Group Symmetry and non-Gaussian Covariance Estimation

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Abstract—We consider robust covariance estimation with group symmetry constraints. Non-Gaussian covariance estimation, e.g., Tyler scatter estimator and Multivariate Generalized Gaussian distribution methods, usually involve non-convex minimization problems. Recently, it was shown that the underlying principle behind their success is an extended form of convexity over the geodesics in the manifold of positive definite matrices. A modern approach to improve estimation accuracy is to exploit prior knowledge via additional constraints, e.g., restricting the attention to specific classes of covariances which adhere to prior symmetry structures. In this paper, we prove that such group symmetry constraints are also geodesically convex and can therefore be incorporated into various non-Gaussian covariance estimators. Practical examples of such sets include: circulant, persymmetric and complex/quaternion proper structures. We provide a simple numerical technique for finding maximum likelihood estimates under such constraints, and demonstrate their performance advantage using synthetic experiments.

Index Terms—geodesic convexity, non-Gaussian covariance estimation.

I. INTRODUCTION

Covariance estimation is a fundamental problem in the field of statistical signal processing. Many algorithms for detection and estimation rely on accurate covariance matrix estimation [1], [2]. Roughly speaking, the problem is tractable as long as the global maximum likelihood solution can be efficiently found (or approximated). Thus, it is important to understand whether the associated negative-loglikelihood minimization problem is convex. Following this line of thought, we combine two ideas. First, there is an increasing interest in covariance estimation in non-Gaussian distributions which are typically non-convex but have been shown to be geodesically convex [6], [8]. Second, many problems adhere to known symmetry constraints which can be exploited in the estimation. Recently, [9] addressed such structures in the Gaussian setting. In this paper, we will consider them in non-Gaussian covariance estimation using the theory of geodesic convexity.

In many applications, the assumption of normal data is not realistic [3], [12]. In such scenarios, improved performance may be obtained by resorting to more general distributions, such as Generalized Gaussian and Elliptical distributions [20], [21]. The associated Maximum Likelihood optimization usually do not lead to closed form solutions and iterative algorithms are required [6], [12]. One of the most prominent robust methods is the Tyler's method for covariance matrix estimation in scaled Gaussian models, which has been successfully applied to different practical applications ranging from array processing to sensor networks [10]. It has been extended to other settings involving regularization and incomplete data [3] - [7]. Recently, it was shown that the underlying principle behind these successful non-convex optimizations is the geodesic convexity [13], [18]. This principle provides more insight on the analysis and design of robust covariance estimation methods, and paves the road to numerous extensions based on *g*-convexity, e.g., regularization [6] and their combination with Kronecker structures [13].

Over the last years, many works have been developed in the area of estimating covariance matrices possessing some additional knowledge such as sparsity or structure [22]. Our work is motivated by [9] which considered group symmetry structures. In particular, [9] addressed symmetry constraints in random fields of physical phenomena, Bayesian models and cyclostationary processes. In addition, it is well known that circulant matrices are invariant to shifts [17], [19]. Symmetric persymmetric (bisymmetric) matrices are invariant under the exchange-operator [15], [16]. Proper complex normal distributions are defined via their invariance to rotations with respect to the real and imaginary axis [14]. Proper quaternion distributions follow invariances with respect to isoclinic rotations [11], [23]. All of these properties have been successfully exploited in covariance estimation in the multivariate Gaussian distribution. Many of them have also been considered in non-Gaussian distributions via problem-specific fixed point iterations and algorithmdependent existence, uniqueness and convergence proofs.

The main result in this paper is that the set of positive definite matrices which are invariant under a conjugation action of a subgroup of orthogonal transformations is g-convex on their respective manifold. Together with the g-convexity of various non-Gaussian negative-log-likelihoods, this implies that the global constrained maximum likelihood solution can be efficiently found using standard descent algorithms. This provides a unified framework for robust covariance estimation with group symmetry constraints. Unlike previous approaches, our results are not specific to any distribution, symmetry set or even numerical algorithm. As a byproduct, we provide a few results on specific symmetry groups and reformulate proper complex and quaternion structures using a finite number of rotation-invariant constraints. For completeness, we also propose a simple numerical method for solving these problems, although we emphasize that other descent algorithms can be used instead. Finally, we demonstrate the performance advantage of our framework

via synthetic simulations in a non-Gaussian proper quaternion environment.

