langle B_{y}\right\rangle$; and
(3) $\mathcal{T}$ is lower-minor-closed.

Point (3) implies the existence of a faithful $G\left\{B_{x}\right\}$-model $\mathcal{M}_{x}:=\left\{G_{v}: v \in B_{x}\right\}$ in $G$ for each $x \in V(T)$. It is worth noting that, for each such $G\left\{B_{x}\right\}$-model and each $v \in B_{x}$, $V\left(G_{v}\right) \cap B_{x}=\{v\}$ and $V\left(G_{v}\right) \cap B_{z} \subseteq\{v\}$ for any node $z$ that is not a descendant of $x$. The first of these facts follows from the faithfulness of $\mathcal{M}_{x}$, so that $v^{\prime}$ is a vertex of $G_{v^{\prime}}$ and therefore not a vertex of $G_{v}$ for any $v^{\prime} \in B_{x} \backslash\{v\}$. The second follows from the fact that, if $y$ is the parent of $x$, then each vertex $B_{x} \cap B_{y}$ is in the top of $G\left\langle B_{x}\right\rangle$. If $v \in B_{x} \cap B_{y}$, then $G_{v}=(\{v\}, \varnothing)$. Otherwise, the vertices in $B_{x} \cap B_{y}$ (none of which are in $\left.G_{v}\right)$ separate $v$ from every vertex in $B_{z} \backslash\left(B_{x} \cap B_{y}\right)$.

### 2.2 Pre-Curtains

As illustrated in Figure 3, a graph $G$ is a $(k, \ell)$-pre-curtain if it has a rooted treedecomposition $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ in which each torso $G\left\langle B_{x}\right\rangle$ is a $(k, \ell)$-almostembedded graph, and for each edge $x y$ of $T$ where $y$ is the parent of $x, B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$ and in the near-top of $G\left\langle B_{y}\right\rangle$. We say that $G$ is a $(k, \ell)$-pre-curtain described by $\mathcal{T}$. The top of $G$ is the union of the tops of the torsos of $\mathcal{T}$ and the near-top of $G$ is the union of the near-tops of the torsos of $\mathcal{T}$. Note that any vertex of $G$ that is not in the top of $G$ appears in exactly one bag of $\mathcal{T}$. We recall that the root torso of $G$ is the torso associated the root bag of $\mathcal{T}$.

A graph $G$ is a tree of $(k, \ell)$-pre-curtains if it has a rooted tree-decomposition $\mathcal{T}:=$ ( $\left.B_{x}: x \in V(T)\right)$ such that:

- for each $x \in V(T), G\left\langle B_{x}\right\rangle$ is a $(k, \ell)$-pre-curtain, described by some rooted treedecomposition $\mathcal{T}_{x}$,
- for each edge $x y$ of $T$ where $y$ is the parent of $x, B_{x} \cap B_{y}$ has size at most $k+3$ and is contained in the top of the root torso of $G\left\langle B_{x}\right\rangle$ in $\mathcal{T}_{x}$.


Figure 3: $\mathrm{A}(k, \ell)$-pre-curtain

Again, we say that $G$ is a tree of $(k, \ell)$-pre-curtains described by $\mathcal{T}$.
Let $G$ be a tree of pre-curtains described by the tree-decomposition $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$. For each $x \in V(T)$, the lower torso $G\left\{B_{x}\right\}$ is the supergraph of $G\left[B_{x}\right]$ obtained by adding each edge $u v \in E\left(G\left\langle B_{x}\right\rangle\right)$ if neither $u$ nor $v$ is in the top of $G\left\langle B_{x}\right\rangle$. The tree of pre-curtains $G$ is lower-minor-closed if, for each $x \in V(T), G$ contains a faithful $G\left\{B_{x}\right\}$-model $\left\{G_{v}: v \in B_{x}\right\}$ such that, for each vertex $v$ in the top of $G\left\langle B_{x}\right\rangle, G_{v}$ consists only of the vertex $v$. Note that the wording of these definitions is identical to that of lower torso and lower-minor-closed in a tree decomposition whose torsos are almost-embedded graphs; the difference is that each torso $G\left\langle B_{x}\right\rangle$ in a tree-of-pre-curtains is a pre-curtain described by a tree decomposition $\mathcal{T}_{x}$. The top of the pre-curtain $G\left\langle B_{x}\right\rangle$ is the union of the tops of the (almost-embedded) torsos of $\mathcal{T}_{x}$.

Lemma 13. For every integer $k \geqslant 1$ and every $k$-apex graph $X$ there exists an integer $\ell$ such that every $X$-minor-free graph $G$ is a lower-minor-closed tree of $(k-1, \ell)$-precurtains.

Proof. Let $\mathcal{T}_{0}:=\left(B_{x}: x \in V\left(T_{0}\right)\right)$ be the rooted tree-decomposition of $G$ guaranteed by Lemma 12. For an edge $x y$ of $T_{0}$ where $y$ is the parent of $x$, we say that $x y$ is $k$-heavy if $B_{x} \cap B_{y}$ is contained in the near-top of $G\left\langle B_{y}\right\rangle$, and call it $k$-light otherwise. Observe that, by property (2b) of Lemma $12,\left|B_{x} \cap B_{y}\right| \leqslant k+2$ for each $k$-light edge $x y$ of $T_{0}$.
By removing all $k$-light edges of $T_{0}$ we obtain a forest with component trees $T_{1}, \ldots, T_{p}$ rooted at $r_{1}, \ldots, r_{p}$, respectively. For each $i \in\{1, \ldots, p\}$, let $B_{i}:=\bigcup_{x \in V\left(T_{i}\right)} B_{x}$, let $G_{i}:=G\left[B_{i}\right]$ and let $\mathcal{T}_{i}:=\left(B_{x}: x \in V\left(T_{i}\right)\right)$. Since each edge $x y$ in $T_{i}$ is $k$-heavy, if $y$ is the parent of $x$, then by definition, $B_{x} \cap B_{y}$ is contained in the near-top of $G\left\langle B_{y}\right\rangle$. By property (2a) of Lemma $12, B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$. Therefore, for each $i \in\{1, \ldots, p\}, G_{i}$ is a $(k-1, \ell)$-pre-curtain described by $\mathcal{T}_{i}$.

Let $T$ be the tree with vertex set $V(T):=\{1, \ldots, p\}$ in which $i j \in E(T)$ if and only if some $k$-light edge of $T_{0}$ has an endpoint in $V\left(T_{i}\right)$ and an endpoint in $V\left(T_{j}\right)$. Root $T$ at the index $r \in V(T)$ such that $T_{r}$ contains the root of $T_{0}$. Let $\mathcal{T}:=\left(B_{i}: i \in V(T)\right)$. We claim that $G$ is a tree of $(k-1, \ell)$-precurtains described by $\mathcal{T}$, in which each torso $G\left\langle B_{i}\right\rangle$ is a $(k-1, \ell)$-pre-curtain described by $\mathcal{T}_{i}$. Each edge $i j$ of $T$ with $j$ the parent of $i$ corresponds to a $k$-light edge $x y$ of $T_{0}$ with $y \in V\left(T_{j}\right)$ and $x \in V\left(T_{i}\right)$. Since $T$ is rooted at $r, y$ is the parent of $x$ in $T_{0}$. By property (2a) of Lemma 12, $B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$, which is the root torso of the curtain $G\left\langle B_{i}\right\rangle$. Thus $\mathcal{T}$ satisfies the first requirement for a tree of $(k-1, \ell)$-curtains. Since $x y$ is $k$-light, $\left|B_{i} \cap B_{j}\right|=\left|B_{x} \cap B_{y}\right| \leqslant k+2$. Thus $\mathcal{T}$ satisfies the second requirement for a tree of ( $k-1, \ell$ )-curtains.

It remains to show that the tree of $(k-1, \ell)$-pre-curtains $G$ described by $\mathcal{T}$ is lower-minor-closed. That is, we must show that for each $i \in\{1, \ldots, p\}$, there is a faithful $G\left\{B_{i}\right\}$-model $\mathcal{M}_{i}:=\left\{G_{v}: v \in B_{i}\right\}$ in $G$, and for any vertex $v$ in the top of $G\left\langle B_{i}\right\rangle$, $G_{v}$ consists only of the vertex $v$ in $\mathcal{M}_{i}$. The model $\mathcal{M}_{i}:=\left\{G_{v}: v \in B_{i}\right\}$ is defined as follows. By property (3) of Lemma 12 , for each $x \in V\left(T_{0}\right)$, there exists a faithful $G\left\{B_{x}\right\}$-model $\mathcal{M}_{x}:=\left\{G_{v}^{x}: v \in B_{x}\right\}$ in $G$ such that for any vertex $v$ in the top of $G\left\langle B_{x}\right\rangle, G_{v}^{x}$ only contains the vertex $v$. For each $v \in B_{i}$, if $v$ is in the top of $G\left\langle B_{i}\right\rangle$, then define $G_{v}$ to only consist of the vertex $v$. Otherwise, define $G_{v}$ to be the component of $G_{v}^{x}-\left(B_{i} \backslash\{v\}\right)$ that contains $v$, where $B_{x}$ is the unique bag in $\mathcal{T}_{i}$ that contains $v$. Now we verify that $\mathcal{M}_{i}$ is a faithful $G\left\{B_{i}\right\}$-model in $G$. Obviously $\mathcal{M}_{i}$ is faithful because $\mathcal{M}_{x}$ is faithful for each $x \in V\left(T_{i}\right)$.

Next we verify that the branch-sets of $\mathcal{M}_{i}$ are pairwise vertex-disjoint. Let $v$ and $v^{\prime}$ be distinct vertices in $B_{i}$, and let $x$ and $x^{\prime}$ be the minimum-depth nodes of $T_{i}$ such that $v \in B_{x}$ and $v^{\prime} \in B_{x^{\prime}}$. Then $G_{v} \subseteq G_{v}^{x} \backslash\left(B_{i} \backslash\{v\}\right)$ and $G_{v^{\prime}} \subseteq G_{v^{\prime}}^{x^{\prime}} \backslash\left(B_{i} \backslash\left\{v^{\prime}\right\}\right)$. If $x=x^{\prime}$ then $G_{v}$ and $G_{v^{\prime}}$ are vertex-disjoint since they are each subgraphs of the branch sets $G_{v}^{x}$ and $G_{v^{\prime}}^{x}$ in the $G\left\{B_{x}\right\}$-model $\mathcal{M}_{x}$. If $x \neq x^{\prime}$ then we may assume without loss of generality that $x$ is not an ancestor of $x^{\prime}$. Let $y$ be the parent of $x$. Since $T_{i}$ is a (connected) tree that contains $x$ and $x^{\prime}, y$ is in $T_{i}$. By the definition of $x, v \notin B_{y}$, so $v \notin B_{x} \cap B_{y}$. Therefore the component of $G-\left(B_{x} \cap B_{y}\right)$ that contains $v$ does not contain $v^{\prime}$. If $v^{\prime} \notin B_{x} \cap B_{y}$ then this implies that $G_{v^{\prime}}$ and $G_{v}$ are vertex-disjoint since they are contained in different components of $G-\left(B_{x} \cap B_{y}\right)$. If $v^{\prime} \in B_{x} \cap B_{y}$ then $v^{\prime}$ is in the top of (the almost-embedded graph) $G\left\langle B_{x}\right\rangle$, so $v^{\prime}$ is in the top of (the curtain) $G\left\langle B_{i}\right\rangle$, so $G_{v^{\prime}}$, which only contains the vertex $v^{\prime}$, is also vertex-disjoint from $G_{v}$.

Finally, we verify that for each edge $v w \in G\left\{B_{j}\right\}, G_{v}$ and $G_{w}$ are adjacent in $G$. If $v w \in E\left(G\left[B_{j}\right]\right)$, then this follows immediately from the fact that $\mathcal{M}_{i}$ is faithful. Otherwise, $v w \in E\left(G\left\{B_{j}\right\}\right) \backslash E\left(G\left[B_{j}\right]\right)$, which implies that $T$ contains an edge $i j$ with $B_{i} \cap B_{j} \supseteq\{v, w\}$. The existence of $i j \in E(T)$ implies that there exists a $k$ -
light edge $x y \in E\left(T_{0}\right)$ where $x \in V\left(T_{i}\right), y \in V\left(T_{j}\right)$, and $\{v, w\} \subseteq B_{x} \cap B_{y}$. Since $v w \in E\left(G\left\{B_{j}\right\}\right) \backslash E\left(G\left[B_{j}\right]\right)$, neither $v$ nor $w$ are in the top of the curtain $G\left\langle B_{j}\right\rangle$. Therefore $y$ is the parent of $x$ in $T_{0}$. Furthermore, $B_{y}$ is the only bag of $\mathcal{T}_{j}$ that contains $v$ and $B_{y}$ is the only bag of $\mathcal{T}_{j}$ that contains $w$. Therefore $G_{v}=G_{v}^{y} \backslash\left(B_{j} \backslash\{v\}\right)=G_{v}^{y}$ and $G_{w}=G_{w}^{y} \backslash\left(B_{j} \backslash\{w\}\right)=G_{w}^{y}$. Therefore $G_{v}$ and $G_{w}$ are adjacent in $G$ since $\mathcal{M}_{y}$ is a $G\left\{B_{y}\right\}$-model in $G$.

Say that a colouring of a graph $G$ properly extends a precolouring of a set $Y \subseteq V(G)$ if no monochromatic component of $G$ has a vertex in $Y$ and a vertex in $V(G) \backslash Y$; that is, each monochromatic component is a subgraph of $G[Y]$ or of $G-Y$. In particular this happens if and only if each vertex in $N_{G}(Y)$ is coloured differently to all its neighbours in $Y$. Lemma 13 allows the problem of colouring an $X$-minor-free graph to be reduced to the problem of colouring a tree of pre-curtains. We now show that for this task, it is sufficient to be able to properly extend precolourings in pre-curtains.

Lemma 14. Let $G$ be a graph that is a tree of $(k, \ell)$-pre-curtains described by $\mathcal{T}:=\left(B_{x}\right.$ : $x \in V(T))$ and such that, for any $(k, \ell)$-pre-curtain $G\left\langle B_{x}\right\rangle$ of $G$, any precolouring of a set $S$ of at most $k+3$ vertices in the top of the root torso of $G\left\langle B_{x}\right\rangle$ can be properly extended to an m-colouring of $G\left[B_{x}\right]$ with clustering at most $c$. Then $G$ has an m-colouring with clustering at most $c$.

Proof. Let $\left(B_{x}: x \in V(T)\right)$ be a tree-decomposition of $G$ rooted at a node $r \in V(T)$ as in the definition of a tree of $(k, \ell)$-curtains. Let $x_{0}, \ldots, x_{r}$ be the nodes of $T$, ordered so that $x_{0}$ is the root of $T$ and, for each $i \geqslant 1$, the parent of $x_{i}$ is among $x_{0}, \ldots, x_{i-1}$. By definition of a tree of $(k, \ell)$-pre-curtains, $G\left\langle B_{x_{0}}\right\rangle$ is a $(k, \ell)$-pre-curtain. By assumption, $G\left[B_{x_{0}}\right]$, as a spanning subgraph of $G\left\langle B_{x_{0}}\right\rangle$, has an $m$-colouring with clustering at most c. Suppose now that $G\left[B_{x_{0}} \cup \cdots \cup B_{x_{i-1}}\right]$ has an $m$-colouring with clustering at most $c$, and let $x_{j}$ be the parent of $x_{i}$. By the definition of $(k, \ell)$-pre-curtain, $S_{i}:=B_{x_{j}} \cap B_{x_{i}}$ has size at most $k+3$ and is contained in the top of the root torso of the $(k, \ell)$-pre-curtain $G\left\langle B_{x_{i}}\right\rangle$. Treat $S_{i}$ as a precoloured set in the top of the root torso of $G\left\langle B_{x_{i}}\right\rangle$, which by assumption can be properly extended to an $m$-colouring of $G\left[B_{x_{i}}\right]$ with clustering at most $c$. In this way, we properly extend the colouring of $G\left[B_{x_{0}} \cup \cdots \cup B_{x_{i-1}}\right]$ to an $m$-colouring of $G\left[B_{x_{0}} \cup \cdots \cup B_{x_{i}}\right]$. Since this is a proper extension, any monochromatic component in this colouring is contained in $G\left[B_{x_{0}} \cup \cdots \cup B_{x_{i-1}}\right]$ or is contained in $G\left[B_{x_{i}}\right]$. In either case, each monochromatic component has size at most $c$. Doing this for $i=1, \ldots, r$ in order gives an $m$-colouring of $G$ with clustering at most $c$.

### 2.3 A Digression

For expository purposes, we now pause to show how the notion of $(k, \ell)$-pre-curtains can be used, with some existing results, to quickly establish that the clustered chromatic number of $K_{h}$-minor-free graphs is at most $h+4$. This is weaker than Theorem 1, but stronger than all previous results on clustered colouring of $K_{h}$-minor-free graphs except for those of Liu and Wood [44]. The purpose of this proof is to introduce some of the techniques used to prove Theorem 7 which are then expanded upon to prove Theorem 4. We need the following result, which is a consequence of Corollary 18 by Liu and Wood [42]. Lemma 26, proven in Section 3.1, is an extension of this result.

Lemma 15. For every $K_{s, t}$-subgraph-free graph $G$ of treewidth at most $k$ and every $P \subseteq V(G)$ such that each vertex in $V(G) \backslash P$ has at most $s$ neighbours in $P$, any precolouring of $P$ with $s+1$ colours can be properly extended to an $(s+1)$-colouring of $G$ with clustering at most some function $f(k, s, t,|P|)$.

Proposition 16. For every $h \geqslant 5$, every $K_{h}$-minor-free graph has an $(h+4)$-colouring with clustering at most some function $f(h)$.

Proof. Let $H$ be a $K_{h}$-minor-free graph. Since $K_{4}$ is planar, $K_{h}$ is $(h-4)$-apex. Therefore, by Lemma $13, H$ is a tree of $(h-5, \ell)$-pre-curtains described by a tree-decomposition $\mathcal{T}:=\left(B_{i}: i \in V(T)\right)$. Fix some $i \in V(T)$, let $\mathcal{T}_{i}:=\left(B_{x}: x \in V\left(T_{i}\right)\right)$ be the treedecomposition that describes the $(h-5, \ell)$-pre-curtain $H\left\langle B_{i}\right\rangle$, and let $G:=H\left[B_{i}\right]$. Let $S$ be a set of at most $h-2$ precoloured vertices contained in the top of the root torso of $\mathcal{T}_{i}$. By Lemma 14, it suffices to show that the precolouring of $S$ can be properly extended to an $(h+4)$-colouring of $G$. Since $H$ is $K_{h}$-minor-free and $G$ is a subgraph of $H, G$ is also $K_{h}$-minor-free. Contracting a matching of size $h-2$ in $K_{h-1, h-1}$ produces $K_{h}$. So $G$ is $K_{h-1, h-1}$-minor-free, implying $G$ is $K_{h-1, h-1}$-subgraph-free.
Let $G^{\prime}$ be the subgraph of $G$ induced by the near-top of the curtain $G\left\langle B_{i}\right\rangle$. The treedecomposition $\mathcal{T}_{i}$ induces a tree-decomposition $\mathcal{T}_{i}^{\prime}:=\left(B_{x}^{\prime}: x \in V\left(T_{i}\right)\right)$ of $G^{\prime}$, where $B_{x}^{\prime}:=B_{x} \cap V\left(G^{\prime}\right)$. We now show that each torso $G^{\prime}\left\langle B_{x}^{\prime}\right\rangle$ in this decomposition has treewidth at most some $k:=k(h)$. The graph $G^{\prime}\left\langle B_{x}^{\prime}\right\rangle-\hat{A}_{x}$ can be written as the union of a graph $H_{0}$ embedded on some surface $\Sigma$ of Euler genus at most $\ell$, and vortices $H_{1}, \ldots, H_{r}$ as in the definition of an almost-embedding (with $r \leqslant \ell$ and such that each $H_{i}$ has a path-decomposition of width at most $\ell$ ), with the additional property that each vertex of $H_{0}$ lies at distance at most 1 from the boundary of a vortex in $H_{0}$ (by the definition of a near-top). Let $H_{0}^{\prime}$ be the supergraph of $H_{0}$ obtained as follows: For each vortex $H_{i}$ with $G_{0}$-clean disc $D_{i}$, add a vertex $v_{i}$ in the interior of $D_{i}$ that is adjacent to each vertex on the boundary of $H_{i}$. Next, add edges so that each pair of consecutive vertices encountered while traversing the boundary of $D_{i}$ are adjacent.

Then $H_{0}^{\prime}$ can be embedded on $\Sigma$ and thus has Euler genus at most $\ell$. Moreover, each connected component of $H_{0}^{\prime}$ has diameter at most $4 r \leqslant 4 \ell$. Eppstein [23] proved that graphs of bounded Euler genus and bounded diameter have bounded treewidth, thus $H_{0}^{\prime}$ has treewidth at most $k^{\prime}:=k^{\prime}(h)$. Thus $H_{0}$, which is a subgraph of $H_{0}^{\prime}$, also has treewidth at most $k^{\prime}$. Dvořák and Thomas [21, Lemma 10] proved that in this case $\bigcup_{i=0}^{r} H_{i}=G^{\prime}\left\langle B_{x}^{\prime}\right\rangle$ has treewidth at most $k^{\prime \prime}:=\ell\left(k^{\prime}+1\right)-1$, where $\ell$ accounts for the fact that the vortices $H_{1}, \ldots, H_{r}$ all have a path-decomposition of width at most $\ell$. Therefore, $G^{\prime}\left\langle B_{x}^{\prime}\right\rangle-\hat{A}_{x}$ has a tree-decomposition $\mathcal{T}_{x}$ of width at most $k^{\prime \prime}$, so $G^{\prime}\left\langle B_{x}^{\prime}\right\rangle$ has a tree-decomposition of width at most $k^{\prime \prime}+\left|\hat{A}_{x}\right| \leqslant k^{\prime \prime}+\ell=: k$. For each $x y \in E(T)$, some bag of $\mathcal{T}_{x}$ contains $B_{x}^{\prime} \cap B_{y}^{\prime}$ and some bag of $\mathcal{T}_{y}$ contains $B_{x}^{\prime} \cap B_{y}^{\prime}$. Joining these two bags by an edge, for each $x y \in E(T)$ gives a tree-decomposition of $G^{\prime}$ having width at most $k$. Therefore the treewidth of $G^{\prime}$ is at most $k$.

Since $S$ is a subset of the top of the root torso of $G\left\langle B_{i}\right\rangle, S$ is a subset of the top (and the near-top) of $G\left\langle B_{i}\right\rangle$. Therefore $S \subseteq V\left(G^{\prime}\right)$. By Lemma 15, the precolouring of $S$ can be properly extended to an $h$-colouring of $G^{\prime}$ with clustering $f(k, h-1, h-$ $1, h-2)$. Without loss of generality, we may assume this colouring of $G^{\prime}$ uses colours $\{1, \ldots, h\}$. Now, consider the graph $G-V\left(G^{\prime}\right)$. Each component of $G-V\left(G^{\prime}\right)$ is a graph of Euler genus at most $\ell$. By Theorem $8, G-V\left(G^{\prime}\right)$ has a 4-colouring using colours $\{h+1, \ldots, h+4\}$ with clustering $f^{\prime}(\ell)$, for some function $f^{\prime}$. The resulting colouring of $G$ is clearly a proper extension of the precolouring of $S$ and has clustering $\max \left\{f(k, h-1, h-1, h-2), f^{\prime}(\ell)\right\}$.

We remark that the proof of Proposition 16 is easily adapted to prove the following result (whose statement is more technical, but which is stronger and immediately implies Proposition 16):

Proposition 17. For every $k$-apex graph $X$, every $X$-minor-free $K_{s, t}$-subgraph-free graph is $\max \{k+7, s+5\}$-colourable with clustering at most some function $f(k, s, t)$.

### 2.4 Layered Partitions and Curtains

We now need the following definitions from the literature. A partition of a graph $G$ is a collection $\mathcal{P}$ of non-empty sets of vertices in $G$ such that each vertex of $G$ is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a part. The quotient of $\mathcal{P}$ is the graph, denoted by $G / \mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G / \mathcal{P}$ if and only if some vertex in $A$ is adjacent in $G$ to some vertex in $B$. A partition of $G$ is connected if the subgraph induced by each part is connected. In this case, the quotient is the minor of $G$ obtained by contracting each part into a single vertex. It is often convenient to omit singleton sets when defining partitions and quotient graphs, and only
require that $\mathcal{P}$ is a set of disjoint subsets of $V(G)$. In this case, $G / \mathcal{P}:=G / \mathcal{P}^{\prime}$ where $\mathcal{P}^{\prime}:=\mathcal{P} \cup\left\{\{v\}: v \in V(G) \backslash \bigcup_{P \in \mathcal{P}} P\right\}$.

Partitions and clustered colouring are intimately related, since a graph $G$ is $k$-colourable with clustering $c$ if and only if $G$ has a partition $\mathcal{P}$ such that every part of $\mathcal{P}$ has at most $c$ vertices and the quotient $G / \mathcal{P}$ is properly $k$-colourable.

A layering of a graph $G$ is an ordered partition $\mathcal{L}=\left(L_{1}, L_{2}, \ldots\right)$ of $V(G)$ such that for every edge $v w \in E(G)$, if $v \in L_{i}$ and $w \in L_{j}$, then $|i-j| \leqslant 1$. For any integer $i$, we use the shorthands $L_{\leqslant i}:=\bigcup_{j \leqslant i} L_{j}$ and $L_{\geqslant i}:=\bigcup_{j \geqslant i} L_{j}$. We say that $\mathcal{L}$ is upward-connected if for every $i \geqslant 2$ every vertex in $L_{i}$ has a neighbour in $L_{i-1}$. A typical example of a layering is a Breadth-First-Search (BFS) layering: set a root vertex $v_{j}$ in each connected component and for each $i \geqslant 1$, let $L_{i}$ be the set of vertices at distance exactly $i-1$ from one of the vertices $v_{j}$. Every BFS layering is upward-connected. We use the following useful property of upward-connectivity

Observation 18. If $\mathcal{L}=\left(L_{1}, L_{2}, \ldots\right)$ is an upward-connected layering of a graph $G$, then, for any integer $i \geqslant 1$, the number of components of $G\left[L_{\leqslant i}\right]$ is at most the number of components of $G\left[L_{1}\right]$.

If $\mathcal{L}$ is a layering of a graph $G$, then a set $X$ of vertices in $G$ is $\ell$-skinny with respect to $\mathcal{L}$ if $|X \cap L| \leqslant \ell$ for each $L \in \mathcal{L}$. We will use the following lemma to transform a colouring in which each monochromatic component is $\ell$-skinny to a colouring in which each monochromatic component has bounded size. A monochromatic component $C$ in some colouring $\varphi$ of a graph $G$ is called a $\varphi$-monochromatic component in $G$ (this notation is helpful to avoid confusion when we consider several colourings of a given graph, as in the proof of the next result).

Lemma 19. Let $G$ be a graph and let $\mathcal{L}:=\left(L_{1}, L_{2}, \ldots\right)$ be a layering of $G$. Let $\varphi: V(G) \rightarrow\{1, \ldots, c\}$ be a c-colouring of $G$ such that every $\varphi$-monochromatic component of $G$ is $\ell$-skinny with respect to $\mathcal{L}$, and for each component $C$ of $G\left[L_{\geqslant 6}\right]$, there exists a colour $a_{C} \in\{1, \ldots, c\} \backslash\{\varphi(v): v \in V(C)\}$ that is not used in $C$. Then $G$ has a $c$-colouring with clustering at most $(2 c+5) \ell$.

Proof. As illustrated in Figure 4, define a colouring $\varphi^{\prime}$ of $G$ as follows: For each component $C$ of $G\left[L_{\geqslant 6}\right]$, each colour $a \in\{1, \ldots, c\} \backslash\left\{a_{C}\right\}$, each $i \in\{7+2(a-1)+2 j c$ : $j \geqslant 0\}$, and each $v \in\left\{w \in V(C) \cap L_{i}: \varphi(w)=a\right\}$, set $\varphi^{\prime}(v):=a_{C}$. For any $v \in V(G)$ not defined by this rule, set $\varphi^{\prime}(v):=\varphi(v)$. Clearly, $\varphi^{\prime}$ is a $c$-colouring of $G$. We now show that $\varphi^{\prime}$ has clustering at most $(2 c+5) \ell$.

Claim. For each edge $v w$ of $G$, if $\varphi(v) \neq \varphi(w)$ then $\varphi^{\prime}(v) \neq \varphi^{\prime}(w)$.


Figure 4: The proof of Lemma 19 with $c=3$ colours

Proof. Suppose for the sake of contradiction that $\varphi(v) \neq \varphi(w)$ and $\varphi^{\prime}(v)=\varphi^{\prime}(w)$. Without loss of generality, $\varphi(v) \neq \varphi^{\prime}(v)$. By construction, $v \in V(C) \cap L_{i}$ where $C$ is a component of $G\left[L_{\geqslant 6}\right]$, and $i=7+2(\varphi(v)-1)+2 j c$ for some $j \geqslant 0$, and $\varphi^{\prime}(v)=a_{C}$. Say $w \in L_{i^{\prime}}$. Since $\mathcal{L}$ is layering, $i^{\prime} \geqslant i-1 \geqslant 6$. Hence $w$ is also in $C$. If $\varphi^{\prime}(w)=\varphi(w)$, then $a_{C}=\varphi^{\prime}(v)=\varphi^{\prime}(w)=\varphi(w)$, which contradicts the assumption that $a_{C}$ is not used in $C$. Thus $\varphi^{\prime}(w) \neq \varphi(w)$. By construction, $i^{\prime}=7+2(\varphi(w)-1)+2 j^{\prime} c$ for some $j^{\prime} \geqslant 0$. Hence $1 \geqslant\left|i-i^{\prime}\right|=\left|2\left(\varphi(v)-\varphi(w)+j c-j^{\prime} c\right)\right|$, implying $\varphi(v)-\varphi(w)+j c-j^{\prime} c=0$ and $\varphi(v)-\varphi(w)=c\left(j^{\prime}-j\right)$. Since $\varphi(v) \neq \varphi(w)$, we have $j^{\prime} \neq j$ and $\left|c\left(j^{\prime}-j\right)\right| \geqslant c$. On the other hand, $|\varphi(v)-\varphi(w)| \leqslant c-1 \operatorname{since} \varphi(v), \varphi(w) \in\{1, \ldots, c\}$. This contradiction shows that $\varphi^{\prime}(v) \neq \varphi^{\prime}(w)$.

It follows from the claim that each $\varphi^{\prime}$-monochromatic component $A^{\prime}$ is contained in some $\varphi$-monochromatic component $A$. Consider the following two cases:

1. $A^{\prime}$ and $A$ have different colours $a^{\prime}$ and $a$, respectively. That is, $a=\varphi(v) \neq \varphi^{\prime}(v)=$ $a^{\prime}$ for each $v \in V\left(A^{\prime}\right)$. In this case $A^{\prime}$ is contained in some component $C$ of $G\left[L_{\geqslant 6}\right]$, $a^{\prime}=a_{C}$, and $A^{\prime}=A\left[L_{i}\right]$ for some $i \in\{7+2(a-1)+2 j c: j \geqslant 0\}$ (since only vertices in odd-numbered layers change their colour). Therefore, $\left|V\left(A^{\prime}\right)\right|=\left|V\left(A\left[L_{i}\right]\right)\right| \leqslant \ell$, since $A$ is $\ell$-skinny with respect to $\mathcal{L}$.
2. $A^{\prime}$ and $A$ have the same colour $a$. In other words, $\varphi^{\prime}(v)=\varphi(v)=a$ for each $v \in V\left(A^{\prime}\right)$. Consider some component $C$ of $G\left[L_{\geqslant 6}\right]$. If $a_{C}=a$, then $A^{\prime}$ does not intersect $C\left[L_{\geqslant 6}\right]$ by the definition of $a_{C}$. Let $C_{1}, \ldots, C_{r}$ be the components of $G\left[L_{\geqslant 6}\right]$ for which $a_{C_{i}} \neq a$. Then, for each $j \in\{1, \ldots, r\}, C_{j}\left[L_{i}\right]$ has no vertex of colour $a$ for any $i \in\{7+2(a-1)+2 j c: j \geqslant 0\}$. Therefore the vertices of
$A^{\prime}\left[L_{>7}\right]$ are contained in $2 c-1$ consecutive layers of $\mathcal{L}$. Since $A^{\prime}$ is contained in $A$, and $A$ is $\ell$-skinny with respect to $\mathcal{L}$, this implies that $\left|V\left(A^{\prime}\left[L_{\geqslant 7}\right]\right)\right| \leqslant(2 c-1) \ell$. Furthermore, $\left|V\left(A^{\prime}\left[L_{\leqslant 6}\right]\right)\right|=\left|V\left(A\left[L_{\leqslant 6}\right]\right)\right| \leqslant 6 \ell$. Therefore, $\left|V\left(A^{\prime}\right)\right| \leqslant(2 c+5) \ell$.
Therefore, $\varphi^{\prime}$ is a $c$-colouring of $G$ with clustering at most $(2 c+5) \ell$.

