On a Certain Type of Unary Operators

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Abstract—In our study we give a general form for modifiers that includes negation, different types of hedge and the sharpness operators. We will show that the four operators have a common form in the Pliant system and they will be called modifier operators. By changing the parameter value of a modifier we get the modalities, negation and the sharpness operators.

Index Terms—modalities, sharpness operator, negation, Pliant system

I. INTRODUCTION

The concept of hedge and modifiers appears at the very beginning of fuzzy set theory. They are related to an attempt to model meanings like "very", "more or less", "somewhat", "rather" and "quite". A hedge modifies the shape of the fuzzy set, inducing a change in the membership function. Thus a hedge transforms one fuzzy set into another fuzzy set. Here we will deal with strictly monotonously increasing and decreasing membership function.

A. Historical background

In the early 1970s, Zadeh [1] introduced a class of powering modifiers. He proposed computing with words as an extension of fuzzy sets and logic theory (Zadeh [2]). As pointed out by Zadeh [3]–[5], linguistic variables and terms are closer to human thinking (which emphasise importance more than certainty) and are used in everyday life. For this reason, words and linguistic terms can be used to model human thinking systems. Zadeh [1] said that a proposition such as "The sea is very rough" can be interpreted as "It is very true that the sea is rough." A number of studies [6], [7] have been conducted that discuss fuzzy logic and fuzzy reasoning with linguistic truth values. Basic notions of linguistic variables were formalized in different works by Zadeh in the mid 1970s [3]–[5]. These papers sought to provide a mathematical model for linguistic variables.

II. THE PLIANT OPERATOR SYSTEM

Here, we will be concerned with strict operators (strict t-norms and t-conorms).

Using the general representation theorem, we have for the strict t-norm (conjunctive operator) and the strict t-conorm (disjunctive operator).

$$c(x,y) = f_c^{-1}(f_c(x) + f_c(y)), d(x,y) = f_d^{-1}(f_d(x) + f_d(y)). \quad (1)$$

Here $f_c(x) : [0,1] \to [0,\infty]$ and $(f_d(x) : [0,1] \to [0,\infty])$ are continuous and strictly decreasing (increasing) monotone

functions and they are the generator functions of the strict t-norms and strict t-conorms.

Those familiar with fuzzy logic theory will find that the terminology used here is slightly different from that used in standard texts [8]–[12].

In the Pliant system, we look for a class of operators with infinitely many negation operators.

Definition 1: If

$$f_c(x)f_d(x) = 1, \quad x \in [0,1] \quad (2)$$

then we call the generated connectives a Pliant system.

Theorem 2: $c(x,y)$ and $d(x,y)$ build a DeMorgan system for $\eta_{\nu_*}(x)$ where $\eta_{\nu_*}(\nu_*) = \nu_*$ for all $\nu_* \in (0,1)$ if and only if

$$f_c(x)f_d(x) = 1. \quad (3)$$

Proof: See [13]. ∎

The general form of the multiplicative Pliant system is

$$o_\alpha(x,y) = f^{-1}\left((f^\alpha(x) + f^\alpha(y))^{1/\alpha} \right) \quad (4)$$

where $f(x)$ is the generator function of the strict t-norm operator and $f : [0,1] \to [0,\infty]$ is a continuous and strictly decreasing function.

$$\begin{aligned} &\text{If} \quad \alpha > 0, \quad \text{than} \quad o_\alpha(x,y) \quad \text{conjunctive operator (t-norm).}\\ &\text{If} \quad \alpha < 0, \quad \text{than} \quad o_\alpha(x,y) \quad \text{disjunctive operator (t-conorm).} \end{aligned} \quad (5)$$

The corresponding negation operator is

$$\eta_\nu(x) = f^{-1}\left(f(\nu_0)\frac{f(\nu)}{f(x)} \right) \qquad \text{or} \qquad (6)$$

$$\eta_{\nu_*}(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)} \right), \quad (7)$$

where ν_* is the fix point of the negation, i.e. $\eta(\nu_*) = \nu_*$ and when we fix a certain ν_0 threshold then ν takes this value, i.e. $\eta(\nu) = \nu_0$.