The paper is organized in the following form. First, we give an outline of *g*-convexity and matrix group symmetry. Then the main result is formulated and examples of symmetry matrix classes are given. Finally, we provide a computational algorithm and numerical results.

II. GEODESIC CONVEXITY

Geodesic convexity is a generalization of the notion of convexity in linear spaces. We therefore begin with a brief review on g-convexity on the manifold $\mathcal{P}(p)$ of positive definite matrices $p \times p$. More details are available in [18], [7]. With each $\mathbf{Q}_0, \mathbf{Q}_1 \in \mathcal{P}(p)$ we associate the following geodesic

$$\mathbf{Q}_{t} = \mathbf{Q}_{0}^{\frac{1}{2}} \left(\mathbf{Q}_{0}^{-\frac{1}{2}} \mathbf{Q}_{1} \mathbf{Q}_{0}^{-\frac{1}{2}} \right)^{t} \mathbf{Q}_{0}^{\frac{1}{2}}, \quad t \in [0, 1].$$
(1)

Definition 1. A set $\mathcal{N} \in \mathcal{P}(p)$ is g-convex if for any $\mathbf{Q}_0, \mathbf{Q}_1 \in \mathcal{N}$ the geodesic \mathbf{Q}_t lies in \mathcal{N} .

Definition 2. Given a g-convex subset $\mathcal{N} \subset \mathcal{P}(p)$, we say that a function f is g-convex on \mathcal{N} if for any two points $\mathbf{Q}_0, \mathbf{Q}_1 \in \mathcal{N}, f(\mathbf{Q}_t) \leq tf(\mathbf{Q}_0) + (1-t)f(\mathbf{Q}_1), \forall t \in [0, 1].$

The advantage of g-convexity stems from the following result [7]

Proposition 1. Any local minimum of a g-convex function over a g-convex set is a global minimum.

Finding local minimum is usually easy and hence *g*-convexity guarantees that a global solution can also be efficiently found.

Recently, it was shown that the negative-log-likelihoods of many popular non-Gaussian distributions are *g*-convex. Two examples are:

• Tyler's [6]

$$L\left(\{\mathbf{x}_i\}_{i=1}^n; \mathbf{Q}\right) = \frac{p}{n} \sum_{i=1}^n \log(\mathbf{x}_i^T \mathbf{Q}^{-1} \mathbf{x}_i) + \log|\mathbf{Q}|, \quad (2)$$

• Mutlivariate Generalized Gaussian Distribution [8]

$$L\left(\{\mathbf{x}_i\}_{i=1}^n; \mathbf{Q}\right) = \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i^T \mathbf{Q}^{-1} \mathbf{x}_i)^\beta + \log|\mathbf{Q}|, \quad (3)$$

where β is the shape parameter.

Together with Proposition 1 above, [6], [8] proved that simple descent algorithm converge to the global estimate in these distributions. In the next section, we will show that this is also true when using symmetry invariance constraints which are also *g*-convex.

III. MATRIX GROUP SYMMETRY

In order to improve the accuracy of covariance estimators it is common to add constraints based on prior knowledge. Of course, this priors can only be exploited if the constraints are convex and the associated optimization can be efficiently solved. Recently, [9] proposed the use of group symmetry constraints which are indeed convex (actually linear) and can be incorporated into a Gaussian setting. The main result in this paper is that such sets are also *g*-convex and can also be utilized in non-Gaussian settings. Let \mathcal{K} be a set¹ of orthogonal matrices. Following [9], we formally assume that this set is actually a multiplicative group. Associated with \mathcal{K} , we define the fixed-point subset $\mathcal{F} \subset \mathcal{P}(p)$ of matrices that are invariant with respect to the conjugation by each element of \mathcal{K} :

$$\mathcal{F}(\mathcal{K}) = \{ \mathbf{Q} \in \mathcal{P}(p) | \mathbf{Q} = \mathbf{L} \mathbf{Q} \mathbf{L}^T, \forall \mathbf{L} \in \mathcal{K} \}.$$
(4)

Theorem 1. The set $\mathcal{F}(\mathcal{K})$ in (4) is g-convex.

