The Empirical Variance of a Set of Fuzzy Intervals

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***Abstract*— The profile method gives a tool to perform fuzzy interval computation under a condition of local monotony of considered functions. This is a plain extension of interval analysis to fuzzy intervals, viewed as pairs of fuzzy bounds. This method yields exact results without applying interval analysis to α-cuts. After a refresher on the notion of profile and its use in fuzzy interval analysis, we adapt the profile method to the computation of the empirical variance of a tuple of fuzzy intervals. To this end, we first reconsider results obtained by Ferson et al. on computation of the empirical variance of a set of intervals. Finally we apply our results to the definition of the variance of a single fuzzy interval,viewed as a family of its α-cuts , and compare this definition to previous ones.**

I. Introduction

In interval computation, the basic problem is the following: given a function $f(x_1, \cdots, x_n)$ and n intervals $[x_i^-, x_i^+]$, find the interval range of the variable $y = f(x)$ such that $x \in \times_i [x_i^-, x_i^+]$ [1]. The goal of interval computation is to find the minimum and the maximum of the function when the different possible values of the variables x_i range in their intervals $[x_i^-, x_i^+]$. Some methods are based on finding a finite set of points (called configurations or poles) on which this minimum and maximum is attained [2]. This is the idea of the vertex method [3].

Modeling possible values of variables by means of real intervals accounts for some uncertainty interpreted as a lack of specificity, but we can be more refined by modeling uncertainty on a variable x_i by means of a fuzzy interval X_i [4]. The traditional method for computing the possible fuzzy range Y of y is to decompose the problem in terms of α-cuts and then to apply a standard interval analysis method to each tuple of cuts of level α for a selected finite subset of levels. This process has drawbacks: it computes only an approximation of Y, and for each α-cut, the interval algorithm has to be completely executed. In general case, the basic problem of interval computation is NP-Hard [5] the complexity of the approach using decomposition by α-cut cannot be lower.

We have generalized the vertex method to the fuzzy case without resorting to α-cuts [6], viewing fuzzy intervals, as pairs of fuzzy bounds. When the function to be computed possesses n arguments, we avoid exploring at most $k \cdot 2^n$ configurations, where k is the number of α-cuts. We explore at most 2^n *fuzzy* configurations, a fuzzy configuration being a n-tuple of fuzzy bounds called profiles. This method gives exact results expressible in an analytical way up to the enumeration of fuzzy configurations for analytically expressible functions having a local monotonicity property, general enough to encompass well-known examples like the fuzzily-weighted average.

In this paper, after recalling the profile method, we extend this approach to the computation of the empirical variance of a set of fuzzy intervals. This application deserves a special study as the variance function does not enjoy the proper local monotonicity property needed for a straightforward application of the profile method. The last part of the paper applies the previous results to the definition of the potential variance of a single fuzzy interval, viewed as its set of α-cuts.

II. Background

A. A Refresher On Classical Interval Computation

Consider n intervals $[x_i^-, x_i^+]$, we call *real configuration* an element of the set $\mathcal{X} = \times_i [x_i^-, x_i^+]$. Among configurations of $\mathcal{X}$, let us distinguish the *extreme* ones, ie the set $\mathcal{H} = \times_i \{x_i^-, x_i^+\}$. The notion of configuration has been proposed by Buckley for the fuzzy scheduling problem [7], but, in the literature of fuzzy intervals, extreme configurations are also called poles [2].

Under suitable monotonicity assumptions, the supremum of a real-valued function f over $\mathcal{X}$ is actually equal to the maximum of f on $\mathcal{H}$, or on a subset $\mathcal{C} \subseteq \mathcal{H}$. An element $\omega \in \mathcal{H}$ has the form $\omega = (x_1^{\varepsilon_1}, \cdots, x_n^{\varepsilon_n})$, with $\varepsilon_i \in \{+, -\}$. Two forms of monotonicity are of interest.

Definition 1: f is said (globally) *monotonic with respect to each* x_i if for each variable x_i, the restricted function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$ is either increasing according to x_i for all n-tuples $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$ or decreasing likewise.

Definition 2: f is said *locally monotonic with respect to each* x_i if for each variable x_i, for all n-tuple $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$ the restricted function $f(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$ is monotonic.

In these definitions, f is a function from $\mathbb{R}^n$ to $\mathbb{R}$. In the last definition f can be increasing for one tuple $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ and decreasing for another. Therefore, a locally monotonic function is not monotonic in the usual sense (Definition 1). For a function to be locally monotonic with respect to x_i, it is enough if the sign of its partial derivative $\frac{\partial f}{\partial x_i}$ does not depend on x_i.

We can now state a well-known proposition:

Proposition 1: Let $x = (x_1, x_2, \cdots, x_n)$ be a tuple of n variables such that $x_i \in [x_i^-, x_i^+]$, and $y = f(x_1, \cdots, x_n) = [y^-, y^+]$.

