COMPRESSIVE TIME DELAY ESTIMATION USING INTERPOLATION

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ABSTRACT

Time delay estimation has long been an active area of research. In this work, we show that compressive sensing with interpolation may be used to achieve good estimation precision while lowering the sampling frequency. We propose an Interpolating Band-Excluded Orthogonal Matching Pursuit algorithm that uses one of two interpolation functions to estimate the time delay parameter. The numerical results show that interpolation improves estimation precision and that compressive sensing provides an elegant tradeoff that may lower the required sampling frequency while still attaining a desired estimation performance.

Index Terms— Compressive sensing, parameter estimation, time delay estimation, interpolation.

1. INTRODUCTION

Time Delay Estimation (TDE) of one or more known signal waveforms from sampled data is of interest in several fields such as radar, sonar, wireless communications, audio, speech and medical signal processing. The classical problem is to obtain good precision of the estimate while keeping the sampling frequency and computational complexity low.

A popular method for TDE is to find the peak of the crosscorrelation function. As this must often be done on sampled data, the cross-correlation function is sampled on a discrete grid. This corresponds to multiplying the received sampled signal onto a dictionary of reference signals, each a delayed version of the known waveform. The problem then reduces to finding the maximum nonzero components in the crosscorrelation vector. If a high estimation precision is desired, this requires a high sampling rate, which may be costly to attain. Since the dimension of the sampled signal may often be much higher than the number of delays to be estimated in the signal, the cross-correlation vector may be assumed sparse and it is possible to lower the sampling rate by employing compressive sensing (CS) [1, 2]. With CS, we seek to recover signals and parameters from an under-determined system of linear equations by assuming sparsity in a known dictionary. A common problem in CS is that the observed signals may not be sparsely representable in the dictionary. This problem also occurs in TDE as the delay parameter of the received waveforms is a continuous parameter.

To remedy such lack of sparsity, we show that interpolation may be used in the CS framework to improve estimation precision. In this work, we focus on extending previous work on interpolation-based TDE and bridging it with CS. Thereby, we attain good estimation precision while keeping the sampling frequency low. We use a redundant dictionary of delayed waveforms as this improves the estimation. Such redundancy, however, introduces coherence, and so we use a coherence-inhibiting greedy algorithm for signal recovery. The coherence-inhibition is similar to the model-based CS approach in [3, 4, 5]. We compare the performance of four compressive delay estimators: 1) an unmodified, coherenceinhibiting greedy algorithm that uses no interpolation and therefore operates on a discrete grid, 2) an algorithm that uses simple parabolic interpolation on the cross-correlation function, 3) an algorithm that uses polar interpolation and 4) an algorithm that first reconstructs the full Nyquist- sampled signal using ℓ_1 -minimization as in classical CS [1, 2] and then estimates the delays using the MUSIC algorithm. For all four estimators, we investigate their performance in terms of estimator precision with and without Gaussian noise added. Furthermore, we briefly examine the computational complexity of the examined estimators. While we use a simple chirp signal in our numerical experiments, the proposed approach may also be extended to more complex systems, e.g., symbol synchronization in wireless communication systems.

2. PROBLEM FORMULATION AND BACKGROUND

Let the received time-domain analog signal be defined as

$$f(t; \boldsymbol{\alpha}, \boldsymbol{\tau}) = \sum_{i=1}^{K} \alpha_i \cdot g(t - \tau_i) + n(t), \quad (1)$$

where $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_K\}$ are the unknown signal amplitudes, $\tau = \{\tau_1, \tau_2, \dots, \tau_K\}$ are the unknown signal delays in time, g(t) is a known signal waveform and n(t) is the noise. The task of the estimation algorithm is then to estimate α and τ from a sampled version of (1). Depending on the bandwidth

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of g(t), the required sampling rate to estimate the delays to a sufficient precision may be high. If we assume that only a few signal components are active, i.e. K is small, we may use CS to achieve the desired precision at a lower sampling rate. With a CS receiver, the received signal has the form $\mathbf{y} = \Phi(f)$, where $\Phi(\cdot)$ represents a CS sampling structure such as Random Demodulator [6] or Modulated Wideband Converter [7], which takes as input a bandlimited signal such as f(t) in (1). These CS sampling structures are designed so the sampling operation in the analog domain is equivalent to a matrix-vector operation in the discrete domain, $\mathbf{y} = \Phi \mathbf{f}$, where $\mathbf{f} \in \mathbb{C}^N$ is the Nyquist sampled version of (1), $\mathbf{y} \in \mathbb{C}^M$ is the received signal and $\Phi \in \mathbb{R}^{M \times N}$ is the discrete equivalent of $\Phi(\cdot)$ with N and M the number of Nyquist and measurement samples, respectively.