A characterization of this operator class can be found in [13].

We can introduce the aggregative operator (uninorm) consistent with the conjunctive and disjunctive operators and

negations [14]. Here, we use the multiplicative form of a solution of the associative functional equation.

$$a(\mathbf{x}) = f^{-1}\left(\prod_{i=1}^{n} f(x_i)\right). \tag{8}$$

where $f(x)$ is the generator function of either conjunctive or disjunctive operator. In the Pliant system the aggregative operator is unique [14]. The pan operators of the conjunctive and disjunctive operators have the same form as the Pliant system.

III. MODIFIERS AND CONNECTIVES

A. Modifiers based on connectives

We will start with the definition of the substantiating, and the weakening modifier. These modifiers are compositional modifiers.

Definition 3: The substantiating operator of grade m induced by the conjunctive operator is

$$\tau_S^{(m)}(x) = c(\underbrace{x, \ldots, x}_{m}), \tag{9}$$

and the corresponding dual modifier i.e. the weakening modifier is

$$\tau_W^{(m)}(x) = d(\underbrace{x, \ldots, x}_{m}). \tag{10}$$

Using the representation theorem we get:

$$\tau_S^{(m)}(x) = f_c^{-1}\left(m f_c(x)\right), \tag{11}$$
$$\tau_W^{(m)}(x) = f_d^{-1}\left(m f_d(x)\right). \tag{12}$$

We generalize the operator that m is a positive real valued number.

Instead of m we will characterize the modifiers by ν_0 and ν values. First choose ν_0 and we define the unary operators in terms of ν, i.e. ν_0 is the fixed threshold.

$$\nu_0 = \tau(\nu) \tag{13}$$

The unary operators can be characterized by ν_S and ν_W. Now $\tau_S(x)$ and $\tau_W(x)$ can be expressed in terms of ν_0, ν_S and ν_W.

So we have:

$$\tau_{\nu_S, \nu_0}(x) = \tau_S(x) = f_c^{-1}\left(f_c(\nu_0)\frac{f_c(x)}{f_c(\nu_S)}\right), \tag{14}$$

$$\tau_{\nu_W, \nu_0}(x) = \tau_W(x) = f_d^{-1}\left(f_d(\nu_0)\frac{f_d(x)}{f_d(\nu_W)}\right). \tag{15}$$

Special cases

Product case: As a special case of the strict monotonous the operator if $c(x, y) = xy$, we get

$$\tau_S^{(m)}(x) = x^m \tag{16}$$
$$\tau_W^{(m)}(x) = 1 - (1 - x)^m \tag{17}$$

Using the ν_0 and ν values

$$\tau_{\nu_S}(x) = x^{\frac{\ln(\nu_0)}{\ln(\nu_S)}} \tag{18}$$

and

$$\tau_{\nu_W}(x) = 1 - (1 - x)^{\frac{\ln(1-\nu_0)}{\ln(1-\nu_W)}} \tag{19}$$

Eq.(16) is the Zadeh case, but $\tau_W^{(m)}(x)$ differs from the previously defined weakening operator.
We call this modifier system the product modifier system.

Dombi operator case:

$$f_c(x) = \left(\frac{1-x}{x}\right)^{\alpha} \quad \text{and} \quad f_d(x) = \left(\frac{x}{1-x}\right)^{\alpha}$$

$$\tau_{\nu_S}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\left(\frac{\nu_S}{1-\nu_S}\frac{1-x}{x}\right)}$$

$$\tau_{\nu_W}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\left(\frac{\nu_W}{1-\nu_W}\frac{1-x}{x}\right)}$$

Theorem 4: In the Pliant operator case (14) and (15) have the same form and we can drop the c and d indices.

