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Constructing Membership Function Based on Fuzzy Shannon Entropy and Human's Interval Estimation

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Abstract—**This paper develops a new constructing approach of an appropriate membership function to integrate a given probability density function and fuzzy Shannon entropy extending the statistical theory into the heuristic method based on the human cognitive behavior and subjectivity. The proposed approach is formulated as a more general mathematical programming problem than previous approaches due to using a general S-curve function and the fuzzy Shannon entropy. Then, performing deterministic equivalent transformations to the initial problem, the optimal condition of parameters is obtained. Furthermore, in order to show the appropriate membership function using the proposed approach, some probability density functions are provided as numerical examples.**

Keywords-constructing membership function; fuzzy entropy; S-curve function; mathematical programming

I. INTRODUCTION

In mathematical formulations and designs of human cognitive behavior, utility, and subjectivity in a decision making problem using the fuzzy theory, the most crucial step is to determine and construct the appropriate membership function. There are many guidelines on developing the membership functions for fuzzy sets (for instance, a survey of Gottwald [7]). Nevertheless, to choose appropriate membership function's shape and values statistically is not usually an easy task, and so heuristic methods have been used in previous studies, which are not mathematically and statistically guaranteed. In order to overcome the disadvantages of constructing appropriate membership functions, some researchers have proposed more rigorous approaches. Until now, some approaches have adopted a transformation from a probability distribution to a possibility distribution (for instance, Bharathi and Sarma [2]). Particularly, Civanlar and Trussell [6] proposed a criteria which results in a more rigorous method of defining membership functions for a certain group of fuzzy sets which are statistically based upon the probability density function. Their approach is mainly based on maximizing a fuzzy entropy measure. Recently, Cheng and Cheng [5] and Nieradka and Butkiewicz [8] proposed constructing approaches introducing an S-curve function and fuzzy index as well as Civanlar and Trussell's approach, and applied to an image processing field. In fuzzy entropy-based approaches, the membership function is obtained using a mathematical programming problem based on the statistical theory. However, human cognitive behavior and subjectivity may not be sufficiently reflected on the constructed membership function, because almost all parameters are automatically determined by solving optimization problems.

On the other hand, some researchers have compared some heuristic approaches to find more appropriate method of constructing the membership function. In Chameau and Santamarina's study [3], they prepared a set of questionnaires to compare four elicitation methods: point estimation, interval estimation, membership function exemplification, and pairwise comparison as possible candidates for practical applications. Then, Yoshikawa [12] discussed the influence of procedures for an interactive identification method on forms of identified membership functions. His proposed approach is also based on the interval estimation and questionnaires to the degree of membership. In these studies, the method based on the estimation of intervals was found to offer a number of advantages that make it very suitable for practical applications. However, these interval estimation approaches are included in heuristic methods, because ranges where an event is entirely included or never included are subjectively determined by the human.

Therefore, in order to overcome each disadvantage of fuzzy entropy-based approach and heuristic method, we construct a new constructing approach of appropriate membership

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functions using both the fuzzy Shannon entropy and the interval estimation derived from human cognitive behavior and subjectivity. The main part of our proposed approach is to solve the mathematical programming problem maximizing the fuzzy entropy under a given probability density function (pdf) and some constraints based on Civanlar and Trussell's study [6], and to find the optimal condition of parameters.

This paper is organized as follows. In Section II, we introduce S-curve functions as a general form of membership function to the L-R fuzzy numbers. Then, we also introduce a fuzzy entropy extending standard statistical entropy, particularly fuzzy Shannon entropy. In Section III, we formulate a mathematical programming problem maximizing the fuzzy Shannon entropy with a constraint to the total average membership value derived from the given pdf. Then, we perform deterministic equivalent transformations, and obtain the nonlinear programming problem. In Section IV, using our constructing approach and introducing some pdfs, we show each appropriate membership function. Finally, in Section V, we conclude this paper and discuss future researches.

II. MATHEMATICAL DEFINITION

A. S-curve function

The shape of S-curve function is commonly used for the representation of several types of human cognitive behavior and subjectivity in social science, decision making, image processing, etc.. The S-curve function was originally introduced by Zadeh [13]. Then, in order to deal with S-curve function more flexibly, the other definition of S-curve function was proposed by Cheng and Cheng [4] as follows:

$$S_L(x;a,b,c)=\begin{cases} 0 & x\le a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & a<x\le b \\ 1-\dfrac{(x-c)^2}{(c-b)(c-a)} & b<x\le c \\ 1 & c\le x \end{cases} \tag{1}$$

where x is a variable, and a, b, and c are parameters determining the shape of S-curve function. In this definition, b can be any point between a and c. One example of this S-curve function is shown in Figure 1.