The number 6 in the statement of Lemma 19 might seem arbitrary, since the same proof works if we replace 6 by any value $t$ (with resulting clustering $(2 c+(t-1)) \ell$ ). It turns out that taking $t=6$ will be enough for all the applications of this lemma in the paper.

A layered partition $(\mathcal{P}, \mathcal{L})$ of a graph $G$ consists of a partition $\mathcal{P}$ and a layering $\mathcal{L}$ of $G$. Adjectives used to modify $\mathcal{P}$ or $\mathcal{L}$ have the obvious meaning when applied to ( $\mathcal{P}, \mathcal{L}$ ). For example, $(\mathcal{P}, \mathcal{L})$ is connected if $\mathcal{P}$ is connected and $(\mathcal{P}, \mathcal{L})$ is upward-connected if $\mathcal{L}$ is upward-connected. We say that $(\mathcal{P}, \mathcal{L})$ is a $(w, \ell)$-layered partition of $G$ if each $P \in \mathcal{P}$ is $\ell$-skinny with respect to $\mathcal{L}$ and the quotient graph $G / \mathcal{P}$ has treewidth at most $w$. We write $\ell$-layered partition instead of $(\ell, \ell)$-layered partition for simplicity.

If $\mathcal{E}:=\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{r}\right)$ is an almost-embedding of a graph $G$, then a layering $\mathcal{L}$ of $G-A$ is neat with respect to $\mathcal{E}$ if the first layer of $\mathcal{L}$ consists of exactly $(\hat{A} \backslash A) \cup$ $V\left(G_{1} \cup \cdots \cup G_{r}\right)$. In other words, the first layer of $\mathcal{L}$ consists of the top of $G$ minus the major apex vertices of $G$. This implies that the first two layers of $\mathcal{L}$ contain the near-top of $G$ minus the major apex vertices of $G$ (possibly plus other vertices).

The next lemma by Dujmović et al. [17] is one of the key tools that distinguishes our proof from those of Liu and Wood [42, 43, 44]. In particular, we use layered partitions, whereas Liu and Wood [42, 43, 44] use layered treewidth.

Lemma 20 ([17, Lemma 28]). For every connected ( $a, \ell$ )-almost-embedded graph $G$ with major apex vertex set $A, G-A$ has a $(13 \ell, 6 \ell)$-layered partition $(\mathcal{P}, \mathcal{L})$ that is connected, upward-connected, and neat with respect to the almost-embedding.

Since Lemma 20 is not stated explicitly in [17], we now explain the small modifications to the proof of [17, Lemma 28] that are needed to deduce it. The proof of Lemma 28 in [17] starts by considering the embedded part $G_{0}$ of the almost-embedding of $G$. Let $F_{1}, \ldots, F_{r}$ be the set of faces of $G_{0}$ that contain the vortices. Choose an arbitrary vertex $v \in V\left(F_{1}\right)$ and add edges so that $v$ is adjacent to each vertex on the boundary of $F_{i}$, for each $i \in\{1, \ldots, r\}$. Since the number, $r$, of vortices is bounded, the resulting graph has bounded Euler genus. The authors of [17] then consider a BFS layering starting at $v$ in the resulting graph, and apply a specific result on graphs on surfaces. This produces an upward-connected layered partition of $G_{0}$ in which all the vertices on the boundary of a vortex appear on the first two layers of the resulting layering, and the proof continues by adding all vortex vertices and apex vertices to the first layer.

To obtain the result stated in Lemma 20 we start this process slightly differently. Introduce a new vertex $x^{*}$ adjacent to every vertex of $G_{0}$ that is on the boundary of a vortex (the resulting graph still has bounded Euler genus). Consider a BFS layering of $G_{0}$ starting at $x^{*}$, and then apply the result for graphs on surfaces, as before. This gives a layered partition in which $x^{*}$ is in the first layer and the second layer contains precisely the vertices lying on the boundary of a vortex. After adding the non-major apex vertices and vortex vertices to the second layer and discarding the first layer (that is, $\left\{x^{*}\right\}$ ), we obtain a neat layering of $G-A$. Moreover, since we started with a BFS layering, each vertex outside the first layer has a neighbour in the previous layer, so the resulting layering is upward-connected.

A $(k, \ell)$-almost-embedded graph $D$ with major apex vertex set $A$ that is equipped with a neat $\ell$-layered partition $(\mathcal{P}, \mathcal{L})$ of $D-A$ is called a $(k, \ell)$-drape, and we say that $D$ is described by $(\mathcal{P}, \mathcal{L}) .{ }^{5}$ A $(k, \ell)$-pre-curtain $G$ described by a tree-decomposition $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ is a $(k, \ell)$-curtain if for each $x \in V(T)$, the torso $G\left\langle B_{x}\right\rangle$ is a ( $k, \ell$ )-drape. A $(k, \ell)$-curtain $G$ described by $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ is upward-connected if, for each $x \in V(T)$, the $\ell$-layered partition $\left(\mathcal{P}_{x}, \mathcal{L}_{x}\right)$ that describes the drape $G\left\langle B_{x}\right\rangle$ is upward-connected.

A tree of curtains is defined similarly as a tree of pre-curtains, except that each precurtain is now replaced by a curtain. Explicitly, a graph $G$ is a tree of $(k, \ell)$-curtains if it has a rooted tree-decomposition $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ such that:

- for each $x \in V(T), G\left\langle B_{x}\right\rangle$ is a $(k, \ell)$-curtain,
- for each edge $x y$ of $T$ where $y$ is the parent of $x, B_{x} \cap B_{y}$ has size at most $k+3$ and is contained in the top of the root torso of $G\left\langle B_{x}\right\rangle$.

Again, we say that $G$ is a tree of $(k, \ell)$-curtains described by $\mathcal{T}$. If every torso in a tree of $(k, \ell)$-curtains is upward-connected, then we say that it is a tree of upward-connected ( $k, \ell$ )-curtains.

The following version of Lemma 14 for curtains has the exact same proof as that of Lemma 14 (replacing each occurrence of $(k, \ell)$-pre-curtain in the proof by $(k, \ell)$-curtain or upward-connected $(k, \ell)$-curtain, depending on the version).

Lemma 21. Let $G$ be a graph that is a tree of $(k, \ell)$-curtains described by $\mathcal{T}:=\left(B_{x}\right.$ : $x \in V(T))$ and such that, for any $(k, \ell)$-curtain $G\left\langle B_{x}\right\rangle$ of $G$, any precolouring of a set $S$ of at most $k+3$ vertices in the top of the root torso of $G\left\langle B_{x}\right\rangle$ can be properly extended to an m-colouring of $G\left[B_{x}\right]$ with clustering at most $c$. Then $G$ has an m-colouring with clustering at most $c$.

[^5]The following theorem, which is an immediate consequence of Lemmas 13 and 20, is a new structural description of graphs excluding a fixed minor, which we believe is of independent interest and might have further applications.

Theorem 22. For any integer $k \geqslant 1$ and for any $k$-apex graph $X$, there is an integer $\ell \geqslant 2$ such that every $X$-minor-free graph $G$ is a lower-minor-closed tree of upwardconnected ( $k-1, \ell$ )-curtains.


Figure 5: A curtain (above) and a raised curtain (below). We think of the layers as drawn from top to bottom (with the top of the root torso of the curtain being depicted in green). All torsos of the decomposition are glued on a subset of their first and second layers (and on a subset of their major apex vertices, which are depicted in blue-or green, for the major apex vertices $A_{r}$ of the root torso-on the figure and are not part of the layerings).

From a $(k, \ell)$-curtain $G$, we obtain the raised curtain $G_{\uparrow}$ as follows (see Figure 5): For each $x \in V(T)$, let $\left(\mathcal{P}_{x}, \mathcal{L}_{x}\right)$ be the $\ell$-layered partition that describes the $(k, \ell)$-drape $G\left\langle B_{x}\right\rangle$ and write $\mathcal{L}_{x}=\left(L_{1}^{x}, L_{2}^{x}, \ldots\right)$. For each $x \in V(T)$, let $G_{\uparrow}^{x}$ be the graph obtained from the induced subgraph $G\left[B_{x}\right]$ by contracting each component $C$ of $G\left[P \cap L_{\geqslant 6}^{x}\right]$ into a single vertex $v_{C}$, for each $P \in \mathcal{P}_{x}$. Let $L_{\uparrow}^{x}:=V\left(G_{\uparrow}^{x}\right) \backslash\left(\bigcup_{i=1}^{5} L_{i}^{x}\right)$ be the set of vertices obtained from these contractions.

For each edge $x y \in E(T)$ where $y$ is the parent of $x$, the adhesion set $B_{x} \cap B_{y}$ is contained in $\left(A_{x} \cup L_{1}^{x}\right) \cap\left(A_{y} \cup L_{1}^{y} \cup L_{2}^{y}\right)$. So for distinct $x, y \in V(T)$, the connected subgraphs contracted to create $G_{\uparrow}^{x}$ and the connected subgraphs contracted to create $G_{\uparrow}^{y}$ have no
vertices in common. The raised curtain is the graph $G_{\uparrow}:=\bigcup_{x \in V(T)} G_{\uparrow}^{x}$ obtained from $G$ by applying all the above edge contractions (for each $x \in V(T)$ ).

Lemma 23. Let $G$ be a $(k, \ell)$-curtain and let $G_{\uparrow}$ be the corresponding raised curtain. Then $G_{\uparrow}$ is a minor of $G, G_{\uparrow}$ is a $(k, \ell)$-curtain with 6 layers in each drape, and the treewidth of $G_{\uparrow}$ is at most $6 \ell(\ell+1)+k-1$.

Proof. It is immediate from the definition that $G_{\uparrow}$ is a minor of $G$ and that it is a $(k, \ell)$-curtain with 6 layers per drape.

For each $x \in V(T)$, let $H_{x}:=\left(G\left\langle B_{x}\right\rangle-A_{x}\right) / \mathcal{P}_{x}$ and write $\left\{C_{y}: y \in V\left(H_{x}\right)\right\}:=\mathcal{P}_{x}$ to highlight the relationship between $H_{x}$ and $\mathcal{P}_{x}$, so $C_{y}$ is the part of $\mathcal{P}_{x}$ whose contraction creates the vertex $y$ in $H_{x}$. For each $y \in V\left(H_{x}\right)$, let $D_{y}$ be the set consisting of the at most $5 \ell$ vertices in $C_{y} \cap L_{\leqslant 5}^{x}$, plus the vertices $v_{C} \in V\left(G_{\uparrow}^{x}\right)$ obtained by contracting a component $C$ of $G\left[C_{y} \cap L_{\geqslant 6}^{x}\right]$. There are at most $\ell$ such components, since if $G\left[C_{y} \cap L_{\geqslant 6}^{x}\right]$ is not connected, $C_{y}$ must intersect $L_{\leqslant 5}^{x}$ and all the connected components $C$ of $G\left[C_{y} \cap L_{\geqslant 6}^{x}\right]$ must intersect $L_{6}^{x}$. So $\left|D_{y}\right| \leqslant 6 \ell$. Let $\mathcal{Q}_{x}:=\left\{D_{y}: y \in V\left(H_{x}\right)\right\}$ (and recall that when we define partitions of graphs we can omit singletons in the description for convenience). It follows that $G_{\uparrow}^{x}$ is isomorphic to a subgraph of $\left(G\left\langle B_{x}\right\rangle-A_{x}\right) / \mathcal{Q}_{x}$. Therefore, in any tree-decomposition of $H_{x}$, each occurrence of $y \in V\left(H_{x}\right)$ in a bag can be replaced by the contents of $D_{y}$, to obtain a tree-decomposition of $G_{\uparrow}^{x}-A_{x}$ in which each bag increases in size by a factor of at most $6 \ell$. Since $\operatorname{tw}\left(H_{x}\right) \leqslant \ell$, this implies that $G_{\uparrow}^{x}-A_{x}$ has a tree-decomposition in which each bag has at most $6 \ell(\ell+1)$ vertices. Adding the vertices in $A_{x}$ to every bag shows that $\operatorname{tw}\left(G_{\uparrow}^{x}\right) \leqslant 6 \ell(\ell+1)+\left|A_{x}\right|-1 \leqslant 6 \ell(\ell+1)+k-1$.

For each edge $x y \in E(T), B_{x} \cap B_{y}$ is a clique in $G\left\langle B_{x}\right\rangle\left[A_{x} \cup L_{1}^{x}\right]$ and in $G\left\langle B_{y}\right\rangle\left[A_{y} \cup L_{1}^{y} \cup L_{2}^{y}\right]$. Thus no vertex in $B_{x} \cap B_{y}$ is in an edge contraction in the construction of $G_{\uparrow}^{x}$ or $G_{\uparrow}^{y}$. Hence $\left(B_{x} \cap B_{y}\right) \backslash A_{x}$ appears in some bag in the tree-decomposition of $G_{\uparrow}^{x}-A_{x}$, implying that $B_{x} \cap B_{y}$ appears in some bag in the tree-decomposition of $G_{\uparrow}^{x}$. Similarly, $B_{x} \cap B_{y}$ appears in some bag in the tree-decomposition of $G_{\uparrow}^{y}$. Adding an edge between the nodes corresponding to these two bags (for each $x y \in E(T)$ ) creates a tree-decomposition of $G_{\uparrow}$. Thus $\operatorname{tw}\left(G_{\uparrow}\right) \leqslant 6 \ell(\ell+1)+k-1$.

Lemma 24. Let $G$ be a lower-minor-closed tree of $(k, \ell)$-curtains described by a treedecomposition $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$. Then, for each $x \in V(T)$, the graph $H_{\uparrow}$ obtained from $G\left\{B_{x}\right\}$ by performing the same contractions used to raise the curtain $G\left\langle B_{x}\right\rangle$ is a minor of $G$.

Proof. Let $(\mathcal{L}, \mathcal{P})$ be the layered partition of $G\left\langle B_{x}\right\rangle-A_{x}$, where $\mathcal{L}:=\left(L_{1}, L_{2}, \ldots\right)$, let $G\left\langle B_{x}\right\rangle_{\uparrow}$ be the graph obtained by raising the curtain $G\left\langle B_{x}\right\rangle$, and let $\mathcal{L}_{\uparrow}:=\left(L_{1}, \ldots, L_{5}, L_{\uparrow}\right)$ be the resulting layering of $G\left\langle B_{x}\right\rangle_{\uparrow}$. Each vertex $v \in L_{\uparrow}$ is obtained by contracting


Figure 6: Adding the vertex $\alpha$ adjacent to the bottom layer of $H_{\uparrow}$.
a connnected subgraph $X_{v}$ of $G\left\langle B_{x}\right\rangle\left[L_{\geqslant 6}\right]$. Since $V\left(X_{v}\right) \subseteq L_{\geqslant 6}$, every edge of $X_{v}$ in $G\left\langle B_{x}\right\rangle$ is also an edge of $G\left\{B_{x}\right\}$. Thus, $H_{\uparrow}$ is a graph that can be obtained from $G\left\{B_{x}\right\}$ by contracting connected subgraphs. Therefore $H_{\uparrow}$ is a minor of $G\left\{B_{x}\right\}$. Since $G$ is lower-minor-closed, $G\left\{B_{x}\right\}$ is a minor of $G$, so $H_{\uparrow}$ is a minor of $G$.

### 2.5 Theorem 7: Proof for $s \geqslant 4$

Recall Theorem 7 which states that for any apex graph $X$, any $X$-minor-free $K_{s, t^{-}}$ subgraph-free graph is $(s+1)$-colourable with clustering at most some function $f(X, s, t)$. Theorem 7 holds for all $t \geqslant s \geqslant 3$. Here we prove it for $t \geqslant s \geqslant 4$. The case $s=3$ requires some more sophisticated tools that are also used to establish Theorems 1, 2 and 4 , so will be proven later, in Section 4.3.

Proof of Theorem 7 for $s \geqslant 4$. Let $J$ be a $X$-minor-free $K_{s, t}$-subgraph-free graph. By Theorem 22, $J$ is a a tree of $(0, \ell)$-curtains for some $\ell:=\ell(X)$. We will prove that, for each curtain $G:=J\left\langle B_{\tau}\right\rangle$ of $J$, any $(s+1)$-colouring of $s$ vertices in the top of the root torso of $G$ can be properly extended to an $(s+1)$-colouring of $H:=J\left[B_{\tau}\right]$ with clustering at most some function $c(s, t)$. Lemma 21 then implies that $J$ is $(s+1)$-colourable with clustering $c(s, t)$.

Let $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ be the tree-decomposition that describes $G$. For each $x \in V(T)$, let $\left(\mathcal{P}_{x}, \mathcal{L}_{x}\right)$ be the $\ell$-layered partition that describes the ( $0, \ell$ )-drape $G\left\langle B_{x}\right\rangle$, where $\mathcal{L}_{x}=:\left(L_{1}^{x}, L_{2}^{x}, \ldots\right)$. For each integer $i \geqslant 1$, let $L_{i}:=\bigcup_{x \in V(T)} L_{i}^{x}$. Observe that $\left(L_{1}, L_{2}, \ldots\right)$ is an upward-connected layering of $G$. Let $G_{\uparrow}$ be the graph obtained by raising $G$, let $L_{\uparrow}:=V\left(G_{\uparrow}\right) \backslash V(G)$, and let $\mathcal{L}_{\uparrow}:=\left(L_{1}, \ldots, L_{5}, L_{\uparrow}\right)$ be the resulting layering of $G_{\uparrow}$. Let $H_{\uparrow}$ be the graph obtained from $H$ by performing the same contractions used to obtain $G_{\uparrow}$ from $G$. Let $H_{\uparrow}^{+}$be the graph obtained from $H_{\uparrow}$ by adding a single vertex $\alpha$ adjacent to every vertex in $L_{\uparrow}$, as illustrated in Figure 6. Assign the colour $a:=s+1$ to $\alpha$.

We will apply Lemma 15 to $H_{\uparrow}^{+}$with the precoloured set $P:=S \cup\{\alpha\}$. To do this, we
must verify that $H_{\uparrow}^{+}$has bounded treewidth and is $K_{s, t^{-}}$-subgraph-free. Observe that each vertex of $L_{\uparrow}$ has only one neighbour, $\alpha$, in $P$. Each vertex in $V(G) \backslash L_{\uparrow}$ has at most $s$ neighbours (all contained in $S$ ) in $P$. Consider the induced graph $H_{\uparrow}\left[L_{5} \cup L_{\uparrow}\right]$. Each component of this graph is embedded in a surface of Euler genus at most $g=g(X)$. It follows from Euler's Formula that $H_{\uparrow}\left[L_{5} \cup L_{\uparrow}\right]$ is $K_{3,2 g+3 \text {-subgraph-free. Since } s \geqslant 4 \text {, }}$, this implies that $H_{\uparrow}^{+}$is $K_{s, \max \{t, 2 g+3\}}$-subgraph-free. By Lemma $23, H_{\uparrow}$ has treewidth at most $k:=k(X)$ and the addition of the vertex $\alpha$ to make $H_{\uparrow}^{+}$increases the treewidth by at most 1 . Therefore, $H_{\uparrow}^{+}$satisfies all the requirements of Lemma 15, so $H_{\uparrow}^{+}$has an $(s+1)$-colouring that properly extends the precolouring of $P$ and has clustering at most $c:=f(k+1, s, \max \{t, 2 g+3\}, s+1)$.

Extend the colouring of $H_{\uparrow}$ to a colouring of $H$ in the obvious way: For each vertex $v \in V\left(H_{\uparrow}\right) \backslash V(G)$ there is a connected subgraph $X_{v}$ of $G$ that was contracted to create $v$. Assign each vertex in $X_{v}$ the colour of $v$. The resulting colouring of $H$ does not necessarily have bounded clustering because $X_{v}$ can be arbitrarily large. However, by the definition of $(\cdot, \ell)$-curtain, $V\left(X_{v}\right)$ is $\ell$-skinny with respect to $\mathcal{L}$. This implies that each monochromatic component of $H$ is $c \ell$-skinny with respect to $\mathcal{L}$. Furthermore, since our colouring properly extends the precolouring $P=S \cup\{\alpha\}$, each vertex in $L_{\geqslant 6}$ avoids the colour $a=s+1$ of $\alpha$. Therefore, by Lemma 19, $H$ has an $(s+1)$-colouring with clustering at most $(2 s+7) c \ell$.

The proof of Theorem 7 uses many of the elements that are ultimately used in our proof of Theorem 4. Specifically, the proof of Theorem 4 uses the same strategy of decomposing our graph into a tree of curtains, and then colouring each curtain $G$ by first introducing extra vertices to the raised curtain $G_{\uparrow}$ whose colours are used to break long skinny monochromatic components into short skinny monochromatic components (Lemma 19). The main difference is that, in the $\mathcal{J}_{s, t}$-minor-free setting, $G$ is an $(s-3, \ell)$-curtain, so each drape of $G$ has up to $s-3$ major apex vertices.

The presence of major apex vertices causes considerable complications. Even a single major apex vertex $a$ in a torso ruins the component breaking strategy used in the proof of Lemma 19. To avoid this, we will colour the curtain $G_{\uparrow}$ so that, for each component $C$ of $G_{\uparrow}\left[L_{5} \cap L_{\uparrow}\right]$, the colour assigned to any vertex $v$ in $V(C) \cap L_{\uparrow}$ is distinct from the colour assigned to each neighbour of $v$ in $A_{x}$, as well one additional extra vertex $\alpha_{C}$, whose colour is different from all vertices in $A_{x}$. This way, the colour of $\alpha_{C}$ can later be used as a breaker. This motivates the introduction of admissible sets that appear in the next section.

However, this highlights another difficulty we will eventually encounter. This difficulty is most obvious in the context of $K_{h}$-minor-free graphs, which Theorem 1 promises to colour with $h-1$ colours. In this setting, the major apex set $A_{x}$ of the drape $G\left\langle B_{x}\right\rangle$ that
contains $C$ can have size up to $h-5$. After excluding the colours used by $A_{x} \cup\left\{\alpha_{C}\right\}$ this may leave only $(h-1)-(h-5)-1=3$ colours available for each vertex in $C$. In general, this would not be possible since the graph $C\left[L_{\uparrow}\right]$ may not have a 3 -colouring with bounded clustering. For example, all graphs in the graph class $\mathcal{G}_{3}$ described in Proposition 9 are planar and have treewidth 3 , so $G_{\uparrow}\left[L_{\uparrow}^{x}\right]$ could contain arbitrarily large members of $\mathcal{G}_{3}$. In this case, we will be forced to produce a 3 -colouring of $G_{\uparrow}\left[L_{\uparrow}^{x}\right]$ that has arbitrarily large monochromatic components. Using some new and old tricks, we show that each monochromatic component in $G_{\uparrow}$ corresponds to a monochromatic component in $G$ that is skinny with respect to $\mathcal{L}$, so it can be broken into bounded size monochromatic components using Lemma 19.

## 3 Bounded Treewidth Lemmas

This section proves two lemmas about bounded treewidth graphs that exclude certain subgraphs. Both lemmas depend on the following sequence of definitions.

A list-assignment of a graph $G$ is a function $L$ such that $L(v)$ is a set of 'colours' for each vertex $v \in V(G)$. An $L$-colouring is a colouring of $G$ where each vertex $v \in V(G)$ is assigned a colour in $L(v)$.

Fix an integer $s \geqslant 1$. For a list-assignment $L$ of a graph $G$, let

$$
\begin{aligned}
& P(L):=\{v \in V(G):|L(v)|=1\} \text { and } \\
& Q(L):=\left\{v \in V(G) \backslash P(L):\left|N_{G}(v) \cap P(L)\right| \in\{1,2, \ldots, s-1\}\right\}
\end{aligned}
$$

The vertices in $P(L)$ are said to be precoloured by $L$. We say $L$ is $(s, p)$-good if:
(g1) $|P(L)| \leqslant p$,
(g2) $|L(v)| \geqslant s+1-\left|N_{G}(v) \cap P(L)\right|$ for all $v \in Q(L)$;
(g3) $|L(v)| \geqslant 2$ for all $v \in N_{G}(P(L)) \backslash Q(L)$;
(g4) $|L(v)| \geqslant s+1$ for all $v \in V(G) \backslash N_{G}[P(L)]$; and
(g5) $L(v) \cap L(u)=\varnothing$ for all $v \in Q(L)$ and $u \in N_{G}(v) \cap P(L)$.
Note that by the definition of $P(L)$, property (g3) above could be equivalently stated as $L(v) \neq \varnothing$ for all $v \in N_{G}(P(L)) \backslash Q(L)$, or equivalently (using the other properties) as $L(v) \neq \varnothing$ for all $v \in V(G)$.

For two list-assignments $L$ and $L^{\prime}$ of $G$, we say that $L^{\prime}$ is a specialization of $L$ if $L(v) \supseteq L^{\prime}(v)$ for each $v \in V(G)$, written as $L \supseteq L^{\prime}$. In this case, any $L^{\prime}$-colouring of $G$ is also an $L$-colouring of $G$.

As illustrated in Figure 7, for an integer $k \geqslant 0$, a set $\mathcal{S}:=\left\{\left(\alpha_{1}, A_{1}\right), \ldots,\left(\alpha_{r}, A_{r}\right)\right\}$ is $k$-admissible in a graph $G$ if:
( t 1 ) $A_{j}$ is a set of at most $k$ vertices in $G$, for each $j \in\{1, \ldots, r\}$,
(t2) $\alpha_{j}$ is a vertex of $G$ and $N_{G}\left[\alpha_{j}\right] \cap A_{j}=\varnothing$, for each $j \in\{1, \ldots, r\}$,
(t3) $\left(\bigcup_{j=1}^{r} A_{j}\right) \cap\left(\bigcup_{j=1}^{r} S_{j}\right)=\varnothing$ where $S_{j}:=N_{G-A_{j}}^{2}\left[\alpha_{j}\right]$, and
(t4) $S_{1}, \ldots, S_{r}$ are pairwise disjoint.


Figure 7: A $k$-admissible set.
To avoid any possible confusion, we emphasise that the neighbourhoods in (t2) and (t3) above are defined in different graphs ( $G$ in ( t 2 ) and $G-A_{j}$ in ( t 3 )).

We call $S_{j}$ the $j$-trigger set, since in the colouring procedure below, colouring the first vertex in $S_{j}$ triggers the colouring of all uncoloured vertices in $A_{j} \cup\left\{\alpha_{j}\right\}$.

A list-assignment $L$ and a $k$-admissible set $\mathcal{S}=\left\{\left(\alpha_{1}, A_{1}\right), \ldots,\left(\alpha_{r}, A_{r}\right)\right\}$ for a graph $G$ are compatible if for each $j \in\{1, \ldots, r\}$ :
(c1) if $S_{j} \cap P(L) \neq \varnothing$, then $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P(L)$;
(c2) if $S_{j} \cap P(L) \neq \varnothing$, then $L(a) \cap L\left(\alpha_{j}\right)=\varnothing$ for each $a \in A_{j}$; and
(c3) if $S_{j} \cap P(L) \neq \varnothing$, then $L(a) \cap L(x)=\varnothing$ for each $a \in A_{j} \cup\left\{\alpha_{j}\right\}$ and each $x \in N_{G}\left(\alpha_{j}\right) \cap N_{G}(a)$.
In words, whenever a vertex of a trigger set $S_{j}$ is precoloured, we require that $\alpha_{j}$ and all the vertices of $A_{j}$ are also precoloured (with $\alpha_{j}$ having a colour distinct from the colours of the vertices from $A_{j}$ ), and moreover the colour of each vertex of $A_{j} \cup\left\{\alpha_{j}\right\}$ does not appear in the list of its neighbours from $N_{G}\left(\alpha_{j}\right)$. In our applications, $A_{j}$ will be the set of major apex vertices of an almost-embedded graph, and $\alpha_{j}$ is a dummy vertex added to a raised curtain arising from the embedded part of the almost-embedded graph, such that $\alpha_{j}$ is adjacent to no apex vertex, and $S_{j}$ is adjacent to no vortex and no non-major apex vertex. The colour assigned to $\alpha_{j}$ (which by (c2) and (c3) is not used by a vertex in $\left.N_{G}\left(\alpha_{j}\right) \cup A_{j}\right)$ then serves as an extra colour that can be used in Lemma 19 to break up large (but skinny) monochromatic components when we lower the curtain.

We use the following result by Liu and Wood [43], which has a simple proof implicit in the earlier work of Ossona de Mendez et al. [52].

Lemma 25 ([43, Lemma 6]). For every integer $\ell \geqslant 2$, every $s \in\{1, \ldots, \ell-1\}$, every
integer $t \geqslant s$, there exists an integer $c:=c(s, t, \ell) \geqslant 1$, such that for every graph $G$ with treewidth less than $\ell$ and with no $K_{s, t}$ subgraph, and for every $P \subseteq V(G)$,

$$
\left|\left\{u \in V(G) \backslash P:\left|N_{G}(u) \cap P\right| \geqslant s\right\}\right| \leqslant c|P|
$$

## 3.1 $K_{s, t^{-}}$Subgraph-Free Bounded Treewidth Graphs

We now reach the first of our two lemmas for bounded treewidth graphs. Say a function $f: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$is reasonable if $f(x) \geqslant x$ for all $x \in \mathbb{N}, f$ is increasing, and $\lim _{x \rightarrow \infty} x / \log f(x)=\infty$. The following lemma is the technical extension of Lemma 15 mentioned earlier.

Lemma 26. For any integers $s, t, \ell \geqslant 2$, there is a reasonable function $f: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ such that for every integer $p \geqslant f(1)$ the following holds: Let $G$ be a graph with treewidth less than $\ell$ and with no $K_{s, t}$ subgraph. Let $L$ be an $(s, p)$-good list-assignment for $G$ that is compatible with an $(s-2)$-admissible set $\mathcal{S}:=\left\{\left(\alpha_{1}, A_{1}\right), \ldots,\left(\alpha_{r}, A_{r}\right)\right\}$ in $G$. Then $G$ has an L-colouring $\varphi: V(G) \rightarrow \bigcup_{v \in V(G)} L(v)$ with clustering $f(p)$, such that:
(a) the union of the monochromatic components that intersect $P(L)$ has at most $f(|P(L)|)$ vertices;
(b) $\varphi(a) \neq \varphi\left(\alpha_{j}\right)$ for each $j \in\{1, \ldots, r\}$ and $a \in A_{j}$.
(c) $\varphi(a) \neq \varphi(x)$ for each $j \in\{1, \ldots, r\}, a \in A_{j} \cup\left\{\alpha_{j}\right\}$ and $x \in N_{G}\left(\alpha_{j}\right) \cap N_{G}(a)$.

Liu and Wood [42] prove a result similar to Lemma 26 with $r=0$ and in the more general setting of tangles (rather than bounded treewidth). Rewriting their result without reference to tangles (assuming bounded treewidth instead) results in a lemma statement equivalent to Lemma 26 with $r=0$, and a proof that has roughly the same structure as the one given below. The main differences between the two are the technical modifications required to achieve (b) and (c) when $r>0$.