B. On the equivalence of conjunctive and disjunctive modifiers
The mathematical model

We will introduce the modifiers induced by connectives:

$$\tau_c(x) = f_c^{-1}(k_c f_c(x))$$
$$\tau_d(x) = f_d^{-1}(k_d f_d(x))$$

where $k_c, k_d \epsilon (0, \infty)$.

It is interesting to ask under what in conditions the following is valid

$$\tau_c(x) = \tau_d(x) \tag{20}$$

for a certain choice k_c and k_d.
Theorem 5:

$$\tau_c(x) = f_c^{-1}\left(k_c f_c(x)\right) = f_d^{-1}\left(k_d f_d(x)\right) = \tau_d(x)$$

if and only if

$$f_d(x) = f_c^l(x), \qquad k_d = k_c^l \quad \text{and} \quad l \in R^- \quad l \neq 0 \tag{21}$$

Remark 6: If $l = -1$ we get the pliant system.
Proof: From Eq.(20), we get

$$f_c^{-1}\left(k_c f_c(x)\right) = f_d^{-1}\left(k_d f_d(x)\right)$$

and from this

$$F(k_c x) = k_d F(x) \qquad (22)$$

where

$$F(x) = f_d\left(f_c^{-1}(x)\right)$$

is a strictly continuously decreasing function.

Let $k_c = y$ and we wish to find a k_d that depends on k_c. We have to solve the functional equation.

$$F(xy) = k(y)F(x) \qquad (23)$$

Let $x = 1$, then

$$F(y) = k(y)F(1)$$

where $F(1) \neq 0$, otherwise $F(y) = 0$ and Eq.(20) has no solution, so

$$k(y) = AF(y)$$

where

$$A = \frac{1}{F(1)}$$

Substituting this into Eq.(20), we have

$$F(xy) = AF(y)F(x)$$

We have to solve

$$F(xy) = AF(x)F(y) \qquad (24)$$

Utilising Eq.(24)

$$F(x(yz)) = AF(x)F(yz) = A^2 F(x)F(y)F(z) \qquad (25)$$

The generalization of this equation

$$F\left(\prod_{i=1}^{n} x_i\right) = A^{n-1} \prod_{i=1}^{n} F(x_i) \qquad (26)$$

Let $G(x) = AF(x)$ then using Eq.(26) we get

$$G\left(\prod_{i=1}^{n} x_i\right) = AF\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n}(AF(x_i)) = \prod_{i=1}^{n} G(x_i) \qquad (27)$$

The solution of Eq.(27) is

$$G(x) = x^l$$

From this

$$F(x) = Ax^l$$

From the definition of $F(x)$

$$f_d\left(f_c^{-1}(x)\right) = Ax^l$$

and so $f_d(x) = Af_c^l(x)$.

Because the generator function is determined up to a constant multiplicative factor we get

$$f_d(x) = f_c^l(x)$$

l should be negative otherwise the conjunctive operator would become a disjunctive like operator which is contradiction.

Now substituting this result into the original problem

$$f_d^{-1}\left(k_d f_d(x)\right) = f_c^{-1}\left((k_d f_c^l(x))^{1/l}\right) = f_c^{-1}\left(k_c f_c(x)\right)$$

we get

$$k_d = k_c^l, \quad l \in R^-, l \neq 0$$

This tell us that equality of $\tau_c(x)$ and $\tau_d(x)$ holds when Eq.(21) is valid. ∎

Remark 7: If $l = -1$, we get the Pliant operators.

IV. The sharpness operator

As we saw previously, modifiers can be introduced by repeating the arguments of conjunctive and disjunctive operators n-times. The next step is that n is extended to any real number.

We will introduce the sharpness operator by repeating the arguments of the aggregation operator. Because in the Pliant system we have [14]

$$a(x_1, x_2, \ldots, x_n) = f_a^{-1}\left(\prod_{i=1}^{n} f_a(x_i)\right) \qquad \text{and}$$

$$a(\underbrace{x, x, \ldots, x}_{n}) = f_a^{-1}\left(f_a^n(x)\right),$$

we can introduce the following definition.