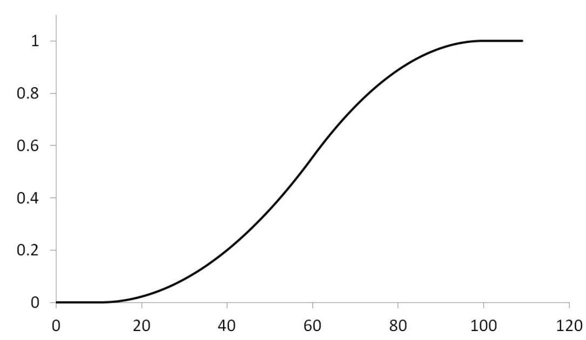

Figure 1. Example of S-curve function with a=10, b=60, and c=100

Most recently, Peidro and Vasant [10] proposed the following modified S-curve membership function as a particular case of the logistic function:

$$S_L(x)=\begin{cases} 1 & x^b<x \\ w^b & x^b=x \\ 1-\dfrac{B_1}{1+C_1e^{\alpha_1 x}} & x^a<x<x^b \\ w^a & x^a=x \\ 0 & x<x^a \end{cases} \tag{2}$$

where B_1, C_1, and $\alpha_1>0$ are parameters determining the shape of modified S-curve membership function. Then, w^a and w^b are constant values initially set by the examinee under $x=x^a$ and $x=x^b$, which are very close to 1 and 0, respectively. One example of this S-curve function is shown in Figure 2.

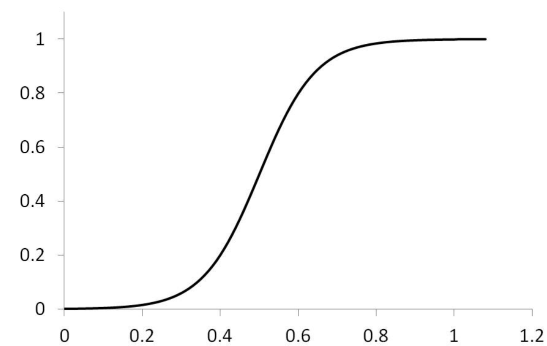

Figure 2. Example of modified S-curve membership function with $x^a=0, x^b=1, w^a=0.001, w^b=0.999,\ S_L(0.5)=0.5$

Two membership functions in Figures 1 and 2 are almost the same shapes, and so in this paper, we focus on the modified S-curve membership function. In formula (2), we define $S_1(x)=1-\dfrac{B_1}{1+C_1e^{\alpha_1 x}}$. Figure 2 shows that formula (2) is given the only left-hand membership function of general L-R fuzzy numbers. Therefore, in a way similar to formula (2), we introduce the following formulation to right-hand membership function of general L-R fuzzy numbers.

$$S_R(x)=\begin{cases} 1 & x<x^c \\ w^c & x^c=x \\ \dfrac{B_2}{1+C_2e^{\alpha_2 x}} & x^c<x<x^d \\ w^d & x^d=x \\ 0 & x^d<x \end{cases} \tag{3}$$

where B_2, C_2, and $\alpha_2>0$ are parameters determining the shape of modified S-curve membership function. Then, w^c

and w^d is constant values initially set by the examinee under $x = x^c$ and $x = x^d$, which are very close to 1 and 0, respectively. In formula (3), we define $S_2(x) = \frac{B_2}{1 + C_2 e^{\alpha_2 x}}$.

B. Fuzzy entropy

Based on the statistical theory, many definitions of fuzzy entropy have been proposed (for instance, Al-sharhan and Karray [1], Nieradka and Butkiewicz [8], Pal and Bezdek [9]). Let I be a set with randomly occurring events $\{x_1, x_2, ..., x_n\}$ in an experiment, and p_i is a corresponding probability of each event x_i. As one of the most standard entropy in the statistical theory, Shannon entropy is formulated as $-\sum_{i=1}^{n} p_i \log(p_i)$. In order to extend the Shannon entropy to uncertainty derived from the fuzziness of the fuzzy set, the fuzzy Shannon entropy of membership values μ_i corresponding to events x_i is defined as follows:

$$-\sum_{i=1}^{n} \left\{ \mu_i \log \mu_i + (1-\mu_i) \log(1-\mu_i) \right\} \tag{4}$$

This mathematical form is very similar to the standard Shannon entropy in the case of binary discrete random variable. In this paper, we focus on this fuzzy Shannon entropy and construct the appropriate membership function.