If f is locally monotonic with respect to each argument, then $y^- = min_{\omega\in\mathcal{H}}(f(\omega))$ and $y^+ = max_{\omega\in\mathcal{H}}(f(\omega))$

Proposition 1 enables computations on functions to be performed under a condition of local monotony. This proposition is the basis of the (FWA) Algorithm [3] which computes the weighted averages with fuzzy weights. We recall another result which decreases the number of configurations used for the computation of a function f with stronger monotony conditions:

Proposition 2: Under the assumption of Proposition 1, if f is locally monotonic with respect to each argument, and $\forall j \in E_1$, f is increasing according to x_j and $\forall j \in E_2$, f is decreasing according to x_j, then

$$y^- = min_{\omega\in\mathcal{H}}\left(f(\omega)| \begin{array}{l}\forall j\in E_1, \omega_j = x_j^- \\ \forall j \in E_2, \omega_j = x_j^+\end{array}\right)$$

and $$y^+ = max_{\omega\in\mathcal{H}}\left(f(\omega)| \begin{array}{l}\forall j\in E_1, \omega_j = x_j^+ \\ \forall j \in E_2, \omega_j = x_j^-\end{array}\right)$$

Now, we recall the notion of fuzzy configuration and its application to fuzzy interval analysis.

B. Fuzzy Interval Analysis Using Profiles

A closed interval is isomorphic to an ordered pair of real values. It is thus natural to consider a fuzzy interval as a pair of fuzzy bounds. This is a *vertical* view of a fuzzy interval, as opposed to the horizontal view as a set of nested intervals. However a fuzzy bound is *not a fuzzy set*. We need to define a special fuzzy object (called profile) to handle the fuzzy bounds of a fuzzy interval in the way precise bounds of interval behave, and also operations between such objects. Intuitively, they are genuine extensions of numbers that convey some fuzziness (to quote Zadeh, they are gradual instead of abrupt), but no imprecision. The following definition was proposed by the authors [6]:

Definition 3: A *profile* is an application Φ from $(0,1]$ to $\mathbb{R}$.

A real number r is a special type of profile Φ, such that $\forall\lambda \in (0,1]$ $\Phi(\lambda) = r$. Note that a profile is not requested to be monotonic and Φ may not be defined everywhere in $(0,1]$. We can now precisely define the increasing and the decreasing part of a fuzzy interval I. In the following, $[s^-, s^+]$ will represent the closure of the support of a fuzzy interval M. This definition is given for fuzzy intervals which have an USC (Upper Semi Continuous) membership function, because this type of fuzzy sets is the most employed.

Definition 4: Let M be an USC fuzzy interval. We call *left profile of M* (denoted M^-) the profile defined as follows:

$$\begin{array}{llll} I^-: & (0,1] & \longrightarrow & \mathbb{R} \\ & \lambda & \longmapsto & M^-(\lambda) = inf\{x|\mu_I(x) \geq \lambda,\ x \geq s^-\}\end{array}$$

We call *right profile of M* (denoted M^+) the profile defined as following:

$$\begin{array}{llll} I^+: & (0,1] & \longrightarrow & \mathbb{R} \\ & \lambda & \longmapsto & M^+(\lambda) = sup\{x|\mu_M(x) \geq \lambda, x \leq s^+\}\end{array}$$

When M has a continuous membership function with core $[c^-, c^+]$, M^- is the converse of the membership function μ_M on $(s^-, c^-]$, and M^+ is the converse of the membership function μ_M on $[c^+, s^+)$. An USC fuzzy interval can be entirely defined by its left profile and its right profile [6]. The definition of a fuzzy interval as a pair of profiles is akin to the so-called graded numbers of Herencia [8]. It is clear that profiles induced by fuzzy intervals are monotonic: increasing (the left profile of M is its fuzzy lower bound) or decreasing (the right profile of M is its fuzzy fuzzy upper bound).

This type of monotonic profiles was already proposed as natural fuzzy generalizations of real numbers by mathematicians in fuzzy topology (Rodabaugh [9], Höhle [10], Lowen [11]). Moreover a decreasing profile on the positive integers corresponds to the fuzzy-valued cardinality of a fuzzy set, as pointed out by Rocacher [12]. However arithmetic operations on profiles do not preserve monotonicity (as shown by the fuzzy relative integers and fuzzy rational numbers of Rocacher). In particular, computations with fuzzy intervals may lead to non-monotonic profiles as intermediary results. That is why profiles are defined as any functions from $(0,1]$ to $\mathbb{R}$, which associate for each possibility level $\lambda \in (0,1]$ a single abscissa $\Phi(\lambda)$.

Now, a fuzzy extreme configuration induced by fuzzy intervals $X_1, \cdots, X_n$ restricting a tuple of n independent variables $x = (x_1, x_2, \cdots, x_n)$ is a n-tuple of left or right profiles $\Omega = (X_1^{\varepsilon_1}, X_2^{\varepsilon_2}, \cdots, X_n^{\varepsilon_n})$, where $\varepsilon_i \in \{+,-\}$. A fuzzy configuration is a kind of fuzzy vertex of a Cartesian product of fuzzy intervals.