Definition 8: The sharpness operator is

$$\chi^{(\lambda)}(x) = f^{-1}\left(f^\lambda(x)\right) \qquad \lambda \in R \qquad (28)$$

Theorem 9: Properties of the sharpness operator when $f(\nu_0) = 1$

1. $\chi^{(\lambda)}(0) \quad = 0$

2. $\chi^{(\lambda)}(1) \quad = 1$

3. If $\lambda = 1$ then $\chi(x) = x$

4. $\left(\chi^{(\lambda)}(\nu_0)\right)' = \lambda$

V. General form of modifiers: the kappa function

A. Modifiers in the Pliant operator case

Three types of modifiers were introduced earlier. These are , in the Pliant concept case:

1) The negation operator:

$$\eta_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(\nu)}{f(x)}\right) \qquad (29)$$

2) The hedge operator (necessity and possibility operator):

$$\tau_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right) \qquad (30)$$

3) The sharpness operator:

$$\chi^{(\lambda)}(x) = f^{-1}\left(f^\lambda(x)\right) \qquad (31)$$

These three types of operators can be represented in a common form.

Definition 10: The general form of the modifier operators is

$$\kappa_{\nu,\nu_0}^{(\lambda)}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^\lambda\right) \qquad (32)$$

Theorem 11: The negation (29), the hedge (30) and the sharpness (31) are special cases of the modifier operators 32.

Proof:

$$
\begin{array}{rcll}
\lambda & = & -1 & \text{is the negation} \\
\lambda & = & 1 & \text{is the hedge} \\
f(\nu_0) = f(\nu) & = & 1 & \text{is the sharpness operator}
\end{array}
$$
∎

Remark 12: The sharpness operator in the fuzzy concept is called the dispersion value transformation.

VI. KAPPA FUNCTION BASED ON DIFFERENTIAL EQUATION

We saw previously that $\kappa(x)$ is closely related to the generator function namely it is an isomorphic mapping of the abstract space of object to the real line. If we have different types of mapping, then we will have different operators. If we change the isomorphic function f we get conjunctive, disjunctive or aggregation operators. See Figure 1.
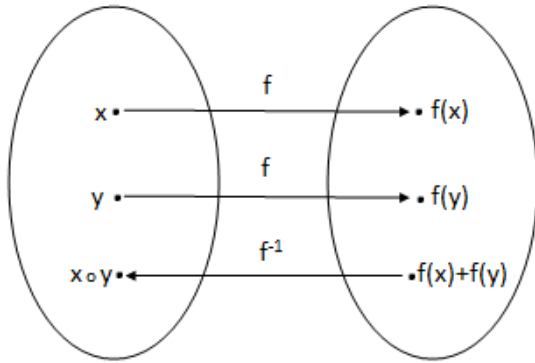

Fig. 1. Interpretation of $x \circ y = f^{-1}(f(x) + f(y))$

We will characterize $\kappa(x)$ by this f function.

Let us introduce the relative effectiveness of $f(x)$ by defining the following

$$r(x) = \frac{f'(x)}{f(x)} \qquad (33)$$

Definition 13: We say that the effectiveness of $\kappa(x)$ is adequate if

$$r(\kappa(x)) = \lambda r(x) \qquad (34)$$

and the boundary condition holds

$$\nu_0 = \kappa(\nu) \qquad (35)$$

Theorem 14: Let $\kappa(x)$ be an arbitrary, continuous and differentiable modifier on $[0,1]$ with the property

$$\kappa(\nu) = \nu_0. \qquad (36)$$

$\kappa(x)$ is adequate iff

$$\kappa(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^\lambda\right) \qquad (37)$$

where $\lambda \neq 0$. This is the general form of the modifier operator.