Proof. Define $f$ by $f(x):=x^{\log _{4 / 3} 2} 12 \ell s(c+1)^{12 \ell s}$, where $c=c(t, \ell, s)$ is defined as in Lemma 25. So $f$ is reasonable. Fix $p \geqslant f(1)$. We prove this result by induction on $(|V(G)|,|V(G)|-|P(L)|)$ in lexicographical order. Since the result is vacuous when $V(G)=\varnothing$, we may assume that $|V(G)| \geqslant 1$. Let $P:=P(L)$ and $Q:=Q(L)$. There are two easy cases to deal with before arriving at the two main cases.

Case A. $P=\varnothing$ : If $A_{j} \neq \varnothing$ for some $j \in\{1, \ldots, r\}$, then let $v$ be any vertex in $A_{j}$. Otherwise, if $r \geqslant 1$, then let $v:=\alpha_{1}$. Otherwise, let $v$ be any vertex of $G$. Set $P^{\prime}:=\{v\}$ and define $L^{\prime}(v)$ to be a 1-element subset of $L(v)$. For each $w \in N_{G}(v)$, let
$L^{\prime}(w):=L(w) \backslash L^{\prime}(v)$. Since each $L(w)$ has size at least $s+1,\left|L^{\prime}(w)\right| \geqslant s$, as desired. For each vertex $x \in V(G) \backslash N_{G}[v]$, let $L^{\prime}(x):=L(x)$. Note that $L^{\prime}$ is an $(s, p)$-good list-assignment, with $P\left(L^{\prime}\right)=\{v\}$ and either $Q\left(L^{\prime}\right)=\varnothing$ (if $s=1$ ) or $Q\left(L^{\prime}\right)=N_{G}(v)$ (if $s \geqslant 2$ ).

We now show that $L^{\prime}$ is compatible with $\mathcal{S}$. If $r=0$ then (c1)-(c3) are vacuous. Otherwise, $v \in A_{j} \cup\left\{\alpha_{j}\right\}$ for some $j \in\{1, \ldots, r\}$. If $v=\alpha_{j}$ then, by the choice of $v, A_{j}=\varnothing($ and $j=1)$, so $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq\{v\}=P\left(L^{\prime}\right)$ which satisfies (c1); (c2) is vacuous; and $Q\left(L^{\prime}\right)=N_{G}(v)$ so (g5) implies (c3). Otherwise $v \in A_{j}$ so, by (t3), $P\left(L^{\prime}\right) \cap S_{j}=\{v\} \cap S_{j}=\varnothing$, so (c1)-(c3) are vacuous.

Since $\left|P^{\prime}\right|>|P|$, by induction, $G$ has an $L^{\prime}$-colouring $\varphi$ that satisfies the requirements of the lemma for $\left(G, L^{\prime}, \mathcal{S}\right)$. Since $L \supseteq L^{\prime}$ and $P=\varnothing, \varphi$ is an $L$-colouring that satisfies the requirements of the lemma for $(G, L, \mathcal{S})$, as desired.

Case B. $N_{G}(P)=\varnothing$ : Let $G^{\prime}:=G-P$. Since Case A does not hold, $P \neq \varnothing$ and $\left|V\left(G^{\prime}\right)\right|<|V(G)|$. By induction, there is a colouring $\varphi$ of $G^{\prime}$ satisfying the conditions of the lemma for $\left(G^{\prime}, L, \mathcal{S}\right)$ (where we consider the natural restrictions of $L$ and $\mathcal{S}$ to $\left.G^{\prime}\right)$. Extend $\varphi$ to $G$ by assigning to each vertex $v \in P$ the unique colour in $L(v)$. Since $N_{G}(P)=\varnothing$, each monochromatic component intersecting $P$ in $G$ is contained in $P$, and thus the union of the monochromatic components intersecting $P$ has size at most $|P| \leqslant p=f(1) \leqslant f(|P|)$, while the size of the remaining monochromatic components is the same in $G$ and $G^{\prime}$. Therefore $\varphi$ satisfies requirement (a). Since $L$ is compatible with $\mathcal{S}$, (c2) and (c3) imply that $\varphi$ satisfies (b) and (c), respectively. Therefore, we obtain an $L$-colouring that satisfies the requirements of the lemma for $(G, L, \mathcal{S})$, as desired.

Case C. $|P|<12 \ell s$ : Let $\hat{p}:=|P|$ and let $v_{1}, \ldots, v_{\hat{p}}$ be the vertices of $P$. Define $P_{0}:=P$ and $Q_{0}:=Q$. We will define a sequence of good list-assignments $L=L_{0} \supseteq L_{1} \supseteq$ $\cdots \supseteq L_{\hat{p}}=: L^{\prime}$ that are all compatible with $\mathcal{S}$. We will ensure that $\left|P\left(L^{\prime}\right)\right|>|P(L)|$ so that we can apply induction to $\left(G, L^{\prime}, \mathcal{S}\right)$, and obtain an $L^{\prime}$-colouring $\varphi$ of $G$ that satisfies (b) and (c). Of course, $\varphi$ also satisfies (a) with respect to $L^{\prime}$, but this is not sufficient to ensure that it satisfies (a) with respect to $L$ since $\left|P\left(L^{\prime}\right)\right|>|P(L)|$. The main difficulty, therefore, is to construct $L_{1}, \ldots, L_{\hat{p}}$ so that the union of the monochromatic components that intersect $P$ has size at most $f(|P|)$.
We will construct a sequence of precoloured sets $P_{1} \subseteq \cdots \subseteq P_{\hat{p}}$ and three related sequences $\left(U_{i}^{-}\right)_{i \in[\hat{p}]},\left(U_{i}\right)_{i \in[\hat{p}]}$, and $\left(Q_{i}\right)_{i \in[\hat{p}]}$ of vertices of $G$ and these will be used to define $L_{1}, \ldots, L_{\hat{p}}$. In order to ensure that $U_{1}^{-}$is non-empty, it is treated slightly differently than the other sets $U_{i}^{-}$: If $N_{G}(P)$ contains at least one vertex that is adjacent to at least $s$ vertices of $P=P_{0}$, then $U_{1}^{-}:=\left\{u \in N_{G}(P):\left|N_{G}(u) \cap P_{0}\right| \geqslant s\right\}$. Otherwise,
$U_{1}^{-}:=\{u\}$ where $u$ is an arbitrary vertex in $N_{G}\left(P_{0}\right)$. The set $U_{1}^{-}$is well-defined and non-empty because $P$ is non-empty (otherwise Case A would apply) and $N_{G}(P)$ is non-empty (otherwise Case B would apply). Consider the following sets (illustrated in Figure 8 and explained below):

$$
\begin{aligned}
J_{1} & :=\left\{j \in\{1, \ldots, r\}: U_{1}^{\prime} \cap S_{j} \neq \varnothing, P_{i-1} \cap S_{j}=\varnothing\right\} \\
U_{1} & :=U_{1}^{-} \cup \bigcup_{j \in J_{i}}\left(A_{j} \cup\left\{\alpha_{j}\right\} \backslash P_{0}\right) \\
P_{1} & :=P_{0} \cup U_{1} \\
Q_{1} & :=\left\{q \in N_{G}\left(P_{1}\right):\left|N_{G}(q) \cap P_{1}\right| \in\{1,2, \ldots, s-1\}\right\} .
\end{aligned}
$$

For each $i=2,3, \ldots, \hat{p}$, let

$$
\begin{align*}
U_{i}^{-} & :=\left\{u \in N_{G}\left(P_{i-1}\right):\left|N_{G}(u) \cap P_{i-1}\right| \geqslant s\right\}  \tag{1}\\
J_{i} & :=\left\{j \in\{1, \ldots, r\}: U_{i}^{-} \cap S_{j} \neq \varnothing, P_{i-1} \cap S_{j}=\varnothing\right\}  \tag{2}\\
U_{i} & :=U_{i}^{-} \cup \bigcup_{j \in J_{i}}\left(A_{j} \cup\left\{\alpha_{j}\right\} \backslash P_{i-1}\right)  \tag{3}\\
P_{i} & :=P_{i-1} \cup U_{i}  \tag{4}\\
Q_{i} & :=\left\{q \in N_{G}\left(P_{i}\right):\left|N_{G}(q) \cap P_{i}\right| \in\{1,2, \ldots, s-1\}\right\} .
\end{align*}
$$

Note that the $i=1$ case only differs from the $i \in\{2, \ldots, \hat{p}\}$ case in the definition of $U_{i}^{-}$.


Figure 8: An illustration of the different sets defined in Case C.
These definitions warrant some explanation. Suppose we have already constructed ( $s, p$ )good list assignments $L_{0} \supseteq L_{1} \supseteq \cdots \supseteq L_{i-1}$ where $P_{j}=P\left(L_{j}\right)$ and $L_{j}$ is compatible with $\mathcal{S}$ for each $j \in\{0, \ldots, i-1\}$. Now consider the construction of $L_{i} \subseteq L_{i-1}$ and $P_{i}=P\left(L_{i}\right)=P_{i-1} \cup U_{i}$. At this point, the set $P_{i-1}$ is precoloured. The set $U_{i}^{-}$consists of those non-precoloured vertices with at least $s$ precoloured neighbours (step (1)), with some exceptions in the $i=1$ case. We are about to precolour all the vertices in $U_{i}^{-}$, hence they are added to $P_{i}$ (step (4)). If the $j$-trigger set $S_{j}$ includes some vertex in
$U_{i}^{-}$and no vertex in the $j$-trigger set is currently precoloured, then $j$ is included in $J_{i}$ (step (2)), and we say that $j$ is triggered. In this case, $A_{j} \cup\left\{\alpha_{j}\right\} \backslash P_{i-1}$ is added to $U_{i}$ and $P_{i}$ (step (4)); subsequently we will also precolour the vertices in $A_{j} \cup\left\{\alpha_{j}\right\}$ that are not already precoloured. In particular, $\alpha_{j}$ is added to $P_{i}$, implying $P_{i} \cap S_{j} \neq \varnothing$ and $j$ is triggered at most once (that is, $J_{1}, \ldots, J_{\hat{p}}$ are pairwise disjoint). This algorithm ensures that the precoloured set $P_{i}$ satisfies (c1); that is, $A_{j} \subseteq P_{i}$ whenever $P_{i} \cap S_{j} \neq \varnothing$.

Now it is time to assign colours to the vertices of $U_{i}=P_{i} \backslash P_{i-1}$. For each $j \in J_{i}$ and each $a \in A_{j} \cap U_{i}$ choose any colour in $L_{i-1}(a)$ distinct from the colour of $v_{i}$. This is possible since $\left|L_{i-1}(a)\right| \geqslant 2$ by (g3), (g2) and (g4). Next, for each $j \in J$ choose a colour for $\alpha_{j}$ that is different from every colour used by any of the at most $(s-2)+1=s-1$ vertices in $A_{j} \cup\left\{v_{i}\right\}$. This is possible since $j \in J_{i}$ implies that $N_{G}\left(\alpha_{j}\right) \cap P_{i-1}=\varnothing$, so $\left|L_{i-1}\left(\alpha_{j}\right)\right| \geqslant s+1$ by (g4). This ensures that $L_{i}$ satisfies (c2).
Before colouring the remaining vertices of $U_{i}$ we pause to consider each vertex $x \in N_{G}\left(\alpha_{j}\right)$ for each $j \in J_{i}$. By definition, $N_{G}(x) \subseteq A_{j} \cup\left\{\alpha_{j}\right\} \cup S_{j}$ and, since $j \in J_{i}, S_{j} \cap P_{i-1}=\varnothing$. Therefore, at this moment the only neighbours of $x$ that have been assigned their final colour are in $A_{j} \cup \alpha_{j}$. We will remove all these colours from $L_{i-1}(x)$ before continuing, which will ensure that $L_{i}(x)$ satisfies $(\mathrm{c} 2)$ and still leaves at least $s+1-(s-1)=2$ colours in $L_{i-1}(x)$. Finally, we assign colours to all the remaining vertices of $U_{i}$. For each such vertex $v$, (g3), (g2) and (g4) imply that $\left|L_{i-1}(v)\right| \geqslant 2$ and this allows us to choose a colour for $v$ that is distinct from $v_{i}$. Lastly, we remove colours from the lists of vertices in $Q_{i}$ so that they avoid their newly-precoloured neighbours in $U_{i}$.
Shortly, we will give a more careful definition of the list-assignments $L_{1}, \ldots, L_{\hat{p}}$. We will want to prove that each of these is an $(s, p)$-good list-assignment for $G$ (that is, a list-assignment satisfying (g1)-(g5) above). Before explicitly defining the contents of these lists, we can already verify that they will satisfy ( g 1 ) by bounding the size of $P_{i}=P\left(L_{i}\right)$ for each $i \in\{1, \ldots, \hat{p}\}$. By Lemma $25,\left|U_{i}^{-}\right| \leqslant c\left|P_{i-1}\right|$. By (t4), the trigger sets $S_{1}, \ldots, S_{r}$ are pairwise disjoint, so each element of $U_{i}^{-}$is responsible for adding the contents of at most one set $A_{j} \cup\left\{\alpha_{j}\right\}$ (of size at most $s-1$ ) to $U_{i}$. Therefore $\left|U_{i}\right| \leqslant s\left|U_{i}^{-}\right| \leqslant c s\left|P_{i-1}\right|$. (We think of each $u \in U_{i}^{-}$as contributing itself and up to $s-1$ elements of $A_{j} \cup\left\{\alpha_{j}\right\}$ to $U_{i}$; for a total of $s$ elements.) Hence $P_{i}$ satisfies the recurrence $\left|P_{i}\right|=\left|P_{i-1}\right|+\left|U_{i}\right| \leqslant(c s+1)\left|P_{i-1}\right|$. Therefore $\left|P_{i}\right| \leqslant\left|P_{0}\right|(1+c s)^{i}<12 \ell s(c s+1)^{12 \ell s}=$ $f(1) \leqslant p$.
We are now ready to carefully define $L_{1}, \ldots, L_{\hat{p}}$. Let $L_{0}:=L$ and for each $i \in\{1, \ldots, \hat{p}\}$, let $c_{i}$ be the unique element in $L\left(v_{i}\right)$. While defining $L_{1}, \ldots, L_{\hat{p}}$ we will prove that each is $(s, p)$-good and compatible with $\mathcal{S}$. By assumption, $L_{0}$ is $(s, p)$-good and compatible with $\mathcal{S}$. Therefore, in the following definition of $L_{i}$, we may inductively assume that $L_{i-1}$ is $(s, p)$-good and compatible with $\mathcal{S}$. We start by defining the lists of the vertices
in $P_{i}$, and then we consider vertices in $Q_{i}$, vertices in $N_{G}\left(P_{i}\right) \backslash Q_{i}$, and finally vertices in $V(G) \backslash N_{G}\left[P_{i}\right]$.

1. For $v \in P_{i-1}$ set $L_{i}(v):=L_{i-1}(v)$. Consider a vertex $v \in U_{i}=P_{i} \backslash P_{i-1}$. We need $L_{i}(v)$ to be a singleton set, so that $P\left(L_{i}\right)=P_{i}$. We distinguish three cases, which we handle in the following order:
1.1 If $v \notin \bigcup_{j=1}^{r} N_{G}\left[\alpha_{j}\right]$ then let $L_{i}(v)$ be a singleton set consisting of an arbitrary element of $L_{i-1}(v) \backslash\left\{c_{i}\right\}$. Since $L_{i-1}$ is $(s, p)$-good and $v \notin P_{i-1}$, (g2)-(g4) imply that $\left|L_{i-1}(v)\right| \geqslant 2$. Therefore $L_{i-1}(v) \backslash\left\{c_{i}\right\}$ is non-empty and $L_{i}(v)$ is well-defined.
1.2 If $v=\alpha_{j}$ for some $j \in\{1, \ldots, r\}$ then (c1) implies that $U_{i} \cap S_{j} \neq \varnothing$. So, $A_{j} \subseteq P_{i-1} \cup U_{i}$ by the definition of $U_{i}$. In particular, for each $a \in A_{j}$, $\left|L_{i}(a)\right|=1$ since $a \in P_{i-1}$ (so $\left|L_{i-1}(a)\right|=1$ ) or $a \in U_{i}$ (so $L_{i}(a)$ was assigned in 1.1, above). If $v$ is adjacent to some vertex $u \in P_{i-1}$, then $u \in N_{G}\left(\alpha_{j}\right) \subseteq S_{j}$, and $v \in P_{i-1}$ by (c1), which is a contradiction. Thus $v \notin N_{G}\left(P_{i-1}\right)$. By (g4), $\left|L_{i-1}(v)\right| \geqslant s+1$. In this case, let $L_{i}(v)$ be a singleton set consisting of an arbitrary colour in $L_{i-1}(v) \backslash\left(\cup_{a \in A_{j}} L_{i}(a)\right) \backslash\left\{c_{i}\right\}$. This is well-defined because $\left|A_{j}\right| \leqslant s-2$ by (t1), and $\left|L_{i}(a)\right|=1$ for each $a \in A_{j}$. Furthermore, this choice of $L_{i}(v)$ ensures that $L_{i}$ satisfies (c2).
1.3 Otherwise, $v \in N_{G}\left(\alpha_{j}\right)$ for some $j \in\{1, \ldots, r\}$ and we must take extra care to ensure that $L_{i}(v)$ satisfies (c3). In particular, each time we set $L_{i}(v)$ for some $v \in N_{G}\left(\alpha_{j}\right) \cap U_{i}$ we ensure that $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P_{i}$ and that $L_{i}(v) \cap L_{i}(a)=\varnothing$ for each $a \in N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$. This ensures that (c3) is satisfied. There are two cases to consider:
(i) The 'typical' case: Some $u \in N_{G}\left(P_{i-1}\right)$ has at least $s$ neighbours in $P_{i-1}$. In this case, the inclusion of $v \in U_{i}$ implies that $\left|N_{G}(v) \cap P_{i-1}\right| \geqslant s$. On the other hand, $\left|N_{G}(v) \cap P_{i-1}\right| \leqslant\left|A_{j} \cup\left\{\alpha_{j}\right\}\right|+\left|S_{j} \cap P_{i-1}\right| \leqslant s-1+\mid S_{j} \cap$ $P_{i-1} \mid$ by (t1). Therefore, $\left|S_{j} \cap P_{i-1}\right| \geqslant 1$. By (c1) and (c3), $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq$ $P_{i-1}$ and $L_{i-1}(a) \cap L_{i-1}(v)=\varnothing$ for each $a \in\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap N_{G}(v)$. Choose $L_{i}(v)$ to be a singleton set that contains an arbitrary element of $L_{i-1}(v) \backslash\left\{c_{i}\right\}$ As in the previous case, (g2)-(g4) ensure that this is welldefined. By (c1) and (c3), $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P_{i-1}$ and $L_{i-1}(a) \cap L_{i-1}(v)=\varnothing$ for each $a \in\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap N_{G}(v)$. Therefore, $L_{i}(v)$ satisfies (c3).
(ii) The 'exceptional' case: Every $u \in N_{G}\left(P_{i-1}\right)$ has at most $s-1$ neighbours in $P_{i-1}$. If $i \geqslant 2$, this implies that $U_{i}^{-}, J_{i}$ and $U_{i}$ are empty, and thus $L_{i}=L_{i-1}$ and there is nothing to prove. Otherwise, $i=1$, $U_{1}^{-}:=\{v\}, J_{1}=\{j\}$, and $\left|N_{G}(v) \cap P_{0}\right| \in\{1, \ldots, s-1\}$. In this case, $U_{1}=\left(\{v\} \cup A_{j} \cup\left\{\alpha_{j}\right\}\right) \backslash P_{0}$. This implies that $v \in Q_{0}$, so (g5) implies that $L_{0}(v) \cap L_{0}(u)=\varnothing$ for each $u \in N_{G}(v) \cap P_{0}$. If $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P_{0}$ then
$L_{0}(a) \cap L_{0}(v)=\varnothing$ for each $a \in N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$ by (g5), and we can again let $L_{1}(v)$ be a singleton set that contains an arbitrary element of $L_{0}(v) \backslash\left\{c_{i}\right\}$. Now assume $A_{j} \cup\left\{\alpha_{j}\right\} \nsubseteq P_{0}$. Then $S_{j} \cap P_{0}=\varnothing$ by (c1). Since $N_{G}(v) \subseteq A_{j} \cup S_{j}$, we have $N_{G}(v) \cap P_{0}=N_{G}(v) \cap P_{0} \cap A_{j}$. By (g2), $\left|L_{0}(v)\right| \geqslant s+1-\left|N_{G}(v) \cap P_{0} \cap A_{j}\right|$. Since $v \in P_{1}$ and $v \in N_{G}\left(\alpha_{j}\right) \subseteq S_{j}$, we have $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq U_{1} \subseteq P_{1}$. For each $a \in A_{j} \backslash P_{0}$, ( t 3$)$ implies that $a \notin$ $\bigcup_{j^{\prime}=1}^{r} N_{G}\left[\alpha_{j^{\prime}}\right]$, so $L_{1}(a)$ is assigned in Case (1.1) above. Similarly, $L_{1}\left(\alpha_{j}\right)$ is assigned in Case (1.2) above. Thus $\left|L_{1}(a)\right|=1$ for each $a \in A_{j} \cup\left\{\alpha_{j}\right\}$. Therefore, $\left|L_{0}(v) \backslash \bigcup_{a \in A_{j} \cup\left\{\alpha_{j}\right\}} L_{1}(a)\right| \geqslant(s+1)-\left|A_{j} \cup\left\{\alpha_{j}\right\}\right| \geqslant 2$ by (t1). Let $L_{1}(v)$ be any singleton subset of $\left(L_{0}(v) \backslash \bigcup_{a \in A_{j} \cup\left\{\alpha_{j}\right\}} L_{1}(a)\right) \backslash\left\{c_{i}\right\}$.
We now show that (c3) holds for each $x \in N_{G}\left(\alpha_{j}\right)$ for each $j \in\{1, \ldots, r\}$. If $x \notin N_{G}\left(P_{i}\right)$ then (c3) holds since $L_{i-1}$ and $\mathcal{S}$ are compatible. Now assume that $x \in N_{G}\left(\alpha_{j}\right) \cap N_{G}\left(P_{i}\right)$. If $P_{i} \cap S_{j}=\varnothing$, then (c3) is vacuous for $L_{i}$, so we can assume that $P_{i} \cap S_{j} \neq \varnothing$, which by (3) implies that $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P_{i}$. We need to ensure that $L_{i}(x) \cap L_{i}(a)=\varnothing$ for each $a \in N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$.

- If $x \in Q_{i}$, then the steps taken in Step 2 below to ensure that $L_{i}$ satisfies (g5) will ensure that $L_{i}(x) \cap L_{i}(a)=\varnothing$ for each $a \in N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$.
- Otherwise $x \in N_{G}\left(P_{i}\right) \backslash Q_{i}$. If $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P_{i-1}$ then since $L_{i-1}$ satisfies (c3) and (g5) we have $L_{i}(x) \cap L_{i}(a)=\varnothing$ for each $a \in N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$. Otherwise $\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i} \neq \varnothing$, in which case (c3) for $L_{i}$ will also be ensured when defining $L_{i}(x)$ in Step 3 below.
Thus $L_{i}$ will satisfy (c3). We have already argued above that $L_{i}$ satisfies (c1) and (c2), so $L_{i}$ will be compatible with $\mathcal{S}$.
At this point, we have already defined $L_{i}(v)$ for each $v \in U_{i}$ so that $\left|L_{i}(v)\right|=1$. This ensures that $P_{i} \subseteq P\left(L_{i}\right)$. In the following, any further lists we define will have size at least 2 , so $P_{i}=P\left(L_{i}\right)$.

2. For each $q \in Q_{i}$, define $L_{i}(q):=L_{i-1}(q) \backslash \bigcup_{v \in U_{i} \cap N_{G}(v)} L_{i}(v)$. The fact that $L_{i-1}$ satisfies (g5) and that $P_{i}:=P_{i-1} \cup U_{i}$ ensures $L_{i}$ satisfies (g5). To check that $L_{i}$ satisfies (g2), first observe that $q \notin N_{G}\left(P_{i-1}\right) \backslash Q_{i-1}$ since this would imply that $q \in$ $U_{i} \subseteq P_{i}$. Therefore, one of (g2) or (g4) implies that $\left|L_{i-1}(q)\right| \geqslant s+1-\left|N_{G}(q) \cap P_{i-1}\right|$. Since $P_{i-1}$ and $U_{i}$ are disjoint, $\left|N_{G}(q) \cap P_{i}\right|=\left|N_{G}(q) \cap P_{i-1}\right|+\left|N_{G}(q) \cap U_{i}\right|$. Therefore, $\left|L_{i}(v)\right| \geqslant\left|L_{i-1}(v)\right|-\left|N_{G}(v) \cap U_{i}\right|=s+1-\left|N_{G}(v) \cap P_{i}\right|$, so $L_{i}$ satisfies (g2).
3. The purpose of this step is to ensure that (c3) will eventually be satisfied for vertices $v \in N_{G}\left(P_{i}\right) \backslash Q_{i}$ (even though we are not ready to precolour such vertices). We consider two cases:
3.1 If $v \in N_{G}\left(\alpha_{j}\right)$ and $N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i} \neq \varnothing$, then set

$$
L_{i}(v):=L_{i-1}(v) \backslash \bigcup_{a \in N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i}} L_{i}(a) .
$$

As discussed above, this ensures that $L_{i}$ satisfies (c3). We simply need to check that $L_{i}(v)$ still satisfies the conditions of an $(s, p)$-good list-assignment. All the lists $L_{i}(a)$, for $a \in N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i}$, have been set at Step 1.1 or Step 1.2 and therefore have size 1. It follows that $\left|L_{i}(v)\right| \geqslant\left|L_{i-1}(v)\right|-$ $\left|N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i}\right|$.
If $v \notin Q_{i-1}$ then $v \notin N_{G}\left(P_{i-1}\right) \backslash Q_{i-1}$ since $v \notin U_{i}$. Thus, $\left|L_{i-1}(v)\right| \geqslant s+1$ by (g4), and thus $\left|L_{i}(v)\right| \geqslant s+1-\left|A_{j} \cup\left\{\alpha_{j}\right\}\right| \geqslant 2$.
Now assume that $v \in Q_{i-1}$. Then $\left|L_{i-1}(v)\right| \geqslant s+1-\left|N_{G}(v) \cap P_{i-1}\right|$ by (g2). If $P_{i-1} \cap S_{j} \neq \varnothing$, then $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq P_{i-1}$ by (c1), contradicting $\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i} \neq \varnothing$. Thus $P_{i-1} \cap S_{j}=\varnothing$. Since $N_{G}(v) \subseteq S_{j} \cup A_{j}$, we have $N_{G}(v) \cap P_{i-1} \subseteq A_{j}$. Since $P_{i-1} \cap U_{i}=\varnothing$, it follows that

$$
\begin{aligned}
\left|L_{i}(v)\right| & \geqslant\left|L_{i-1}(v)\right|-\left|N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i}\right| \\
& \geqslant s+1-\left|N_{G}(v) \cap P_{i-1}\right|-\left|N_{G}(v) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap U_{i}\right| \\
& \geqslant s+1-\left|A_{j} \cup\left\{\alpha_{j}\right\}\right| \\
& \geqslant 2
\end{aligned}
$$

so $L_{i}$ satisfies (g3).
3.2 Otherwise, set $L_{i}(v):=L_{i-1}(v)$. Since $v \notin P_{i-1}$, (g2)-(g4) imply that $\left|L_{i-1}(v)\right| \geqslant 2$. Therefore, $L_{i}(v) \geqslant 2$, so $L_{i}$ satisfies (g3).
4. For each $v \in V(G) \backslash N_{G}\left[P_{i}\right]$, define $L_{i}(v):=L_{i-1}(v)$. Since $N_{G}\left[P_{i}\right] \supseteq N_{G}\left[P_{i-1}\right]$, (g4) implies that $\left|L_{i-1}(v)\right| \geqslant s+1$. Therefore $\left|L_{i}(v)\right|=\left|L_{i-1}(v)\right| \geqslant s+1$ and $L_{i}$ satisfies (g4).

Let $L^{\prime}:=L_{\hat{p}}$ and let $P^{\prime}:=P\left(L^{\prime}\right)=P_{\hat{p}}$. We previously established that $L^{\prime}$ is $(s, p)$-good and is compatible with $\mathcal{S}$. Since $U_{1}^{-} \subseteq U_{1} \subseteq P_{1} \subseteq P^{\prime}$ is non-empty, $\left|P^{\prime}\right| \geqslant|P|+1$. Therefore, by induction applied to $\left(G, L^{\prime}, \mathcal{S}\right), G$ has an $L^{\prime}$-colouring $\varphi: V(G) \rightarrow$ $\cup_{v \in V(G)} L(v)$ that satisfies the requirements of the lemma for $\left(G, L^{\prime}, \mathcal{S}\right)$. Since $L^{\prime}$ is a specialization of $L, \varphi$ is also an $L$-colouring that satisfies (c).

It remains to show that $\varphi$ satisfies (a). To accomplish this, we now prove that any monochromatic component that intersects $P$ is contained in $P^{\prime}$. Consider the monochromatic component $C_{i}$ of $G$ that contains $v_{i}$ for an arbitrary $i \in\{1, \ldots, \hat{p}\}$, so every vertex in $C_{i}$ is coloured $c_{i}$. We claim that $V\left(C_{i}\right) \subseteq P_{i-1}$. If not, there are neighbours $w \in P_{i-1}$ and $v \in N\left(P_{i-1}\right)$ such that $v, w \in V\left(C_{i}\right)$. Since $v \in N\left(P_{i-1}\right), v$ is either in $U_{i}$ or in $Q_{i-1}$. If $v \in U_{i}$ then, by definition $\varphi(v) \neq c_{i}$. Thus, $v \in Q_{i-1}$ and by (g5), $L_{i-1}(v) \cap L_{i-1}(w)=\varnothing$. Since $\varphi(v) \in L_{i-1}(v), \varphi(w) \in L_{i-1}(w)$, and $\varphi(w)=\varphi(w)$. This contradiction shows that $C_{i}$ is contained in $P_{i-1} \subseteq P^{\prime}$. By (g1), the union of all monochromatic components intersecting $P$ has size at most $\left|P^{\prime}\right| \leqslant p=f(1) \leqslant f(|P|)$, as required.

Therefore $\varphi$ is an $L$-colouring of $G$ that satisfies the requirements of the lemma, as desired.

Case D. $|P| \geqslant 12 \ell s$ : Robertson and Seymour [56, (2.6)] proved that for any graph $G$ of treewidth less than $\ell$ and for any $P \subseteq V(G)$ there exists $V_{1}, V_{2} \subseteq V(G)$ such that $G=G\left[V_{1}\right] \cup G\left[V_{2}\right]$ and:

$$
\left|V_{1} \cap V_{2}\right| \leqslant \ell, \quad\left|P \backslash V_{2}\right| \leqslant 2\left|P \backslash V_{1}\right| \quad \text { and } \quad\left|P \backslash V_{1}\right| \leqslant 2\left|P \backslash V_{2}\right|
$$

Apply this result to $G$ and the set $P:=P(L)$ of precoloured vertices. Note that

$$
\begin{equation*}
\left|P \backslash V_{1}\right| \leqslant \frac{2}{3}\left(\left|P \backslash V_{1}\right|+\left|P \backslash V_{2}\right|\right) \leqslant \frac{2}{3}|P| \tag{5}
\end{equation*}
$$

Similarly, $\left|P \backslash V_{2}\right| \leqslant \frac{2}{3}|P|$. Let $Z:=V_{1} \cap V_{2}$.
Let $J:=\left\{j \in\{1, \ldots, r\}: S_{j} \cap Z \backslash P \neq \varnothing\right\}$; that is, $j \in J$ if there is a non-precoloured vertex in the $j$-trigger set and in $Z$. By ( t 4 ), $|J| \leqslant|Z| \leqslant \ell$. Let

$$
P^{\prime}:=P \cup Z \cup \bigcup_{j \in J}\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)
$$

We now define a list-assignment $L^{\prime}$ of $G$ that is compatible with $\mathcal{S}$, and such that $P\left(L^{\prime}\right)=P^{\prime}$. By the definition of $J$ and $P^{\prime}$, and since $P\left(L^{\prime}\right)=P^{\prime}$, the first requirement (c1) for $L^{\prime}$ to be compatible with $\mathcal{S}$ is satisfied.