Proof: First note that $\mathbf{Q} = \mathbf{L}\mathbf{Q}\mathbf{L}^T$ is equivalent to $\mathbf{Q}\mathbf{L} = \mathbf{L}\mathbf{Q}$. Now, assume $\mathbf{Q}_0, \mathbf{Q}_1 \in \mathcal{F}(\mathcal{K})$. Let us show that the geogesic (1) lies in $\mathcal{F}(\mathcal{K})$. Choose $\mathbf{L} \in \mathcal{K}$, $\mathbf{L}\mathbf{Q}_0 = \mathbf{Q}_0\mathbf{L}$, $\mathbf{L}\mathbf{Q}_1 = \mathbf{Q}_1\mathbf{L}$. Let \mathbf{M} be a diagonalizable matrix and fa smooth function, then we can think of $f(\mathbf{M})$ as of f acting on the eigenvalues of \mathbf{M} in the orthonormal eigenbasis of \mathbf{M} . For any diagonalizable matrix \mathbf{M} it commutes with \mathbf{P} iff $f(\mathbf{M})$ commutes with \mathbf{P} for any smooth function f, also if two matrices \mathbf{M}_1 and \mathbf{M}_2 commute with \mathbf{P} , then their product $\mathbf{M}_1\mathbf{M}_2$ commutes with \mathbf{P} . This implies that $\mathbf{Q}_0^{-\frac{1}{2}}\mathbf{Q}_1\mathbf{Q}_0^{-\frac{1}{2}}$ commutes with \mathbf{L} , thus $\left(\mathbf{Q}_0^{-\frac{1}{2}}\mathbf{Q}_1\mathbf{Q}_0^{-\frac{1}{2}}\right)^t$ also commutes with \mathbf{L} and the whole \mathbf{Q}_t commutes with \mathbf{L} . Thus the geodesic (1) lies in $\mathcal{F}(\mathcal{K})$ and the set $\mathcal{F}(\mathcal{K})$ is g-convex.

IV. EXAMPLES AND APPLICATIONS

In this section we provide examples of group symmetry constraints which appear in real world covariance estimation problems.

A. Circulant

A common class of symmetry constrained covariances is the set of positive definite circulant matrices:

$$\mathbf{C} = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}.$$

Such matrices are typically used as an approximation to Toeplitz structured matrices which are associated with signal processing in stationary environments [17], [19]. It is easy to see that the set of circulant matrices can be expressed as $\mathcal{F}(\mathcal{K})$ with \mathcal{K} being the cyclic group of order n which acts on the rows of the matrix by shifts. Thus, an immediate corollary of Theorem 1 it that the set of circulant matrices is *g*-convex.

B. Persymmetric

Another class of symmetry constrained covariances is the set of positive definite persymmetric matrices, i.e., matrices which are symmetric in the northeast-to-southwest diagonal $\mathbf{PJ}_n = \mathbf{J}_n \mathbf{P}^T$, where \mathbf{J} is the exchange $n \times n$ matrix containing ones only on the northeast-to-southwest diagonal. Since we deal with symmetric matrices the constraint

¹Here we treat the case of finite \mathcal{K} , but the result can be easily generalized to the infinite case.

becomes $\mathbf{PJ}_n = \mathbf{J}_n \mathbf{P}$ and the matrix form is:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{12} & p_{22} & \dots & p_{1n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n-1} & p_{2n-1} & \dots & p_{12} \\ p_{1n} & p_{1n-1} & \dots & p_{11} \end{pmatrix}$$

Such matrices are commonly encountered in radar systems using a symmetrically spaced linear array with constant pulse repetition interval [16]. This structure information could be exploited to improve detection performance [15], [16]. This set can be expressed as $\mathcal{F}(\mathcal{K})$ with \mathcal{K} consisting of \mathbf{I}_n and \mathbf{J}_n . Thus, an immediate corollary of Theorem 1 it that the set of persymmetric matrices is also *g*-convex.

Recently, [16] extended the Tyler's covariance estimator to the case of persymmetric matrices, proposed and analyzed the asymptotic behaviour of the fixed point estimator. Theorem 1 generalizes this result to other *g*-convex optimizations, independent of the algorithm that finds the local minimum.

