On Fuzzy Ideals of Fuzzy Lattice

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***Abstract*—We characterize a fuzzy lattice through a fuzzy partial order relation, define a fuzzy ideal and fuzzy filter of fuzzy lattice, characterize a fuzzy ideal of fuzzy lattice using its level set and its support and show that a subset of a fuzzy lattice is a fuzzy ideal if and only if its support is a crisp ideal. Similarly, we show the same for its level set.**

I. Introduction

The concept of a fuzzy set was first introduced by Zadeh [18] and this concept was adapted by Goguen [12] and Sanchez [15] to define and study fuzzy relations. The notion of partial and lattice order goes back to 19th century investigations in logic. Yuan and Wu [16] introduced the concepts of fuzzy sublattices and fuzzy ideals of a lattice. Ajmal and Thomas [1] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. In 2000, Attallah [2] defined a fuzzy ideal and fuzzy filter of lattice, characterized in terms of meet and join operations. In 2008, Koguep, Nkumi and Lele [13] studied the notion of fuzzy prime ideal and highlighted the difference between fuzzy prime ideal and prime fuzzy ideal of a lattice. Recently, Chon [7] characterized a fuzzy partial order relation using its level set and defined a fuzzy lattice as a fuzzy relation, developed some basic properties of fuzzy lattices, characterized a fuzzy lattice using its level set, and showed that a fuzzy totally ordered set is a distributive fuzzy lattice. As a continuation of these studies, we define a fuzzy ideal of fuzzy lattice using the fuzzy partial order relation and fuzzy lattice defined by Chon.

In section 2, we report preliminary results on some basic concepts of ideal, filter and lattice of both the algebraic point of view and as partial order. In section 3, we define a fuzzy partial order relation using a fuzzy relation of a set and its fuzzy partial ordered set. Also we define a fuzzy lattice as a partial order relation and develop some properties of fuzzy lattice. In section 4, we define fuzzy ideal and fuzzy filter of fuzzy lattices, characterize a fuzzy ideal of a fuzzy lattice using its support and its level set.

II. Preliminaries

In this section, we will briefly review some basic concepts of lattices, ideals and filters both under the algebraic point of view and as partial order as necessary for the development of other sections. This text is quite introductory and can be found in many books and articles like in [5] and [9]. Any people familiar with the matter may proceed to the next section.

Many important properties of an ordered set P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. Two of the most important classes of ordered sets defined in this way are lattices and complete lattices. Here we present the basic theory of such ordered sets, and also consider lattices as algebraic structures.

It is a fundamental property of the real numbers, $\mathbb{R}$, that if I is a closed and bounded interval in $\mathbb{R}$, then every subset of I has both a least upper bound (or supremum) and a greatest lower bound (or infimum) in I. These concepts pertain to any ordered set.

Definition 2.1: Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an upper bound of S if $s \leq x$ for all $s \in S$. A lower bound is defined dually. The set of all upper bounds of S is denoted by S^u and the set of all lower bounds by S^l:

$$\begin{aligned} S^u &:= \{x \in P : (\forall s \in S)\ s \leq x\} \text{ and} \\ S^l &:= \{x \in P : (\forall s \in S)\ s \geq x\}. \end{aligned}$$

Since $\leq$ is transitive, S^u is always an up-set and S^l a down-set. If S^u has a least element x, then x is called the least upper bound of S. Equivalently, $x \in P$ is the least upper bound of S if
(i) x is an upper bound of S, that is, $x \in S^u$ and
(ii) $x \leq y$ for all upper bounds $y \in S^u$.

Dually, if S^l has a greatest element x, then x is called the greatest lower bound of S. Since least elements and greatest elements are unique, least upper bounds and greatest lower bounds are unique when they exist.

We use the following notation: we write $x \vee y$ in place of $sup\{x, y\}$ when it exists and $x \wedge y$ in place of $inf\{x, y\}$ when it exists. Similarly, we write $\bigvee S$ and $\bigwedge S$ instead of $sup\ S$ and $inf\ S$ when these exist.