We denote $\widetilde{\mathcal{H}}$ the set of all fuzzy extreme configurations: $\widetilde{\mathcal{H}} = \times_i\{X_i^-, X_i^+\}$ $(|\widetilde{\mathcal{H}}| = 2^n)$

We denote Ω_i the i^{th} profile of configuration Ω. For any $\Omega \in \widetilde{\mathcal{H}}$, let $\Omega(\lambda)$ denote the classical configuration obtained at level λ. $\Omega(\lambda) = (\Omega_1(\lambda), \Omega_2(\lambda), \cdots, \Omega_n(\lambda)) \in \mathbb{R}^n$ is a vertex of the hyper-rectangle $\times_i[X_i]_\lambda$.

Real-valued functions can be extended to profile-valued arguments using the composition of functions:

Definition 5: Let f be a function of arity n. Let us denote $\dot{f}$ the extension of f applicable to profiles: for any n-tuple of profiles $\Omega = (\Omega_1, \Omega_2, \cdots, \Omega_n)$, $\dot{f}(\Omega)$ is the profile defined as follows: $\forall\lambda \in (0,1]$

$$\begin{array}{lll}\dot{f}(\Omega)(\lambda) & = & f(\Omega(\lambda)) \\ & = & f(\Omega_1(\lambda), \Omega_2(\lambda), \cdots, \Omega_n(\lambda))\end{array}$$

Now, let us define a set $\xi \subseteq \{-,+\}^n$ such that for all fuzzy intervals $X_1, \cdots, X_n$, ξ defines a set of fuzzy configurations: $\widetilde{\mathcal{H}_\xi} = \{(X_1^{\varepsilon_1}, \cdots, X_n^{\varepsilon_n})|(\varepsilon_1, \cdots, \varepsilon_n) \in \xi\}$. Let Y is the fuzzy set of the possible values of the variable $y = f(x)$.

If there is a set $\xi \subseteq \{(\varepsilon_1, \cdots, \varepsilon_n), \varepsilon_i \in \{-,+\}\}$, such that for all α-cuts f attains its maximum and minimum on $\mathcal{X}_\alpha = \times_i[X_i]_\alpha$ for a configuration in $\mathcal{H}_{\mathcal{X}_\alpha,\xi}$,

then $Y^+ = \dot{max}_{\Omega\in\widetilde{\mathcal{H}_\xi}}\{\dot{f}(\Omega)\}$

and $Y^- = \dot{min}_{\Omega\in\widetilde{\mathcal{H}_\xi}}\{\dot{f}(\Omega)\}$.

As in the interval case, if f is locally monotonic with respect to each argument,

then $Y^- = \dot{min}_{\Omega\in\widetilde{\mathcal{H}}}(\dot{f}(\Omega))$

and $Y^+ = \dot{max}_{\Omega\in\widetilde{\mathcal{H}}}(\dot{f}(\Omega))$

The fuzzy extension of the interval-based result for locally monotonic function then holds:

Corollary 1: If f is locally monotonic with respect to each argument, and $\forall j \in E_1$, f is increasing according to x_j and $\forall j \in E_2$, f is decreasing according to x_j, then

$$Y^- = \dot{min}_{\Omega \in \widetilde{\mathcal{H}}} \left(\dot{f}(\Omega) | \begin{array}{l} \forall j \in E_1, \Omega_j = X_j^- \\ \forall j \in E_2, \Omega_j = X_j^+ \end{array} \right)$$

and $Y^+ = \dot{max}_{\Omega \in \widetilde{\mathcal{H}}} \left(\dot{f}(\Omega) | \begin{array}{l} \forall j \in E_1, \Omega_j = X_j^+ \\ \forall j \in E_2, \Omega_j = X_j^- \end{array} \right)$

This last corollary was in fact known for strictly increasing functions [4]. In the remainder of this paper, we will not tell f from its extension applicable to the profiles $\dot{f}$.

C. Example of Application

Let h be the function such as $h(x,y) = x + y - xy$. h is of course locally monotonic (and not globally monotonic). So applying the profile method, we get the following equations:
$h(A,B)^- = min_{\varepsilon_1,\varepsilon_2 \in \{-,+\}}(h(A^{\varepsilon_1}, B^{\varepsilon_2}))$
$h(A,B)^+ = max_{\varepsilon_1,\varepsilon_2 \in \{-,+\}}(h(A^{\varepsilon_1}, B^{\varepsilon_2}))$
Now, consider the two fuzzy intervals A and B, defined by Figure 1. The profiles of A and B are defined as follows:

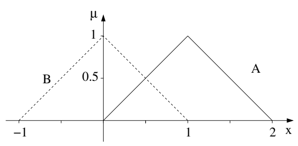

Fig. 1. Possibility distribution of two fuzzy intervals A and B

$A^-(\lambda) = \lambda$, $A^+(\lambda) = 2 - \lambda$,
$B^-(\lambda) = \lambda - 1$, $B^+(\lambda) = 1 - \lambda$
Then we get: $h(A^-, B^-)(\lambda) = -\lambda^2 + 3\lambda - 1$
$h(A^-, B^+)(\lambda) = \lambda^2 - \lambda + 1$
$h(A^+, B^-)(\lambda) = \lambda^2 - 3\lambda + 3$
$h(A^+, B^+)(\lambda) = -\lambda^2 + \lambda + 1$

The computed profile and the result $C = h(A,B)$ are shown on figure 2. The above calculations are in the style of graded numbers [8] but some profiles obtained as partial results are not monotonic.