Proof: Let f be a strictly monotonous transformation of $\kappa(x)$.

$$r(\kappa(x)) = \frac{f(\kappa(x))'}{f(\kappa(x))} = \lambda\frac{f'(x)}{f(x)} = r(x) \qquad (38)$$

or writing this equation, we can have

$$\frac{f(\kappa(x))'}{f'(x)} = \lambda\frac{f(\kappa(x))}{f(x)},$$

i.e. the proportion of the speed of the transformed value $\kappa(x)$ and the speed of x are the same as the proportion of the transformed value and the value x multiplied by a constant λ.

Recall that

$$(ln(f(x)))' = \frac{f'(x)}{f(x)},$$

so Eq.(38) can be written in the following form:

$$(\ln(f(\kappa(x))))' = (\lambda\ln(f(x)))'$$

By integrating both sides, we get:

$$\ln(f(\kappa(x))) = \lambda\ln(f(x)) + C$$

and

$$f(\kappa(x)) = Cf^\lambda(x)$$

Expressing this in terms of $\kappa(x)$ we get:

$$\kappa(x) = f^{-1}\left(Cf^\lambda(x)\right)$$

Using the boundary condition $\kappa(\nu) = \nu_0$

$$f(\nu_0) = Cf^\lambda(\nu) \quad \Rightarrow \quad C = \frac{f(\nu_0)}{f^\lambda(\nu)},$$

which is the desired result. ∎

Corollary 15: From (38), we find that

$$\frac{f'(\kappa(x))\kappa'(x)}{f(\kappa(x))} = \lambda\frac{f'(x)}{f(x)}$$

and so

$$\kappa'(x) = \lambda\frac{f(\kappa(x))}{f'(\kappa(x))}\frac{f'(x)}{f(x)}$$

Special cases:

1) When $\lambda = 1$, then $r(\kappa(x)) = r(x)$ and the modifier is a hedges modifier
$$\kappa(x) = \tau(x)$$

2) When $\lambda = -1$, then $r(\kappa(x)) = -r(x)$ and the modifier is a negation modifier
$$\kappa(x) = \eta(x)$$

VII. SIGMOID AND KAPPA FUNCTION

The sigmoid function plays a very important role in economie, biology, chemistry, etc. A key feature of the sigmoid function is that the solution of the differential equation $\sigma'(x) = \lambda \sigma(x)(1 - \sigma(x))$ is the sigmoid function.

Let us replace $\sigma(x)$ function by $\frac{\kappa(x)}{x}$ and $1 - \sigma(x)$ by $\frac{1-\kappa(x)}{1-x}$, then with this substitution, we can characterize the kappa function. Here, $\sigma(x)$ is the value and $\frac{\kappa(x)}{x}$ is the relative value and similarly $1 - \sigma(x)$ is the value and $\frac{1-\kappa(x)}{1-x}$ is the relative value.

Theorem 16: Let $\kappa(x)$ be an arbitrary continuous and differentable function on $[0, 1]$ with the properties
$$\kappa(\nu) = \nu_0 \qquad \text{and} \qquad \kappa'(x) = \lambda \frac{\kappa(x)(1 - \kappa(x))}{x(1 - x)}.$$

If the above conditions hold, then the $\kappa(x)$ function is
$$\kappa(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0} \left(\frac{1-\nu}{\nu} \frac{x}{1-x} \right)^\lambda} \tag{39}$$

Proof: Make use of (39), we can write
$$y' = \kappa'(x) = \lambda \frac{1}{x(1-x)} \frac{1}{\frac{1}{\kappa(x)(1-\kappa(x))}}$$
$$= \lambda \frac{1}{x(1-x)} \frac{1}{\frac{1}{y(1-y)}}$$
$$= \frac{f_1(x)}{f_2(y)}$$

where
$$f_1(x) = \frac{\lambda}{x(1-x)} \qquad f_2(x) = \frac{1}{y(1-y)}$$

The differential equation
$$y' = \frac{f_1(x)}{f_2(y)}$$

is separable, so the solution is:
$$\int f_2(y)dy = \int f_1(x)dx + c$$

In our case:
$$\int \frac{1}{y(1-y)}dy = \lambda \int \frac{1}{x(1-x)}dx + c$$

so
$$ln(y) - ln(y-1) = \lambda\, ln(x) - ln(x-1) + ln(a)$$

We get:
$$\frac{y}{1-y} = a \left(\frac{x}{1-x} \right)^\lambda,$$

so the unary operator is
$$y = \frac{1}{1 + a \left(\frac{1-x}{x} \right)^\lambda}$$