C. Proper Complex

An important class of matrices is known as proper complex, or circularly symmetric covariance matrices. In most radar and communication problems it is typical to work with complex valued random variables which are invariant to rotations. A *p*-dimensional complex vector can be expressed as a 2*p*-dimensional real valued vector. Due to the symmetries, the associated $2p \times 2p$ covariances belong to $\mathcal{F}(\mathcal{K})$ with \mathcal{K} being an infinite set of rotations of the form [14]

$$\mathbf{L}_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \otimes \mathbf{I}_{p}, \tag{5}$$

which must hold for any θ . This result already shows that the set is *g*-convex. However, in order to efficiently exploit it, we also need a finite characterization.

Proposition 2. The set of proper complex $2p \times 2p$ covariance matrices is equivalent to $\mathcal{F}(\mathcal{K})$ with \mathcal{K} consisting of $\mathbf{L}_0 = \mathbf{I}_{2p}$ and $\mathbf{L}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{I}_p$.

Proof: This is a particular case of the Proposition 3 below.

Thus, *g*-convex maximum likelihood problems with proper complex constraints can be globally and efficiently solved. As special cases this includes proper complex versions of Tyler's estimator and MGGD solutions. We note that this result is not surprising. Recently, most of these complex multivariate settings have been analyzed [24], [25]. However, previous approaches were highly specific, and relied on defining new complex distributions. Our framework allows a unified treatment based on the real valued distributions with a single additional *g*-convex constraint.

D. Proper quaternion

Another modern class of covariance matrices is known as proper quaternion [11]. Quaternions are a generalization of complex numbers and is a 4-dimensional vector space over reals, so that a length p quaternion vector can be dealt with as a length 4p real vector. Typical applications are complex electromagnetic signals with two polarizations [23], [26]. Similarly to the complex case, here too it is common to consider proper distributions, which are invariant to specific quaternion rotations. A $4p \times 4p$ proper quaternion covariance belongs to $\mathcal{F}(\mathcal{K})$ with \mathcal{K} being an infinite set of rotations of the form

$$\mathbf{L}_{\theta\alpha\beta\gamma} = \begin{bmatrix} \cos(\theta) & \alpha\sin(\theta) & \beta\sin(\theta) & \gamma\sin(\theta) \\ -\alpha\sin(\theta) & \cos(\theta) & -\gamma\sin(\theta) & \beta\sin(\theta) \\ -\beta\sin(\theta) & \gamma\sin(\theta) & \cos(\theta) & -\alpha\sin(\theta) \\ -\gamma\sin(\theta) & -\beta\sin(\theta) & \alpha\sin(\theta) & \cos(\theta) \end{bmatrix} \otimes \mathbf{I}_{p}, \quad (6)$$

which must hold for θ , α , β , γ satisfying $\alpha^2 + \beta^2 + \gamma^2 = 1$. The next result characterizes this set using a finite number of constraints.

Proposition 3. The set of proper quaternion $4p \times 4p$ covariance matrices is equivalent to $\mathcal{F}(\mathcal{K})$ with \mathcal{K} consisting of $L_0 = \mathbf{I}_{4p}, L_1 = \mathbf{R}_1 \otimes \mathbf{I}_p, L_2 = \mathbf{R}_2 \otimes \mathbf{I}_p, L_3 = \mathbf{R}_3 \otimes \mathbf{I}_p,$ $L_4 = -L_0, L_5 = -L_1, L_6 = -L_2 \text{ and } L_7 = -L_3, \text{ where}$ $\mathbf{R}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbf{R}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbf{R}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$ (7)

Proof: The matrices \mathbf{L}_i for i = 0, ..., 7 are particular cases of (6), so the necessity is obvious. Assume now that \mathbf{Q} is invariant under \mathbf{L}_i conjugation, meaning that \mathbf{Q} commutes with them: $\mathbf{Q}\mathbf{L}_i = \mathbf{L}_i\mathbf{Q}$ and we are given some matrix \mathbf{R} of the form (6). Take the equalities $\mathbf{Q}\mathbf{L}_i = \mathbf{L}_i\mathbf{Q}, i = 0, 1, 2, 3$, multiply them by $\cos(\theta), \alpha \sin(\theta), \beta \sin(\theta), \gamma \sin(\theta)$ correspondingly and add them up to get: $\mathbf{Q}\mathbf{R} = \mathbf{R}\mathbf{Q}$.