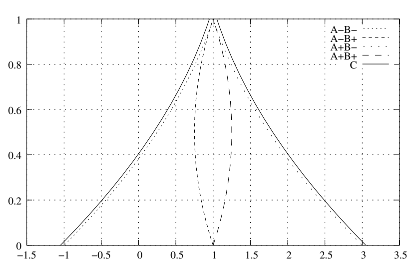

Fig. 2. $C = h(A,B)$ and the computed profiles

Non-monotonic profiles can thus appear in the intermediate computations, but hopefully, the final result is always a classical fuzzy interval. The above example is meant for illustration of non-monotonic profiles.

III. Variance of a Tuple of Fuzzy Intervals

In the scope of fuzzy random variables, an interesting question is to define the counterpart of a variance. The mean value of a set of fuzzy intervals is already known for a long time, but the notion of variance has received less attention. Yet, several definitions already exist by Körner [13], Feng et al [14], Carlsson and Fuller [15]. They are point-valued variances. Here we consider a fuzzy-valued variance, understood as the application of the extension principle to the notion of empirical variance, based on a sampling of fuzzy numbers.

A. Crisp Interval Case

In this section we are going to see how to calculate exact bounds on sample variance of interval data. The algorithms presented in this section come from [16]. We propose a new presentation of the algorithms, which is more tutorial.

1) Problem Definition: Consider n measurement results $x_1, \ldots, x_n$. Their sample average $e(x_1, \cdots, x_n)$ and their empirical variance $v(x_1, \cdots, x_n)$ are defined by:

$$e(x_1, \cdots, x_n) = \frac{x_1 + \cdots + x_n}{n} \quad (1)$$

$$V(x_1, \cdots, x_n) = \frac{(x_1 - e(x))^2 + \cdots + (x_n - e(x))^2}{n-1} \quad (2)$$

In practice, the measurement of a variable x_i is not perfect and is better described as an interval X_i, such that $x_i \in X_i = [x_i^-, x_i^+]$. X_i is the interval of the possible values of the measured variable x_i.

In the remainder of this section, x denotes the tuple $x = (x_1, \cdots, x_n)$ and $X = (X_1, \cdots, X_n)$.

The computation of the sample average is easy since Equation 1 is still valid, with elementary operations changed by interval arithmetic operations[1]:

$$E(X_1, \cdots, X_n) = \frac{X_1 \oplus \cdots \oplus X_n}{n} \quad (3)$$

The potential sample variance is an interval, given by:

$$V(X) = \{V(x) | \forall i,\ x \in \times_i X_i\} \quad (4)$$

Computing $V(X)$ is difficult because the simple use of the Equation 2 with interval arithmetics operations yields too wide an interval for the variance. This comes from the fact that in Equation 2, x_i and E are linked by Equation 1. Moreover, the variance is not a locally monotonic function.

In the rest of this section, we present a polynomial algorithm already outlined by Ferson et al [16] to compute the GLB of the variance, and an exponential algorithm for the computation of the LUB of the variance.

2) GLB of the Interval Variance: Suppose that $V(.)$ reaches its lower bound on $\times_i X_i$ for a n-tuple $x = (x_1, \cdots, x_n)$ of known average e. Then we can find the exact value of x for which this lower bound is attained:

- if $e \in X_i$ then $x_i = e$
- if $e > X_i$ then $x_i = x_i^+$
- if $e < X_i$ then $x_i = x_i^-$

And in this case, we can give the expression of the minimum of the variance:

[1] The operators defined on interval are such that $[a,b] \oplus [c,d] = [a+c, b+d]$, and $\frac{[a,b]}{n} = [\frac{a}{n}, \frac{b}{n}]$

$min(V(X)) =$

$$min_{e \in E(X)} \left(V(x) | x_i = \begin{cases} e & \text{if } e \in X_i \\ x_i^+ & \text{if } e > X_i \\ x_i^- & \text{if } e < X_i \end{cases} \right)$$

Now, suppose that we know an interval $E_j \in E$ such that E_j is contained or is disjoint from each interval X_i. So, $\forall e', e'' \in E_j$, $\{i | e' \in X_i\} = \{i | e'' \in X_i\}$. If this interval is included in all X_i, then it is easy to see that the minimum of the variance is zero since we can choose $x_1 = x_2 = \cdots = x_n = e \in E_j$. If E_j is not included in all X_i, and if $V(X)$ reaches its lower bound on a n-tuple of average $e \in E_j$, then this average must be:

$$e(E_j) = \frac{\sum_{X_i \geq E_j} x_i^- + \sum_{X_i \leq E_j} x_i^+}{\| \{X_i | X_i \geq E_j \ or \ X_i \leq E_j\} \|}.$$