To enable reconstruction, CS requires a sparsifying dictionary $\Psi \in \mathbb{C}^{N \times N}$. In the case of TDE, the dictionary is a circulant matrix, consisting of delayed waveforms:

$$\Psi = \begin{bmatrix} \psi_0 & \psi_1 & \cdots & \psi_{N-1} \end{bmatrix}$$
$$= \begin{bmatrix} g[0] & g[N-1] & \cdots & g[1] \\ g[1] & g[0] & \ddots & g[2] \\ \vdots & \vdots & \ddots & \vdots \\ g[N-1] & g[N-2] & \cdots & g[0] \end{bmatrix}, \quad (2)$$

where $\mathbf{g} = \begin{bmatrix} g[0] & g[1] & \cdots & g[N] \end{bmatrix}^T$ is the Nyquistsampled version of g(t) in (1). We use the term *atom* to signify one column in this dictionary, so that arbitrary signals are composed of atoms from the dictionary. With this dictionary, the sampled cross-correlation function may be obtained as $\hat{R}_f[n] = |\langle \mathbf{y}, \psi_n \rangle|$. Since the delay parameter is continuous, the received signal may not be perfectly representable by the sparsifying dictionary, and the peak of the cross-correlation function then falls between its sampled values.

Prior work on this problem includes [8], which uses a gradient descent approach to approximate solutions off the grid for a generic greedy algorithm. This approach is similar to one of the two algorithms proposed in [9], one using a firstorder Taylor expansion, the other a form of polar interpolation. The authors show that polar interpolation outperforms Taylor expansion. In our work we extend upon the polar interpolation approach. In [4, 5], the authors use both a redundant dictionary with coherence rejection and second order polynomial interpolation to better estimate the solution. Similarly, in [10] the authors introduce algorithms that inhibit coherent atoms in the recovery algorithms. Time delay estimation with CS has been treated before in [11], where Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) is used to retrieve the time delays after processing the signal with a specially designed filter bank. Their algorithm relies on periodic sequences of delayed signals and specially tailored analog filters that ensure stable inversion and is

applicable to specific types of waveforms q(t). In contrast, our approach processes the signal in the digital domain, provides a generic acquisition framework compatible with arbitrary waveforms g(t), and does not require custom hardware filters or periodicity. To compare our interpolation algorithms with a framework similar to that in [11], but for discrete, nonperiodic sequences, we use an algorithm in the digital domain that first reconstructs the Nyquist-sampled signal using the Basis Pursuit (BP) algorithm [12] for noise-less experiments or the Basis Pursuit Denoising (BPDN) algorithm [12] for noisy experiments. Then, we find the delay taps as $\hat{\mathbf{h}} = \Psi^{\dagger} \hat{\mathbf{f}}$, where $\hat{\mathbf{f}}$ is the reconstructed signal. If all the delays are on the grid the delay tap vector, h, has only K active taps. However, as is also seen for frequency sparse signals, h has side lobes when the delay parameters are off the grid. A standard solution to solving this problem for frequency estimation is to use a high resolution algorithm, such as the Multiple Signal Classification (MUSIC) algorithm [13]. We therefore transform the time delay estimation problem to a frequency estimation problem by taking the inverse Fourier transform of h and use MUSIC to estimate the tap positions, which corresponds to the delay estimates. We term this algorithm TDE MUSIC.

3. INTERPOLATION IN TIME DELAY ESTIMATION

Our contribution is bridging the work on CS and interpolation to improve estimator precision in TDE while keeping the sampling frequency and computational complexity low. This is achieved by proposing a new greedy algorithm with an interpolation step. In each iteration of the algorithm, after finding the strongest correlating atom (i.e. the largest absolute value of $\hat{R}_f[n]$), we propose to use an interpolation function to improve the estimation precision:

$$\tilde{\tau}_n = \mathcal{T}(\mathbf{y}, \boldsymbol{\Psi}, i_n), \tag{3}$$

where $\tilde{\tau}_n$ is the new *n*th estimate of the delay, **y** is the received signal, Ψ is the dictionary and i_n is the index for the atom in the dictionary that features the strongest correlation with the signal. There are many possible choices of interpolation functions. In this work we compare two interpolation functions: second order polynomial and polar.