When they exist, the top and bottom elements of P are denoted by $\top$ and $\bot$ respectively. It is easy to see that if P has a top element, then $P^u = \{\top\}$ in which case $sup\ P = \top$. When P has no top element, we have $P^u = \emptyset$ and hence $sup\ P$ does not exist. By duality, $inf\ P = \bot$ wherever P has a bottom element.

Definition 2.2: Let P be a non-empty ordered set.
(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a lattice.
(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a complete lattice.

We introduced lattices as ordered sets of a special type. However, we may adopt an alternative viewpoint. Given a lattice $\underline{L} = \langle L, \leq \rangle$, we may define binary operations *join* and *meet* on the nonempty set L by

$$a \vee b = sup\{a, b\} \text{ and } a \wedge b = inf\{a, b\} \qquad (a, b \in L).$$

Now, we view a lattice as an algebraic structure $\underline{L} = \langle L, \vee, \wedge \rangle$. We amplify the connection between $\vee, \wedge$ and $\leq$.

Lemma 2.1: Let $\underline{L}$ be a lattice and let $a, b \in L$. Then the following are equivalent:
(i) $a \leq b$;
(ii) $a \vee b = b$;
(iii) $a \wedge b = a$.

Proof. See in [8] Lemma 2.8. ■

Theorem 2.1: Let $\underline{L}$ be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c \in L$,
(L1.1) $(a \vee b) \vee c = a \vee (b \vee c)$ (associative laws)
(L1.2) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(L2.1) $a \vee b = b \vee a$ (commutative laws)
(L2.2) $a \wedge b = b \wedge a$
(L3.1) $a \vee a = a$ (idempotency laws)
(L3.2) $a \wedge a = a$
(L4.1) $a \vee (a \wedge b) = a$ (absorption laws)
(L4.2) $a \wedge (a \vee b) = a$.

Proof. See in [8] Theorem 2.9. ■

Theorem 2.2: Let $\langle L, \vee, \wedge \rangle$ be a non-empty set equipped with two binary operations which satisfy all the conditions of Theorem 2.1.
(i) For all $a, b \in L$, we have $a \vee b = b$ iff $a \wedge b = a$.
(ii) Define $\leq$ on L by $a \leq b$ if $a \vee b = b$. Then, $\leq$ is an order relation.
(iii) With $\leq$ as in (ii), $\langle L, \leq \rangle$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in L$,

$$a \vee b = sup\{a, b\} \text{ and } a \wedge b = inf\{a, b\}.$$

Proof. See in [8] Theorem 2.10. ■

We have shown that lattices can be completely characterized in terms of the join and meet operations. We may henceforth say 'let $\underline{L}$ be a lattice', replacing $\underline{L}$ by $\langle L, \leq \rangle$ or by $\langle L, \vee, \wedge \rangle$ if we want to emphasize that we are thinking of it as a special kind of ordered set or as an algebraic structure.

It may happen that $\langle L, \leq \rangle$ has top and bottom elements. When thinking of $\underline{L}$ as $\langle L, \vee, \wedge \rangle$, it is appropriate to view these elements from a more algebraic standpoint.

Definition 2.3: Let $\underline{L}$ be a lattice and L a nonempty set. We say $\underline{L}$ has a top element if there exists $1 \in \underline{L}$ such that $a = a \wedge 1$ for all $a \in \underline{L}$. Dually, we say $\underline{L}$ has a bottom element if there exists $0 \in \underline{L}$ such that $a = a \vee 0$ for all $a \in \underline{L}$. The lattice $\langle L, \vee, \wedge \rangle$ has a 1 iff $\langle L, \leq \rangle$ has a top element $\top$ and, in that case, $1 = \top$. A dual statement holds for 0 and $\bot$. A lattice $\langle L, \vee, \wedge \rangle$ possessing 0 and 1 is called bounded.