Where $A \leq B$ means $sup(A) \leq inf(B)$. We use this order relation because $E_j \subseteq X_i$ or $E_j \cap X_i = \emptyset$. Now we can state the following proposition:

Proposition 3: Let $\mathcal{E} = \{E_1, E_2, \cdots, E_k\}$ be a partition of $E(X)$ such that $\forall i = 1..n$, $E_j \subseteq X_i$ or $E_j \cap X_i = \emptyset$. Let $c_j = \| \{X_i | X_i \geq E_j \ or \ X_i \leq E_j\} \|$.
Then $min(V(X)) = min_{j=1..k}(V^{E_j})$ such that

- if $c_j = 0$, $V^{E_j} = 0$
- otherwise let $e(E_j) = \frac{\sum_{X_i \geq e} x_i^- + \sum_{X_i \leq e} x_i^+}{c_j}$.

and if $e(E_j) \notin E_j$, $V^{E_j} = +\infty$

otherwise $V^{E_j} = \frac{\sum_{X_i \geq E_j} (x_i^- - e(E_j))^2 + \sum_{X_i \leq E_j} (x_i^+ - e(E_j))^2}{n-1}$

To obtain the partition $\mathcal{E}$ of $E(X)$ we just have to compute $E(X) = [e_X^-, e_X^+]$ and to sort all $2n+2$ values x_i^+, x_i^-, e_X^-, e_X^+ into a sequence $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(2n+2)}$. $\mathcal{E}$ is constituted by the intervals $[x_{(k)}, x_{(k+1)}] \subseteq E_X$.

The resulting algorithm computes $min(V(X))$ in complexity of $O(n^2)$: The calculation of $E(X)$ can be made in linear time, sorting the $2n+2$ values can be done in $O(n * log(n))$, and the computation part is $O(n)$ for each interval $[x_{(k)}, x_{(k+1)}]$.

3) LUB of the Interval Variance: Computing the LUB of the interval variance is NP-hard [16].

Proposition 4: The LUB of the variance is attained on a vertex of the hyper-rectangle $\times X_i$.

Proof: This comes from that $E(.)$ is a quadratic function with positive coefficients for the terms in x_i^2. It is well-known that such functions attain their maximum on intervals on the boundaries of their definition domains. □

A brute-force algorithm to get the LUB of E is then to compute the variance of the 2^n vertex of $\times X_i$, keeping the greatest computed value. This algorithm is of complexity $0(n \cdot 2^n)$.

We can find a better algorithm using Proposition 5. It presupposes that for some values of the average, we can find the best values of particular x_i:

Proposition 5: Let $\mathcal{E} = \{E_1, E_2, \cdots, E_k\}$ be a partition of $E(X)$ such that $\forall i = 1..n$, $E_j \subseteq X_i$ or $E_j \cap X_i = \emptyset$. Then $max(V_X) =$

$$max_{E_i \in \mathcal{E}} \left\{ V(x) | x_i = \begin{cases} x_i^- & \text{if } X_i \leq E_j \\ x_i^+ & \text{if } X_i \geq E_j \\ x_i^- \text{ or } x_i^+ & \text{otherwise} \end{cases} \right)$$

So owing to this last proposition, we can write an algorithm, of time complexity $O(n \cdot 2^l)$ which computes $max(V(X))$ if l if the maximal number of overlapping intervals X_i. In particular, the maximal complexity is attained when the X_i are nested ($l = n$).

4) Example of Variance Computation: Let $X_1 = [0,2]$, $X_2 = [2,5]$ and $X_3 = [4,8]$ be three intervals representing the possibles values of three variables x_1, x_2 and x_3. The interval average is $E(X_1, X_2, X_3) = [2,5]$. Now, we can partition $E(X)$ in two intervals E_1 and E_2, for which $E(X) = E_1 \cup E_2$, and for $i \in \{1,2,3\}$, $j \in \{1,2\}$, $E_j \subseteq X_i$ or $E_j \leq X_i$ or $E_j \geq X_i$. Figure 3 gives a graphical representation of the intervals X_1, X_2, X_3, $E(X)$, E_1 and E_2.

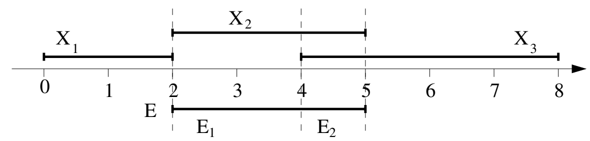

Fig. 3. Example of average and variance computation

Now, for the sample variance, if $min(V(X))$ is attained for a tuple of average $e_1 \in E_1$ then the optimal configuration is $\Omega_1 = (2, e_1, 4)$, with $e_1 = E(2,4) = 3$. $e_1 \in E_1$ so we can compute the variance of the tuple Ω_1 by:

$$\begin{array}{rcl} V(\Omega_1) & = & \frac{1}{2}\left((2-3)^2 + (4-3)^2\right) \\ & = & 1 \end{array}$$

On the other hand, if we suppose that the minimum of the variance is attained for a tuple of average $e_2 \in E_2$ then this tuple must be $\Omega = (2, e_2, e_2)$, with $e_2 = E(2) = 2$. $e_2 \notin E_2$ so the minimum of the variance can not be attained for a tuple of average in E_2, and so we know that $min(V(X)) = 1$.