Polynomial interpolation is a common method to increase the TDE precision for sampled data [14, 15, 16]. The simplest and most often used polynomial interpolation is fitting a parabola around the correlation peak. In some cases, it is possible to improve the estimation by using different polynomial interpolation techniques for different problems, see, e.g., the references in [17]. In this work, we use the Direct Correlator estimator from [15] for parabolic interpolation:

$$\tau_i = -\frac{\Delta}{2} \frac{\hat{R}_f[(n+1)\Delta] - \hat{R}_f[(n-1)\Delta]}{\hat{R}_f[(n+1)\Delta] - 2\hat{R}_f[n\Delta] + \hat{R}_f[(n-1)\Delta]} + n\Delta$$

where τ_i is the delay to estimate, Δ is the spacing in time between samples of the discrete cross-correlation function \hat{R}_f , and n is the index of the largest absolute entry in \hat{R}_f .

Polar interpolation is proposed in [9] and obtains improved estimates of off-the-grid atoms. This is done for shift-invariant systems, where it can be assumed that the signal manifold of possible delayed waveforms lies on a hypersphere [9], which is also the case for time delay estimation. Such assumption is supported by the fact that the magnitude $\|\mathbf{f}\|_2$ of a delayed signal, \mathbf{f} , is the same regardless of the value of the parameter τ ; hence, the magnitude becomes the radius of the modeling hypersphere.

Instead of interpolating using the cross-correlation function, polar interpolation is based on the received signal itself and three atoms from the dictionary; the strongest correlating atom, $\psi_{p,i}$ and its two adjacent neighbours in the dictionary, ψ_{p-1} and ψ_{p+1} . With these three vectors we may approximate a small part of the signal manifold's hypersphere with a circle arc. The interpolation is obtained as follows; define the function \mathbf{f}_i as the sampled *i*th signal component $\alpha_i \cdot g(t - \tau_i)$ in (1). Then \mathbf{f}_i may be approximated as

$$\mathbf{f}_{i} \approx \mathbf{c}^{\mathrm{T}} \mathbf{A} \begin{bmatrix} \boldsymbol{\psi}_{p-1} \\ \boldsymbol{\psi}_{p} \\ \boldsymbol{\psi}_{p+1} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \alpha_{i} \\ \alpha_{i} r \cos\left(\frac{|\tau_{i} - n\Delta|\theta}{\Delta}\right) \\ \alpha_{i} r \sin\left(\frac{|\tau_{i} - n\Delta|\theta}{\Delta}\right) \end{bmatrix}, \quad (4)$$
$$\mathbf{A} = \begin{bmatrix} 1 & r \cos(\theta) & -r \sin(\theta) \\ 1 & r & 0 \\ 1 & r \cos(\theta) & r \sin(\theta) \end{bmatrix}^{-1}, \quad (5)$$

where $r = \|\psi_i\|_2$ is the magnitude of a signal waveform and radius of the hypersphere, and θ is the angle between the vectors ψ_p and either ψ_{p-1} or ψ_{p+1} . Notice that r is identical for all choices of τ , hence the hypersphere assumption. In this formula, **A** rotates the three ψ vectors to form a new, general basis for the circle arc and **c** scales the vectors in that basis to estimate the received signal. Given a signal **f** and the atom in the dictionary that correlates the strongest with the signal ψ_p , we may solve (4) as a linear least squares problem with **c** as the unknown. From $\hat{\mathbf{c}} = \{\hat{c}_1, \hat{c}_2, \hat{c}_3\}$, we may obtain an estimate of τ_i by taking the inverse tangent of \hat{c}_3/\hat{c}_2 .

4. INTERPOLATING BAND-EXCLUDED ORTHOGONAL MATCHING PURSUIT

We use a redundant, circulant dictionary which introduces coherence between atoms. To remedy the coherence effect we leverage the Band-Excluded Orthogonal Matching Pursuit (BOMP) algorithm [10] instead of classical orthogonal matching pursuit, as it inhibits atoms in the recovered signal from being too closely spaced. In this work we compare: 1) the original BOMP algorithm [10] and 2) Interpolating Band-Excluded Orthogonal Matching Pursuit (IBOMP), which uses interpolation to estimate time delays in between the sampling grid. The interpolation is done using parabolic or polar interpolation. IBOMP is an extension to the BOMP algorithm and is defined in Algorithm 1. First, the best correlating atom index i_n is found by generating a proxy for the sparse signal. This proxy is trimmed using a band exclusion function [10]:

$$B_{\eta}(S) = \bigcup_{k \in S} B_{\eta}(k), \tag{6}$$

$$B_{\eta}(k) = \{i \mid \mu(i,k) > \eta\}, \ \mu(i,k) = |\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_k \rangle|,$$
(7)

where $\mu(i, k)$ is the coherence between two atoms in the dictionary, $B_{\eta}(k)$ is the η -coherence band of the index k, and $B_n(S)$ is the η -coherence band of the index set S. In this work, we set $\eta = 0$, as we assume that the signal waveforms are well spaced so that signal components present in the received signal are orthogonal to each other. Therefore, the band-exclusion does not inhibit two pulses from interfering, but inhibits the algorithm from finding the same pulse again due to a large remaining residual y_{res} . The selected atom is then input to the interpolation function $T(\mathbf{y}, \boldsymbol{\Psi}, i_n)$, cf. (3), which finds an estimate of the *n*th time delay, $\tilde{\tau}_n$. Using that time delay estimate and the original parametric signal model, we create a new atom for a signal dictionary, **B**, which is used to find the basis coefficients a using linear least squares. Finally, a new residual is calculated and n and S are updated. When exiting the loop the signal is reconstructed. The stopping criteria may be based on a noise floor estimate or, if the sparsity K is known, the loop is set to run K times.

Algorithm 1 Interpolating Band-Excluded Orthogonal Matching Pursuit Algorithm (IBOMP)

Input: Compressed signal \mathbf{y} , interpolation function $T(\mathbf{y}, \mathbf{\Psi}, i_n)$, dictionary $\mathbf{\Phi}$ and measurement matrix $\mathbf{\Psi}$ Output: Reconstructed signal $\tilde{\mathbf{x}}$ and delay estimates $\tilde{\tau}_n$ Initialize: $\mathbf{y}_{res} = \mathbf{y}, \mathbf{B} = \emptyset, n = 1$ and $S^n = \emptyset$ **repeat** $i_n = \arg \max_i |\langle \mathbf{y}_{res}, \mathbf{\Psi} \boldsymbol{\phi}_i \rangle|, i \notin B_0(S^{n-1})$ $\tilde{\tau}_n = T(\mathbf{y}_{res}, \mathbf{\Psi}, i_n)$ Include sampled version of $f(t - \tilde{\tau}_n)$ as new atom in \mathbf{B} $\mathbf{a} = (\mathbf{\Psi} \mathbf{B})^{\dagger} \mathbf{y}$ $\mathbf{y}_{res} = \mathbf{y} - \mathbf{\Psi} \mathbf{B} \mathbf{a}$ $n = n + 1, S^n = S^{n-1} \cup \{i_n\}$ **until** stop-criterion = True $\tilde{\mathbf{x}} = \mathbf{B} \mathbf{a}$

5. NUMERICAL SIMULATIONS

To evaluate the proposed reconstruction algorithms, we have performed two numerical experiments. The documentation and code for these experiments are made freely available at http://www.sparsesampling.com/tde, following the principle of Reproducible Research [18].

For the numerical experiments, we let g(t) in (1) be a



Fig. 1. τ -MSE or variance versus subsampling ratio κ . See Fig. 2 for legend.

chirp signal defined as

$$g(t) = \frac{1}{\sqrt{\mathcal{E}_q}} \cdot e^{j2\pi(f_0 + \frac{\Delta f}{2T}(t - T/2))(t - T/2)} \cdot p(t), \tag{8}$$

$$p(t) = \begin{cases} \frac{T}{2} (1 + \cos(2\pi(t - T/2)/T)), & t \in (0, T) \\ 0, & \text{otherwise} \end{cases}, \quad (9)$$

where $f_0 = 1$ MHz is the center frequency, $\Delta f = 40$ MHz is the sweeped frequency, and $T = 1\mu s$ is the duration of the chirp in time. The chirp is limited in time by a raised cosine pulse and normalized to unit energy. We assume well-spaced pulses so that no two pulses overlap. Each signal is composed of K = 3 pulses, with K known to the algorithms.