1. For each $v \in\left(Z \cup \bigcup_{j \in J} A_{j}\right) \backslash\left(P \cup \bigcup_{j \in 1}^{r} N_{G}\left[\alpha_{j}\right]\right)$, set $L^{\prime}(v)$ to be an arbitrary singleton subset of $L(v)$. This is analogous to Case C (Step 1.1), above.
2. For each $j \in J$, let $L^{\prime}\left(\alpha_{j}\right)$ be an arbitrary singleton subset of $L\left(\alpha_{j}\right) \backslash \bigcup_{a \in A_{j}} L^{\prime}(a)$. As in the analysis of Case C (Step 1.2) above, this is possible because $\left|L\left(\alpha_{j}\right)\right| \geqslant s+1$, (after the completion of Step 1) $\left|L^{\prime}(a)\right|=1$ for each $a \in A_{j}$, and $\left|A_{j}\right| \leqslant s-2$. This step ensures that $L^{\prime}$ satisfies (c2).
3. For each $j \in\{1, \ldots, r\}$ and each $x \in N_{G}\left(\alpha_{j}\right) \cap Z \backslash P$, let $L^{\prime}(x)$ be an arbitrary singleton subset of $L(x) \backslash \bigcup_{a \in N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)} L^{\prime}(a)$. To see that this is possible, note that $j \in J$ since $x \in S_{j} \cap Z \backslash P$, and after Steps 1 and $2,\left|L^{\prime}(a)\right|=1$ for each $a \in A_{j} \cup\left\{\alpha_{j}\right\}$. Now, as in the analysis of Case C (Step 1.3), (c1)-(c3) and (g2) and (g4) ensure that $|L(x)| \geqslant s+1-\left|N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap P\right|$ and (g5) implies $L(x) \cap L(a)=\varnothing$ for each $a \in\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap P$, so $\left|L(x) \backslash \bigcup_{a \in N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)} L^{\prime}(a)\right| \geqslant$ $s+1-\left|A_{j} \cup\left\{\alpha_{j}\right\}\right| \geqslant 2$ by (t1). Furthermore, this ensures that $L^{\prime}$ satisfies (c3) so that $L^{\prime}$ is compatible with $\mathcal{S}$.
4. Let $Q^{\prime}:=\left\{v \in V(G) \backslash P^{\prime}:\left|N_{G}(v) \cap P^{\prime}\right| \in\{1, \ldots, s-1\}\right\}$. For each $q \in Q^{\prime}$, set $L^{\prime}(q):=L(q) \backslash \bigcup_{v \in N_{G}(q) \cap P^{\prime}} L^{\prime}(v)$. As in the analysis of Case C (Step 2) above, this ensures that $L^{\prime}$ satisfies (g2) and (g5) and does not make $L^{\prime}$ incompatible with $\mathcal{S}$.
5. For each $j \in J$ and $x \in\left(N_{G}(P) \cap N_{G}\left(\alpha_{j}\right)\right) \backslash\left(Z \cup Q^{\prime} \cup P\right)$, set $L^{\prime}(x):=L(x) \backslash$ $\bigcup_{v \in N_{G}(x) \cap\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)} L^{\prime}(v)$. As in the analysis of Case C (Step 3) above, this ensures that $L^{\prime}$ satisfies (g4) and does not make $L^{\prime}$ incompatible with $\mathcal{S}$.
6. For any vertex $v \in V(G)$ not covered by any of the preceding cases, set $L^{\prime}(v):=$ $L(v)$. This ensures that $L^{\prime}$ satisfies (g3) and (g4) and does not make $L^{\prime}$ incompatible with $\mathcal{S}$.

Thus $L^{\prime}$ is a list-assignment for $G$ that satisfies (g2)-(g5) and is compatible with $\mathcal{S}$. However, $L^{\prime}$ is not necessarily $(s, p)$-good for $G$ since $P^{\prime}=P\left(L^{\prime}\right)$ may be too large to satisfy (g1). Instead, we use divide and conquer. Let

$$
P_{1}:=\left(P \backslash V_{2}\right) \cup Z \cup \bigcup_{j \in J}\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \quad \text { and } \quad P_{2}:=\left(P \backslash V_{1}\right) \cup Z \cup \bigcup_{j \in J}\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)
$$

Note that $P_{1} \cup P_{2}=P^{\prime} . \mathrm{By}(\mathrm{t} 1)$ and (5),

$$
\left|P_{1}\right| \leqslant\left|P \backslash V_{2}\right|+|Z|+\sum_{j \in J}\left|A_{j} \cup\left\{\alpha_{j}\right\}\right| \leqslant \frac{2}{3}|P|+\ell+\ell(s-1)=\frac{2}{3}|P|+\ell s
$$

Similarly, $\left|P_{2}\right| \leqslant \frac{2}{3}|P|+\ell s$. Since $\ell s \leqslant \frac{|P|}{12}$, for each $i \in\{1,2\}$,

$$
\left|P_{i}\right| \leqslant \frac{3}{4}|P|<|P| \leqslant p
$$

For $i \in\{1,2\}$, let $G_{i}:=G\left[V_{i} \cup P_{i}\right]$. Thus $L^{\prime}$ satisfies $(\mathrm{g} 1)$ in $G_{i}$. Since $L^{\prime}$ satisfies (g2)-(g5) and is compatible with $\mathcal{S}$ in $G$, the same properties hold in $G_{i}$. Hence $L^{\prime}$ restricted to $G_{i}$ is $(s, p)$-good.
Since $\left|P_{1}\right|<|P|$ and $P_{1} \supseteq P \cap V_{1}$, there exists at least one vertex $v \in P \cap V_{2}$ that is not in $Z \cup \bigcup_{j \in J}\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$, and thus $v$ is not in $G_{1}$. Therefore, by induction applied to $\left(G_{1}, L^{\prime}, \mathcal{S}\right)$ and $\left(G_{2}, L^{\prime}, \mathcal{S}\right)$, each of $G_{1}$ and $G_{2}$ has an $L^{\prime}$-colouring, $\varphi_{1}$ and $\varphi_{2}$, respectively, that satisfies (a)-(c). Since $\left|L^{\prime}(v)\right|=1$ for each $v \in Z, \varphi_{1}(v)=\varphi_{2}(v)$ for each $v \in Z$, so these two colourings of $G$ can be combined to produce an $L^{\prime}$-colouring $\varphi$ of $G$ that satisfies (b) and (c).

It remains to show that $\varphi$ also satisfies (a) and that monochromatic components of $G$ have size at most $f\left(12 \ell s(c+1)^{12 \ell s}\right)$. We start by establishing (a). Let $C$ denote the union of the monochromatic components intersecting $P^{\prime}$ in $G$, and for $i \in\{1,2\}$, let $C_{i}$ be the union of the monochromatic components intersecting $P_{i}^{\prime}$ in $G_{i}$. Then $V(C) \subseteq V\left(C_{1}\right) \cup V\left(C_{2}\right)$, so

$$
\begin{aligned}
|C| \leqslant\left|C_{1}\right|+\left|C_{2}\right| \leqslant f\left(\left|P_{1}\right|\right)+f\left(\left|P_{2}\right|\right) \leqslant 2 f\left(\frac{3}{4}|P|\right) & \leqslant 2 \cdot\left(\frac{3}{4}|P|\right)^{\log _{4 / 3} 2} 12 \ell s(c+1)^{12 \ell s} \\
& =|P|^{\log _{4 / 3} 2} 12 \ell s(c+1)^{12 \ell s} \\
& =f(|P|) .
\end{aligned}
$$

Thus, $\varphi$ is an $L$-colouring of $G$ that satisfies (a).
Finally, we verify the bound on the clustering. Since $\varphi_{1}$ and $\varphi_{2}$ satisfy (b) and (c), each monochromatic component of $G$ that is completely contained in $G_{i}$, for some $i \in\{1,2\}$, has size at most $f(p)$. Any monochromatic component of $G$ not contained entirely in $G_{1}$ or in $G_{2}$ contains a vertex in $Z \subseteq P^{\prime}$, and we have argued above that each such component has size at most $f(|P|) \leqslant f(p)$. Thus, $\varphi$ is an $L$-colouring of $G$ that satisfies all the requirements of the lemma, which concludes the proof.

## $3.2 \mathcal{J}_{s, t}$-Minor-Free Bounded Treewidth Graphs

Recall that $k$-admissible sets $\left\{\left(\alpha_{1}, A_{1}\right), \ldots,\left(\alpha_{r}, A_{r}\right)\right\}$ were defined at the beginning of Section 3, and that we defined $S_{j}:=N_{G-A_{j}}^{2}\left[\alpha_{j}\right]$ for each $j \in\{1, \ldots, r\}$.

The following is our second lemma for bounded treewidth graphs. In comparison with Lemma 26, this lemma makes a weaker admissibility assumption, but adds several new requirements related to $K_{2, t}$ and $K_{3, t}$ subgraphs in (or close to) $G\left[S_{j}\right]$. Each of the sets $S_{j}$ will come from the larger numbered layers in a layering of some torso, which means that $S_{j}$ is contained in the embedded part of some torso. This limits the size of $K_{3, t}$-subgraphs, but not the size of $K_{2, t}$-subgraphs. Therefore, before we can apply Lemma 27 we will, in Section 4.1, develop some tools for eliminating $K_{2, t}$ subgraphs in embedded graphs.

Lemma 27. For any integers $\ell \geqslant 1, s \geqslant 3, t \geqslant s+2$, and $k \geqslant s-2$ there exists an integer $p_{0}$ and a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that the following holds for every integer $p \geqslant p_{0}$. Let $G$ be a graph with treewidth less than $\ell$. Let $\mathcal{S}:=\left\{\left(\alpha_{1}, A_{1}\right), \ldots,\left(\alpha_{r}, A_{r}\right)\right\}$ be an $(s-3)$-admissible set in $G$ such that:
(x1) $G-\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}$ is $\mathcal{J}_{s, t}$-minor-free;
(x2) for each $j \in\{1, \ldots, r\}, G\left[S_{j} \backslash\left\{\alpha_{j}\right\}\right]$ is connected;
(x3) for each $j \in\{1, \ldots, r\}, G\left[S_{j} \backslash\left\{\alpha_{j}\right\}\right]$ is $K_{2, k-s+3 \text {-subgraph-free; and }}$
(x4) for each $j \in\{1, \ldots, r\}, G\left[N_{G-A_{j}}^{4}\left[\alpha_{j}\right] \backslash\left\{\alpha_{j}\right\}\right]$ is $K_{3, k-s+3 \text {-subgraph-free. }}$
Let $L$ be an $(s, p)$-good list-assignment for $G$ that is compatible with $\mathcal{S}$. Then $G$ has an L-colouring $\varphi$ with clustering $f(p)$ such that:
(a) the union of the monochromatic components that intersect $P(L)$ has at most $f(|P(L)|)$ vertices;
(b) $\varphi(a) \neq \varphi\left(\alpha_{j}\right)$ for each $j \in\{1, \ldots, r\}$ and $a \in A_{j}$; and
(c) $\varphi(a) \neq \varphi(x)$ for each $j \in\{1, \ldots, r\}, a \in A \cup\left\{\alpha_{j}\right\}$ and $x \in N_{G}\left(\alpha_{j}\right) \cap N_{G}(a)$.

Proof. This proof in the $r=0$ case closely follows the proof of Liu and Wood [44,

Lemma 18], which is a statement about $K_{h}$-minor-free graphs. Here, various modifications are needed to establish (b) and (c), and because this theorem is about $\mathcal{J}_{s, t}$-minor-free graphs.

Let $p_{0}$ be the minimum integer satisfying $p_{0} \geqslant s+t-1$ and one other inequality below. Let $f_{0}: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_{0}(x):=t^{x /(t-s-1)}$ and let $f(x):=f_{0}\left(\max \left\{p_{0}, x\right\}\right)$. We prove the result by induction on $|V(G)|$, with this choice of $f$ fixed. Let $P:=P(L)$, so $|P| \leqslant p$. Let $t^{\prime}:=\max \{k, t(t-1)\}$.

First consider the easy case in which $G$ is $K_{s, t^{\prime}}$-subgraph-free. Since $\mathcal{S}$ is ( $s-3$ )-admissible, it is also $(s-2)$-admissible. Since $L$ is $(s, p)$-good and compatible with $\mathcal{S}$, we can apply Lemma 26 and obtain an $L$-colouring $\varphi$ of $G$ that satisfies (b) and (c), has clustering at most $f^{\prime}(p)$, and for which the union of monochromatic components that intersect $P$ has size at most $f^{\prime}(|P|)$, where $f^{\prime}: \mathbb{R} \rightarrow \mathbb{R}$ is the reasonable function that appears
 logarithms, we see that $f_{0}(x) \geqslant f^{\prime}(x)$ provided that $x / \log \left(f^{\prime}(x)\right) \geqslant(t-s-1) / \log t$. Since $f^{\prime}$ is reasonable, the left-hand side of this equation is increasing without bound as $x$ increases, so there exists some $p_{0}:=p_{0}(s, t, \ell)$ such that this inequality is satisfied for all $x \geqslant p_{0}$. It follows that $f(|P|) \geqslant f^{\prime}(|P|)$ for all $|P|$, as required.

We can now assume that $G$ contains a $K_{s, t^{\prime}}$ subgraph. Consider any copy $H$ of $K_{s, y}$ with $y \geqslant t^{\prime}$, implying $y \geqslant k \geqslant k-s+3$ and $y \geqslant t(t-1)$. Let $\{X, Y\}$ be the bipartition of $H$ with $|X|=s$ and $|Y|=y$.

The following claim is critical in order to establish that requirements (b) and (c) do not interfere with the structure of the original proof of Liu and Wood [44].
Claim 27.1. For each $j \in\{1, \ldots, r\},(X \cup Y) \cap S_{j}=\varnothing$.
Proof. Since $\mathcal{S}$ is $(s-3)$-admissible, $\left|A_{j}\right| \leqslant s-3$ by (t1).
First suppose that $A_{j} \cup\left\{\alpha_{j}\right\} \subseteq X$. Then, since $\alpha_{j} \in X, Y \subseteq N_{G}\left(\alpha_{j}\right)$. By (t2) and (t3), $X \subseteq N_{G}(Y) \subseteq N_{G}^{2}\left[\alpha_{j}\right] \subseteq S_{j} \cup A_{j}$ and $\left|X \cap S_{j}\right|=\left|X \backslash A_{j}\right| \geqslant|X|-\left|A_{j}\right| \geqslant s-(s-3)=3$, which implies that $G\left[S_{j} \backslash\left\{\alpha_{j}\right\}\right]$ contains a $K_{2, y}$-subgraph (since $S_{j} \cap A_{j}=\varnothing$ by (t3)), contradicting (x3) (since $y \geqslant k-s+3$ ).

Now suppose that $A_{j} \cup\left\{\alpha_{j}\right\} \nsubseteq X$ and assume, for the sake of contradiction, that $(X \cup Y) \cap S_{j} \neq \varnothing$. Since $A_{j} \cup\left\{\alpha_{j}\right\} \nsubseteq X$, we have $\left|\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap X\right| \leqslant\left|A_{j} \cup\left\{\alpha_{j}\right\}\right|-1 \leqslant s-3$. Let $X^{\prime}:=X \backslash\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$ and $Y^{\prime}:=Y \backslash\left(A_{j} \cup\left\{\alpha_{j}\right\}\right)$. Thus $\left|X^{\prime}\right|=|X|-\left|\left(A_{j} \cup\left\{\alpha_{j}\right\}\right) \cap X\right| \geqslant$ $s-(s-3)=3$ and $\left|Y^{\prime}\right| \geqslant|Y|-\left|A_{j} \cup\left\{\alpha_{j}\right\}\right| \geqslant y-(s-2) \geqslant k-s+3$. Thus $G\left[X^{\prime} \cup Y^{\prime}\right]$ contains a $K_{3, k-s+3}$-subgraph. Since $X \cup Y$ intersects $S_{j}$, and $S_{j} \cap A_{j}=\varnothing$ by (t3), we have that $X^{\prime} \cup Y^{\prime}$ intersects $S_{j}$. By construction, $\alpha_{j} \notin X^{\prime} \cup Y^{\prime}$. Thus $G\left[N_{G-A_{j}}^{2}\left[S_{j}\right] \backslash\left\{\alpha_{j}\right\}\right]$ contains a $K_{3, k-s+3}$ subgraph, which contradicts (x4).

The next claim shows that $X$ does an excellent job of separating $Y$. The proof of this simple result is the only place where the specific structure of graphs in $\mathcal{J}_{s, t}$ is used.

Claim 27.2. Each component of $G-X$ contains at most $t-1$ vertices in $Y$.
Proof. Suppose, for the sake of contradiction, that some component $C$ of $G-X$ has at least $t$ vertices in $Y$. Choose a tree $T$ in $C$ with $|V(T) \cap Y| \geqslant t$ to lexicographically minimise $\left(\left|V(T) \cap\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}\right|,|V(T)|\right)$. This is well-defined since any spanning tree $T$ of $C$ satisfies $|V(T) \cap Y| \geqslant t$.

We claim that $V(T) \cap\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}=\varnothing$. Otherwise $T$ contains $\alpha_{j}$ for some $j \in\{1, \ldots, r\}$. By Claim 27.1, $\alpha_{j} \notin Y$. Now, $N_{T}\left(\alpha_{j}\right)$ is contained in $S_{j} \backslash\left\{\alpha_{j}\right\}$, which induces a connected subgraph of $G$ by $(\mathrm{x} 2)$. So $G\left[\left(V(T) \cup S_{j}\right) \backslash\left\{\alpha_{j}\right\}\right]$ is connected. Let $T^{\prime}$ be a spanning tree of $G\left[\left(V(T) \cup S_{j}\right) \backslash\left\{\alpha_{j}\right\}\right]$. By Claim 27.1, $S_{j} \cap X=\varnothing$, so $T^{\prime}$ is a subtree of $C$. By construction, $V(T) \cap Y \subseteq V\left(T^{\prime}\right) \cap Y$, so $\left|V\left(T^{\prime}\right) \cap Y\right| \geqslant t$. By ( t 4$), S_{j} \backslash\left\{\alpha_{j}\right\}$ avoids $\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}$. So $T^{\prime}$ has fewer vertices in $\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}$ than $T$, contradicting the choice of $T$. Hence $V(T) \cap\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}=\varnothing$.

If a leaf $x$ of $T$ is not in $Y$, then $T-x$ contradicts the minimality of $T$. So every leaf $x$ of $T$ is in $Y$. If $|V(T) \cap Y|>t$ and $x$ is any leaf of $T$, then $T-x$ again contradicts the minimality of $T$. So $|V(T) \cap Y|=t$.

Let $T_{1}, \ldots, T_{t}$ be a partition of $T$ into $t$ pairwise disjoint subtrees, each with exactly one vertex in $Y$. Contract each $T_{i}$ into a vertex $z_{i}$. The graph induced by $\left\{z_{1}, \ldots, z_{t}\right\}$ is connected and thus contains a spanning tree. Since $|V(T) \cap Y|=t$ and $|Y| \geqslant t+s$, there is a matching $M=\left\{e_{1}, \ldots, e_{s}\right\}$ between $X$ and $Y \backslash V(T)$, which avoids $\alpha_{1}, \ldots, \alpha_{r}$ by Claim 27.1. Contract each $e_{i}$ into a vertex $x_{i}$. In the resulting graph, the subgraph induced by $\left\{x_{1}, \ldots, x_{s}, z_{1}, \ldots, z_{t}\right\}$ contains a graph in $\mathcal{J}_{s, t}$. Since $V(T) \cap\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}=$ $\varnothing$ and $M$ avoids $\alpha_{1}, \ldots, \alpha_{r}$, we have found a graph in $\mathcal{J}_{s, t}$ as a minor of $G-\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}$. This contradiction shows that every component of $G-X$ has at most $t-1$ vertices in $Y$.

For the remainder of the proof, consider a copy $H$ of $K_{s, y}$ in $G$ that maximises $y$ (in particular, $y \geqslant t^{\prime}$ and the two claims above apply). We again write the bipartition of $H$ as $\{X, Y\}$, with $|X|=s$ and $|Y|=y$.

Let $C_{1}, \ldots, C_{q}$ be the components of $G-X$. By Claim 27.2, each of $C_{1}, \ldots, C_{q}$ has at most $t-1$ vertices in $Y$. So $q \geqslant \frac{|Y|}{t-1}=\frac{y}{t-1} \geqslant t \geqslant s+2$. We now consider two cases.

Case A. Some component $C_{i}$ of $\boldsymbol{G}-\boldsymbol{X}$ contains no vertex in $\boldsymbol{P}$ : Write $C:=C_{i}$ for brevity. Let $G^{\prime}:=G-V(C)$, and let $\mathcal{S}^{\prime}$ be the restriction of $\mathcal{S}$ to $G^{\prime}$. Since $G^{\prime}$ is a subgraph of $G$, it satisfies (x1), (x3) and (x4). By Claim 27.1 and (x2), $G\left[S_{j}\right]=G^{\prime}\left[S_{j}\right]$
or $G^{\prime}\left[S_{j}\right]$ is empty for each $j \in\{1, \ldots, r\}$. In the former case, $G^{\prime}$ satisfies (x2). In the latter case, $\left(\alpha_{j}, A_{j}\right)$ plays no role in $\mathcal{S}^{\prime}$, so $G^{\prime}$ also satisfies (x2). Since $G^{\prime}$ has fewer vertices than $G$, we may apply induction to ( $G^{\prime}, L^{\prime}, \mathcal{S}^{\prime}$ ), so $G^{\prime}$ has an $L^{\prime}$-colouring $\varphi^{\prime}$ that satisfies (a), (b), and (c). We show how to extend $L^{\prime}$ to a colouring of $G$. To ensure that monochromatic components in $G^{\prime}$ do not grow, our colouring of $C$ will properly extend $\varphi^{\prime}$ to $G$.

Let $G_{C}:=G[V(C) \cup X]$. Then $G_{C}$ satisfies (x1)-(x4) for the same reasons $G^{\prime}$ does. Let $U_{C}:=\left\{v \in V(C):\left|N_{G}(v) \cap X\right|=s\right\}=Y \cap V(C)$ by the maximality of $y$. By Claim 27.2, $\left|U_{C}\right| \leqslant t-1$ and, by Claim 27.1, $U_{C} \cap S_{j}=\varnothing$ for each $j \in\{1, \ldots, r\}$. Let $P_{C}:=X \cup U_{C}$. Observe that, for each $v \in V(C), N_{G}(v) \cap P \subseteq X$. So $|L(v)| \geqslant s+1-|P \cap X|$ by (g2)-(g4). This is helpful in verifying the claims about $\left|L_{C}(v)\right|$ for the list-assignment $L_{C}$ that we now define:

- For each $x \in X$, let $L_{C}(x):=\left\{\varphi^{\prime}(x)\right\}$.
- For each of the at most $t-1$ vertices $v \in U_{C}$, let $L_{C}(v)$ be a singleton subset of $L(v) \backslash \bigcup_{x \in X \cap N_{G_{C}}(v)} L_{C}(x)$. This is possible since $v$ has exactly $s$ neighbours in $X$.
- Let $Q_{C}:=\left\{v \in N_{G_{C}}\left(P_{C}\right):\left|N_{G_{C}}(v) \cap P_{C}\right| \in\{1, \ldots, s-1\}\right\}$. For each $v \in Q_{C}$, let $L_{C}(v):=L(v) \backslash \bigcup_{w \in N_{G_{C}}(v) \cap P_{C}} L_{C}(w)$. The fact that $L$ satisfies (g2), (g4) and (g5) implies that $\left|L_{C}(v)\right| \geqslant s+1-\left|N_{G_{C}}(v) \cap P_{C}\right|$ for each $v \in Q_{C}$.
- For each $v \in N_{G_{C}}\left(P_{C}\right) \backslash Q_{C}$, let $L_{C}(v):=L(v) \backslash \bigcup_{x \in X \cap N_{G}(v)} L_{C}(x)$. Observe that $v \notin P_{C}$, so $v$ is adjacent to at most $s-1$ vertices in $X$. Since $L$ satisfies (g2)-(g5) and $\left|L_{C}(x)\right|=1$ for each $x \in X$, this implies that $\left|L_{C}(v)\right| \geqslant 2$ for each $v \in N_{G_{C}}\left(P_{C}\right) \backslash Q_{C}$.
- For each $v \in V(C) \backslash N_{G_{C}}\left[P_{C}\right]$, let $L_{C}(v):=L(v)$. Since $L$ satisfies (g4), $\left|L_{C}(v)\right|=$ $|L(v)| \geqslant s+1$.

The only vertices assigned singleton lists in $L_{C}$ are in $X \cup U_{C}=P_{C}$, so $P_{C}=P\left(L_{C}\right)$ and, by definition, $Q_{C}=Q\left(L_{C}\right)$. We now verify that, for any $p \geqslant p_{0} \geqslant s+1, L_{C}$ is an $(s, p)$-good list-assignment for $G_{C}$.
(g1): $\left|P_{C}\right|=|X|+\left|U_{C}\right| \leqslant s+t-1 \leqslant p_{0} \leqslant p$;
(g2): $\left|L_{C}(v)\right| \geqslant s+1-\left|N_{G}(v) \cap P_{C}\right|$ for each $v \in Q_{C}$, as discussed above;
(g3): $\left|L_{C}(v)\right| \geqslant 2$ for all $v \in N_{G_{C}}\left(P_{C}\right) \backslash Q_{C}$, as discussed above;
(g4): $\left|L_{C}(v)\right| \geqslant s+1$ for all $v \in V\left(G_{C}\right) \backslash N_{G_{C}}\left[P_{C}\right]$ since $L_{C}(v)=L(v)$; and
(g5): $L_{C}(v) \cap L_{C}(u)=\varnothing$ for all $v \in Q_{C}$ and $u \in N_{G_{C}}(v) \cap P_{C}$.
Each of the rules above explicitly ensures that $L_{C}(v) \cap L_{C}(x)=\varnothing$ for each $x \in X \cap N_{G_{C}}(v)$. Therefore, any $L_{C}$-colouring of $G_{C}$ properly extends the precolouring of $X$ given by $\varphi^{\prime}$. Let $\mathcal{S}_{C}$ be the restriction of $\mathcal{S}$ to $G_{C}$. By Claim 27.1, $\left(X \cup U_{C}\right) \cap S_{j}=\varnothing$ for each $j \in\{1, \ldots, r\}$. So $S_{j} \cap P\left(L_{C}\right) \neq \varnothing$ if and only if $S_{j} \cap P(L) \neq \varnothing$ and $S_{j} \subseteq V(C)$. So $L_{C}$ trivially satisfies the requirements for being compatible with $\mathcal{S}_{C}$.

Since $G-X$ has at least two components, $\left|V\left(G_{C}\right)\right|<|V(G)|$. By induction applied to $\left(G_{C}, L_{C}, \mathcal{S}_{C}\right)$, there exists an $L_{C^{-}}$-colouring $\varphi_{C}$ of $G_{C}$ that satisfies (a), (b), and (c). For each $x \in X, \varphi_{C}(x)=\varphi^{\prime}(x)$, so $\varphi_{C}$ and $\varphi^{\prime}$ can be combined to give a colouring $\varphi$ of $G$. The colouring $\varphi$ inherits (b) and (c) from $\varphi^{\prime}$ and $\varphi_{C}$. It also satisfies (a) since each vertex in $N_{G_{C}}(X)$ is assigned a colour distinct from the colours of its neighbours in $X$, so the monochromatic components that intersect $P$ are contained in $G^{\prime}$. Therefore, the colouring $\varphi$ of $G$ satisfies all requirements of the lemma.

Case B. Each component $C_{i}$ of $G-X$ contains at least one vertex in $P$ : In this case, $X$ plays a similar role to that of the separator $Z$ used in Case D of the proof of Lemma 26. For each $i \in\{1, \ldots, q\}$, let $G_{i}:=G\left[V\left(C_{i}\right) \cup X\right]$ and let $\mathcal{S}_{i}$ be the restriction of $\mathcal{S}$ to $G_{i}$. Since $G_{i}$ is a subgraph of $G$ it satisfies (x1), (x3) and (x4) and, by Claim 27.1 and (x2), $G_{i}$ and $L_{i}$ also satisfy (x2).

Let $L_{1}$ be the restriction of $L$ to $G_{1}$. Then $L_{1}$ is $(s, p)$-good for $G_{1}$ (since $L$ is $(s, p)$-good for $G$ ) and $L_{1}$ is compatible with $\mathcal{S}_{1}$ (since $L$ is compatible with $\mathcal{S}$ ). Let $P_{1}:=P\left(L_{1}\right)$. Observe that $\left|P_{1}\right| \leqslant|P|-q+1$ since $P_{1}$ avoids at least one vertex of $P$ in each component of $G-X$ other than $C_{1}$. Recall that $q \geqslant 2$, so $G_{1}$ has fewer vertices than $G$. Thus we may apply induction to $\left(G_{1}, L_{1}, \mathcal{S}_{1}\right)$, so $G_{1}$ has a colouring $\varphi_{1}$ that satisfies (b) and (c) and such that the union of monochromatic components of $G_{1}$ that intersect $P_{1}$ has size at most $f\left(\left|P_{1}\right|\right) \leqslant f(|P|-q+1)$.

We now define a list-assignment $L_{i}$ for each $i \in\{2, \ldots, q\}$. This is similar to the listassignment $L_{C}$ used in the proof of Case A except that we cannot prevent monochromatic components in $P$ from entering $C_{i}$ (since $P$ already contains vertices of $C_{i}$ ). Instead, we use the fact that $P_{i}:=P\left(L_{i}\right)$ is much smaller than $P$ to control the growth of these components.

For each $i \in\{2, \ldots, q\}$, let $P_{i}:=\left(P \cap V\left(G_{i}\right)\right) \cup X$, let $Q_{i}:=\left\{v \in N_{G_{i}}\left(P_{i}\right):\left|N_{G_{i}}(v) \cap P_{i}\right| \in\right.$ $\{1, \ldots, s-1\}\}$, and define the list-assignment $L_{i}$ as follows:

- For each $x \in X$, let $L_{i}(x):=\left\{\varphi_{1}(x)\right\}$.
- For each $v \in Q_{i}$, let $L_{i}(v):=L(v) \backslash \bigcup_{w \in P_{i} \cap N_{G_{i}}(v)} L_{i}(w)$. The fact that $L$ satisfies (g2), (g4) and (g5) ensures that $\left|L_{i}(v)\right| \geqslant s+1-\left|P_{i} \cap N_{G_{i}}(v)\right|$.
- For each $v \in V\left(G_{i}\right) \backslash\left(P_{i} \cup Q_{i}\right)$, let $L_{i}(v):=L(v)$.

From these definitions, it follows immediately that $P_{i}:=P\left(L_{i}\right)$ and $Q_{i}:=Q\left(L_{i}\right)$. We now verify that $L_{i}$ is an $(s, p)$-good list-assignment for $G_{i}$.
(g1): $\left|P_{i}\right| \leqslant|P|-q+|X|+1=|P|+s-q+1 \leqslant|P| \leqslant p$ (since $q \geqslant s+1$ );
(g2): $\left|L_{i}(v)\right| \geqslant s+1-\left|N_{G_{i}}(v) \cap P_{i}\right|$ for each $v \in Q_{i}$;
(g3): $\left|L_{i}(v)\right| \geqslant 2$ for each $v \in N_{G_{i}}\left(P_{i}\right) \backslash Q_{i}$, since $L_{i}(v)=L(v)$;
(g4): $\left|L_{i}(v)\right| \geqslant s+1$ for all $v \in V\left(G_{i}\right) \backslash N_{G_{i}}\left[P_{i}\right]$, since $L_{i}(v)=L(v)$; and (g5): $L(v) \cap L(u)=\varnothing$ for all $v \in Q_{i}$ and $u \in N_{G_{i}}(v) \cap P_{i}$, by definition.