For the computation of the maximum of $V(X)$, suppose that this maximum is attained in E_1. If so, then the possibles tuple are $\Omega_3 = (0,2,8)$ and $\Omega_4 = (0,5,8)$. If the maximum is attained in E_2, then the tuples can be Ω_3, Ω_4, $\Omega_5 = (0,2,4)$ and $\Omega_6 = (0,5,4)$. So we finally got:

$$\begin{array}{rcl} max(V(X)) & = & max(V(\Omega_3), V(\Omega_4) V(\Omega_5) V(\Omega_6)) \\ & = & max(17.33, 16.33, 6.33, 7) \\ & = & 17.33 \end{array}$$

B. Fuzzy Case

In this section, $X = (X_1, \cdots, X_n)$ is a n-tuple of fuzzy intervals. First of all, $E(.)$ is increasing according to each argument, so $E(.)$ can be computed by the following formulas:
$E^- = E(X_1^-, \cdots, X_n^-)$ $E^+ = E(X_1^+, \cdots, X_n^+)$
In the calculation of the crisp interval variance, we divided the average interval such as for all E_j in the partition, E_j is contained or is disjoint from each domain X_i. Before dividing the fuzzy average $E(X)$, we need to ensure that the profiles of fuzzy intervals X_i and $E(X)$ can be compared. We say that a function f is smaller than g (from D in $\mathbb{R}$) if $\forall x \in D$, $f(x) \leq g(x)$. So let $\Phi^{(\lambda_1, \lambda_2]}$ be the restriction of a profile Φ on $(\lambda_1, \lambda_2] \subseteq (0,1]$. Now, we must divide the interval $(0,1]$ representing the possibility degrees into k domains $P = \{(\lambda_i^-, \lambda_i^+] | i \in [1,k]\}$ such that for each element $p \in P$ the restriction of two profiles of X_i or $E(X)$ can be compared two by two ($\forall p \in P$, $\forall I_1, I_2 \in \{X_1^-, \cdots, X_n^-, X_1^+, \cdots, X_n^+, E^-, E^+\}$

we need $I_1^p \leq I_2^p$ or $I_2^p \leq I_1^p$).

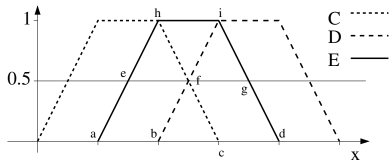

Fig. 4. Two fuzzy intervals C and D and their average $E(C,D)$

In the example of Figure 4, we need to divide the interval $(0,1]$ into two parts $(0,0.5]$ and $(0.5,1]$. The restrictions of the profiles of C, D and $E(C,D)$ do not intersect on these two sub-intervals. Now we can decompose the fuzzy average $E(X)$. In the crisp interval case, the decomposition of an interval gives intervals. In the fuzzy case, a fuzzy interval (represented in practice by an area), is partitioned into sub-areas. Each sub-area A_j must be contained or be disjoint from each fuzzy interval X_i. In the example of Figure 4, six different areas appear: $A_1 = (aefb)$, $A_2 = (bfc)$, $A_3 = (cfgd)$, $A_4 = (ehf)$, $A_5 = (hfi)$ and $A_6 = (fig)$. Let A_j be such an area, we note A_j^- the left profile of A_j, and A_j^+ its right profile (for example, for $A_1 = (aefb)$, A_1^- is the segment $(a:e)$, and A_1^+ is the segment $(b:f)$).

Now, as in the interval case, we can discuss the possible values of the variables X_i when we suppose that the average Φ_E of the tuple which minimize the variance is in an area A_j, representing a part of $E(X)$ on the possibility degrees in $p \in P$. According to the crisp interval case, the only fuzzy configuration which can minimize the variance for average Φ_E in A_j is:

$\Omega^{(\lambda_1,\lambda_2]} = (\Omega_1, \cdots, \Omega_n)^{(\lambda_1,\lambda_2]}$ such that:

$$\Omega_i = \begin{cases} X_i^+ & \text{if } X_i^+ \leq A_j^- \\ X_i^- & \text{if } X_i^- \geq A_j^+ \\ \Phi_E & \text{otherwise} \end{cases}$$

Now, we can compute the possible average Φ_E of $\Omega^{(\lambda_1,\lambda_2]}$:

if $\| \{\Omega_i^{(\lambda_1,\lambda_2]} | \Omega_i^{(\lambda_1,\lambda_2]} \subseteq A_j \| \neq \varnothing$ then

$$\Phi_E = E(\Omega^{(\lambda_1,\lambda_2]}) = \frac{\sum_{\Omega_i^{(\lambda_1,\lambda_2]} \subseteq A_j} \Omega_i^{(\lambda_1,\lambda_2]}}{\| \{\Omega_i^{(\lambda_1,\lambda_2]} | \Omega_i^{(\lambda_1,\lambda_2]} \subseteq A_j \|}$$

otherwise, $\Phi_E = 0$.