A finite lattice is automatically bounded, with $1 = \bigvee L$ and $0 = \bigwedge L$.

For more detailed study we refer to [8].

Now, we define an ideal in the algebraic viewpoint. In [13] is defined the ideal and filter of a lattice $\underline{L}$ as follows. Let L be a nonempty set and $\underline{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ stand for a bounded distributive lattice.

Definition 2.4: A nonempty subset J of L is called an ideal of $\underline{L}$ if for all $x, y \in L$
(i) if $x \in J$ with $y \leq x$, then $y \in J$.
(ii) $x, y \in J$ implies $x \vee y \in J$.

Definition 2.5: A nonempty subset F of L is called a filter of $\underline{L}$ if for all $x, y \in L$
(i) if $x \in F$ with $x \leq y$, then $y \in F$.
(ii) $x, y \in F$ implies $x \wedge y \in F$.

III. Fuzzy Lattices

In this section, we define a fuzzy lattice as a fuzzy partial order relation and develop some properties of fuzzy lattices.

Let X be a nonempty set. A function $A : X \times X \longrightarrow [0, 1]$ is called a *fuzzy equivalence relation* in X if A is reflexive, transitive and symmetric. A fuzzy relation A in X is *reflexive* iff $A(x, x) = 1$ for all $x \in X$, A is *transitive* iff $A(x, z) \geq sup_{y \in X} min[A(x, y), A(y, z)]$, and A is *antisymmetric* iff $A(x, y) > 0$ and $A(y, x) > 0$ implies $x = y$.

A fuzzy relation A is a *fuzzy partial order relation* if A is reflexive, antisymmetric and transitive. A fuzzy partial order relation A is a *fuzzy total order relation* iff $A(x, y) > 0$ or $A(y, x) > 0$ for all $x, y \in X$. If A is a fuzzy partial order relation on a set X, then (X, A) is called a *fuzzy partially ordered set* or *fuzzy poset*. If A is a fuzzy total order relation in a set X, then (X, A) is called *fuzzy totally ordered set* or a *fuzzy chain*.

For more detailed study we refer to [7].

In the literature there are several other ways to define a fuzzy relation reflexive, symmetric and transitive as for example in [4], [10], [11]. Also, we can find several other forms to define fuzzy partial order relations, as we can see in [5], [6], [7], [17].

Remark 3.1: When A is reflexive, then the transitivity can be rewritten by replacing the "$\geq$" by "=". In other words, A is *transitive* iff $A(x,z) = sup_{y\in X} min[A(x,y), A(y,z)]$, for all $x, y, z \in X$.

The statement that is claimed in the last remark can be easily proved. First, we know that $A(x,z) \geq sup_{y\in X} min[A(x,y), A(y,z)]$ and second, trivially, $sup_{y\in X} min[A(x,y), A(y,z)] \geq min[A(x,x), A(x,z)] = min[1, A(x,z)] = A(x,z)$. Therefore, we have that $A(x,z) = sup_{y\in X} min[A(x,y), A(y,z)]$. ■

We define a fuzzy lattice as a fuzzy partial order relation and develop some properties of fuzzy lattices.

Definition 3.1: Let (X, A) be a fuzzy poset and let $Y \subseteq X$. An element $u \in X$ is said to be an upper bound for a subset Y iff $A(y,u) > 0$ for all $y \in Y$. An upper bound u_0 for Y is the least upper bound (or supremum) of Y iff $A(u_0, u) > 0$ for every upper bound u for Y. An element $v \in X$ is said to be a lower bound for a subset Y iff $A(v,y) > 0$ for all $y \in Y$. A lower bound v_0 for Y is the greatest lower bound (or infimum) of Y iff $A(v, v_0) > 0$ for every lower bound v for Y.

The least upper bound of Y will be denoted by $sup\ Y$ or $LUB\ Y$ and the greatest lower bound by $inf\ Y$ or $GLB\ Y$. We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

Remark 3.2: Since A is antisymmetric, then the least upper (greatest lower) bound, if it exists, is unique.