■

VIII. ODDS AND KAPPA FUNCTION

Let us denote the odds of the input variable by X
$$X = \frac{X}{1 - X} \tag{40}$$

and by Y the odds of the output (transformed) value, i.e.
$$Y = \frac{\kappa(x)}{1 - \kappa(x)}. \tag{41}$$

Let $F(X)$ be the corresponding function between the input and output odds
$$Y = F(X). \tag{42}$$

It is natural to assume that:
$$F(X_1 X_2) = C_1 F(X_1) F(X_2) \tag{43}$$

or
$$F(\frac{X_1}{X_2}) = C_2 \frac{F(X_1)}{F(X_2)}, \tag{44}$$

where $C_i > 0, \quad i = 1, 2$.

Theorem 17: The general solution of (42) and (43) is:
$$F(X_1, X_2) = \frac{1}{C_1} X^\lambda \tag{45}$$

$$F(X_1, X_2) = C_2 X^\lambda \tag{46}$$

Then (45) and (46) have the same form, i.e. $C_1 = \frac{1}{C_2}$.

Proof: See [15]. ■

Theorem 18: If (43) (or (44)) is true, then $\kappa(x)$ has the form
$$\kappa(x) = \frac{1}{1 + C(\frac{1-x}{x})}$$

Proof: Using (40), (41) and (45), we get
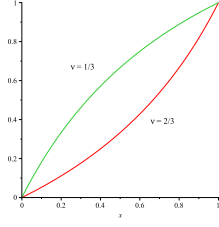$$\frac{\kappa(x)}{1 - \kappa(x)} = \frac{1}{C} \left(\frac{x}{1-x} \right)^\lambda. \tag{47}$$

■

Fig. 2. κ as modifier ($\lambda = 1, \nu = 1/3, 2/3$)

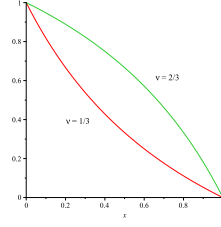Fig. 3. κ as negation ($\lambda = -1, \nu = 1/3, 2/3$)

IX. SOME PROPERTIES OF THE MODIFIER

In the following we will summarize some of the basic properties of the modifier.

1) $\kappa_{\nu,\nu_0}^{(\lambda)}(x)$ is a continuous function.
2) $\kappa_{\nu,\nu_0}^{(\lambda)}(\nu) = \nu_0$
3) $\kappa_{\nu,\nu_0}^{(\lambda)}(x)$ is strictly monotonously increasing if $\lambda > 0$
4) $\kappa_{\nu,\nu_0}^{(\lambda)}(x)$ is strictly monotonously decreasing if $\lambda < 0$
5) If $\lambda > 0$ then

$$\kappa_{\nu,\nu_0}^{(\lambda)}(0) = 0 \qquad \text{and} \qquad \kappa_{\nu,\nu_0}^{(\lambda)}(1) = 1$$

6) If $\lambda < 0$ then

$$\kappa_{\nu,\nu_0}^{(\lambda)}(0) = 1 \qquad \text{and} \qquad \kappa_{\nu,\nu_0}^{(\lambda)}(1) = 0$$