If Φ_E does not intersect A_j then the minimum of the variance can not be obtained for some profiles of average in A_j. On the contrary, if Φ_E intersects A_j for some possibility degrees $\lambda \in (\lambda_1, \lambda_2]$, for those degrees (and only those one), we can compute the variance of $\Omega^{(\lambda_1,\lambda_2]}$ by the formula in Proposition 3 applied to restricted profiles. We note $D_{\phi_E \in A_j}$ the set of possibility degrees λ such that $\Phi_E(\lambda) \in A_j$. If $D_{\phi_E \in A_j}$ is the empty set, then the variance can not reach its lower bound on any possibility degree for a n-tuple of average in A_j. Otherwise, we compute the variance of $\Omega^{(\lambda_1,\lambda_2]}$, and we only keep its restriction on $D_{\phi_E \in A_j}$. When we have done this computation for all areas composing $E(X)$, we need to keep the minimum of the partial profiles for all possibility degrees in $(0,1]$. The result is then the left profile of $V(X)$.

Let us illustrate this method on our example: in the remainder of this section, we will not tell the partial profiles from the segments which represent them on Figure 4 (for example $(D^-)^{(0,0.5]} = (b:f)$). Suppose that the variance reaches its minimum for possibility degrees in $(0,0.5]$ on a pair of profiles of average Φ_E included in area $A_1 = (aefb)$. Then this pair must be $\Omega_1 = (\Phi_E, (b:f))$. Its average is the profile $\Phi_E = (b,f) \in A_1$. So we can compute its variance $V(\Omega_1) = 0$. The minimal variance is then equal to zero for all possibility degrees in $(0,0.5]$ and we don't need to consider areas (bfc) and $(cfgd)$. For possibility degrees in $(0.5,1]$, we can first suppose that the variance reaches its minimum for a pair of profiles of average Φ'_E included in area $A_4 = (ehf)$. Then this pair must be $\Omega_4 = (\Phi'_E, (f:i))$. Its average is then $\Phi'_E = (f,i) \notin A_4$. So the variance can not reach its lower bound on a pair of profiles included in A_4. If we suppose that the variance reaches its minimum for a pair of profiles of average Φ'_E included in area $A_5 = (hfi)$, then this pair must be $\Omega_5 = ((h:f),(f:i))$, of average $\Phi'_E = 1,25$ $(\forall\lambda, \Phi'_E(\lambda) = 1,25)$. $\Phi'_E \in A_5$, so we can compute its variance:

$$\begin{aligned} V(\Omega_5)(\lambda) &= (1.5 - 0.5\lambda - 1.25)^2 \\ &\quad + (1 + 0.5\lambda - 1.25)^2 \\ &= 2(0.25 - 0.5\lambda)^2 \end{aligned}$$

For the area A_6, the revelant pair of profiles is $\Omega_6 = ((f:h), \Phi'_E)$ of average $\Phi'_E = (f:h) \notin A_6$, so the variance can not reach its lower bound on a pair of profiles included in A_6. Now we can reconstruct the entire profile V^- on $(0,1]$ which is:

$$V^-(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq 0.5 \\ 2(0.25 - 0.5\lambda)^2 & \text{otherwise} \end{cases}$$

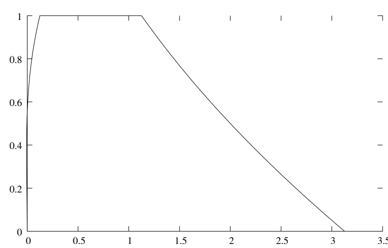

Fig. 5. $V(C,D)$

For the right hand side profile of $V(C,D)$, we can make the same decomposition of $E(C,D)$, and discuss its possible values with proposition 5. But for this example, it is obvious that the maximum of the interval variance will be attained for the fuzzy extreme configuration (C^-, D^+) of average $\Phi = 1,25$:

$$\begin{aligned} V^+(\lambda) &= V(A^-(\lambda), B^+(\lambda)) \\ &= (0.5\lambda - 1.25)^2 + (1.25 - 0.5\lambda)^2 \end{aligned}$$

So we obtain the fuzzy variance of our pair of intervals on Figure 5.

IV. The Potential Variance of a Symmetric Fuzzy Interval

An intuitive way to define the potential variance of a fuzzy interval M is to compute the interval variance of a set of n α-cuts equally distributed in the interval $(0,1]$, where n may increase to infinity. Of course the minimal potential variance of a fuzzy interval is null. This is due to the fact that α-cuts are nested.