The statement claimed in the above remark is easily proved. Suppose that u_0 and u_1 are two least upper bounds in a subset $Y \subseteq X$. Then, by definition, u_0 and u_1 are upper bounds of Y. Thus as u_0 is a LUB, then $A(u_0, u_1) > 0$ and as u_1 is a LUB, then $A(u_1, u_0) > 0$. Therefore, by the antisymmetry of A, $u_0 = u_1$. Similarly, we prove that GLB is unique. ■

Example 3.1: Let $X = \{x, y, z, w\}$ and let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation such that $A(x,x) = A(y,y) = A(z,z) = A(w,w) = 1, A(x,y) = A(x,z) = A(x,w) = A(y,z) = A(y,w) = A(z,w) = 0, A(y,x) = 0.3, A(z,x) = 0.5, A(w,x) = 0.8, A(z,y) = 0.2, A(w,y) = 0.4$, and $A(w,z) = 0.1$. Then it is easily checked that A is a fuzzy total order relation. Also, $x \vee y = x$, $x \vee z = x$, $x \vee w = x$, $y \vee z = y$, $y \vee w = y$, $z \vee w = z$, $x \wedge y = y$, $x \wedge z = z$, $x \wedge w = w$, $y \wedge z = z$, $y \wedge w = w$, and $z \wedge w = w$.
Now, let $Y = \{z, w\}$ be a subset of X. Then, x, y and z are upper bounds of Y and how $A(z,w) = 0$ and $A(w,z) > 0$, the LUB is z and the GLB is w. ■

But, not every set of elements of a fuzzy poset has a least upper (greatest lower) bound as can be seen in the following example.

Example 3.2: Let $X = \{x, y, z, w\}$ and let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation such that $A(x,x) = A(y,y) = A(z,z) = A(w,w) = 1, A(x,y) = A(y,x) = A(x,z) = A(x,w) = A(y,z) = A(y,w) = A(z,w) = 0, A(z,x) = 0.5, A(w,x) = 0.8, A(z,y) = 0.2, A(w,y) = 0.4$, and $A(w,z) = 0.1$. Then it is easily checked that A is a fuzzy partial order relation. Also, $x \vee z = x$, $x \vee w = x$, $y \vee z = y$, $y \vee w = y$, $z \vee w = z$, $x \wedge z = z$, $x \wedge w = w$, $y \wedge z = z$, $y \wedge w = w, x \wedge y = z$ and $z \wedge w = w$. Notice that $x \vee y$ does not exist. ■

Definition 3.2: Let (X, A) be a fuzzy poset. Then, (X, A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$. We denote by $\mathcal{L}$ the fuzzy lattice (X, A).

The example 3.1 is an example of fuzzy lattice whereas the example 3.2, is not a fuzzy lattice.

Proposition 3.1: Let (X, A) be a fuzzy poset (or chain) and $Y \subseteq X$. If $B = A|_{Y\times Y}$, that is, B is a fuzzy relation on Y such that for all $x, y \in Y$, $B(x,y) = A(x,y)$, then (Y, B) is a fuzzy poset (or chain).

Proof. Straightforward. ■

Definition 3.3: Let $\mathcal{L}$=(X, A) be a fuzzy lattice. (Y, B) is a fuzzy sublattice of $\mathcal{L}$ if $Y \subseteq X$, $B = A|_{Y\times Y}$ and (Y, B) is a fuzzy lattice.

We define for any $p \in (0,1]$ the p-level set $A_p = \{(x,y) \in X \times X : A(x,y) \geq p\}$ of a fuzzy relation A in a set X and characterize a relation on X and we define the support of a fuzzy relation A in X by $S(A) = \{(x,y) \in X \times X : A(x,y) > 0\}$.

The following Proposition was transcribed from paper [7] Proposition 2.4 and its proof is included because the proof the transitivity has been modified.