Let $\mathcal{S}_{i}$ be the restriction of $\mathcal{S}$ to $G_{i}$. Since $P_{i} \backslash P \subseteq X$, Claim 27.1 implies that any $j$-trigger set intersected by $P_{i}$ is also intersected by $P$, so $L_{i}$ is compatible with $\mathcal{S}_{i}$ because $L$ is compatible with $\mathcal{S}$.

We are almost done. By assumption, $P$ contains at least one vertex in each of $C_{1}, \ldots, C_{q}$, so $\left|P_{i}\right| \leqslant|P|+s+1-q$. Since $G_{i}$ has fewer vertices than $G$, we may apply induction to $\left(G_{i}, L_{i}, \mathcal{S}_{i}\right)$, so $G_{i}$ has an $L_{i}$-colouring $\varphi_{i}$ that satisfies (b) and (c). Furthermore, the union of monochromatic components of $G_{i}$ has size at most $f(|P|+s+1-q)$.
Since $\varphi_{i}(x)=\varphi_{j}(x)$ for each $i, j \in\{1, \ldots, q\}$ and each $x \in X$, the colourings $\varphi_{1}, \ldots, \varphi_{q}$ can be combined to give a colouring $\varphi$ of $G$ that satisfies (b) and (c). In this colouring, each monochromatic component that intersects $P$ intersects $P_{i}$ for some $i \in\{1, \ldots, q\}$. Therefore, the total number of vertices in all monochromatic components that intersect $P$ is at most

$$
\begin{array}{rlr}
q \cdot f(|P|+s+1-q) & =q \cdot f_{0}\left(\max \left\{p_{0},|P|\right\}+s+1-q\right) \\
& =q \cdot t^{\frac{\max \left\{p_{0},|P|\right\}+s+1-q}{t-s-1}} \\
& =t^{\frac{\max \left\{p_{0},|P|\right\}+s+1-q}{t-s-1}+\frac{\log q}{\log t}} \\
& =t^{\frac{\max \left\{p_{0},|P|\right\}}{t-s-1}+\frac{s+1-q}{t-s-1}+\frac{\log q}{\log t}} \\
& \leqslant t^{\frac{\max \left\{p_{0},|P|\right\}}{t-s-1}} \\
& =f_{0}\left(\max \left\{p_{0},|P|\right\}=f(|P|) .\right. & \\
\text { (proved below) }
\end{array}
$$

The above inequality is proved as follows. Since $q \geqslant t \geqslant s+2 \geqslant 5$, we have $t(q-s-1) \geqslant$ $q(t-s-1)$ and $\frac{q}{\log q} \geqslant \frac{t}{\log t}$, which together imply $\frac{q-s-1}{t-s-1} \geqslant \frac{q}{t} \geqslant \frac{\log q}{\log t}$ and $\frac{s+1-q}{t-s-1}+\frac{\log q}{\log t} \leqslant 0$. Thus $\varphi$ is an $L$-colouring of $G$ that satisfies (a), (b), and (c), as desired.

## 4 Proofs of the Main Results

This section completes the proofs of Theorem 4 and Theorem 7 in the $s=3$ case. A key ingredient is the next section on $K_{2, t}$-subgraphs.

## 4.1 $\quad K_{2, t}$-Subgraphs in Embedded Graphs

This section introduces some old and some new tools for eliminating $K_{2, t}$ subgraphs in surface embedded graphs, so that we can eventually apply Lemma 27.

A $d$-island of a graph $G$ is a non-empty subset $I$ of $V(G)$ such that $\left|N_{G}(v) \backslash I\right| \leqslant d$ for each $v \in I$. A $d$-island $I$ of $G$ is a $(d, c)$-island of $G$ if $|I| \leqslant c$. The concept of islands was introduced in the context of clustered colouring by Esperet and Ochem [24], who made the following simple but important observation.

Observation 28 ([24]). For any (d, c)-island I of a graph $G$, any $(d+1)$-colouring of $G-I$ with clustering at most $c$ can be properly extended to $a(d+1)$-colouring of $G$ with clustering at most $c$ by colouring each vertex $v \in I$ by a colour not used on $N_{G}(v) \backslash I$.

Recall Theorem 8 by Dvořák and Norin [20], which says that every graph of bounded Euler genus has a 4-colouring with bounded clustering. To prove this, Dvořák and Norin [20] showed that graphs of bounded Euler genus have 3-islands of bounded size and applied Observation 28 inductively (see [69, Section 3.5] for a concise exposition). Their proof uses strongly sublinear separators, and we adapt it to show the following.

Lemma 29. There is a constant $c>0$ such that for every graph $G$ of Euler genus at most $g$, and for every set $Y \subseteq V(G)$ such that $|Y| \geqslant 4\left|N_{G}(Y)\right|+4 g$, there exists a $(3, c(g+1))$-island $I$ of $G$ with $I \subseteq Y$.

Proof. The following lemma is folklore (see Lemma 15 in [69]): Let $G$ be a graph such that for fixed $\alpha>0$ and $\beta \in(0,1)$, every subgraph $H$ of $G$ has a balanced separator of size at most $\alpha|V(H)|^{1-\beta}$. Then for every $\varepsilon \in(0,1]$ there exists $S \subseteq V(G)$ of size at most $\varepsilon|V(G)|$ such that each component of $G-S$ has at most $f(\alpha, \varepsilon, \beta):=\left\lceil 2\left(\frac{\alpha}{\varepsilon\left(2^{\beta}-1\right)}\right)^{1 / \beta}\right\rceil$ vertices. Here a balanced separator in a graph $G$ is a set $S \subseteq V(G)$ such that every component of $G-S$ has at most $\frac{1}{2}|V(G)|$ vertices.

Let $G$ and $Y$ be as defined in the lemma. We may delete vertices not in $N_{G}[Y]$. Now assume that $V(G)=N_{G}[Y]$. The above lemma with $\beta=\frac{1}{2}$ and $\alpha=\Theta(\sqrt{g+1})$ is applicable to $G$ and its subgraphs by a separator result of Gilbert, Hutchinson, and Tarjan [27]. We now apply the method of Dvořák and Norin [20]. By the above lemma applied to $G[Y]$ with $\varepsilon:=\frac{1}{16}$, there exists $X \subseteq Y$ of size at most $\frac{1}{16}|Y|$ such that if $K_{1}, \ldots, K_{r}$ are the components of $G[Y \backslash X]$, then each $K_{i}$ has at most $f\left(\alpha, \frac{1}{16}, \frac{1}{2}\right)$ vertices, which is at most $c(g+1)$ for some constant $c$. Let $e\left(K_{i}\right)$ be the number of edges of $G$ with at least one endpoint in $K_{i}$. By Euler's formula,

$$
\sum_{i=1}^{r} e\left(K_{i}\right) \leqslant|E(G)|<3(|V(G)|+g)=3\left(|Y|+\left|N_{G}(Y)\right|+g\right) \leqslant \frac{15}{4}|Y| \leqslant 4|Y \backslash X|
$$

$$
=4 \sum_{i=1}^{r}\left|V\left(K_{i}\right)\right| .
$$

Hence $e\left(K_{i}\right)<4\left|V\left(K_{i}\right)\right|$ for some $i$. Repeatedly remove vertices from $K_{i}$ with at least four neighbours outside $K_{i}$. Doing so maintains the property that $e\left(K_{i}\right)<4\left|V\left(K_{i}\right)\right|$. Thus the final set $I$ is non-empty. By construction, $I \subseteq Y$ and $I$ is a 3-island of $G$ of size at most $\left|V\left(K_{i}\right)\right| \leqslant c(g+1)$.

Consider the following loose interpretation of Lemma 29 obtained by setting $X:=N_{G}(Y)$. If $X$ is small compared to $G-X$, then $V(G-X)$ contains a small 3-island of $G$. Unfortunately, a few parts in a layered partition $(\mathcal{L}, \mathcal{P})$ of a bounded genus graph $G$ may contain the vast majority of the vertices of $G$. The following lemma shows that removing a constant number of (arbitrarily large) parts from a layered partition ( $\mathcal{P}, \mathcal{L}$ ) of a bounded genus graph $G$, leaves components that contain $(3, c)$-islands of $G$ or that are skinny with respect to $\mathcal{L}$.

Lemma 30. Let $\ell \geqslant 1$ and $k, g \geqslant 0$ be integers, and let $c$ be the constant from Lemma 29. Let $G$ be a graph of Euler genus $g$, let $(\mathcal{P}, \mathcal{L})$ be a $(w, \ell)$-layered partition of $G$, and let $P_{1}, \ldots, P_{k} \in \mathcal{P}$. Let $C$ be a component of $G-\bigcup_{i=1}^{k} P_{i}$ such that $V(C)$ contains no $(3, c(g+1))$-island of $G$. Then $V(C)$ is $(44 \ell k+4 g)$-skinny with respect to $\mathcal{L}$.

Proof. Assume for the sake of contradiction that $|V(C) \cap L| \geqslant 44 \ell k+4 g$ for some layer $L \in \mathcal{L}$. For any set $R \subseteq \mathcal{L}$, let $C_{R}$ be the set of all vertices in $C$ that are in layers in $R$. Let $R$ be a consecutive set of layers with

$$
\begin{equation*}
\left|C_{R}\right| \geqslant 4 \ell k(|R|+10)+4 g \tag{6}
\end{equation*}
$$

and $|R|$ maximum. This is well-defined since $R=\{L\}$ satisfies this condition, since $\left|C_{\{L\}}\right| \geqslant 4 \ell k(1+10)+4 g$.

If $\left|C_{R}\right| \geqslant 4\left|N_{G}\left(C_{R}\right)\right|+4 g$ then $C_{R}$ contains a 3-island of $G$ with size at most $c(g+1)$ by Lemma 29. So $\left|C_{R}\right|<4\left|N_{G}\left(C_{R}\right)\right|+4 g$. If $R=\mathcal{L}$ then

$$
\frac{1}{4}\left|C_{R}\right|-g<\left|N_{G}\left(C_{R}\right)\right| \leqslant\left|\bigcup_{i=1}^{k} P_{i}\right| \leqslant k \ell|R|,
$$

and thus $\left|C_{R}\right|<4 \ell k|R|+4 g$, which contradicts the choice of $R$. So $R \neq \mathcal{L}$. Let $R^{\prime}$ be the set of layers in $R$ plus the layers before and after $R$ (at least one of these exist). By the choice of $R, R^{\prime}$ fails (6). Thus

$$
\left|C_{R^{\prime}}\right|<4 \ell k\left(\left|R^{\prime}\right|+10\right)+4 g \leqslant 4 \ell k(|R|+12)+4 g \leqslant\left|C_{R}\right|+8 \ell k .
$$

Since $C$ is a component of $G-\bigcup_{i=1}^{k} P_{i}$,

$$
\begin{aligned}
\left|C_{R}\right|+8 \ell k>\left|C_{R^{\prime}}\right| & \geqslant\left|C_{R}\right|+\left|N_{G}\left(C_{R}\right)\right|-\left|N_{G}\left(C_{R}\right) \cap\left(\bigcup_{i=1}^{k} P_{i}\right)\right| \\
& \geqslant\left|C_{R}\right|+\frac{1}{4}\left|C_{R}\right|-g-\ell k(|R|+2) \\
& =\frac{5}{4}\left|C_{R}\right|-g-\ell k(|R|+2) .
\end{aligned}
$$

Thus $\frac{1}{4}\left|C_{R}\right|<8 \ell k+g+\ell k(|R|+2)$ and $\left|C_{R}\right|<4 g+4 \ell k(|R|+10)$, which contradicts the choice of $R$. Hence $|V(C) \cap L|<44 \ell k+4 g$ for each layer $L \in \mathcal{L}$.

For any surface $\Sigma$ and any region $\Delta \subseteq \Sigma$, let $\partial \Delta$ denote the boundary of $\Delta$ and let $\operatorname{Int}(\Delta):=\Delta \backslash \partial \Delta$ denote the interior of $\Delta$. We say that two subsets $\Delta_{1}$ and $\Delta_{2}$ of $\Sigma$ overlap if $\operatorname{Int}\left(\Delta_{1}\right) \cap \operatorname{Int}\left(\Delta_{2}\right) \neq \varnothing$, and $\Delta_{1}$ and $\Delta_{2}$ strictly overlap if they overlap, but neither $\operatorname{Int}\left(\Delta_{1}\right)$ nor $\operatorname{Int}\left(\Delta_{2}\right)$ contains the other.

In the following, we treat the vertices and edges of a graph $G$ embedded in a surface $\Sigma$ as points and curves in $\Sigma$, respectively. Consequently, paths in $G$ are treated as simple curves in $\Sigma$ and cycles in $G$ are treated as simple closed curves in $\Sigma$. For a region $R \subseteq \Sigma$, let $G[R]:=G[V(G) \cap R]$ and $G-R:=G[V(G) \backslash R]$. Here we include an edge $u v$ in $G[R]$ if and only if both $u$ and $v$ are in $R$ (the edge $u v$ is not necessarily contained in $R$ ). A $d$-disc $\Delta$ in $G$ (with respect to $\Sigma$ ) is a closed subset of $\Sigma$ that is homeomorphic to a disc and whose boundary is a cycle of length $d$ in $G$. Note that each contractible cycle of length $d$ in $G$ bounds at least one $d$-disc.

Lemma 31. For any integer $g \geqslant 0$ and for any embedding of $K_{2,3 g+2}$ in a surface $\Sigma$ of Euler genus g, there exists a 4-disc in $K_{2,3 g+2}$ with respect to $\Sigma$.

Proof. It is known that if $x$ and $y$ are two vertices in a graph $G$ that is cellularly embedded in $\Sigma$, and $P_{1}, \ldots, P_{k}$ are internally disjoint paths with ends $x$ and $y$, such that for any $1 \leqslant i<j \leqslant k$, the cycle $P_{i} \cup P_{j}$ is non-contractible in $\Sigma$, then $k \leqslant 3 g+1$ (see [46, Proposition 4.2.7]).

Now consider any (not necessarily cellular) embedding of a copy of $K_{2,3 g+2}$ in $\Sigma$. Let $\left\{\{x, y\},\left\{v_{1}, \ldots, v_{3 g+2}\right\}\right\}$ be the bipartition of $K_{2,3 g+2}$. Assume for the sake of contradiction that there is no pair $i<j$ such that the embedding contains a 4 -disc bounded by $x v_{i} y v_{j}$. Triangulate all faces of the embedding that are not homeomorphic to a disc, so that each resulting triangular face bounds a disc (see for example [46, Section 3.1]).

The resulting supergraph $G$ is cellularly embedded in $\Sigma$. The copy of $K_{2,3 g+2}$ under consideration is a subgraph of $G$, with the property that the paths $P_{i}:=x v_{i} y$ of $K_{2,3 g+2}$ are internally disjoint, and $P_{i} \cup P_{j}$ is a non-contractible cycle for any $i<j$, which contradicts the above property.


Figure 9: The graphs $G, G_{\circ}$ and $G_{\bullet}^{-}$in Lemmas 32 and 35.

The following lemma, illustrated in Figure 9, will allow us to eliminate $K_{2, t}$ subgraphs that contain vertices in the bottom layer $L_{\uparrow}^{x}$ of the raised drape $G_{\uparrow}^{x}$ without disturbing the upper layers $L_{1}^{x}, \ldots, L_{5}^{x}$. (This lemma will be applied with $F:=F^{x}:=L_{\leqslant 4}^{x}$ so that $\left.N_{G_{\uparrow}}[F]=L_{\leqslant 5}^{x}.\right)$
Consider a graph $G$ embedded in a surface $\Sigma$ and a subset $F \subseteq V(G)$. We say that a $d$-disc $\Delta$ in $G$ with respect to $\Sigma$ is $F$-avoiding if $\Delta$ has no vertex of $F$ on its boundary and no vertex of $N_{G}[F]$ in its interior. An $F$-simplification $G_{\circ}$ of $G$ is obtained as follows: let $\mathcal{C}$ be a set of closed discs in $\Sigma$ such that:
(a) each $\Delta \in \mathcal{C}$ is a 3 -disc or a 4 -disc in $G$ with respect to $\Sigma$;
(b) no pair of discs in $\mathcal{C}$ overlap;
(c) every disc in $\mathcal{C}$ is $F$-avoiding;
(d) $\left|V(G) \cap\left(\cup_{\Delta \in \mathcal{C}} \operatorname{Int}(\Delta)\right)\right|$ is maximised.

Then $G_{\circ}:=G-\left(\bigcup_{\Delta \in \mathcal{C}} \operatorname{Int}(\Delta)\right)$ is an $F$-simplification of $G$.
Lemma 32. Let $G$ be a graph embedded in a surface $\Sigma$ of Euler genus $g$, let $F \subseteq V(G)$ be such that $G[F]$ has at most c connected components, and let $G_{\circ}$ be an $F$-simplification of $G$. Then there exists an integer $t:=t(g, c)$ such that $G_{\circ}-F$ is $K_{2, t}$-subgraph-free.

Proof. A chord in a $d$-disc $\Delta$ of $G$ having a boundary cycle $v_{1}, \ldots, v_{d}$ is an edge $v_{i} v_{j}$ of $G$ with endpoints in $\left\{v_{1}, \ldots, v_{d}\right\}$ such that the interior of $v_{i} v_{j}$ is contained in the interior of $\Delta$. A $d$-disc is chord-free if it has no chords. We may assume that each 4 -disc in $\mathcal{C}$ is chord-free since each non-chord-free 4 -disc $\Delta$ can be replaced by two interior disjoint 3 -discs $\Delta_{1}$ and $\Delta_{2}$ whose union is $\Delta$ and whose intersection is a chord of $\Delta$. This operation preserves (a)-(d) and does not change $V(G) \cap\left(\cup_{\Delta \in \mathcal{C}} \operatorname{Int}(\Delta)\right)$.

We may assume that $\bigcup_{\Delta \in \mathcal{C}} \Delta$ contains every $F$-avoiding chord-free 4-disc of $G$. Indeed, the only reason not to include such a disc $\Delta$ would be that it overlaps one or more other discs in $\mathcal{C}$. Since all discs in $S$ are 3 -discs or chord-free 4-discs, no disc in $\mathcal{C}$ strictly
overlaps $\Delta .{ }^{6}$ Therefore, all discs in $\mathcal{C}$ that overlap $\Delta$ are contained in $\Delta$ so we can remove them all and replace them with $\Delta$. This does not change $V(G) \cap\left(\cup_{\Delta \in \mathcal{C}} \operatorname{Int}(\Delta)\right)$ since this set is already of maximum size, and thus it does not change the simplification $G_{0}$.

Suppose that $G_{\circ}-F$ contains a subgraph $Z$ isomorphic to $K_{2, t}$. Let $X:=\{v, w\}$ and $Y:=\left\{y_{1}, \ldots, y_{t}\right\}$ be the two parts in the bipartition of $V(Z)$. Define a graph $J$ with vertex set $V(J):=Y$ that contains an edge $y_{i} y_{j}$ if and only if the cycle $v y_{i} w y_{j}$ is contractible. By Ramsey's Theorem [54], for any integer $k$ there exists a sufficiently large integer $t:=R(2 c+6,3 g+2)$ such that the graph $J$ contains a clique of size $2 c+6$ or a stable set of size $3 g+2$. In the latter case, this stable set $Y^{\prime}$ defines a subgraph of $G_{0}\left[X \cup Y^{\prime}\right]$ isomorphic to $K_{2,3 g+2}$ such that no 4-cycle in this subgraph bounds a 4-disc. This contradicts Lemma 31. Therefore we may assume that $J$ contains a clique of size $2 c+6$ with vertex set $Y^{\prime}$.

Let $k:=2 c+6$. Then $G_{\circ}\left[X \cup Y^{\prime}\right]$ contains a subgraph $L$ isomorphic to $K_{2, k}$ in which every 4 -cycle bounds a 4 -disc of $L$ with respect to $\Sigma$. By Lemma 10, it follows that $L$ is embedded in a closed 4 -disc $\Delta$ of $L$ with respect to $\Sigma$. Let $Y^{\prime}:=\left\{y_{1}^{\prime}, \ldots, y_{k}^{\prime}\right\}$ where the boundary of $\Delta$ is the cycle $v y_{1}^{\prime} w y_{k}^{\prime}$ and, for each $i \in\{1, \ldots, k-1\}$, the 4-cycle $v y_{i} w y_{i+1}$ bounds a face of $L$.

For each $i \in\{1, \ldots,\lfloor(k-1) / 2\rfloor\}$, consider the 4 -disc $\Delta_{2 i}$ bounded by $v y_{2 i-1} w y_{2 i+1}$ that contains $y_{2 i}$ in its interior. Since $y_{2 i} \in V\left(G_{\circ}\right), \Delta_{2 i} \notin \mathcal{C}$. Therefore, $\Delta_{2 i}$ contains a vertex of $N_{G}[F]$ in its interior (and is therefore not $F$-avoiding) or $\Delta_{2 i}$ contains the edge $v w$ (and is therefore not chord-free). ${ }^{7}$

- Since $G$ is a simple graph, there is at most one value of $i$ such that $\Delta_{2 i}$ contains $v w$.
- If $\Delta_{2 i}$ contains a vertex of $N_{G}[F]$ in its interior then, since $V(Z) \cap F=\varnothing$, the interior of $\Delta_{2 i}$ contains an entire component of $F$. Therefore there are at most $c$ values of $i$ for which $\Delta_{2 i}$ contains a vertex of $N_{G}[F]$.

Therefore, $\lfloor(k-1) / 2\rfloor \leqslant c+1$ which implies that $k<2 c+6$, a contradiction. Therefore, $G_{\circ}-F$ does not contain a subgraph isomorphic to $K_{2, t}$ for $t \geqslant R(2 c+6,3 g+1)$.

For distinct vertices $v$ and $w$ in a graph $G$, we say that $v$ is dominated by $w$ (in $G$ ) if $N_{G}(v) \subseteq N_{G}(w)$. (Note that by this definition a vertex cannot dominate its neighbour). If $v$ is dominated in $G$ by $w$, then $w$ dominates $v$ in $G$. We say that $v$ is dominated in $G$

[^6]if there exists $w \in V(G) \backslash\{v\}$ that dominates $v$ in $G$. Two properties of this definition are immediate: (i) If $v$ is a dominated vertex of a connected graph $G$, then $G-v$ is also connected. (ii) If $v$ is dominated by $w$ in $G$ then $v$ is dominated by $w$ in any induced subgraph of $G$ that includes both $v$ and $w$. For any $R \subseteq V(G)$ and any $v \in V(G)$ we say that $v$ is $R$-dominated in $G$ if there exists some $w \in R \backslash\{v\}$ that dominates $v$ in $G$.

Observation 33. Let $G$ be a graph embedded in the plane having an outer face bounded by a cycle $F$ in $G$. Let $R:=V(G) \backslash V(F)$ be such that $N_{G}(x) \subseteq V(F)$ for each $x \in R$. Then any $R$-dominated vertex in $R$ has degree at most 2 .

Proof. If $x \in R$ has three or more neighbours in $F$ then the only face of $G[V(F) \cup\{x\}]$ with three neighbours of $x$ on its boundary is the outer face. Thus, no vertex $y \in R$ can dominate $x$.

Observation 34. Let $G$ be a graph embedded in the plane having an outer face bounded by a cycle $F$ in $G$ and such that $N_{G}(x) \subseteq V(F)$ for each $x \in V(G) \backslash V(F)$. Let $v$ and $w$ be distinct vertices of $F$, and let $R:=N_{G}(v) \cap N_{G}(w) \backslash V(F)$. Then $|R| \leqslant 2$ or some vertex of $R$ is $R$-dominated in $G$.

Proof. Suppose that $R$ contains three vertices $x, y$, and $z$. By definition, each of these vertices is adjacent to both $v$ and $w$, and thus has at least two neighbours in $F$. If any of these vertices has exactly two neighbours in $F$ then it is dominated by the other two and we are done, so assume each of $x, y, z$ has a neighbour in $F$ that is distinct from $v$ and $w$. By symmetry, we can assume that $y$ lies in the interior of the 4 -disc bounded by $v x w z$. Since this cycle only intersects $F$ in $\{v, w\}$, it follows from the Jordan Curve Theorem that $y$ has no neighbour in $F$ distinct from $v$ and $w$, which is a contradiction.

For an embedded graph $G$ and $F \subseteq V(G)$, an $F$-contraction $G_{\bullet}$ of $G$ is obtained as follows: Let $G_{\circ}$ be an $F$-simplification of $G$ obtained by a set $\mathcal{C}$ of 3 - and 4 -discs. In the original graph $G$, contract each component $C$ of $G-V\left(G_{\circ}\right)$ into a single vertex $v_{C}$, and call the resulting graph $G_{\bullet}$. An $F$-compression $G_{\bullet}^{-}$of $G$ is obtained as follows: Let $R \subseteq V\left(G_{\bullet}\right) \backslash V\left(G_{\circ}\right)$ be such that, for each $\Delta \in \mathcal{C}$, no vertex of $R \cap \Delta$ is $(R \cap \Delta)$-dominated in $G_{\bullet}$. Then $G_{\bullet}^{-}:=G\left[V\left(G_{\circ}\right) \cup R\right]$ is an $F$-compression of $G$.

To make the end of this section more concrete, we explain how the notions of $F$ contraction and $F$-compression will be used in Section 4.2. Once we have raised a curtain, our aim is to clean its embedded part to avoid large copies of $K_{2, t}$. It is safe to remove components $C$ that are attached to at most two vertices of the embedded part of the raised curtain, because these can be coloured easily using our strategy of breaking skinny components using one additional colour. These components $C$ correspond to the vertices of $G_{\bullet}-V\left(G_{\bullet}^{-}\right)$(to which they have been contracted), and Observations 33
and 34 tell us that they are indeed attached to the embedded part by at most two vertices. We now prove that once these vertices are removed, the maximum size of a copy of $K_{2, t}$ disjoint from $F$ is bounded.

Lemma 35. Let $G$ be a graph embedded in a surface $\Sigma$ of Euler genus $g$, let $F \subseteq V(G)$ be such that $G[F]$ has at most c connected components, and let $G_{\bullet}^{-}$be an $F$-compression of $G$. Then there exists an integer $t:=t(g, c)$ such $G_{\bullet}^{-}-F$ is $K_{2, t}$-subgraph-free.

Proof. Let $G_{\circ}$ be the $F$-simplification of $G$ used to create $G_{\bullet}^{-}$, obtained from some set $\mathcal{C}$ of 3 and 4-discs, and let $R:=V\left(G_{\bullet}^{-}\right) \backslash V\left(G_{\circ}\right)$. The vertices in $R$ form an independent set in $G_{\bullet}^{-}$and each vertex in $R$ is adjacent to at most four vertices on the boundary cycle of some 3 - or 4 -disc in $\mathcal{C}$. Therefore, each vertex in $R$ has degree at most 4 . Let $Z$ be a $K_{2, t}$ subgraph in $G_{\bullet}^{-}$with parts $X:=\{v, w\}$ and $Y:=\left\{y_{1}, \ldots, y_{t}\right\}$. Then each vertex in $X$ has degree at least $t$. Therefore, $\{v, w\} \subset V\left(G_{\circ}\right)$ or $t \leqslant 4$. In the latter case, we are done, so now assume that $\{v, w\} \subset V\left(G_{\circ}\right)$.

Suppose that $V(Z) \cap F=\varnothing$. By Lemma 32, $\left|Y \cap V\left(G_{0}\right)\right| \leqslant t^{\prime}$ for some $t^{\prime}:=t^{\prime}(g, c)$. Therefore, $|Y \cap R| \geqslant t-t^{\prime}$. By Observation 34, for each disc $\Delta \in \mathcal{C}$ there are at most two vertices of $R$ in the interior of $\Delta$. Therefore, there are at least $\left(r-t^{\prime}\right) / 2$ discs in $\mathcal{C}$ that contain both $v$ and $w$ on their boundary cycles. There are at most two such discs in which $v$ and $w$ are consecutive in the boundary cycles. Therefore there are $d \geqslant\left(r-t^{\prime}\right) / 2-24$-discs in $\mathcal{C}$ with $v$ and $w$ on, but not consecutive on, their boundary cycles. Each of these $d$ discs defines two length-2 paths in $G_{\circ}$ from $v$ to $w$ and each of these length- 2 paths is shared by at most two discs. Therefore $G_{\circ}-F$ contains a $K_{2, d}$ subgraph, so $d<t^{\prime}$. Thus, $\left(t-t^{\prime}\right) / 2-2 \leqslant d<t^{\prime}$ so $t<3 t^{\prime}+4$. Therefore $G_{\bullet}^{-}-F$ does not contains a $K_{2, t}$ subgraph for $t:=3 t^{\prime}+4$.

### 4.2 Theorem 4: $\mathcal{J}_{s, t}$-Minor-Free Graphs

This section finished the proof of our main theorem, Theorem 4 , which says that $\mathcal{J}_{s, t^{-}}$ minor-free graphs are $(s+1)$-colourable with bounded clustering. We use the same general strategy used in the proof of Theorem 7 (for $s \geqslant 4$ ) given in Section 2.5, with some additional steps. The following lemma shows that we can restrict our attention to trees of $(s-3, \ell)$-curtains.

Lemma 36. For integers $s \geqslant 3$ and $t \geqslant 2$, every $\mathcal{J}_{s, t}$-minor-free graph is a lower-minorclosed tree of $(s-3, \ell)$-curtains for some $\ell=\ell(s, t)$.

Proof. Let $K_{s} \oplus P_{t}$ be the complete join of $K_{s}$ and a $t$-vertex path, which is an element of $\mathcal{J}_{s, t}$. Observe that $K_{2} \oplus P_{t}$ is planar, implying $K_{s} \oplus P_{t}$ is $(s-2)$-apex. By definition,
every $\mathcal{J}_{s, t}$-minor-free graph $G$ is $\left(K_{s} \oplus P_{t}\right)$-minor-free. Therefore, by Theorem 22, $G$ is a lower-minor-closed tree of $(s-3, \ell)$-curtains for some $\ell=\ell(s, t)$.

By Lemma 21, we can focus on the case where $G$ is a single ( $s-3, \ell$ )-curtain with some set $S$ of at most $s$ precoloured vertices in the top of its root. As a first step, we will use Observation 28 along with the following lemma to eliminate small 3-islands in the lower layers of each drape.

Lemma 37. Let $G$ be an $(s-3, \ell)$-drape described by $(\mathcal{P}, \mathcal{L})$ where $\mathcal{L}=:\left(L_{1}, L_{2}, \ldots\right)$. If $I \subseteq L_{\geqslant 6}$ is a 3 -island in $G\left[L_{\geqslant 5}\right]$, then $I$ is an s-island in $G$.

Proof. Let $A$ be the major apex set of $G$. For each $v \in I, N_{G}(v) \subseteq L_{\geqslant 5} \cup A$. Therefore, $\left|N_{G}(v) \backslash I\right| \leqslant\left|N_{G[L \geqslant 5]}(v) \backslash I\right|+|A| \leqslant 3+(s-3)=s$.

In the following, we work with a drape $G$ and the raised drape $G_{\uparrow} .{ }^{8}$ The contractions performed in $G$ to obtain $G_{\uparrow}$ involve edges in the surface-embedded part $G_{0}$ of $G$. Equivalently, each vertex $v$ in $V\left(G_{\uparrow}\right) \backslash V(G)$ is obtained by contracting a subset $Q_{v} \subseteq$ $V\left(G_{0}\right)$ such that $G_{0}\left[Q_{v}\right]$ is connected. Thus, $\mathcal{Q}:=\left\{Q_{v}: v \in V\left(G_{\uparrow}\right) \backslash V(G)\right\}$ is a set of disjoint subsets of $V\left(G_{0}\right)$ and $G_{\uparrow}:=G / \mathcal{Q}$ is a minor of $G$. Using this interpretation, we will let $\widetilde{G}_{\uparrow}:=G_{0} / \mathcal{Q}$ denote the graph obtained from $G_{0}$ by performing the same set of edge contractions used to create $G_{\uparrow}$. Observe that $\widetilde{G}_{\uparrow}$ is a surface-embedded graph that inherits an embedding from the embedding of $G_{0}$.