III. Constructing Approach of Appropriate Membership Function

Using two definitions of modified S-curve membership functions (2) and (3) and fuzzy Shannon entropy, we propose a new constructing approach of appropriate membership function. In order to integrate human cognitive behavior and subjectivity into S-curve functions, we assume an interval estimation that an examinee initially set two ranges that an event or a condition is entirely included in human's feelings, and never included, respectively. For instance, when we ask the examinee to answer a range of temperatures that the examinee feels "entirely comfortable", she or he will answer from 20 to 24 degrees Celsius. Then, when we also ask the examinee to answer a range of "never comfortable" temperatures, she or he will also answer less than 16 degrees Celsius or more than 30 degrees Celsius. Thus, it is not a burden and not difficult to answer these two ranges. Furthermore, if we introduce these two ranges in the proposed fuzzy Shannon entropy-based model, it is possible to integrate personal cognitive behavior and subjectivity into the statistical theory.

A. Mathematical formulation of constucting approach

The main object of our constructing approach is to maximize the fuzzy Shannon entropy under S-curve function and assumed ranges set by the examinee whose membership values are 1 and 0. In addition, we introduce the average membership value according to the given pdf as a constraint, because the only available quantitative data is the pdf. This concept is based on Civanlar and Trussell's study [6]. However, in Civanlar and Trussell's study, the objective function is minimizing the square of membership functions, which is a special case of fuzzy entropies. Therefore, we extend Civanlar and Trussell's study to the more general case using fuzzy Shannon entropy. In these conditions, we formulate the following mathematical programming problem to our constructing approach:

$$\begin{aligned} &\text{Maximize} \ -\int_{-\infty}^{\infty} \{\mu(x)\log\mu(x) \\ &\qquad\qquad + (1-\mu(x))\log(1-\mu(x))\} dx \\ &\text{subject to} \ E(\mu(x)) = \int_{-\infty}^{\infty} \mu(x) p(x) dx \geq m, \\ &\qquad 0 \leq \mu(x) \leq 1, \ \forall x \end{aligned} \tag{5}$$

where m is the target total average membership value initially determined by the examinee. As a similar formulation of problem (5) which is an integral-based form, we also introduce the following summation-based formulation:

$$\begin{aligned} &\text{Maximize} \ -\sum_{i=1}^{n} \{\mu_i \log\mu_i + (1-\mu_i)\log(1-\mu_i)\} \\ &\text{subject to} \ \sum_{i=1}^{n} \mu_i p(x_i) \geq m, \\ &\qquad 0 \leq \mu(x_i) \leq 1, \ \forall x_i, (i = 1, 2, ..., n) \end{aligned} \tag{6}$$

B. Deterministic equivalent transformations

In this paper, we consider this summation-based problem, because the given pdf is often assumed to be a histogram or histogram-based function derived from numerical data $\{x_1, x_2, ..., x_n\}$. We set the following sets for indexes *i*:

$$\begin{aligned} I_1 &= \{ i \mid x^a < x_i < x^b \} \\ I_2 &= \{ i \mid x^c < x_i < x^d \} \\ I' &= \{ i \mid x^b < x_i < x^c \} \end{aligned} \tag{7}$$

All parameters x^a, x^b, x^c, and x^d are initially set by the examinee when she or he do an interval estimation. Using these sets of indexes, summation-based problem (6) is equivalently transformed into the following form:

$$\begin{aligned} &\text{Maximize} \ -\sum_{i \in I_1} \{ S^1(x_i)\log S^1(x_i) + (1-S^1(x_i))\log(1-S^1(x_i)) \} \\ &\qquad\qquad -\sum_{i \in I_2} \{ S^2(x_i)\log S^2(x_i) + (1-S^2(x_i))\log(1-S^2(x_i)) \} \\ &\text{subject to} \ \sum_{i \in I_1} S^1(x_i) p(x_i) + \sum_{i \in I_2} S^2(x_i) p(x_i) \geq m', \\ &\left(\text{where} \ m' = m - \sum_{i \in I'} p(x_i) \right) \end{aligned} \tag{8}$$