Proposition 3.2: Let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation. Then, A is a fuzzy partial order relation on X iff for each $p \in (0,1]$, the p-level set A_p is a partial order relation in X.

Proof. ($\Rightarrow$) Let A be a fuzzy partial order relation on X and $p \in (0,1]$. Since $A(x,x) = 1$ for all $x \in X$, $(x,x) \in A_p$ for all p such that $0 < p \leq 1$. Suppose $(x,y) \in A_p$ and $(y,x) \in A_p$. Then, $A(x,y) \geq p > 0$ and $A(y,x) \geq p > 0$, and hence, because A is antisymmetric, $x = y$. Suppose $(x,y) \in A_p$ and $(y,z) \in A_p$. Then, $A(x,y) \geq p$ and $A(y,z) \geq p$. Since $A(x,z) \geq sup_{y \in X} min[A(x,y), A(y,z)]$, $A(x,z) \geq min[A(x,y), A(y,z)] \geq p$, that is, $(x,z) \in A_p$.

($\Leftarrow$) Let A_p be a partial order relation for all p such that $0 < p \leq 1$. Then, $(x,x) \in A_p$ for all p such that $0 < p \leq 1$. Thus, $(x,x) \in A_1$, that is, $A(x,x) = 1$. Suppose $A(x,y) > 0$ and $A(y,x) > 0$. Then, $A(x,y) > v > 0$ for some $v \in \mathbb{R}$ and $A(Y,x) > w > 0$ for some $w \in \mathbb{R}$. Let $u = min(v,w)$. Then, $A(x,y) > u > 0$ and $A(y,x) > u > 0$. Thus, $(x,y),(y,x) \in A_u$. Since A_u is antisymmetric, $x = y$. Let $x,y,z \in X$ and $p_y = \min(A(x,y), A(y,z))$. So, $(x,y),(y,z) \in A_{p_y}$ and because A_{p_y} is by hypothesis a partial order, then $(x,z) \in A_{p_y}$. Therefore, $A(x,z) \geq p_y = \min[A(x,y), A(y,z)]$ and hence $A(x,z) \geq sup_{y \in X} \min[A(x,y), A(y,z)]$, that is, A is transitive. ■

Proposition 3.3: Let $A : X \times X \to [0,1]$ be a fuzzy relation and let $A_p = \{(x,y) \in X \times X : A(x,y) \geq p\}$. If (X, A_p) is a lattice for every p with $p \in (0,1]$, then (X, A) is a fuzzy lattice.

Proof. See in [7] Proposition 3.5.

In first view, we might think that the reverse is also true. That is, if (X,A) is a fuzzy lattice, then (X, A_p) is a lattice for every p with $p \in (0,1]$. But, we can see that depending on the α-cut, (X, A_p) may not be a lattice as seen in the following example.

Example 3.3: Let $X = \{x,y,z,w\}$ and let $A : X \times X \to [0,1]$ be a fuzzy relation such that $A(x,x) = A(y,y) = A(z,z) = A(w,w) = 1, A(x,y) = A(x,z) = A(x,w) = A(y,z) = A(z,y) = A(y,w) = A(z,w) = 0, A(y,x) = 0.2, A(z,x) = 0.5, A(w,x) = 0.8, A(w,y) = 0.4$ and $A(w,z) = 0.1$. Then it is easily checked that A is a fuzzy partial order relation and that (X,A) is a fuzzy lattice. But, if we choose the α-cut equal to 0.5, (X, A_p) is not a lattice because the GLB there not exists. ■

We can not even say that there is a α-cut that is a fuzzy lattice (X, A_p) as was claimed in [7] Proposition 3.6. The following example shows this situation.