The next step is to raise the curtain, as in the proof of Theorem 7. However, before immediately applying a bounded treewidth result (Lemma 27), we will perform an $F$-contraction and an $F$-compression on (the surface-embedded part of) each raised drape in order to eliminate $K_{2, t}$-subgraphs in the bottom layers of each raised drape. In order for this to be useful, we require that each vertex of the $F$-contraction be obtained by contracting a connected subgraph of $G$ that is skinny with respect to the layering $\mathcal{L}$. That is what we show in the following lemma.

Lemma 38. Let $G$ be an upward-connected $(s-3, \ell)$-drape with $(s-3, \ell)$-almostembedding $\mathcal{E}:=\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{\ell}\right)$ and described by $(\mathcal{P}, \mathcal{L})$ with $\mathcal{L}=:\left(L_{1}, L_{2}, \ldots\right)$ and such that $L_{\geqslant 6}$ contains no $(3, \ell)$-island of $G$. Let $G_{\uparrow}$ be the $(s-3, \ell)$-drape obtained by raising $G$, let $F:=V\left(G_{0}\right) \cap L_{\leqslant 4}$, and let $\widetilde{G}_{\uparrow}$. be an $F$-contraction of $\widetilde{G}_{\uparrow}$. Then each $v \in V\left(\widetilde{G}_{\uparrow_{\bullet}}\right)$ is the result of contracting a connected subgraph $X_{v}$ in $G$ and $V\left(X_{v}\right)$ is $\ell^{\prime}$-skinny with respect to $\mathcal{L}$, for some $\ell^{\prime}:=\ell^{\prime}(\ell)$.

[^7]Proof. That each $v \in V\left(\widetilde{G}_{\uparrow \bullet}\right) \backslash V\left(\widetilde{G}_{\uparrow}\right)$ is obtained by contracting a connected subgraph $X_{v}$ of $G$ follows immediately from the definitions of $G_{\uparrow}$ (a raised curtain) and $G_{\uparrow}$ ( an $F$-contraction of $G_{\uparrow}$ ).
For $v \in V\left(\widetilde{G}_{\uparrow}\right) \cap V\left(\widetilde{G}_{\uparrow}\right), X_{v}$ is contained in a single part $P_{v} \in \mathcal{P}$, so $V\left(X_{x}\right)$ is $\ell$-skinny with respect to $\mathcal{L}$, since $(\mathcal{P}, \mathcal{L})$ is an $\ell$-layered partition describing $G$.
Let $\widetilde{G}_{\uparrow ॰}$ be the $F$-simplification of $\widetilde{G}_{\uparrow}$ used to create $\widetilde{G}_{\uparrow \bullet}$. For $v \in V\left(\widetilde{G}_{\uparrow \bullet}\right) \backslash V\left(\widetilde{G}_{\uparrow}\right)$, $N_{\widetilde{G}_{\uparrow \bullet}}(v) \subseteq V\left(\widetilde{G}_{\uparrow_{0}}\right)$ and $\left|N_{\widetilde{G}_{\uparrow \bullet}}(v)\right| \leqslant 4$, since $v$ is obtained by contracting a component $C_{v}$ of $\widetilde{G}_{\uparrow}-V\left(\widetilde{G}_{\uparrow \bigcirc}\right)$. In particular, $C_{v}$ is contained in the interior of a 3- or 4-disc of $\widetilde{G}_{\uparrow \bigcirc}$. Therefore, there exists a subset $\mathcal{P}_{v} \subseteq \mathcal{P}$ of size at most 4 such that $N_{G_{0}}\left(X_{v}\right) \subseteq$ $\bigcup_{P \in \mathcal{P}_{v}} P$. In other words, $X_{v}$ is a component of $G_{0}-\bigcup_{P \in \mathcal{P}_{v}} P$. Furthermore, $X_{v} \subseteq L_{\geqslant 6}$ and, by assumption, $L_{\geqslant 6}$ contains no ( $3, \ell$ )-island of $G_{0}$. Therefore Lemma 30 with $\mathcal{P}_{v}=\left\{P_{1}, \ldots, P_{j}\right\}(j \leqslant 4)$ implies that $V\left(X_{v}\right)$ is $\ell^{\prime}$-skinny with respect to $\mathcal{L}$, for some $\ell^{\prime}:=\ell^{\prime}(\ell)$.

We continue with our convention of using a tilde ( $\sim$ ) to distinguish between surfaceembedded minors of $G$ obtained from $G_{0}$ and minors of $G$. In the following, $\widetilde{G}_{\uparrow}^{-}$is an $F$-compression of $\widetilde{G}_{\uparrow}$ obtained from $\widetilde{G}_{\uparrow}$ by edge contractions and vertex deletions, so $G_{\uparrow}^{-}$denotes the graph obtained by performing the same operations in $G_{\uparrow}$.

Lemma 39. Let $G,\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{r}\right),(\mathcal{P}, \mathcal{L}), \mathcal{L}=:\left(L_{1}, L_{2}, \ldots\right)$, $G_{\uparrow}$, and $F$ be defined as in Lemma 38 and let $\widetilde{G}_{\uparrow}^{-}$be an $F$-compression of $\widetilde{G}_{\uparrow}$. Then $G_{\uparrow}^{-}-\left(A \cup L_{\leqslant 4}\right)$ is $K_{2, t^{\prime}}$-subgraph-free for some $t^{\prime}:=t^{\prime}(\ell)$.

Proof. This would immediately follow from Lemma 35 if the number of components of $G_{0}[F]$ were bounded by some $t^{\prime}(\ell)$. This may not be the case, so an extra step is required. Recall that $G_{0}\left[L_{1}\right]$ contains exactly the vertices of $G_{0}$ that participate in the vortices $G_{1}, \ldots, G_{\ell}$ and that each vortex $G_{i}$ is defined in terms of a $G_{0}$-clean disc $D_{i}$. Define the embedded graph $H_{\uparrow}$ by starting with $G_{\uparrow}$ and adding a vertex $v_{i}$ in the interior of $D_{i}$ that is adjacent to each vertex in $V\left(G_{i}\right) \cap V\left(G_{0}\right)$, for each $i \in\{1, \ldots, \ell\}$. Define the graph $H_{\uparrow \bullet}^{-}$analogously, starting from the graph $G_{\uparrow \bullet}^{-}$. Let $L_{0}:=\left\{v_{1}, \ldots, v_{\ell}\right\}$. Then $H_{\uparrow}\left[L_{0} \cup L_{1}\right]$ has at most $\ell$ components and, since $\mathcal{L}$ is upward-connected, Observation 18 implies that $H_{\uparrow}\left[L_{0} \cup F\right]$ has at most $\ell$ components. Furthermore, $H_{\uparrow}^{-}$is an $\left(L_{0} \cup F\right)$-compression of $H_{\uparrow}$. By Lemma 35, $H_{\uparrow \bullet}^{-}-\left(A \cup L_{\leqslant 4}\right)$ is $K_{2, t^{\prime}}$ subgraph-free for some $t^{\prime}:=t^{\prime}(\ell)$. Since $G_{\uparrow_{\bullet}}^{-}-\left(A \cup L_{\leqslant 4}\right)$ is a subgraph of $H_{\uparrow \bullet}^{-}-\left(A \cup L_{\leqslant 4}\right)$, this completes the proof.

Lemma 40. Let $G$, $\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{r}\right),(\mathcal{P}, \mathcal{L}), \mathcal{L}=:\left(L_{1}, L_{2}, \ldots\right), G_{\uparrow}, F, G_{\uparrow}$ • and $G_{\uparrow_{\bullet}}^{-}$be defined as in Lemmas 38 and 39 and let $L_{\uparrow_{\bullet}}:=V\left(\widetilde{G}_{\uparrow_{\bullet}}\right) \backslash L_{\leqslant 5}$. Then, for any component $C$ of $G_{\uparrow_{\bullet}}\left[L_{5} \cup L_{\uparrow_{\bullet}}\right], G_{\uparrow_{\bullet}}^{-}[V(C)]$ is connected.

Proof. This follows from the fact that removing a dominated vertex from a connected graph does not disconnect the graph. The graph $G_{\uparrow \bullet}^{-}$can be obtained from $G_{\uparrow \bullet}$ by repeatedly removing a $L_{\uparrow \bullet}$-dominated vertex. Therefore $G_{\uparrow}^{-}$[ $V(C)$ ] can be obtained from $C$ by repeatedly removing a $L_{\uparrow \bullet}$-dominated vertex.

The next lemma will ensure that vertices added to each drape do not introduce $K_{3, t^{-}}$ subgraphs in the lower layers of the drape.

Lemma 41. Let $G,\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{r}\right),(\mathcal{P}, \mathcal{L}), \mathcal{L}=:\left(L_{1}, L_{2}, \ldots\right), G_{\uparrow}, F, G_{\uparrow \bullet}, G_{\uparrow_{\bullet}}^{-}$, and $L_{\uparrow}$ • be defined as in Lemmas 38 and 39. Let $\widetilde{G}_{\uparrow_{\bullet}}^{-+}$be the graph obtained from $\widetilde{G}_{\uparrow_{\bullet}}^{-}$ by adding, for each component $C$ of $G_{\uparrow}^{-}\left[L_{5} \cup L_{\uparrow}\right]$ a vertex $\alpha_{C}$ adjacent to every vertex in $V(C) \cap L_{\uparrow_{\bullet}}$. Then, for each vertex $\alpha_{C}$ added this way, $\widetilde{G}_{\uparrow \bullet}^{+-}\left[N_{\tilde{G}_{\bullet \bullet}^{+-}}^{4}\left[\alpha_{C}\right] \backslash\left\{\alpha_{C}\right\}\right]$ is $K_{3, k^{\prime}-\text { subgraph-free for some }} k^{\prime}:=k^{\prime}(\ell)$.

Proof. Let $H:=\widetilde{G}_{\uparrow \bullet}^{+-}\left[N_{\widetilde{G}_{\uparrow \bullet}^{+-}}^{4}\left[\alpha_{C}\right] \backslash\left\{\alpha_{C}\right\}\right]$. Then $H$ is $K_{3, k^{\prime} \text {-subgraph-free (for some }}$ $\left.k^{\prime}:=k^{\prime}(\ell)\right)$ because the vertices of $H$ are contained in $\bigcup_{i \in\{3,4,5, \uparrow \bullet\}} L_{i}$, so $H$ is a minor of $G_{0}$. The graph $G_{0}$ is embedded in a surface of genus $g \leqslant \ell$ and is therefore $K_{3,2 g+3^{-}}$ minor-free (by Euler's Formula).

Lemma 42. Let $G$ be a graph of treewidth at most $k$, and let $C_{1}, \ldots, C_{r}$ be pairwise disjoint connected subgraphs in $G$. Let $G^{+}$be the graph obtained from $G$ by adding, for each $i \in\{1, \ldots, r\}$, a new vertex $v_{i}$ whose neighbourhood is a subset of $V\left(C_{i}\right)$. Then $G^{+}$ has treewidth at most $2 k+1$.

Proof. Consider a tree-decomposition $\left(B_{x}: x \in V(T)\right)$ of $G$, such that $\left|B_{x}\right| \leqslant k+1$ for each node $x \in V(T)$. Let $B_{x}^{\prime}:=B_{x} \cup\left\{v_{i}: V\left(C_{i}\right) \cap B_{x} \neq \varnothing\right\}$ for each $x \in V(T)$. Since $C_{i}$ is connected, the set $\left\{x \in V(T): B_{x} \cap C_{i} \neq \varnothing\right\}$ induces a subtree of $T$. Thus $\left(B_{x}^{\prime}: x \in V(T)\right)$ is a tree-decomposition of $G^{+}$. If a new vertex $v_{i}$ is in $B_{x}^{\prime}$, then $B_{x}$ contains a vertex of $C_{i}$. Since $C_{1}, \ldots, C_{\ell}$ are pairwise disjoint, $\left|B_{x}^{\prime}\right| \leqslant 2\left|B_{x}\right| \leqslant 2 k+2$ for each $x \in V(T)$. This shows that $G^{+}$has treewidth at most $2 k+1$, as desired.

We are now ready to prove the following lemma, which is the last big step in the proof of Theorem 4.

Lemma 43. Let $s \geqslant 3, t \geqslant s+2$ and $\ell \geqslant 2$ be integers. Let $J$ be a $\mathcal{J}_{s, t}$-minor-free lower-minor-closed tree of $(s-3, \ell)$-curtains described by $\mathcal{T}_{0}:=\left(B_{\tau}: \tau \in V\left(T_{0}\right)\right)$, let $G:=J\left\{B_{\tau}\right\}$ be the lower torso of a curtain $J\left\langle B_{\tau}\right\rangle$ in $J$, and let $S$ be a set of at most $s$ vertices contained in the top of the root torso of $J\left\langle B_{\tau}\right\rangle$. Then for any $(s+1)$-colouring of $S$, there is an $(s+1)$-colouring of $G$ that properly extends the given precolouring of $S$ and has clustering at most some function $f(s, t, \ell)$.

Proof. Let $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ be the tree-decomposition that describes the curtain $J\left\langle B_{\tau}\right\rangle$. For each $x \in V(T)$, let $\mathcal{E}:=\left(A_{x}, \hat{A}_{x}, G_{0}^{x}, G_{1}^{x}, \ldots, G_{\ell}^{x}\right)$ be the $(s-3, \ell)$-almostembedding of the $(s-3, \ell)$-drape $J\left\langle B_{x}\right\rangle$ and let $\left(\mathcal{P}_{x}, \mathcal{L}_{x}\right)$ be the $\ell$-layered partition that describes $J\left\langle B_{x}\right\rangle$, where $\mathcal{L}^{x}:=\left(L_{1}^{x}, L_{2}^{x}, \ldots\right)$. Since $G$ is a spanning subgraph of $J\left\langle B_{\tau}\right\rangle, \mathcal{T}$ is a tree-decomposition of $G$ and $\mathcal{E}$ is an $(s-3, \ell)$-almost-embedding of $G$. By Lemma 37 and Observation 28, we may assume that, for each drape $G\left\langle B_{x}\right\rangle, G_{0}^{x}\left[L_{\geqslant 6}^{x}\right]$ contains no $(3, \ell)$-island of $G_{0}^{x}$.

Raise the curtain $J\left\langle B_{\tau}\right\rangle$ to obtain the raised ( $s-3, \ell$ )-curtain $J\left\langle B_{\tau}\right\rangle_{\uparrow}$ with layering $\mathcal{L}_{\uparrow}:=\left(L_{1}^{\uparrow}, \ldots, L_{5}^{\uparrow}, L_{\uparrow}\right)$. Let $G_{\uparrow}$ be graph obtained from $G=J\left\{B_{\tau}\right\}$ by applying the same contractions used to obtain $J\left\langle B_{\tau}\right\rangle_{\uparrow}$ from $J\left\langle B_{\tau}\right\rangle$. By Lemma $24, G_{\uparrow}$ is a minor of $J$ and is therefore $\mathcal{J}_{s, t}$-minor free. For each $x \in V(T)$, let $\widetilde{G}_{\uparrow}^{x}$ be the graph obtained from $G_{0}^{x}$ by applying the same contractions used to obtain $G_{\uparrow}^{x}$.
For each $x \in V(T)$, let $F_{x}:=L_{\leqslant 4}^{x}$, and let $\widetilde{G}_{\uparrow}^{x}$ and $\widetilde{G}_{\uparrow \bullet}^{x-}$ be the $F_{x^{-}}$-contraction and $F_{x^{-}}$ compression, respectively, of $\widetilde{G}_{\uparrow}$. For each $x \in V(T)$, apply all of the edge contractions used to obtain $G_{\uparrow} \times$ to $G$ to obtain a graph $G_{\uparrow}$. Define the graph $G_{\uparrow}-$ similarly, by applying the vertex deletions used to obtain each $G_{\uparrow_{\bullet}}^{x-}$ to $G_{\uparrow_{\bullet}}$. For each $x \in V(T)$, let $L_{\uparrow \bullet}^{x}:=V\left(\widetilde{G}_{\uparrow_{\bullet}}\right) \backslash L_{\leqslant 5}^{x}$. For each $i \in \mathbb{N}$, let $L_{i}:=\bigcup_{x \in V(T)} L_{i}^{x}$, and let $L_{i}:=\bigcup_{x \in V(T)} L_{\uparrow \bullet}^{x}$.
Let $R_{1}, \ldots, R_{r}$ be the vertex sets of the connected components of $G_{\uparrow_{\bullet}}\left[L_{5} \cup L_{\uparrow_{\bullet}}\right]$. For each $i \in\{1, \ldots, r\}$, let $x_{i}$ be the unique node of $T$ such that $R_{i}\left[L_{5}\right] \subseteq B_{x_{i}}$ and let $A_{i}:=A_{x_{i}}$. Starting with $G_{\uparrow_{\bullet}}$, define the graph $G_{\uparrow \bullet}^{+}$by introducing vertices $\alpha_{1}, \ldots, \alpha_{r}$, where $\alpha_{i}$ is adjacent to every vertex in $R_{i} \cap L_{\uparrow}$, for each $i \in\{1, \ldots, r\}$. Let $G_{\uparrow_{\bullet}}^{+-}:=$ $G_{\uparrow_{\bullet}}^{+}\left[V\left(G_{\uparrow_{\bullet}}^{-}\right) \cup\left\{\alpha_{1}, \ldots, \alpha_{r}\right)\right]$. In words, $G_{\uparrow_{\bullet}}^{+-}$is obtained from $G_{\uparrow}^{+}$by deleting the vertices that are created in some $F_{x}$-contraction, but are not present in the corresponding
 $S_{j}^{-}:=N_{G_{\uparrow}-A_{j}}^{2}\left[\alpha_{j}\right]=S_{j} \cap V\left(G_{\uparrow \bullet}^{+-}\right)$. Observe that $S_{j}$ and $S_{j}^{-}$agree with the definition of $S_{j}$ in property ( t 3 ) of admissible set for the graphs $G_{\uparrow_{\bullet}}^{+}$and $G_{\uparrow_{\bullet}}^{+-}$, respectively.
The set $\mathcal{S}:=\left\{\left(\alpha_{1}, A_{1}\right), \ldots,\left(\alpha_{r}, A_{r}\right)\right\}$ is $(s-3)$-admissible with respect to $G_{\uparrow \bullet}^{+-}$because:
( t 1 ): $\left|A_{j}\right| \leqslant s-3$, by the definition of $(s-3, \ell)$-curtain;
(t2): $\alpha_{j}$ is a vertex of $G_{\uparrow_{\bullet}}^{+-}$and $N_{G_{\bullet}^{+-}}\left[\alpha_{j}\right] \cap A_{j} \subseteq\left(L_{\uparrow \bullet} \cup\left\{\alpha_{j}\right\}\right) \cap A_{j}=\varnothing$, for each $j \in\{1, \ldots, r\}$.
(t3): We need to show that $\left(\bigcup_{j=1}^{r} A_{j}\right) \cap\left(\bigcup_{j=1}^{r} S_{j}^{-}\right)=\varnothing$. Suppose otherwise, so that $v \in A_{j} \cap S_{j^{\prime}}^{-}$for some $j, j^{\prime} \in\{1, \ldots, r\}$. Then $j \neq j^{\prime}$ and $x_{j} \neq x_{j^{\prime}}$ since $A_{j}=A_{x_{j}}$ does not intersect $L_{5}^{x_{j}} \cup L_{\uparrow_{\bullet}}^{x_{j}} \cup\left\{\alpha_{j}\right\}=S_{j}^{-}$. Therefore, by the definition of curtain, $v \in\left(L_{1}^{x_{j}} \cup A_{x_{j}}\right) \cap\left(L_{1}^{x_{j^{\prime}}} \cup A_{x_{j^{\prime}}}\right)$, which contradicts the assumption that $v \in S_{j^{\prime}} \subseteq$ $L_{5}^{x_{j^{\prime}}} \cup L_{\uparrow}^{x_{j^{\prime}}}$.
(t4): $S_{1}^{-}, \ldots, S_{r}^{-}$are pairwise disjoint because $R_{1}, \ldots, R_{r}$ are pairwise disjoint.

We want to apply Lemma 27 to $G_{\uparrow_{\bullet}}^{+-}$with $\mathcal{S}$ as our $(s-3)$-admissible set. As required by Lemma 27 , the graph $G_{\uparrow_{\bullet}}^{+-}$has treewidth at most some function $c_{1}:=c_{1}(s, \ell)$, by Lemma 23 and since $G_{\uparrow}$. $\left[R_{j}\right]$ is connected for each $j \in\{1, \ldots, r\}$. Lemma 42 shows that the addition of $\alpha_{1}, \ldots, \alpha_{r}$ to $G_{\uparrow}$. creates a graph $G_{\uparrow}^{+}$with treewidth at most $2 c_{1}+1$. The removal of vertices in $S_{j} \backslash S_{j}^{-}$to obtain $G_{\uparrow \bullet}^{+-}$does not increase treewidth.
In addition to having bounded treewidth, Lemma 27 requires that $G_{\uparrow}^{+-}$satisfy several additional conditions, which we now verify.
(x1): $G_{\uparrow_{\bullet}}^{+-}-\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}$ is $\mathcal{J}_{s, t^{-}}$-minor-free because $G_{\uparrow_{\bullet}}^{+-}-\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}=G_{\uparrow_{\bullet}}^{-}$(by definition) and $G_{\uparrow}^{-}$is a minor of $G_{\uparrow}$ which, by Lemma 24 is a minor of $J$, which is $\mathcal{J}_{s, t}$-minor-free.
(x2): For each $j \in\{1, \ldots, r\}, G_{\uparrow_{\bullet}}^{+-}\left[S_{j}^{-} \backslash\left\{\alpha_{j}\right\}\right]$ is connected by Lemma 40.
(x3): For each $j \in\{1, \ldots, r\}, G_{\uparrow \bullet}^{+-}\left[S_{j}^{-} \backslash\left\{\alpha_{j}\right\}\right]$ is $K_{2, t^{\prime}}$-subgraph-free (for some $t^{\prime}=t^{\prime}(\ell)$ ) by Lemma 39.
(x4): For each $\left.j \in\{1, \ldots, r\}, G_{\uparrow \bullet}^{+-}\left[N_{G_{\bullet}}^{4} A_{j}\right]\left[\alpha_{j}\right] \backslash\left\{\alpha_{j}\right\}\right]=\widetilde{G}_{\uparrow \bullet}^{+-}\left[N_{\widetilde{G}_{\bullet}+-}^{4}\left[\alpha_{j}\right] \backslash\left\{\alpha_{j}\right\}\right]$ and is therefore $K_{3, k^{\prime}}$-subgraph-free (for some $k^{\prime}:=k^{\prime}(s, t)$ ) by Lemma 41.

Define the list-assignment $L_{0}: V\left(G_{\uparrow_{\bullet}}^{+-}\right) \rightarrow 2^{\{1, \ldots, s+1\}}$ for $G_{\uparrow_{\bullet}}^{+-}$as follows: For each $v \in S$, let $L_{0}(v)$ be the singleton set that contains the colour assigned to $v$ by the given precolouring of $S$. For each $w \notin S, L_{0}(w):=\{1, \ldots, s+1\} \backslash \cup\left\{L_{0}(v): v \in N_{G_{\uparrow}^{+}}(w) \cap S\right\}$. Since $|S| \leqslant s$, this ensures that each vertex has a non-empty list $L_{0}$ and that any $L_{0^{-}}$ colouring of $G_{\uparrow \bullet}^{+-}$properly extends the precolouring of $S$.
Let $P:=P\left(L_{0}\right)$ be the set of vertices precoloured by $L_{0}$ and let $Q:=Q\left(L_{0}\right)$. We now define a second list-assignment $L$. The need for $L_{0}$ and $L$ comes from the existence of vertices with less than $s$ neighbours in $S$ but at least $s$ neighbours in $P$; we do not want to precolour these vertices, but we require them to avoid colours used by their neighbours in $S$. For each $v \in Q$, let $L(v):=\{1, \ldots, s+1\} \backslash \cup\left\{L(w): w \in N_{G_{+_{\bullet}}}(v) \cap P\right\}$. For each $v \notin Q$, let $L(v):=L_{0}(v)$. Clearly $P=P(L)$ and therefore $Q=Q(L)$. The list-assignment $L$ is a specialization of $L_{0}$, so any $L$-colouring of $G_{\uparrow \bullet}^{+-}$properly extends the precolouring of $S$.

We claim that $L$ is trivially compatible with $\mathcal{S}$ because $P$ does not contain any vertex of any trigger set $S_{j}$. Indeed, $P$ consists of $S$ along with vertices of $G_{\uparrow \bullet}^{+-}$that have $s$ neighbours in $S$. No vertex of any trigger set is in $S$ because $S$ is a subset of $\bigcup_{x \in V(T)}\left(L_{1}^{x} \cup A_{x}\right)$, but vertices of $S_{j}^{-}$are in $L_{5}^{x_{j}} \cup L_{\uparrow}^{x_{j}} \cup\left\{\alpha_{j}\right\}$, for each $j \in\{1, \ldots, r\}$. For each $j \in\{1, \ldots, r\}$, the neighbours of $\alpha_{j}$ are all contained in $S_{j}^{-}$. For each $j \in\{1, \ldots, r\}$, each vertex in $S_{j}^{-}$has at most $s-3$ neighbours in $L_{1}^{x_{j}} \cup A_{x_{j}}$ (all contained in $A_{x_{j}}$ ) and each vertex of $S \cap N_{G_{\uparrow}^{+}}\left(S_{j}^{-}\right)$is in $L_{1}^{x_{j}}$. Therefore, $P \cap S_{j}^{-}=\varnothing$ for each $j \in\{1, \ldots, r\}$. Now we check that $L$ satisfies conditions (g2)-(g5) for being ( $s, p$ )-good.
(g2): $|L(v)| \geqslant s+1-\left|N_{G_{\uparrow}^{+}}(v) \cap P\right|$, by definition, for all $v \in Q(L)$;
(g3): $|L(v)| \geqslant 2$ for all $v \in N_{G_{\dagger_{\bullet}^{+}}^{+}}(P) \backslash Q$ since each such vertex has at most $s-1$ neighbours in $S$ and therefore has a list of size at least $(s+1)-(s-1)=2$;
(g4): $|L(v)|=s+1$ for all $v \in V\left(G_{\uparrow \bullet}^{+-}\right) \backslash N_{G_{\uparrow}^{+-}}[P]$; and
(g5): $L(v) \cap L(u)=\varnothing$ for all $v \in Q$ and $u \in N_{G_{\uparrow}^{+-}}(v) \cap P$ by definition.
We did not establish (g1) - the upper bound on the size of $P$-because the size of $P$ cannot be upper bounded by any function of $s, t$, and $\ell$. However, each vertex in $P$ is either in $S$ or adjacent to each of the $s$ vertices in $S$. Thus, $G_{\uparrow}^{+-}[P]$ contains a $K_{s, q}$ subgraph, where $q:=|P|-s$. This bipartite subgraph has parts $S$ of size $s$ and $Y$ of size $q$. By Claim 27.1 (whose proof relies only on (x3) and (x4) and the admissibility of $\mathcal{S}$ ) and Claim 27.2 (whose proof relies only on (x1) and the admissibility of $\mathcal{S}$ ), this implies that $q \leqslant k$ (where $k:=k(s, t)$ is defined as in Lemma 27) or that each component of $G_{\uparrow_{\bullet}}^{+-}-S$ contains at most $t-1$ vertices of $Y$. In the former case, $|P| \leqslant k+s$ and $L$ satisfies (g1) for $p \geqslant k+s$, so we can apply Lemma 27 to all of $G_{\uparrow \bullet}^{+-}$to obtain an $L$-colouring with clustering at most some function $c_{2}:=c_{2}\left(c_{1}, s, t, \ell\right)$. In the latter case, we can apply Lemma 27 independently on $G_{C}:=G_{\uparrow \bullet}^{+-}[S \cup V(C)]$ for each component $C$ of $G_{\uparrow \bullet}^{+-}-S$. In each application, the size of the precoloured set is at most $s+t-1$ (so $L$ is $(s, s+t-1)$-good) and we obtain an $L$-colouring that properly extends the precolouring of $S .{ }^{9}$ Therefore, each monochromatic component is either contained in $S$ or is contained in a single component of $G_{\uparrow}^{+-}-S$. Therefore, all of these colourings can be combined to provide an $L$-colouring of $G_{\uparrow}^{+-}$with clustering at most some function $c_{3}:=c_{3}\left(c_{1}, s, t, \ell\right)$.
In either case, we obtain an $L$-colouring $\varphi$ of $G_{\uparrow}^{+-}$with clustering at most $c_{4}:=$ $\max \left\{c_{2}, c_{3}\right\}$, and where $\varphi(x) \neq \varphi(a)$ for each $j \in\{1, \ldots, r\}$, each $a \in A_{j} \cup\left\{\alpha_{j}\right\}$ and each $x \in N_{G_{\uparrow}^{+-}}(a)$. Since $G_{\uparrow_{\bullet}}^{-}$is a subgraph of $G_{\uparrow_{\bullet}}^{+-}, \varphi$ is also a colouring of $G_{\uparrow_{\bullet}}^{-}$. We now extend $\varphi$ to obtain a colouring of $G$ :

1. Let $Y_{2}:=V\left(G_{\uparrow_{\bullet}}\right) \backslash V\left(G_{\uparrow_{\bullet}}^{-}\right)$. Each $x \in Y_{2}$ belongs to $N_{G_{\uparrow}}\left(\alpha_{j}\right) \subset L_{\uparrow_{\bullet}}^{x_{j}}$ for exactly one $j \in\{1, \ldots, r\}$. By Observation 33, each $x \in Y_{2}$ has degree at most 2 in $\tilde{G}_{\uparrow_{\bullet}}^{x_{j}}$. Therefore, $\left|N_{G_{\uparrow} \bullet}(x)\right| \leqslant 2+\left|A^{x_{j}}\right| \leqslant s-1$. Set $\varphi(x)$ to an arbitrary colour in $\{1, \ldots, s+1\} \backslash\left\{\varphi(v): v \in N_{G_{\uparrow}}(x) \cup\left\{\alpha_{j}\right\}\right\}$. (Note that $\left.\varphi(x) \neq \varphi\left(\alpha_{j}\right)\right)$.
This extends the colouring $\varphi$ to a colouring of $G_{\uparrow \bullet}$, that still has clustering at most $c_{4}$, since each vertex in $Y_{2}$ has a colour different from all its neighbours. Furthermore, this colouring has the property that $\varphi(x) \neq \varphi(a)$ for each $j \in$ $\{1, \ldots, r\}$, each $x \in N_{G_{\uparrow}^{+}}\left(\alpha_{j}\right)$, and each $a \in\left\{\alpha_{j}\right\} \cup\left(A_{j} \cap N_{G_{\uparrow}}(x)\right)$.
2. Each $v \in V\left(G_{\uparrow}\right) \backslash V(G)$ is obtained by contracting a connected subgraph $X_{v}$ of

[^8]$G$. Extend the colouring $\varphi$ to a colouring of $G$ by setting $\varphi(w):=\varphi(v)$ for each $v \in V\left(G_{\uparrow}\right) \backslash V(G)$ and each $w \in V\left(X_{v}\right)$. Now the $\varphi$-monochromatic components in $G$ may have unbounded size, because $\left|V\left(X_{v}\right)\right|$ is unbounded.