The problem is harder for computing the maximal potential variance. The problem to compute the maximal variance of a n-tuple of fuzzy random variables with values in n nested intervals is known to be NP-hard [16]. The proof is still valid when all intervals are symetric with respect to a real number (defining the mid-point of each interval). In practice, this difficulty is not a problem: as we are going to see, we can bracket the result in a tight way when n is big enougth. Without any loss of generality, we can suppose that all intervals are symmetric with respect to 0 (If not we can translate them without any change of the potential variance). So if we suppose

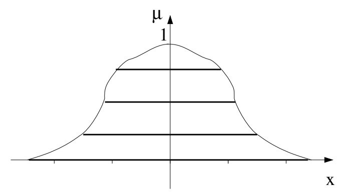

Fig. 6. Decomposition of a interval in α-cuts

that we have n symmetric intervals centred in 0 $Z_i = [-z_i, +z_i]$ ordered as $Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1$, we can approximate the maximal possible variance $\overline{V}_Z$ of n random variables with values in $Z_1, \cdots, Z_n$ due to the following lemma:

Lemma 1: The maximal variance of n random variables with values in $Z_1, \cdots, Z_n$ verifies:
$\frac{\sum_i z_i^2}{n-1} - \frac{z_n^2}{n \cdot (n-1)} \leq \overline{V}_Z \leq \frac{\sum_i z_i^2}{n-1}$

Proof: By definition we have:
$V(x_1, \cdots, x_n) = \frac{\sum_i x_i^2}{n-1} - \frac{(\sum_i x_i)^2}{n \cdot (n-1)}$ And so $V_Z \leq \frac{\sum_i z_i^2}{n-1}$.
Now, $V(.)$ is quadratic with positive terms in x_i^2 so the maximum is reached on a vertex of the hyper-rectangle $\times_i Z_i$ (which is an extreme configuration). For $x_i = z_i$ or $x_i = -z_i$, the term $\frac{\sum_i x_i^2}{n-1}$ is constant ($\frac{\sum_i x_i^2}{n-1} = \frac{\sum_i z_i^2}{n-1}$). Now, we can determine the sign for each variable x_i by the following way: Let $x_1 = +z_1$ and proceed recursively. If the sum $\sum_{j=1..i} x_j$ of already assigned values is positive, we choose $x_{i+1} = -z_{i+1}$ otherwise we choose $x_{i+1} = +z_{i+1}$. When all variables are assigned, their sum ($\sum x_n$) verifies $-z_n \leq \sum x_n \leq z_n$. And so $\frac{\sum_i z_i^2}{n-1} - \frac{z_n^2}{n \cdot (n-1)} \leq V(X)$ □

Now, if we have a symmetric fuzzy interval I centred in 0. We can decompose I into n α-cuts $I^i = [I^-(i/n), I^+(i/n)]$, where $I^-(x) = -I^+(x)$ because the interval is supposed to be symmetric (see Figure 6). We bracket the potential variance of I with those n α-cuts. The more α-cuts we take, the more precie we get. Making n grow to infinity, we obtain the following proposition:

Proposition 6: Let I be a symetric fuzzy interval centred in 0. The maximal potential variance of I is defined by:
$\overline{V} = \int_0^1 \left(\frac{I^+(\lambda) - I^-(\lambda)}{2}\right)^2 d\lambda$

Proof: $lim_{n \to +\infty}(\frac{z_n^2}{n \cdot (n-1)})) = 0$ and $lim_{n \to +\infty}(\frac{\sum_i z_i^2}{n-1}) = \int_0^1 I^+(\lambda)^2 d\lambda = \int_0^1 \left(\frac{I^+(\lambda) - I^-(\lambda)}{2}\right)^2 d\lambda$ because I is symmetric. □

So we have now an exact analytic expression of the potential variance of a symmetric fuzzy interval centred in 0, but the same approach can not be extended to a general fuzzy interval (centered in 0) as the term $\frac{\sum_i x_i^2}{n-1}$ is not constant for all extreme configurations if the intervals are not symetric. This problem should be studied in further work. We conjecture the same result may hold in the general case.

V. Conclusion

The profile method extends interval analysis to fuzzy intervals for locally monotonic functions which reach their maximum and minimum values on the bounds of interval entries. In basic problems of interval computation, this is of course not always the case. The variance is an example of non-monotonic function for which this method can be extended. However, the extension of interval analysis techniques require a partition of the unit interval (of membership values) to be maintained. The complexity of the approach depends on how many elements this partition can attain. This is a topic for further research. This method can be applied to all problems involving fuzzy random variables.

We have defined the potential variance of a symmetric fuzzy interval based on these results. The obtained definition is an interval of the form $[0, V_{max}]$. Interestingly the expression of V_{max} is similar to Carlson and Fullér variance [15]. Two questions remain: is this expression valid for asymmetric fuzzy intervals? Is V_{max} equal to the upper bound of the variance of all random variables included in the probability family induced by the fuzzy interval?

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