Example 3.4: Let $L = ((0,1] \times \{a,b\}) \cup \{0, \top\}$ be a set, let the fuzzy partial order defined by

$$A(x,y) = \begin{cases} 1, & \text{if} \quad x = y \\ \frac{n-m}{2}, & \text{if} \quad x{=}(m,c), y{=}(n,c), c{\in}\{a,b\}, m{\leq}n \\ \frac{n}{2}, & \text{if} \quad x = 0,\ y = (n,c) \text{ and } c \in \{a,b\} \\ 0.5, & \text{if} \quad y = \top \text{ and } x \neq \top \\ 0, & \quad \text{otherwise} \end{cases}$$

Clearly, $\langle L, A \rangle$ is a fuzzy lattice. Nevertheless, for all $p > 0$, A_p is not a lattice because if $x = (p,a)$ and $y = (p,b)$, then $\{x,y\}$ has no lower bound in A_p. Suppose l is a lower bound in A_p for $\{x,y\}$. Then, $A(l,x) \geq p$ and $A(l,y) \geq p$. Thus, by definition of A,
(i) If $l = x$, then $A(x,y) \geq p$, but by definition of A, $A(x,y) = A((p,a),(p,b)) = 0$. Analogously, if $l = y$.
(ii) If $l = (z,a)$, then by definition of A, $A(l,y) = A((z,a),(p,b)) = 0$. Analogously, if $l = (z,b)$.
(iii) If $l = 0$, then $A(l,x) = A(0,(p,a)) = p/2 < p$. Analogously, $A(l,y) = p/2$.
Hence, $\{x,y\}$ has no lower bound in A_p.
The forth and fifth conditions of the equation $A(x,y)$ not occur in this case. ■

But we can build a lattice using the idea of support as follows.

Proposition 3.4: Let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation. If A is a fuzzy partial order relation on X, then $S(A)$ is a partial order relation on X.

Proof. Let A be a fuzzy partial order relation on X. Since $A(x,x) = 1$ for all $x \in X$, $(x,x) \in S(A)$. Suppose $(x,y) \in S(A)$ and $(y,x) \in S(A)$. Then, $A(x,y) > 0$ and $A(y,x) > 0$, and hence, because A is antisymmetric, $x = y$. Suppose $(x,y) \in S(A)$ and $(y,z) \in S(A)$. Then, $A(x,y) > 0$ and $A(y,z) > 0$. Since $A(x,z) \geq sup_{y \in X} min[A(x,y), A(y,z)]$, $A(x,z) \geq min[A(x,y), A(y,z)] > 0$, that is, $(x,z) \in S(A)$. ■

Proposition 3.5: Let (X,A) be a fuzzy lattice and let $x,y,z \in X$. Then

1) $A(x, x \vee y) > 0,\ A(y, x \vee y) > 0,\ A(x \wedge y, x) > 0,\ A(x \wedge y, y) > 0$.
2) $A(x,z) > 0$ and $A(y,z) > 0$ implies $A(x \vee y, z) > 0$.
3) $A(z,x) > 0$ and $A(z,y) > 0$ implies $A(z, x \wedge y) > 0$.
4) $A(x,y) > 0$ iff $x \vee y = y$.
5) $A(x,y) > 0$ iff $x \wedge y = x$.
6) If $A(y,z) > 0$, then $A(x \wedge y, x \wedge z) > 0$ and $A(x \vee y, x \vee z) > 0$.

Proof. See in [7] Proposition 3.3. ■

Corollary 3.1: Let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation. If (X,A) is a fuzzy lattice, then $(X, S(A))$ is a lattice.
Proof. Straightforward from Proposition 3.5 item (1), (2) and (3). ■

IV. Fuzzy Ideal and Fuzzy Filter

In this section, we define a fuzzy ideal and fuzzy filter of a fuzzy lattice and characterize a fuzzy ideal and a fuzzy filter of a fuzzy lattice using its support and its level set. First, we will define a fuzzy ideal and fuzzy filter of a fuzzy lattice as follows:

Definition 4.1: Let (Y, B) be a fuzzy sublattice of $\mathcal{L}$. (Y, B) is a fuzzy ideal of $\mathcal{L}$ if for all $x \in X$ and $y \in Y$,
(i) If $A(x, y) > 0$, then $x \in Y$;
(ii) If $x, y \in Y$, then $x \vee y \in Y$.