We now argue that we are in a position to apply Lemma 19. Let $A$ be a $\varphi$-monochromatic component in $G$ and let $A_{\uparrow}$ • be the corresponding $\varphi$-monochromatic component in $G_{\uparrow}$. Then $V(A)=\bigcup_{v \in V\left(A_{\bullet}\right)} V\left(X_{v}\right)$. Since $A_{\uparrow \bullet}$ is $\varphi$-monochromatic, $\left|V\left(A_{\uparrow}\right)\right| \leqslant c_{4}$. By Lemma 38, each $V\left(X_{v}\right)$ is $c_{5}$-skinny with respect to $\mathcal{L}$, for some $c_{5}:=c_{5}(\ell)$. Therefore, $V(A)$ is $c_{4} c_{5}$-skinny with respect to $\mathcal{L}$. For each component $C$ of $G\left[L_{\geqslant 5}\right]$, none of the vertices of $C\left[L_{\geqslant 6}\right]$ receive the colour $\varphi\left(\alpha_{C}\right)$. Therefore, by Lemma 19, $G$ has an $(s+1)$-colouring with clustering at most $(2 s+7) c_{4} c_{5}$.

We now prove the main theorem.

Proof of Theorem 4. First consider the case $s \geqslant 3$. We may assume that $t \geqslant s+2$. Let $J$ be a $\mathcal{J}_{s, t}$-minor-free graph. By Lemma 36, $J$ is a lower-minor-closed tree of ( $s-3, \ell$ )-curtains for some $\ell=\ell(s, t)$. Lemma 43 implies that for each curtain $J\left\langle B_{\tau}\right\rangle$ of $J$, any $(s+1)$-colouring of $s$ vertices in the top of the root torso of $J\left\langle B_{\tau}\right\rangle$ can be properly extended to an $(s+1)$-colouring of the lower torso $J\left\{B_{\tau}\right\}$ with clustering at most some function $c(s, t)$. Since $J\left[B_{\tau}\right]$ is a subgraph of $J\left\{B_{\tau}\right\}$, Lemma 21 then implies that $J$ is $(s+1)$-colourable with clustering $c(s, t)$.

Now consider the $s=2$ case. Let $G$ be a $\mathcal{J}_{2, t}$-minor-free graph. Since the complete join $K_{2} \oplus P_{t}$ is a planar graph in $\mathcal{J}_{2, t}$, by the Grid Minor Theorem [57], $G$ has treewidth at most some $\ell_{t}$. We want to apply Lemma 27 with $s=2$ and $r=0$. However, Lemma 27 assumes $s \geqslant 3$ (which is required since the lemma assumes $\mathcal{S}$ is ( $s-3$ )-admissible). However, in the case $s=2$ and $r=0$ the admissibility assumption is vacuous, and the proof is correct. Here we sketch the ideas. We may assume that $G$ contains a $K_{2, y}$-subgraph for some large $y$, otherwise Lemma 26 can be applied directly. Claim 27.1 is vacuously true. Claim 27.2 is still true: each component of $G-X$ has at most $t-1$ vertices of $Y$, or we can find a minor of $G$ in $\mathcal{J}_{2, t}$. Cases A and B still work because the only role played by $\mathcal{S}$ is when we argue that $P_{C}$ (or $P_{i}$ ) avoids $\bigcup_{j=1}^{r} S_{j}$, which is now trivially true since $r=0$. It now follows that $G$ is 3 -colourable with clustering bounded by a function of $t$.

For the case $s=1$, we may assume that $G$ is connected. Select an arbitrary vertex $v$ in $G$. Let $L_{i}:=\left\{x \in V(G): \operatorname{dist}_{G}(v, x)=i\right\}$. So $\left(L_{0}, L_{1}, \ldots\right)$ is a layering of $G$. For each $i \geqslant 0$, colour each vertex in $L_{i}$ by $i \bmod 2$. Any monochromatic component $C$ of $G$ is contained in some layer $L_{i}$, and every vertex in $C$ has a path to $v$ whose internal vertices are in $L_{2} \cup \ldots \cup L_{i-1}$. The union of these paths and the component $C$ contains a minor in $\mathcal{J}_{1,|V(C)|}$. Therefore $|V(C)|<t$.

### 4.3 Theorem 7: The $s=3$ Case

We now complete the proof of Theorem 7. Section 2.5 already establishes Theorem 7 for $s \geqslant 4$. Observe that, in this proof, the restriction $s \geqslant 4$ appears because the graph $H_{\uparrow}^{+}$ obtained from $H_{\uparrow}$ by adding a vertex $\alpha$ may contain arbitrarily large $K_{3, t}$-subgraphs. This is because $G\left[L_{5} \cup L_{\uparrow}\right]$ may contain arbitrarily large $K_{2, t}$-subgraphs with the $t$ 'rightpart' vertices in $L_{\uparrow}$. The additional vertex $\alpha$ then completes this into a $K_{3, t}$-subgraph, so Lemma 15 cannot be applied with $s=3$. We encountered a similar issue in the proof of Lemma 43 and used Lemma 39 to eliminate $K_{2, t^{-}}$-subgraphs from $G\left[L_{5} \cup L_{\uparrow}\right]$. The following paragraph explains how the same techniques can be incorporated into the proof of Theorem 7 in order to handle the case $s=3$.

Proof sketch of Theorem 7 for $s=3$. By Lemma 13, the $X$-minor-free $K_{3, t}$-subgraph free graph $J$ that we want to colour is a lower-minor-closed tree of $(0, \ell)$-curtains described by $\mathcal{T}_{0}:=\left(B_{i}: i \in V\left(T_{0}\right)\right)$, for some $\ell:=\ell(X)$. By Lemma 21, it suffices to show that for each curtain $J\left\langle B_{i}\right\rangle$ any precolouring of any set $S$ of at most 3 vertices in the top of the root torso of $J\left\langle B_{i}\right\rangle$ can be properly extended to an $(s+1)$-colouring of $G:=J\left\{B_{i}\right\}$ with clustering at most $c$.

Let $\mathcal{L}:=\left(L_{1}, L_{2}, \ldots\right)$ be the layering of $G$. By Observation 28, we may assume that $L_{\geqslant 6}$ contains no ( $3, \ell$ )-islands of $G$. We raise the ( $0, \ell$ )-curtain $J\left\langle B_{i}\right\rangle$ as in the proof of Theorem 7 and perform the same contractions in $G$ to obtain $G_{\uparrow}$. Next, define the graphs $G_{\uparrow_{\bullet}}$, and $G_{\uparrow_{\bullet}}^{-}$exactly as in the proof of Lemma 43. The graph $G_{\uparrow_{\bullet}}^{+-}$is then obtained from $G_{\uparrow}^{-}$. by adding a single vertex $\alpha$ that dominates $L_{\uparrow \bullet}$. and setting the colour of $\alpha$ to be $m:=4$. By Lemma 39, $G_{\uparrow_{\bullet}}^{-}\left[L_{5} \cup L_{\uparrow \bullet}\right]$ is $K_{2, t^{\prime}}$ subgraph-free for some $t^{\prime}:=t^{\prime}(\ell)$. Therefore $G_{\uparrow_{\bullet}}^{+-}$is $K_{3, \max \left\{t^{\prime}, t\right\}^{-} \text {-subgraph-free. The treewidth of } G_{\uparrow}+-}$ is not more than the treewidth of $G_{\uparrow}$ which, by Lemma 23 is at most $k:=k(\ell)$.

Since $G_{\uparrow_{\bullet}}^{+-}$has bounded treewidth and is $K_{3, \max \left\{t^{\prime}, t\right\}^{-} \text {-subgraph-free, it can be } 4 \text {-coloured }}$ using Lemma 15 with the precoloured set $S \cup\{\alpha\}$. Since no vertex in $G_{\uparrow \bullet}^{+-}$has more than 3 neighbours in $S$, the resulting colouring properly extends the precolouring of $S \cup\{\alpha\}$ and has clustering at most some $f(\ell)$. We extend this to a 4-colouring of $G$ as before: for each connected subgraph $X_{v}$ of $G$ that is contracted to obtain a vertex $v$ of $G_{\uparrow \bullet}^{+-}$, set the colour of each vertex in $X_{v}$ to the colour of $v$. By Lemma 38, each monochromatic component in the resulting colouring of $G$ is $\ell^{\prime}$-skinny with respect to the layering $\mathcal{L}$ of $G$. Each vertex in $L_{\geqslant 6}$ avoids the colour, 4 , of $\alpha$. We finish by applying Lemma 19 to obtain a 4 -colouring of $G$ with clustering at most some $f(\ell)$.

## 5 Algorithms

All of the results in this paper are algorithmic and give polynomial-time algorithms for finding the colourings promised by Theorems 1, 2, 4 and 7 . More specifically, there exists a constant $c$ such that

- the colouring in Theorem 1 can be found in $f(h) \cdot n^{c}$ time for some $f: \mathbb{N} \rightarrow \mathbb{N}$;
- the colourings in Theorems 2 and 4 can be found in $f(s, t) \cdot n^{c}$ time for some $f: \mathbb{N}^{2} \rightarrow \mathbb{N}$; and
- the colouring in Theorem 7 can be found in $f(|X|, s, t) \cdot n^{c}$ time for some $f: \mathbb{N}^{3} \rightarrow \mathbb{N}$.

We now elaborate on the steps required to compute these colourings:

1. The tree decomposition and almost-embeddings of Theorem 11 can be computed in $f(\mid X)) \cdot n^{c}$ time [21].
2. From this decomposition we easily derive the tree-of-curtains decomposition. The decomposition itself is read off from the previous tree decomposition. The layering of each drape is obtained using existing algorithms for product structure of almostembedded graphs without major apex vertices [5, 17].
3. Before raising each curtain $G$, we eliminate small 3-islands from the lower layers of each drape $G\left\langle B_{x}\right\rangle$ of $G$. This is done to ensure that some subgraph that we are about to contract in order to eliminate $K_{2, t}$-subgraphs is not skinny with respect to the layering $\mathcal{L}_{x}$ of the drape $G\left\langle B_{x}\right\rangle$. For the drape $G\left\langle B_{x}\right\rangle$, any such subgraph would be contained in a component of $G_{0}^{x}-R$, where $R$ is the union of at most 4 parts in the partition $\mathcal{P}_{x}$ of $G\left\langle B_{x}\right\rangle$. Thus, we can first compute $\mathcal{P}_{x}$ and iterate over all $O\left(n^{4}\right)$ choices of $R$. For each, we check if any component of $G_{0}^{x}-R$ is not skinny with respect to $\mathcal{L}$. If so, this component contains a small 3 -island that can be found and eliminated using the separator method of Dvořák and Norin [20, Theorem 8] outlined in Lemma 29. This reduces the number of vertices in the drape, so we can repeat the process a total of at most $n$ times to obtain a drape with a layering in which any set of parts we contract to eliminate $K_{2, t}$ subgraphs will be skinny.
4. Raising each curtain is easily done in time linear in the size of the curtain.
5. In each raised curtain we identify a maximal set of disjoint 3 -discs and 4 -discs. These can easily be identified in $O\left(n^{8}\right)$ time by enumerating all of the 3 - and 4 -cycles (each of which defines at most two discs) and making a directed acyclic graph with these discs as vertices in which the direction of each edge is determined by the containment relationship. The set of sources in this graph gives the desired set of discs.
6. The bounded treewidth results in Lemmas 26 and 27 are applied to each simplified
raised curtain. The proofs of these two results are inductive and give easy recursive polynomial-time algorithms.
7. Lowering the curtains, extending the colouring, and breaking long skinny components using Lemma 19 is easy to implement in linear time.

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## A Proof of Lemma 12

In this appendix, we prove Lemma 12. We begin with a version that requires only moderate changes to the tree-decomposition of Theorem 11.

Lemma 44. For every integer $k \geqslant 1$ and every $k$-apex graph $X$ there exists an integer $\ell$ such that every $X$-minor-free graph $G$ has a rooted tree-decomposition ( $B_{x}: x \in V(T)$ ) such that:
(1) for each $x \in V(T)$, the torso $G\left\langle B_{x}\right\rangle$ is a $(k-1, \ell)$-almost-embedded graph; and
(2) for each edge $x y$ of $T$ where $y$ is the parent of $x$;
(a) $B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$,
(b) $B_{x} \cap B_{y}$ is contained in the near-top of $G\left\langle B_{y}\right\rangle$ or $\left|B_{x} \cap B_{y}\right| \leqslant k+2$; and
(c) $B_{x} \cap B_{y}$ contains at most three vertices not in the top of $G\left\langle B_{y}\right\rangle$.

Proof. First, apply Theorem 11 to obtain a tree-decomposition $\left(B_{x}: x \in V(T)\right)$ of $G$ in which each torso $G\left\langle B_{x}\right\rangle$ satisfies (i) and (ii). For each $x \in V(T)$, denote by
$\left(A_{x}, \hat{A}_{x}, G_{0}^{x}, G_{1}^{x}, \ldots, G_{r}^{x}\right)$ the $\left(k-1, \ell_{0}\right)$-almost-embedding of $G\left\langle B_{x}\right\rangle$. For convenience, assume the number of vortices equals $\ell_{0}$. We keep the same tree-decomposition $\left(B_{x}: x \in\right.$ $V(T)$ ). In order to establish (2) we will, for each node $x \in V(T)$, create at most five additional vortices in $G\left\langle B_{x}\right\rangle$ that are vertex-disjoint from each other and from existing vortices. After these modifications, $G\left\langle B_{x}\right\rangle$ is $\left(k-1, \ell_{0}+5\right)$-almost-embeddable, so (1) is satisfied with $\ell=\ell_{0}+5$.

Let $x$ be a non-root node of $T$ and let $y$ be the parent of $x$. Suppose that $B_{x} \cap B_{y}$ is not contained in the top of $G\left\langle B_{x}\right\rangle$. Let $C$ be the set of non-top vertices of $B_{x} \cap B_{y}$. By definition, $C$ induces a complete subgraph $K$ in $G\left\langle B_{x}\right\rangle$ and, since $C$ contains no vertices in the top of $G\left\langle B_{x}\right\rangle$, the vertices and edges of $K$ are all contained in the embedded part $G_{0}^{x}$ of $G\left\langle B_{x}\right\rangle$. By (ii), every 3-cycle in $G_{0}^{x}$ bounds a 2-cell face. Thomassen's 3-Path Property [64, Proposition 3.5] states that if any three internally disjoint paths $P_{1}, P_{2}, P_{3}$ from a vertex $u$ to a vertex $v$ in an embedding ( $G_{0}^{x}$ in our case) are such that two of the three cycles $C_{i, j}:=P_{i} \cup P_{j}(1 \leqslant i<j \leqslant 3)$ are contractible, then all three cycles $C_{i, j}$ are contractible (or equivalently, bound an open disc on the surface). Assume for the sake of contradiction that $K$ contains a cycle, $J$, that does not bound a disc in the embedding of $G_{0}^{x}$ and let $J$ be a shortest such cycle. By (ii), $|J| \geqslant 4$. Since $J$ is a subgraph of the clique $K, J$ has a chord in $K$. The two cycles defined by the chord and the two half-cycles of $J$ are contractible, by the minimality of $J$. Thus by Thomassen's 3-Path Property, $J$ is also contractible, which is a contradiction. It then follows from Lemma 10 that the embedding of $K$ in $G_{0}^{x}$ is in a disc. Since no complete graph on five or more vertices has an embedding in a disc, $|V(K)| \leqslant 4$. For each vertex $v \in C$, define a trivial vortex that contains only $v$. Doing this for each $x \in V(T)$ ensures that (2a) is satisfied and introduces at most four new vortices in each torso $G\left\langle B_{x}\right\rangle$.
To establish (2b), suppose that $B_{x} \cap B_{y}$ contains a vertex $v$ not in the near-top of $G\left\langle B_{y}\right\rangle$. Then $C:=\left(B_{x} \cap B_{y}\right) \backslash A_{y} \subseteq N_{G_{0}^{y}}(v)$. If $|C| \leqslant 3$, then $\left|B_{x} \cap B_{y}\right| \leqslant\left|A_{y}\right|+|C| \leqslant k+2$ and there is nothing more to prove. Suppose, therefore, that $|C| \geqslant 4$. Now $C$ induces a complete subgraph $K$ in $G_{0}^{x}$ so, by the argument in the previous paragraph, $|C|=4$. Since each of the four triangles of $K$ is the boundary of a 2-cell face, $G_{0}^{y}$ is embedded in the sphere, and $V\left(G_{0}^{y}\right)=V(K)=C$. In this case, we create a trivial vortex in $G\left\langle B_{y}\right\rangle$ that contains one vertex of $G_{0}^{y}$. With this new vortex, every vertex of $B_{y}$ is contained in the near-top of $G\left\langle B_{y}\right\rangle$, so (2b) is again trivially satisfied.
To establish (2c), suppose that $B_{x} \cap B_{y}$ contains a set $C$ of four vertices not in the top of $G\left\langle B_{y}\right\rangle$. Since no vertex of $C$ is in the top of $G\left\langle B_{y}\right\rangle$, each edge of the complete graph $G\left\langle B_{y}\right\rangle[C]$ is an edge of $G_{0}^{y}$. By the argument above, this implies that $G_{0}^{y}=K_{4}$ and is embedded on the sphere. As in the previous paragraph we handle this by creating a trivial vortex that contains a single vertex of $C$. (Note that establishing both (2b) and (2c) requires the addition of at most one vertex per torso.)

For an edge $x y$ in a tree $T$, let $T_{x: y}$ be the component of $T-x y$ that contains $x$. For a tree-decomposition $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ of a graph $G$, let $G_{x: y}^{+}:=G\left[\bigcup_{z \in V\left(T_{x: y)}\right.} B_{z}\right]$ and let $G_{x: y}:=G_{x: y}^{+}-\left(B_{x} \cap B_{y}\right)$. The next lemma shows that subgraphs of almost-embedded graphs are also almost-embedded.

Lemma 45. Let $G$ be a graph, let $G^{\prime}$ be a subgraph of $G$, and let $\left(A, \hat{A}, G_{0}, G_{1}, \ldots, G_{r}\right)$ be an $(a, \ell)$-almost-embedding of $G$. Then $G^{\prime}$ has an $(a, 2 \ell+1)$-almost-embedding $\left(A^{\prime}, \hat{A}^{\prime}, G_{0}^{\prime}, G_{1}^{\prime}, \ldots, G_{r}^{\prime}\right)$, on the same surface $\Sigma$ as $G$, where $A^{\prime}=A \cap V\left(G^{\prime}\right)$, $\hat{A}^{\prime}=$ $\hat{A} \cap V\left(G^{\prime}\right)$, and $G_{i}^{\prime}=G_{i}\left[V\left(G^{\prime}\right)\right]$ for each $i \in\{1, \ldots, r\}$.

Proof. First, consider the case when $G^{\prime}$ is an induced subgraph of $G$. The statement of the lemma already defines $A^{\prime}, \hat{A}^{\prime}$, and the graphs $G_{1}^{\prime}, \ldots, G_{r}^{\prime}$ that form the vortices of $G^{\prime}$. There are two related issues: (i) the statement of the lemma does not define the graph $G_{0}^{\prime}$, and (ii) each vortex $G_{i}$ has a path-decomposition whose bags are indexed by $V\left(G_{i}\right) \cap V\left(G_{0}\right)$, so removing vertices in $G_{i}$ makes (the path-decompositions of) $G_{i}^{\prime}$ undefined. We use the freedom provided by (i) to deal with the problem raised by (ii). The only thing we change is that for some vortices $G_{i}$ some interior vertices in $G_{i}-V\left(G_{0}\right)$ will be moved to $G_{0}^{\prime}$ in order to ensure that the vortices $G_{i}^{\prime}$ have an appropriate path-decomposition of width at most $2 \ell+1$.

Let $X:=V(G) \backslash V\left(G^{\prime}\right)$ be the set of vertices that are removed from $G$ to obtain $G^{\prime}$. For each $i \in\{1, \ldots, r\}$, let $X_{i}:=X \cap V\left(G_{i}\right)$ be the vertices removed from vortex $G_{i}$ and let $Y_{i}:=X_{i} \cap V\left(G_{0}\right)$ be the set of vertices removed from the boundary of $G_{i}$.

Fix some $i \in\{1, \ldots, r\}$, let $B_{i}:=\left\{v_{1}, \ldots, v_{p}\right\}:=V\left(G_{i}\right) \cap V\left(G_{0}\right)$ be the vertices on the boundary of $G_{i}$, and let $\left(C_{1}, \ldots, C_{p}\right)$ be the path-decomposition of $G_{i}$, where $C_{j}$ is the bag associated with $v_{j}$ for each $j \in\{1, \ldots, p\}$. First remove the elements of $X_{i} \backslash Y_{i}$ from $G_{i}$ and from each $C_{1}, \ldots, C_{p}$. Now $C_{1}, \ldots, C_{p}$ is a path-decomposition of $G_{i}-\left(X_{i} \backslash Y_{i}\right)$ whose width has not increased. It remains to remove the vertices in $Y_{i}$.

First consider the easy case: If $v_{j} \in Y_{i}$ and $C_{j}$ contains a vertex $w \notin B_{i}$, then redefine $w$ to be a vertex of $G_{0}^{\prime}$ and embed it at the same location as $v_{j}$. Leave the contents of the bag $C_{j}$ unchanged (but $C_{j}$ is now associated with $w$ ). Now $w$ is still a vertex of $G_{i}^{\prime}$, but it is a boundary vertex of $G_{i}^{\prime}$ so it is added to $B_{i}$. Note that each of the edges incident to $w$ belongs to $G_{i}^{\prime}$ or has an endpoint in $\hat{A}$, so $w$ is an isolated vertex of $G_{0}^{\prime}$ placed at the same point as $v_{j}$, which is not a vertex of $G_{0}^{\prime}$. This modification does not increase the width of the path-decomposition $\left(C_{1}, \ldots, C_{p}\right)$.
After handling all of the easy cases described in the previous paragraph, we are in the situation where $C_{j} \subseteq B_{i}$ for each $v_{j} \in Y_{i}$. We handle these deletions in batches. Consider a maximal interval $\{a, a+1, \ldots, b\}$ such that $\left\{v_{a}, v_{a+1}, \ldots, v_{b}\right\} \subseteq Y_{i}$. If $a=1$ and $b=p$ then $G_{i}^{\prime}=G_{i}-Y_{i}$ is the empty graph and we just remove it.

We can therefore assume without loss of generality that $b<p$. If $a=1$ then, for any $j \leqslant b, C_{j} \backslash\left\{v_{1}, \ldots, v_{b}\right\} \subseteq C_{b+1}$ and we can simply remove $v_{1}, \ldots, v_{b}$ from all bags and remove the bags $C_{1}, \ldots, C_{b}$ from the path-decomposition. What remains is still a path-decomposition since, for any edge $v w$ of $G_{i}^{\prime}$ with $v, w \in C_{j}$ and $j \leqslant b, v$ and $w$ are also contained in $C_{b+1}$.

Thus, we may assume $1<a \leqslant b<p$. Then, for any edge $v w$ of $G_{i}^{\prime}$ with $v, w \in C_{j}$ and $a \leqslant j \leqslant b,\{v, w\} \subseteq C_{a-1} \cup C_{b+1}$. To handle this case, we remove $v_{a}, \ldots, v_{b}$ from all bags, we remove the bags $C_{a}, \ldots, C_{b}$ from the path-decomposition, and we set $C_{a-1}:=C_{a-1} \cup C_{b+1}$. This modification ensures that if some bag $C_{i}$ with $a \leqslant i \leqslant b$ contains two vertices $v_{c}, v_{d}$, with $c<a$ and $d>b$, then the new set $C_{a-1}$ contains both $v_{c}$ and $v_{d}$, thus after the removal of the vertices $v_{a}, \ldots, v_{b}$ and bags $C_{a}, \ldots, C_{b}$, the resulting sequence of bags is still a path-decomposition. The bag $C_{a-1}$, which immediately precedes the interval $\{a, \ldots, b\}$, now contains the union of two of the original bags of the decomposition so its size is at most $2(\ell+1)$.

The step described in the preceding paragraph removes the interval $\{a, \ldots, b\}$ and modifies $C_{a-1}$ so that its size is at most $2(\ell+1)$. Repeat this, handling the intervals $\left\{\left\{a_{j}, \ldots, b_{j}\right\}\right\}_{j=1}^{q}$ by increasing order of left endpoint $a_{j}$. After removing $\left\{a_{j}, \ldots, b_{j}\right\}$, $C_{a_{j}-1}$ has size at most $2(\ell+1)$ and $C_{a_{j^{\prime}}}$ has size at most $\ell+1$ for each $j^{\prime} \geqslant j+1$. When this process completes, no bag has size greater than $2(\ell+1)$, so it gives a $(a, 2 \ell+1)$-almost-embedding of $G^{\prime}$ that satisfies the conditions of the lemma.

Finally, suppose that $G^{\prime}$ is not an induced subgraph of $G$. First, suppose $G$ and $G^{\prime}$ differ by the deletion of exactly one edge. Then the given $(a, \ell)$-almost-embedding of $G$ is also an $(a, \ell)$-almost-embedding of $G^{\prime}$ that satisfies the conditions of the lemma. Since the parameters $a$ and $\ell$ of the almost-embedding do not change, we can use this fact repeatedly for any $G^{\prime}$ in which $V\left(G^{\prime}\right)=V(G)$. To handle the (general) case where $V\left(G^{\prime}\right) \neq V(G)$ and $G^{\prime}$ is not an induced subgraph of $G$, we apply the first argument above on the induced graph $G\left[V\left(G^{\prime}\right)\right]$ to obtained an $(a, 2 \ell+1)$-almost-embedding of $G\left[V\left(G^{\prime}\right)\right]$ that satisfies the requirements of the lemma. Then $G^{\prime}$ is a subgraph of $G\left[V\left(G^{\prime}\right)\right]$ with $V\left(G^{\prime}\right)=V\left(G\left[V\left(G^{\prime}\right)\right]\right)$ so the given embedding of $G\left[V\left(G^{\prime}\right)\right]$ is also an embedding of $G^{\prime}$ that satisifies the requirements of the lemma.

The first step in proving Lemma 12 is to ensure that some of the subgraphs $G_{x: y}$ are connected (see $[26,28]$ for related results that are proven using similar techniques).

Lemma 46. Let $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ be a rooted tree-decomposition of a graph $G$. Then $G$ has a rooted tree-decomposition $\mathcal{T}^{\prime}:=\left(B_{x}^{\prime}: x \in V\left(T^{\prime}\right)\right)$ such that:

1. for any $x y \in E\left(T^{\prime}\right)$ with $y$ the parent of $x$,
(a) $G_{x: y}$ is connected and non-empty; and
(b) $N_{G}\left(V\left(G_{x: y}\right)\right)=B_{x}^{\prime} \cap B_{y}^{\prime}$;
2. there is a mapping $\rho: V\left(T^{\prime}\right) \rightarrow V(T)$ such that:
(a) for $x \in V\left(T^{\prime}\right), B_{x}^{\prime} \subseteq B_{\rho(x)}$; and
(b) for each $x y \in E\left(T^{\prime}\right), \rho(x) \rho(y) \in E(T)$.

Proof. If each edge $x y$ of $T$ satisfies (1a) and (1b) then the tree-decomposition $\mathcal{T}$ and the identity mapping $\rho(x):=x$ already satisfies the requirements of the lemma, and we are done. Otherwise, let $y$ be a minimum-depth node in $T$ that has a child $x$ such that $G_{x: y}$ is not connected or $N_{G}\left(V\left(G_{x: y}\right)\right) \neq B_{x} \cap B_{y}$. Let $C_{1}, \ldots, C_{k}$ be the components of $G_{x: y}$. (Note that, if $B_{x} \subseteq B_{y}$, then $G_{x: y}$ has no vertices, so $k=0$, and the subtree $T_{x: y}$ will not contribute anything to $T^{\prime}$.)

For each $i \in\{1, \ldots, k\}$, let $T_{i}$ be a copy of $T_{x: y}$, let $x_{i}$ be the copy of $x$ in $T_{i}$, and let $\mathcal{T}_{i}:=\left(B_{z} \cap N_{G}\left[V\left(C_{i}\right)\right]: z \in V\left(T_{i}\right)\right) .{ }^{10}$ Create a new tree $T^{\prime}$ by joining $T_{y: x}$ to each of $T_{1}, \ldots, T_{k}$ using the edge $x_{i} y$. Then $\mathcal{T}^{\prime}:=\left(B_{z}: z \in V\left(T^{\prime}\right)\right)$ is a tree-decomposition of $G$ and the edges $x_{1} y, \ldots, x_{k} y$ satisfy (1a) and (1b). Note that each node $x^{\prime}$ in $T^{\prime}$ is either a node of $T$ or a copy of some node $x$ of $T$. In the former case we set $\rho\left(x^{\prime}\right):=x^{\prime}$ and in the latter case we set $\rho\left(x^{\prime}\right):=x$. Clearly the mapping $\rho$ satisfies (2a) and (2b). Repeat this step as long as some edge does not satisfy (1a) or (1b).

This process eventually eliminates all edges $y x$ incident to $y$ that do not satisfy (1a) or (1b) since the number of such edges is reduced by 1 at each iteration. The process eventually eliminates all edges whose upper endpoint has the same depth as $y$ because there are a finite number of nodes with the same depth as $y$ and this process does not introduce any new nodes at this depth. After this any subsequent nodes processed have depth greater than $y$. This process eventually terminates because every treedecomposition it produces uses a tree whose height is no more than that of $T$.

We also make use of the following result. For a graph $G$ and distinct vertices $u, v, w \in$ $V(G)$, we say $G$ has a $\{u, v, w\}$-rooted $K_{3}$ minor if there are pairwise disjoint pairwise adjacent connected subgraphs $A, B, C$ in $G$ with $u \in V(A), v \in V(B)$ and $w \in V(C)$.

Lemma 47 ([70]). Let $G$ be a graph with three distinguished vertices $u$, $v$, and $w$. If $G$ does not contain a $\{u, v, w\}$-rooted $K_{3}$ minor then $G$ contains a vertex $q$ such that each component of $G-q$ contains at most one of $u$, $v$, or $w$.

We now prove the main result of this appendix, which we restate for convenience.

[^9]Lemma 12. For every integer $k \geqslant 1$ and every $k$-apex graph $X$ there exists an integer $\ell$ such that every $X$-minor-free graph $G$ has a rooted tree-decomposition $\mathcal{T}:=\left(B_{x}: x \in\right.$ $V(T))$ such that:
(1) for each $x \in V(T)$, the torso $G\left\langle B_{x}\right\rangle$ is a $(k-1, \ell)$-almost-embedded graph;
(2) for each edge $x y$ of $T$ where $y$ is the parent of $x$;
(a) $B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$,
(b) $B_{x} \cap B_{y}$ is contained in the near-top of $G\left\langle B_{y}\right\rangle$ or $\left|B_{x} \cap B_{y}\right| \leqslant k+2$, and
(c) $B_{x} \cap B_{y}$ contains at most three vertices not in the top of $G\left\langle B_{y}\right\rangle$; and
(3) $\mathcal{T}$ is lower-minor-closed.