We perform Monte Carlo experiments and repeat each experiment 1000 times to get an average result. In each experiment, we generate a time signal by sampling the pulse function in (1) N = 500 times with a sampling frequency of $f_s =$ 50MHz. This sampling rate ensures that the corresponding bandwidth of the signal contains more than 99% of its energy. The real and imaginary part of each α_i are drawn from a uniform distribution between -10 and 10 and enforced to have a minimum absolute value of 1. The delays, τ , are drawn from a uniform distribution between 0 and $\frac{N-1}{f_{*}} - T = 8.98 \mu s$. For the CS measurement matrix we choose a Random Demodulator matrix [6], $\Psi \in \{-1, 0, 1\}^{M \times N}$. We set $M = \kappa N$, where $\kappa \in [0,1)$ is the CS subsampling rate. We evaluate the performance of the four estimators by computing the time delay mean squared error (τ -MSE) between the true and estimated value of the time delay. This corresponds to the sample variance of the estimators and is a measure of estimator precision.

For the first experiment, we assume a noise-free signal. We perform this experiment with a range of subsampling ratios κ . Figure 1 shows our results. All four estimators allow for subsampling while maintaining good estimation precision. TDE MUSIC performs best for low κ , while the interpolation algorithms perform best as κ increases. The Polar IBOMP algorithm is the best performing interpolation algorithm. To compare the proposed CS algorithms with a method that does not use CS, we also show an algorithm that directly downsamples the signal by a factor of N/M and then esti-



Fig. 2. τ -MSE or variance versus SNR in decibel for $\kappa = 0.5$.

mates the delays using the MUSIC algorithm. As expected it does not attain the same estimation precision as the CSbased algorithms, due to aliasing. As a note on the y-axis and as validation for our implementation, notice that BOMP converges to approximately $0.3 \cdot 10^{-4} (\mu s)^2$ which is to be expected since the time delay between columns of the dictionary Φ is 1/50MHz = $0.02\mu s$. Therefore the average error squared is $(0.02/4)^2 = 2.5 \cdot 10^{-5} (\mu s)^2$, which corresponds well with the numerical results.

For the second experiment we include additive white Gaussian measurement noise in the signal model. We fix $\kappa = 0.5$ and vary the signal-to-noise ratio (SNR) from -5 to 30 dB. The noise is generated by calculating the noise power as the desired SNR divided by the measurement power, which is found as the squared ℓ_2 -norm of the measurement vector. Fig. 2 shows that the five algorithms are affected by noise, but as the SNR increases they converge to the same performance as in the noiseless case for $\kappa = 0.5$ in Fig. 1.

6. COMPUTATIONAL COMPLEXITY

To compare the computational complexity of the four estimators, we first look at TDE MUSIC. This algorithm consists of two parts: an ℓ_1 minimization problem, in which the solution to the Newton system has complexity $O(N^3)$ [19], where N is the signal length, and the MUSIC algorithm, which is dominated by computing the Singular Value Decomposition (SVD), has complexity of $O(N^3)$ for a square matrix [20]. For the algorithms based on BOMP the most significant term is to find the amplitude coefficients, a in Algorithm 1, where the pseudo-inverse is found by solving a linear least squares (LS) problem, which has cost [20]:

$$\text{Cost}_{\text{LS with SVD}} \sim 2NK^2 + 11K^3 = 2NK^2 + 11K^3.$$
 (10)

This is for the last iteration of the BOMP algorithm, as this is the most costly when all K atoms are used in the LS problem. As we assume $K \ll N$, the term $O(NK^2)$ dominates.

The interpolation in BOMP also adds some complexity. For the polar interpolation, it is possible to compute the radius and angle beforehand and create a dictionary of all possible sets of ψ_{p-1} , ψ_p , ψ_{p+1} for all p. This dictionary may be multiplied with the **A** matrix in (5) beforehand and the remaining steps are then dominated by finding the LS solution to (4), \hat{c} . If the complex LS problem is solved using a complex SVD, the total cost is O(N) and derived as follows: For real-valued matrices/vectors the cost is identical to (10) where K = 3 is the number of variables, i.e. $\text{Cost}_{\text{LS with SVD}} \sim 18N + 297$. As complex arithmetic can be reduced to real arithmetic, we simply say the computational complexity is O(N).

For the parabolic interpolation, we estimate the delay based on the cross correlation function evaluated in three places. The BOMP algorithm has already found these three values and the estimation complexity is therefore independent of the problem size, i.e. O(1). Thus, it is clear that the parabolic interpolation is less computationally demanding than the polar interpolation and that the greedy algorithms are much less computationally demanding than TDE MUSIC.

7. CONCLUSION

We have compared four time delay estimators and shown that all four methods are compatible with CS. Out of the four, TDE MUSIC performs the best for low values of κ , while IBOMP with polar interpolation obtains the best performance with higher values of κ . We also show that the interpolating greedy algorithms are less computationally demanding than TDE MUSIC.

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