Furthermore, from the continuous of functions on $x = x^a, x^b, x^c, x^d$, the following equations holds:

$$\begin{cases} 1 - \dfrac{B_1}{1 + C_1 e^{\alpha_1 x^a}} = w^a \\ 1 - \dfrac{B_1}{1 + C_1 e^{\alpha_1 x^b}} = w^b \\ \dfrac{B_2}{1 + C_2 e^{\alpha_2 x^c}} = w^c \\ \dfrac{B_2}{1 + C_2 e^{\alpha_2 x^d}} = w^d \end{cases} \tag{9}$$

From these equations, we find that B_j and C_j depend on α_j derived from the study of Vasant [11]. Therefore, decision variables of problem (8) are only α_1 and α_2, and so using nonlinear solution algorithm or heuristic approaches such as Particle Swarm Optimization, we obtain the optimal values of parameters in the modified S-curve membership function.

IV. Numerical Examples

In order to show appropriate membership functions from a given pdf using our proposed constructing approach, we provide uniform distribution, and partially uniform distribution based on Civanlar and Trussell's study [6].

A. Uniform distribution

First, we consider a uniform distribution $U[1,6]$ displayed in Figure 3. In Civanlar and Trussell's study [6], the optimal membership function based on the pdf of the uniform distribution is trivial. However, in this paper, since we consider the fuzzy Shannon entropy to S-curve function as well as ranges of membership set by an examinee, the appropriate membership function is not trivial. We introduce the following three different conditions; (i) set $[2,5]$ where the value of membership function is 1, and $(-\infty,0],[7,\infty)$ where the value of membership function is 0, (ii) set $[3,5.5]$, and $(-\infty,0],[7,\infty)$, and (iii) set $[4,6]$, and $(-\infty,0],[8,\infty)$. Under each condition and $m = 0.85$, we solve problem (8) and obtain the membership function shown in Figure 4.

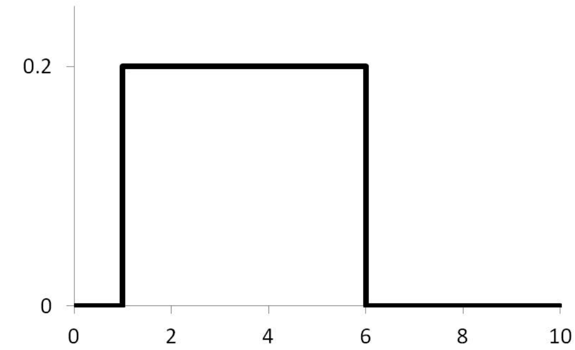

Figure 3. Example of uniform distribution

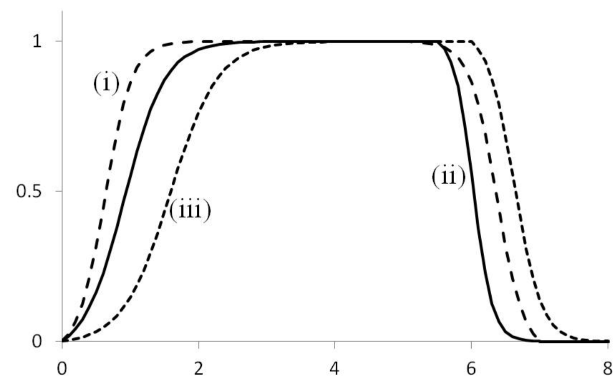

Figure 4. Appropriate memebership functions: (i) dashed line, (ii) solid line, and (iii) dotted line

In Figure 4, first condition (i) is symmetric to the given uniform distribution, and so the appropriate membership function is obtained as symmetric shape. On the other hand, second condition (ii) and third condition (iii) are not symmetric to the given uniform distribution, and so the shape of $S_1(x)$ is different from $S_2(x)$ in each condition. Furethermore, comparing conditions (i) and (ii), since the range in (ii) where the value of membership function is 1 is the right side to the range in (i), the shapes of $S_1(x)$ and $S_2(x)$ in (ii) are more moderate and drastical than in (i), respectivly.