Proof of Lemma 12. Let $\mathcal{T}$ be the rooted tree-decomposition of $G$ guaranteed by applying Lemma 46 to the tree-decomposition given by Lemma 44. Suppose that $\mathcal{T}$ does not satisfy the requirements of the lemma. Therefore, there exists $y \in V(T)$ of minimum depth such that there is no faithful $G\left\{B_{y}\right\}$-model in $G$ in which each vertex in the top of $G\left\langle B_{y}\right\rangle$ is in a branch set of size one. We will repeatedly choose an edge $u v$ in $G\left\{B_{y}\right\}$ and show the existence of a subgraph in $G\left[G_{x: y} \cup\{u, v\}\right]$ that represents the edge $u v$ (and possibly some other edges). The subgraphs assigned this way are pairwise disjoint, except for the vertices in $B_{y}$, so that the union of all such graphs contains a model of $G\left\{B_{y}\right\}$. This process may fail for some child $x$ of $y$, in which case we will make adjustments to a subtree rooted at one of the children $x$ of $y$ that will lift a vertex $q \in V\left(G_{x: y}\right)$ into $B_{y}$. This will introduce new edges (incident to $q$ ) in $G\left\{B_{y}\right\}$. We will be able to represent these new edges and this change will eliminate the edges of the clique $B_{x} \cap B_{y}$ that we are unable to represent.
Consider any edge $u v$ of $G\left\{B_{y}\right\}$ that is not yet represented. If $u v$ is an edge of $G\left[B_{y}\right]$ then we say that $u v$ is represented (by $u v$ ). Note that this applies in particular to any edge $u v$ of $G\left\{B_{y}\right\}$ for which $u$ or $v$ is in the top of $G\left\langle B_{y}\right\rangle$. This will ensure that the branch set $G_{v}$ for any vertex $v$ in the top of $G\left\langle B_{y}\right\rangle$ consists of a single vertex.
Otherwise $u v$ is not an edge in $G\left[B_{y}\right]$, and thus the vertices $u$ and $v$ are part of at least one adhesion set $B_{x} \cap B_{y}$ for some edge $x y$ of $T$ where $y$ is the parent of $x$. For each torso $G\left\langle B_{x}\right\rangle$ in a tree-decomposition, let $\operatorname{top}\left(G\left\langle B_{x}\right\rangle\right)$ denote the top of $G\left\langle B_{x}\right\rangle$. Since neither $u$ nor $v$ is in the top of $G\left\langle B_{y}\right\rangle$, Lemma 44(2c) implies that $B_{x} \cap B_{y} \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right)$ contains $u, v$ and at most one other vertex.

First, suppose that there exists an edge $x y \in E(T)$ such that $B_{x} \cap B_{y} \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right)=$ $\{u, v\}$. Since $G_{x: y}$ is connected and non-empty and $\{u, v\} \subseteq N_{G}\left(V\left(G_{x: y}\right)\right)$, there is a path in $G$ from $u$ to $v$ whose internal vertices are contained in $G_{x: y}$. In this case we say that $u v$ is represented (by $G_{x: y}$ ).
Otherwise, there exists a 3-cycle $u v w$ in $G\left\langle B_{y}\right\rangle \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right)$ and an edge $x y \in E(T)$
such that $B_{x} \cap B_{y} \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right)=\{u, v, w\}$. Suppose that at least one of $v w$ or $u w$ (say $v w)$ is already represented. The edge $v w$ may be represented by itself or by a subgraph $G_{x^{\prime}: y}$ with $B_{x^{\prime}} \cap B_{y} \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right) \supseteq\{v, w\}$. Importantly, $v w$ is not represented by $G_{x: y}$. Since $G_{x: y}$ is connected, non-empty, and adjacent to each of $u, v, w$ we can contract $G_{x: y}$ into a single vertex $q$ that is adjacent to each of $u, v, w$. Then by contracting the edge $q u$ we obtain a graph that contains $u v$ and $u w$. In this case we say that the path $v u w$ is represented (by $G_{x: y}$ ) and that the edges $u v$ and $u w$ are represented (by $G_{x: y}$ ).

If neither $u w$ nor $v w$ are already represented, but there are distinct edges $x y$ and $x^{\prime} y$ of $T$ such that $\{u, v, w\} \subseteq B_{x} \cap B_{y}$ and $\{u, v, w\} \subseteq B_{x^{\prime}} \cap B_{y}$ then the path uvw can be represented by $G_{x: y}$ and the path vuw be represented by $G_{x^{\prime}: y}$. In this case we say that the 3 -cycle $u v w$ and the edges $u v, v w$, and $u w$ are represented (by $G_{x: y}$ and $G_{x^{\prime}: y}$ ).

We are left with the case where none of $u v, v w$, or $u w$ are represented and there is exactly one edge $x y \in E(T)$ such that $B_{x} \cap B_{y} \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right)=\{u, v, w\}$. Consider the graph $H:=G\left[V\left(G_{x: y}\right) \cup\{u, v, w\}\right]$. Since $G_{x: y}$ is connected and $N_{G}\left(V\left(G_{x: y}\right)\right)=B_{x} \cap B_{y}$, $H$ is connected. Since none of the edges of the cycle uvw are represented yet, none of these edges is present in $H$, so $H$ contains at least four vertices. If $H$ contains a $\{u, v, w\}$-rooted $K_{3}$-minor, then we can again contract edges of $H$ to obtain the edges of $u v w$. In this case, we say that the 3 -cycle $u v w$, and the edges $u v$, $v w$, and $u w$ are represented (by $G_{x: y}$ ).

Otherwise, by Lemma 47, $H$ contains a vertex $q$ such that each component of $H-q$ contains at most one of $u, v$, or $w$. Since $G_{x: y}$ is connected and adjacent to each of $u, v, w$ this implies that $q \notin\{u, v, w\}$. In this case, we will lift the vertex $q$ into $B_{y}$ by modifying our tree decomposition $\mathcal{T}$.

We now describe these modifications, first showing how to modify $\mathcal{T}$ to obtain a new tree decomposition $\mathcal{T}^{\prime}:=\left(B_{\tau}^{\prime}: \tau \in V\left(T^{\prime}\right)\right)$ with $q \in B_{y}^{\prime}$ in a way that preserves the properties of Lemma 46. We then show how to obtain almost-embeddings of each torso of $\mathcal{T}^{\prime}$ so that these almost-embeddings collectively satisfy the properties of Lemma 44 (with a slightly larger value of $\ell$ ). After this modification, $q$ will be in $B_{y}^{\prime}$ and $G\left\{B_{y}^{\prime}\right\}$ will include the edges $u q, v q$, and $w q$ but will not necessarily include the edges $u v, v w$, or $u w$. We show that each of the edges $u q, v q$, and $w q$ can be represented by one of the graphs $G_{c_{i}: y}$ where $c_{i}$ is the root of the one of the trees in $T^{\prime}-V(T)$ that was introduced to replaces $x$.

Modified Tree Decomposition: Let $z$ be the minimum-depth node of $T$ with $q \in B_{z}$ and let $z_{0}, \ldots, z_{p}$ be the path in $T$ from $z_{0}:=z$ to $z_{p}:=y$. We define a sequence of tree decompositions $\mathcal{T}_{0}, \mathcal{T}_{1}, \ldots, \mathcal{T}_{p}$ where $\mathcal{T}_{0}=\mathcal{T}, T^{0}:=T$ and $\mathcal{T}_{i}:=\left(B_{x}^{i}: x \in V\left(T^{i}\right)\right)$. Refer to Figure 10 The tree $T^{i}$ and $T^{i-1}$ be almost identical except that the node $z_{i-1}$ that


Figure 10: Adding $q$ to $B_{z_{i}}$ may disconnect the graph $G_{z_{i-1}: z_{i}}$.
appears in $T^{i-1}$ will be replaced with several nodes $\left\{c_{1}, \ldots, c_{k}\right\}=V\left(T_{i}\right) \backslash V\left(T_{i-1}\right)$ in $T^{i}$. Each of the children $a_{1}, \ldots, a_{d}$ of $z_{i-1}$ in $T^{i-1}$ will become the child of a node in $\left\{c_{1}, \ldots, c_{k}\right\}$. For each node $\tau \in V\left(T^{i}\right) \cap V\left(T^{i-1}\right) \backslash\left\{z_{i}\right\}$, we define $B_{\tau}^{i}:=B_{\tau}^{i-1}$. For $z_{i}$ we define $B_{z_{i}}^{i}=B_{z_{i}}^{i-1} \cup\{q\}$.
Let $C_{1}, \ldots, C_{k}$ be the connected components of $G_{z_{i-1}: z_{i}}-\{q\}$ and, for each $j \in\{1, \ldots, k\}$, let $G_{j}:=G\left[N_{G}\left[V\left(C_{j}\right)\right]\right]$. In $T^{i}$ we replace $z_{i-1}$ with $k$ new nodes $c_{1}, \ldots, c_{k}$ and set $B_{c_{j}}:=B_{z_{i-1}} \cap V\left(G_{j}\right)$ for each $j \in\{1, \ldots, k\}$. Let $a_{1}, \ldots, a_{d}$ be the children of $z_{i-1}$ in $T^{i-1}$. For each $j \in\{1, \ldots, d\}$, the graph $G_{a_{j}: z_{i-1}}$ is connected and its vertex set is a subset of $G_{z_{i-1}: z_{i}}-q$. Therefore, $G_{c_{j}: z_{i-1}}$ is an induced subgraph of $C_{s}$ for some $s \in\{1, \ldots, d\}$. In $T_{i}$, we make $c_{j}$ a child of $a_{s}$. This completes the definition of $T_{i}$ and $\mathcal{T}_{i}:=\left(B_{\tau}: \tau \in V\left(T_{i}\right)\right)$.

Connectivity Conditions for $\mathcal{T}_{i}$ : We claim that $\mathcal{T}_{i}$ satisfies properties (2a) and (2b) of Lemma 46. That is, for each node $a \in V\left(T^{i}\right)$ with parent $b \in V\left(T^{i}\right), G_{a: b}$ is connected and $N_{G}\left(V\left(G_{a: b}\right)\right)=\left\{B_{a} \cap B_{b}\right\}$. It suffices to check this condition for $a \in\left\{a_{1}, \ldots, a_{k}\right\} \cup\left\{c_{1}, \ldots, c_{d}\right\} ;$ for any other choice of $a$, the edge $a b$ appears in $T_{i-1}$ and $G_{a: b}$ is identical independent of whether it is defined in terms of $\mathcal{T}_{i-1}$ or $\mathcal{T}_{i}$.

- For each $j \in\{1, \ldots, k\}, C_{j}=G_{c_{j}: z_{i}}$ is connected and $N_{G}\left(C_{j}\right)=V\left(G_{j}\right) \backslash V\left(C_{j}\right)=$ $B_{c_{j}} \cap B_{z_{i}}$.
- Now consider $G_{a_{j}: c_{s}}$ for some $j \in\{1, \ldots, d\}$ and (the relevant) $s \in\{1, \ldots, k\}$. We will show that $B_{c_{j}} \cap B_{z_{i-1}} \subseteq B_{a_{j}} \cap B_{c_{s}} \subseteq B_{c_{j}} \cap B_{z_{i-1}}$, which implies that $B_{a_{j}} \cap B_{c_{s}}=B_{c_{j}} \cap B_{z_{i-1}}$. Since $T_{a_{j}: c_{s}}=T_{a_{j}: z_{i-1}}$ this shows that $G_{a_{j}: c_{s}}$ is connected and that $N_{G}\left(V\left(G_{a_{j}: c_{s}}\right)\right)=B_{a_{j}} \cap B_{c_{s}}$.
By definition, $B_{c_{s}} \subseteq B_{z_{i-1}}$, so $B_{a_{j}} \cap B_{c_{s}} \subseteq B_{a_{j}} \cap B_{z_{i-1}}$. On the other hand,

$$
V\left(G_{s}\right)=N_{G}\left[V\left(C_{s}\right)\right] \supseteq N_{G}\left[V\left(G_{a_{j}: z_{i-1}}\right)\right] \supseteq N_{G}\left(V\left(G_{a_{j}: z_{i-1}}\right)\right) .
$$

Therefore $B_{c_{s}}=V\left(G_{s}\right) \cap B_{z_{i-1}} \supseteq N_{G}\left(V\left(G_{a_{j}: z_{i-1}}\right)\right) \cap B_{z_{i-1}}=B_{a_{j}} \cap B_{z_{i-1}}$. Therefore $B_{a_{j}} \cap B_{c_{s}} \supseteq B_{a_{j}} \cap B_{z_{i-1}}$.
This process completes with a tree decomposition $\mathcal{T}_{p}=\left(B_{\tau}^{p}: \tau \in V\left(T^{p}\right)\right)$ that satisfies conditions (2a) and (2b) of Lemma 46 and in which $q \in B_{y}$.

Almost-Embedding the New Torsos: Now we explain how to almost-embed each torso of $\mathcal{T}_{i}$ so that the torsos collectively satisfy the conditions of Lemma 44. First, consider the torso $G\left\langle B_{z}^{i}\right\rangle=G\left\langle B_{z_{0}}^{i}\right\rangle$. This torso inherits an embedding from $G\left\langle B_{z}^{i-1}\right\rangle$ that includes $q$. If $q$ is not in the top of $G\left\langle B_{z}^{i-1}\right\rangle$ then we make $q$ a trivial vortex in $G\left\langle B_{z}^{i}\right\rangle$, so that $q$ is in the top of $G\left\langle B_{z}\right\rangle$.
Next, consider some torso $G\left\langle B_{z_{i}}^{i}\right\rangle$ for some $i \in\{1, \ldots, p-1\}$. This torso inherits an embedding from $G\left\langle B_{z_{i}}^{i-1}\right\rangle$ that does not include $q$. In the embedding of $G\left\langle B_{z_{i}}^{i}\right\rangle$, we make $q$ a non-major apex vertex and we make each vertex of $B_{z_{i-1}}^{i-1} \cap B_{z_{i}}^{i-1}$ that is not in the top of $G\left\langle B_{z_{i}}^{i-1}\right\rangle$ a top vertex of $G\left\langle B_{z_{i}}^{i}\right\rangle$ by creating a trivial vortex. By Lemma 44(2c), this results in the creation of at most three new vortices. Note: It is important here that and each vertex in $B_{z_{i-1}}^{i-1} \cap B_{z_{i}}^{i}$ (including $q$ ) is in the top of $G\left\langle B_{z_{i}}^{i}\right\rangle$.
Next, consider one of the torsos $G\left\langle B_{c_{j}}^{i}\right\rangle$ where $c_{j}$ is one of the vertices in $V\left(T_{i}\right) \backslash V\left(T_{i-1}\right)$ that replaces $z_{i-1}$. Since $B_{c_{j}}^{i} \subseteq B_{z_{i-1}}^{i-1}$ This torso inherits an embedding from $G\left\langle B_{z_{i-1}}^{i-1}\right\rangle$ that includes $q$ in its top. We use this embedding as is.

Finally, consider the torso $G\left\langle B_{y}^{p}\right\rangle=G\left\langle B_{z_{p}}^{p}\right\rangle$. Recall that uvw is the boundary of a 2-cell face $D$ in the embedded part of $G\left\langle B_{y}\right\rangle$. The torso $G\left\langle B_{y}^{p}\right\rangle$ inherits an embedding from $G\left\langle B_{y}^{p}\right\rangle$ that does not contain $q$. To add $q$, we make a $q$ a vertex in the embedded part that is embedded in the interior of $D$. We also embed each edge $q u$, $q v$, and $q w$ in $D$. This completes the description of the embedding of $G\left\langle B_{\tau}^{i}\right\rangle$ for each $\tau \in V\left(T^{i}\right)$.

The New Torso $\boldsymbol{G}\left\langle\boldsymbol{B}_{y}\right\rangle$ : Now $\mathcal{T}^{\prime}:=\mathcal{T}_{p}$ is a tree decomposition in which each torso is equipped with an almost-embedding. Let $T^{\prime}:=T^{p}$, and let $B_{\tau}^{\prime}:=B_{\tau}^{p}$ for each $\tau \in V\left(T^{\prime}\right)$. Consider the torso $G\left\langle B_{y}^{\prime}\right\rangle$. In $T^{\prime}$, the child $x \in V(T)$ of $y$ has been replaced by nodes $c_{1}, \ldots, c_{k}$ associated with the components $C_{1}, \ldots, C_{k}$ of $G_{x: y}-\{q\}$. Each component $C_{i}$ is adjacent to $q$ and to at most one of $u$, $v$, or $w$. Since $N_{G}\left(B_{x}\right)=B_{x} \cap B_{y}$, either $u q \in E(G)$ or some component $C_{i}$ is adjacent to both $q$ and $u$. In the former case we say that the edge $u q \in E\left(G\left\langle B_{y}\right\rangle\right)$ is represented by itself. In the latter case, $u q$ is represented by $C_{i}:=G_{c_{i}: y}$. Similar comments hold for the edges $v q$ and $w q$ of $G\left\langle B_{y}\right\rangle$.
Since each $C_{i}$ contains at most one of $u, v$, or $w$, none of the edges $y c_{i}$ causes an edge of the cycle $u v w$ to to be in $G\left\{B_{y}\right\}$. Some of these edges may be present in $G\left\langle B_{y}^{\prime}\right\rangle$, but this is caused by other children of $y$. Any such edge will be dealt with when we consider
unrepresented edges of $G\left\{B_{y}^{\prime}\right\}$ with the help of some child $x^{\prime}$ of $y$ that is not one of $c_{1}, \ldots, c_{k}$.

Bounding the Increase in $\ell$ : Let $Z_{y, x}:=V\left(T^{\prime}\right) \backslash V(T)$. (Note that this does not include the node $y=z_{p}$.) Each vertex in $Z_{y, x}$ is obtained from $z_{i-1}$ for some $i \in\{1, \ldots, p\}$ by splitting $B_{z_{i-1}}^{i-1}$ into $B_{c_{1}}^{i}, \ldots, B_{c_{k}}^{i}$, so that $z_{i-1} \in V\left(T^{i-1}\right)$ is replaced by $c_{1}, \ldots, c_{k} \in V\left(T^{i}\right)$. For $i \in\{1, \ldots, p-1\}$, each vertex of $B_{c_{j}}^{i} \cap B_{z_{i}}^{i}$ is contained in the top of $G\left\langle B_{z_{i}}^{i}\right\rangle$. This implies that, for each edge $\tau \tau^{\prime} \in T^{\prime}\left[Z_{y, z}\right], B_{\tau}^{\prime} \cap B_{\tau^{\prime}}^{\prime}$ is in the top of $G\left\langle B_{\tau}^{\prime}\right\rangle$ and in the top of $G\left\langle B_{\tau^{\prime}}^{\prime}\right\rangle$.

By construction, for each $\tau \in Z_{y, x}$, there exists a $\rho(\tau) \in V(T)$ such that $B_{\tau} \subseteq B_{\rho(\tau)} \cup\{q\}$. The almost-embedding of $G\left\langle B_{\tau}\right\rangle$ inherits an almost-embedding from $B_{\rho(\tau)}$ and then does some modifications that increase the number of vortices by at most three and the number of non-major apex vertices by at most one. By Lemma 45, the inherited embedding has at most $k-1$ major apex vertices, at most $\ell$ vortices, each of width at most $2 \ell+1$, and at most $\ell$ apex vertices. Since $\ell+3 \leqslant 2 \ell+1$ for $\ell \geqslant 2$, the final embedding of $G\left\langle B_{\tau}\right\rangle$ is a $(k, 2 \ell+1)$-almost embedding, for each $\tau \in Z_{y, x}$.

The preceding operation modifies the tree-decomposition $\mathcal{T}$ in a way that only affects the subtree $T_{x: y}$ of $T$, and this only occurs at the unique child $x$ of $y$ such that $B_{x} \cap B_{y} \backslash \operatorname{top}\left(G\left\langle B_{y}\right\rangle\right)=\{u, v, w\}$. Therefore, we can perform this operation on each of the children $x_{1}, \ldots, x_{c}$ of $y$ for which it is required, and each such operation will only affect the subtree $T_{x_{i}: y}$. Since $T_{x_{1}: y}, \ldots, T_{x_{c}: y}$ are pairwise vertex-disjoint, $Z_{y, x_{1}}, \ldots, Z_{y, x_{c}}$ are pairwise disjoint. For convenience, we still call the resulting tree $T^{\prime}$ and the resulting tree-decomposition $\mathcal{T}^{\prime}:=\left(B_{x}^{\prime}: x \in V\left(T^{\prime}\right)\right)$. After doing this, every edge of the lower torso $G\left\{B_{y}\right\}$ is represented. ${ }^{11}$ This ensures that $G\left\{B_{y}\right\}$ is a faithful minor of $G$.

Let $Z_{y}:=\bigcup_{i=1}^{c} Z_{y, x_{i}}$. We ran this process at $y$ because it was a minimum-depth node for which $G\left\{B_{y}\right\}$ was not a faithful minor of $G$. If there exists another node $y^{\prime}$ for which $G\left\{B_{y^{\prime}}\right\}$ is not a faithful minor of $G$, then run this process again (for the minimum-depth such $y^{\prime}$ ). Because $y$ was a minimum-depth node, $y^{\prime}$ is not an ancestor of $y$. If $y^{\prime} \notin Z_{y}$ then, again, $Z_{y}$ and $Z_{y^{\prime}}$ will be disjoint because $T^{\prime}\left[Z_{y} \cup\{y\}\right]$ and $T^{\prime}\left[Z_{y^{\prime}} \cup\left\{y^{\prime}\right\}\right]$ are each connected and neither contains the root of the other. If $y^{\prime} \in Z_{y}$ then running the preceding process on $y^{\prime}$ affects subtrees $T_{x_{1}^{\prime}: y^{\prime}}, \ldots, T_{x_{c}^{\prime}: y^{\prime}}$ for one or more children $x_{1}^{\prime}, \ldots, x_{c}^{\prime}$ of $y^{\prime}$. For each $i \in\{1, \ldots, c\}, B_{y^{\prime}} \cap B_{x_{i}^{\prime}}$ contains exactly three vertices not in the top of $G\left\langle B_{y^{\prime}}\right\rangle$ - namely three vertices $u^{\prime}, v^{\prime}, w^{\prime}$ that form a $K_{3}$ in $G\left\{B_{y^{\prime}}\right\}$ but for which $G_{x_{i}^{\prime}: y^{\prime}}$ has no $\left\{u^{\prime}, v^{\prime}, w^{\prime}\right\}$-rooted $K_{3}$ minor. Suppose that $x_{i}^{\prime}=\tau^{\prime}$ for some $i \in\{1, \ldots, c\}$ and $\tau^{\prime} \in Z_{y}$. Since $y^{\prime} \neq y$, the parent $\tau$ of $\tau^{\prime}$ is also in $Z_{y}$. This implies that $x_{i}^{\prime} \notin Z_{y}$ since $B_{\tau} \cap B_{\tau^{\prime}}$ is contained in the top of $G\left\langle B_{\tau}\right\rangle$. Therefore $Z_{y}$ and $Z_{y^{\prime}}$ are

[^10]disjoint.
Thus we can find a sequence of tree-decompositions $\mathcal{T}_{0}=\mathcal{T}, \mathcal{T}_{1}, \ldots, \mathcal{T}_{d}$ where $\mathcal{T}_{i}:=\left(B_{y}^{i}\right.$ : $\left.y \in V\left(T_{i}\right)\right)$ and a sequence $y_{1}, \ldots, y_{d}$ of nodes, where $y_{i} \in \bigcap_{j=i-1}^{d} V\left(T_{j}\right), G\left\{B_{y_{i}}^{i-1}\right\}$ is not a faithful minor of $G$, but $G\left\{B_{y_{i}}^{j}\right\}$ is a faithful minor of $G$ for each $j \in\{i, \ldots, d\}$. By construction, for each $i \in\{1 \ldots, d\}, Z_{y_{i}} \subseteq \bigcap_{j=i}^{d} V\left(T_{j}\right)$ and the sets $Z_{y_{1}}, \ldots, Z_{y_{d}}$ are pairwise disjoint. The end result is a tree-decomposition $\mathcal{T}_{d}:=\left(B_{z}^{d}: z \in V\left(T_{d}\right)\right)$ of $G$. For each $z \in V\left(T_{d}\right)$, the torso $G\left\langle B_{z}\right\rangle$ can be obtained by taking a subgraph of a torso of $\mathcal{T}$ and then adding at most one new non-major apex vertex (referred to as $q$, above) and at most three new vortices (one for each vertex in $N$, above). As discussed above, this implies that, for $\ell \geqslant 2$, the resulting graph is $(k-1,2 \ell+1)$-almost-embedded. For each $x \in V(T)$, the faithful $G\left\{B_{x}\right\}$-model $\left\{G_{v}: v \in B_{x}\right\}$ satisfies the "furthermore" clause of the lemma because the only vertices that are assigned a branch-set $G_{v}$ with more than one vertex (the vertices $u, v$, and $w$, above) are not in the top of the $G\left\langle B_{x}\right\rangle$. Therefore, the tree-decomposition $\mathcal{T}^{\prime}$ satisfies the requirements of the lemma.

The following theorem and its consequences are not used directly in the current paper, but have applications to other problems [16].

Theorem 48. For every apex graph $X$, there exists positive integers $\ell, t$ such that every $X$-minor-free graph $G$ has a tree-decomposition $\left(B_{x}: x \in V(T)\right)$ of adhesion at most 3 such that for every $x \in V(T)$ :
(1) $G\left\{B_{x}\right\}$ is a minor of $G$,
(2) $G\left\{B_{x}\right\}$ has a connected $\ell$-layered partition $\left(\mathcal{L}_{x}, \mathcal{P}_{x}\right)$, and
(3) if $y$ is the parent of $x$ then
(a) every vertex in $B_{x} \cap B_{y}$ is contained in the first layer of $\mathcal{L}_{x}$,
(b) no vertex in $B_{x} \cap B_{y}$ is contained in the first layer of $\mathcal{L}_{y}$, and
(c) each vertex in $B_{x} \cap B_{y}$ is in a singleton part of $\mathcal{P}_{x}$.

Proof. By Theorem 22 with $k=1, G$ is a lower-minor-closed tree of $\left(0, \ell_{0}\right)$-curtains described by $\mathcal{T}:=\left(B_{x}: x \in V(T)\right)$ where each torso $G\left\langle B_{x}\right\rangle$ is equipped with the connected $\ell_{0}$-layered partition $\left(\mathcal{P}_{x}^{0}, \mathcal{L}_{x}\right)$ for some $\ell_{0}:=\ell_{0}(X)$. This already satisfies (1), (2), and (3a). Although not stated in Theorem 22, the construction used in the proof of Theorem 22 only makes $x$ a child of $y$ if $B_{x} \cap B_{y}$ contains a vertex not in the near-top of $G\left\langle B_{y}\right\rangle$ (this is the definition of a $k$-light edge). This implies that $B_{x} \cap B_{y}$ does not contain any vertices in the top of $G\left\langle B_{y}\right\rangle$, which is the first layer in $\mathcal{L}_{y}$. Thus, the construction so far already satisfies (3b) as well.

We now show that a slight modification of $\mathcal{P}_{x}$ satisfies (3c) without greatly increasing the treewidth of $H_{x}:=G\left\{B_{x}\right\} / \mathcal{P}_{x}$. For each $x$ in $V(T)$ with parent $y$ in $V(T)$, each
$v \in B_{x} \cap B_{y}$ is contained in the top of $G\left\langle B_{x}\right\rangle$ and is therefore contained in the first layer of $\mathcal{L}_{x}$. We now modify $\mathcal{P}_{x}:=\left(L_{1}^{x}, L_{2}^{x}, \ldots\right)$ by breaking parts that intersect $B_{x} \cap B_{y}$ so that each $v \in B_{x} \cap B_{y}$ is in a singleton part, and each part is connected. Let $P$ be a part in $\mathcal{P}_{x}$ that contains a vertex of $B_{x} \cap B_{y}$. By definition, $\left|P \cap L_{2}^{x}\right| \leqslant \ell$. Since $G[P]$ is connected, each of the components $C_{1}, \ldots, C_{r}$ of $G[P]-\left(P \cap L_{1}^{x}\right)$ contains at least one vertex in $P \cap L_{2}^{x}$. Therefore $G[P]-\left(P \cap L_{1}^{x}\right)$ has $r \leqslant \ell$ connected components. In $\mathcal{P}_{x}$, replace $P$ with the at most $2 \ell$ parts in $R_{P}:=\left\{V\left(C_{1}\right), \ldots, V\left(C_{r}\right)\right\} \cup\left\{\{v\}: v \in P \cap L_{1}^{x}\right\}$. In any tree-decomposition of $G\left\{B_{x}\right\} / \mathcal{P}_{x}$, the vertex that represents $P$ can be replaced by the vertices that represent each set in $R_{P}$. By definition, before this modification of $\mathcal{P}_{x}, G\left\{B_{x}\right\} / \mathcal{P}_{x}$ had a tree-decomposition of width at most $\ell_{0}$. Therefore, after this modification, $G\left\{B_{x}\right\} / \mathcal{P}_{x}$ has a tree-decomposition of width at most $\ell:=\ell_{0}\left(\ell_{0}+1\right)-1$.

Theorem 48 has several implications:

- For each $x \in V(T), H_{x}:=G\left\{B_{x}\right\} / \mathcal{P}_{x}$ is a minor of $G$ (since $G\left\{B_{x}\right\} / \mathcal{P}_{x}$ is obtained by contracting connected subgraphs in a minor of $G$ ).
- For each $x \in V(T), G\left[B_{x}\right] / \mathcal{P}_{x}$ is a minor of $G$ (since $G\left[B_{x}\right]$ is a subgraph of $\left.G\left\{B_{x}\right\}\right)$.
- $G\left[B_{x}\right]$ is isomorphic to a subgraph of $H_{x} \boxtimes P$, for some path $P$ and some minor $H_{x}$ of $G$ whose treewidth is at most $\ell$.


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[^1]:    ${ }^{1}$ We consider simple, finite, undirected graphs $G$ with vertex-set $V(G)$ and edge-set $E(G)$. See latter sections and [11] for graph-theoretic definitions not given here.

[^2]:    ${ }^{2}$ The result of Wood [68] depended on a result announced by Norin and Thomas [51, 62], which has not yet been fully written.
    ${ }^{3}$ Dvořák and Norin [20] announced in 2017 that a forthcoming paper, which has not yet been fully written, will also prove the Clustered Hadwiger Conjecture.

[^3]:    ${ }^{4}$ Let $P_{s, t}$ be the graph obtained from $K_{s, t}$ by adding a path on the $t$-vertex side. Every tree on $t^{2}$ vertices contains a $K_{1, t}$ minor or a path on $t$ vertices. It follows that every graph in $\mathcal{J}_{s, t^{2}}$ contains $K_{s+1, t}$ or $P_{s, t}$ as a minor. Conversely, $P_{s, t+s}$ and $K_{s+1, t+s}$ contain a graph in $\mathcal{J}_{s, t}$ as a minor (obtained by

[^4]:    contracting a suitable $s$-edge matching). This says that (ignoring dependence on $t$ ) being $\mathcal{J}_{s, t}$-minor-free is equivalent to being $K_{s+1, t}$-minor-free and $P_{s, t}$-minor-free. Therefore Theorem 4 could be stated for graphs that are $K_{s+1, t}$-minor-free and $P_{s, t}$-minor-free. We choose to work with $\mathcal{J}_{s, t}$ for convenience.

[^5]:    ${ }^{5}$ In reality it is rather $D-A$ that is described by $(\mathcal{P}, \mathcal{L})$, but writing $D$ instead of $D-A$ is sometimes convenient, since it allows us to avoid naming the major apex set systematically.

[^6]:    ${ }^{6}$ This uses the fact that if two chord-free 4-discs $\Delta_{1}$ and $\Delta_{2}$ strictly overlap, then $\Delta_{1} \cup \Delta_{2}$ is also a chord-free 4 -disc whose interior contains stricly more vertices than $\operatorname{Int}\left(\Delta_{1}\right) \cup \operatorname{Int}\left(\Delta_{2}\right)$, so $\mathcal{C}$ would contain neither $\Delta_{1}$ nor $\Delta_{2}$.
    ${ }^{7}$ The fact that $\Delta_{2 i}$ cannot contain the edge $y_{2 i-1} y_{2 i+1}$ follows from the Jordan Curve Theorem.

[^7]:    ${ }^{8}$ Technically, we are raising the curtain $G$ whose description uses a tree-decomposition $\mathcal{T}$ that contains a single bag $B:=V(G)$.

[^8]:    ${ }^{9}$ The colouring of each graph $G_{C}$ is similar to the colouring of the graph $G_{C}$ in Case A of the proof of Lemma 27.

[^9]:    ${ }^{10}$ Since each $T_{i}$ is a copy of $T_{x: y} \subseteq T$, we abuse notation slightly here and use vertices of $T_{i}$ as indices for the bags of $\mathcal{T}$.

[^10]:    ${ }^{11}$ The graphs $G\left\langle B_{y}\right\rangle$ and $G\left\{B_{y}\right\}$ are defined here with respect to the new tree-decomposition $\mathcal{T}^{\prime}$.