7) If $\lambda > 0$ and
   a) $\kappa_{\nu,\nu_0}^{(\lambda)}(x) > \nu_0$ then $\nu < x$
   b) $\kappa_{\nu,\nu_0}^{(\lambda)}(x) < \nu_0$ then $x < \nu$
8) If $\lambda < 0$ and
   a) $\kappa_{\nu,\nu_0}^{(\lambda)}(x) > \nu_0$ then $x < \nu$
   b) $\kappa_{\nu,\nu_0}^{(\lambda)}(x) < \nu_0$ then $\nu < x$
9)

$$\lim_{\lambda \to \infty} \kappa_{\nu,\nu_0}^{(\lambda)}(x) = \begin{cases} 0 & \text{if } x < \nu \\ \nu_0 & \text{if } x = \nu \\ 1 & \text{if } x > \nu \end{cases}$$

10)

$$\lim_{\lambda \to -\infty} \kappa_{\nu,\nu_0}^{(\lambda)}(x) = \begin{cases} 1 & \text{if } x < \nu \\ \nu_0 & \text{if } x = \nu \\ 0 & \text{if } x > \nu \end{cases}$$

X. KAPPA FUNCTION IN THE DOMBI OPERATOR CASE

Because the generator function is $f(x) = \left(\frac{1-x}{x}\right)^\alpha$

$$\kappa_\nu^{(\lambda)}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0} \left(\frac{\nu}{1-\nu} \frac{1-x}{x}\right)^\lambda}$$

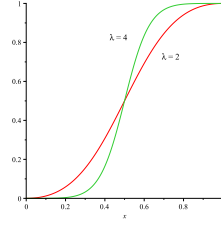In Figure [2 - 5] we plot the different curves of the $\kappa_\nu^{(\lambda)}(x)$ function.

Fig. 4. κ as sharpness ($\nu = 1/2, \lambda = 2, 4$)

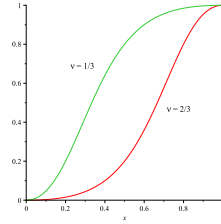Fig. 5. $\kappa_\nu^{(\lambda)}(x)$ function with the parameters: ($\lambda = 2, \nu = 1/3, 2/3$)

XI. EXTENSION OF $\kappa_\nu^\lambda(x)$

A. $K : [a, b] \to [0, 1]$

$\kappa(x)$ is defined on $[0, 1]$. We can extend it into $[a, b]$ interval by a linear transformation, i.e.:

$$x := \frac{x - a}{b - a}$$

and

$$\nu := \frac{x_\nu - a}{b - a}$$

So we get

$$K_{a,b}^{(\lambda)}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0} \left(\frac{x_\nu - a}{b - x_\nu} \frac{b - x}{x - a}\right)^\lambda}$$

$K_{a,b}^{(\lambda)}(x)$ has the following properties:

$$K_{a,b}^{(\lambda)}(x_\nu) = \nu_0$$
$$K_{a,b}^{(\lambda)}(a) = 0 \quad \text{and} \quad K_{a,b}(b) = 1$$

$K_{a,b}$ has the same form as we can find in [16] where we show that this is the general form of a certain class of membership function.

1) $K : [a, b] \to [A, B]$: This $K(x)$ function can be written in the following implicit form:

$$\frac{K(x) - K(a)}{K(b) - K(x)} \frac{K(b) - K(x_\nu)}{K(x_\nu) - K(a)} = \left(\frac{x - a}{b - x}\right)^\lambda \left(\frac{b - x_\nu}{x_\nu - a}\right)^\lambda \tag{48}$$

Let us denote:

$$X := \left(\frac{x - a}{b - x}\right)^\lambda \left(\frac{b - x_\nu}{x_\nu - a}\right)^\lambda \frac{K(x_\nu) - K(a)}{K(b) - K(x_\nu)} \tag{49}$$

Then we get the explicit form of $K(x)$:

$$K(x) = \frac{X K(b) + K(a)}{1 + X} \tag{50}$$

XII. Conclusion

In this article we presented the general form of the negation, weakening, strenghtening and sharpness operators. We extended the Pliant operator system with a kappa function that describes various unary operators.

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