B. Partially uniform distribution

As a partially uniform density function, we introduce the following function displayed in Figure 5 based on Civanlar and Trussell's study [6]. In Civanlar and Trussell's study [6], the shape of membership function is the same as the given partially uniform distribution. On the other hand, in our proposed approach, since we use a modified S-curve membership function, the obtained membership function is not same as the given partially uniform distribution.

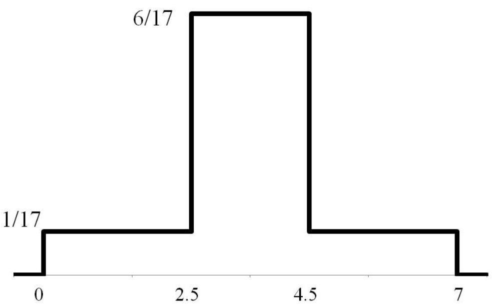

Figure 5. Example of symmetric partially uniform distribution

In this partially uniform distribution, we consider two different conditions; ; (i) set $[2,5]$ where the value of membership function is 1, and $(-\infty,0],[7,\infty)$ where the value of membership function is 0, (ii) set $[3,5.7]$, and $(-\infty,0],[8,\infty)$. Under each condition and $m=0.75$, we solve problem (8) and obtain the membership function shown in Figure 6.

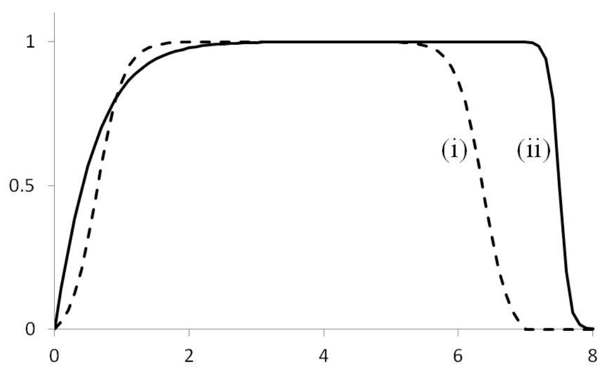

Figure 6. Appropriate memebership functions: (i) dashed line, (ii) solid line

In Figure 6, the shapes of two membership functions are obviously different from the given partially uniform distribution. Furthermore, first condition (i) is also symmetric to the given partially uniform distribution, and so the appropriate membership function is obtained as symmetric shape. On the other hand, second condition (ii) is not symmetric to the given partially uniform distribution. Therefore, $S_1(x)$ is moderately increasing, but on the contrary, $S_2(x)$ is drastically decreasing. Furthermore, comparing ranges $[2,5]$ in (i) and $[3,5.7]$ in (ii) where values of membership functions are 1, the range in (i) entirely includes $[2.5,4.5]$ where the corresponding probability is the highest in the given partially uniform distribution, but the range in (ii) partially includes only $[3.5,4.5]$. It means a difficulty of attainment more than target value $m=0.75$ that the average membership value in condition (ii) tends to be a smaller value than condition (i), because $\sum_{x_i\in[3.5,4.5]} p(x_i) < \sum_{x_i\in[2.5,4.5]} p(x_i)$. Therefore, $S_1(x)$ in (ii) tend to be more drastically increasing than that of (i) due to the possibly large value of $\sum_{x_i\in[0,2]} p(x_i)$ in (ii).

V. Conclusion

In this paper, we have developed a new constructing approach of an appropriate membership function to the given probability density function by integrating fuzzy Shannon entropy extending the statistical theory and the heuristic method based on the human cognitive behavior and subjectivity. The proposed approach have been formulated as a more general mathematical programming problem than previous approaches using a modified S-curve membership function and fuzzy entropy extending Shannon entropy. Then, performing deterministic equivalent transformations to the initial problem, the optimal condition of parameters has been obtained. Furthermore, introducing some examples of probability density function, we have shown appropriate membership functions in our proposed approach. Our approach includes both advantages of statistical theory and heuristic method to do simple questionnaires, and so the obtained membership function will be more statistically and mathematically appropriate as well as more closely fitting to human subjectivity.

On the other hand, our proposed approach has some disadvantages. Particularly, when we solve mathematical programming problem (8) to obtain the membership function, it is difficult and not rapid to solve problem (8) due to the nonlinear programming problem. Then, if the probability density function is complicated, it may be hard to solve problem (8) directly. Therefore, as a future work, we will develop more efficient solution algorithms.

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