Definition 4.2: Let (Y, B) be a fuzzy sublattice of $\mathcal{L}$. (Y, B) is a fuzzy filter of $\mathcal{L}$ if for all $x \in X$ and $y \in Y$,
(i) If $A(y, x) > 0$, then $x \in Y$;
(ii) If $x, y \in Y$, then $x \wedge y \in Y$.

In section II we define an ideal of a lattice $\underline{L}$. So far we have defined a fuzzy ideal (Y, B) from a fuzzy lattice $\mathcal{L}$. We have also defined the set $S(A)$ of a fuzzy relation A in a set X as well as p-level set A_p of a fuzzy relation A in a set X and characterize a relation on X. Then, we can think of a set of ideals from a α-cut, that is, the set of ideals with degree greater than or equal to α or, the set of elements $x \in X$ and $y \in Y$ such that $A(x, y) \geq p$ with $p \in (0, 1]$.

Proposition 4.1: (Y, B) is a fuzzy ideal (fuzzy filter) of fuzzy lattice $\mathcal{L}$ iff Y is an ideal (filter) of $(X, S(A))$.

Proof. $(\Rightarrow)$ Let (Y, B) be a fuzzy ideal of $\mathcal{L}$ and let $y \in Y$. Then,
(i) If $(x, y) \in S(A)$, then $A(x, y) > 0$. So, by Definition 4.1 item (i) $x \in Y$.
(ii) If $x \in Y$ and $y \in Y$, then by Definition 4.1 item (ii), $x \vee y \in Y$.
$(\Leftarrow)$ (i) Let $x \in X$ and $y \in Y$ and suppose that $A(x, y) > 0$, then $(x, y) \in S(A)$ and $x \in Y$.
(ii) Trivially.

Similarly, we can proof that (Y, B) is a fuzzy filter of $\mathcal{L}$ iff Y is a filter of $(X, S(A))$. ■

Let B_p be the p-level set $B_p = \{(x, y) \in Y \times Y : B(x, y) \geq p\}$ for any $p \in (0, 1]$.

Theorem 4.1: (Y, B) is a fuzzy ideal (fuzzy filter) of fuzzy lattice $\mathcal{L}$ iff for each $p \in (0, 1]$, B_p is an ideal (filter) of (Y, A_p).

Proof. $(\Rightarrow)$ Let (Y, B) be a fuzzy ideal of $\mathcal{L}$ and let $y \in Y$. Then,
(i) If $(x, y) \in B_p$, then $A(x, y) \geq p$. So, by Definition 4.1 item (i) $x \in Y$.
(ii) If $x \in Y$ and $y \in Y$, then by Definition 4.1 item (ii), $x \vee y \in Y$.
$(\Leftarrow)$ (i) Let $x \in X$ and $y \in Y$ and suppose that $A(x, y) > 0$, then $(x, y) \in B_p$ and $x \in Y$.
(ii) Trivially.

Similarly, we can proof that (Y, B) is a fuzzy filter of $\mathcal{L}$ iff Y is a filter of (X, A_p). ■

V. Conclusion

We have studied the notion of fuzzy lattice using a fuzzy order relation, the notion of fuzzy ideal and fuzzy filter of fuzzy lattice and established the fuzzy ideal theorem of a fuzzy lattice through its level set. We can found several other forms to define fuzzy partial order relations, as we can see in [5], [6], [7], [17]. The same way, one should observe that the concept of fuzzy partial order, fuzzy partially ordered set and fuzzy lattice can be found in several other forms in the literature.

One of the most promising ideas could be the investigation of a fuzzy prime ideal of a fuzzy lattice and its consequences. As future work we extend the idea of [14] from fuzzy ideal and fuzzy filter. Thus, for further works we hope to think of building bounded interval fuzzy lattice using the idea of [3] from bounded fuzzy lattices.

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