In other words, the set of proper quaternion covariance matrices is *g*-convex. Thus, we can easily extend the *g*convex estimates of Tyler and MGGD to the quaternion case, and guarantee that any descent algorithm will converge to the global solution.

V. MINIMIZATION ALGORITHM

In this section, we address the numerical optimization of the above minimizations. Various numerical techniques can be used to find local minimas. Since the problems are gconvex these local minimas will also be the global solution. The negative-log-likelihoods in (2)-(3) have the form [8]:

$$\mathbf{L}(\mathbf{Q}) = \frac{1}{n} \sum_{i=1}^{n} \rho(\mathbf{s}_{i}^{T} \mathbf{Q}^{-1} \mathbf{s}) + \log |\mathbf{Q}|.$$
(8)

For simplicity, we consider the classical iterative reweighed scheme:

$$\mathbf{Q}_{k+1} = \frac{1}{n} \sum_{i=1}^{n} u(\mathbf{s}_i^T \mathbf{Q}^{-1} \mathbf{s}_i) \mathbf{s}_i \mathbf{s}_i^T,$$
(9)

where $u(x) = \rho'(x)$.

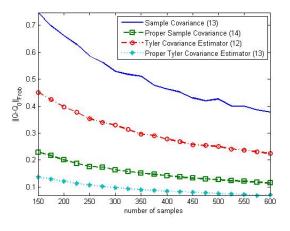
Following [9], we note that adding the g-convex constraints in the form of symmetry is equivalent to replicating the sample measurements. Given n p-dimensional measurements $\{s_i\}_{i=1}^n$ the symmetrization is equivalent to generating synthetically $|\mathcal{K}|$ new measurements from each one, thus getting $|\mathcal{K}|n$ samples $\{\pounds s_i\}_{i=1, \pounds \in \mathcal{K}}^n$ instead of n. This generalizes the iterative scheme as follows:

$$\mathbf{Q}_{k+1} = \frac{1}{|\mathcal{K}|n} \sum_{\mathbf{L}\in\mathcal{K}} \sum_{i=1}^{n} u((\mathbf{L}\mathbf{s}_i)^T \mathbf{Q}^{-1}(\mathbf{L}\mathbf{s}_i))(\mathbf{L}\mathbf{s}_i)(\mathbf{L}\mathbf{s}_i)^T.$$
(10)

A simple minimization majorization argument can be used to show that this iteration leads to a descent method, see for example [6].

VI. NUMERICAL RESULTS

For numerical simulations, we chose Tyler's scatter estimate in proper quaternion distributions. We have generated a proper real covariance matrix \mathbf{Q}_0 and generated elliptically distributed 10-dimensional quaternion random vectors as $s_i = \sqrt{\tau} \mathbf{v}$, where $\tau \sim \chi^2$ and \mathbf{v} is zero-mean normally distributed with covariance matrix \mathbf{Q}_0 . We choose $\rho(x) = p\log(x)$ to get the Tyler's covariance estimator [10].



We compare four different covariance estimators:

• Sample Covariance

$$\mathbf{Q}_{SC} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{s}_i \mathbf{s}_i^T, \qquad (11)$$

• Proper Sample Covariance

$$\mathbf{Q}_{PSC} = \frac{1}{|\mathcal{K}|n} \sum_{\mathbf{k} \in \mathcal{K}} \sum_{i=1}^{n} \mathbf{k} \mathbf{s}_i \mathbf{s}_i^T \mathbf{k}^T, \qquad (12)$$

• Tyler Covariance Estimator Iteration

$$\mathbf{Q}_{k+1} = \frac{p}{n} \sum_{i=1}^{n} \frac{\mathbf{s}_i \mathbf{s}_i^T}{\mathbf{s}_i^T \mathbf{Q}_k^{-1} \mathbf{s}_i}.$$
 (13)

• Tyler Proper Covariance Estimator Iteration

$$\mathbf{Q}_{k+1} = \frac{p}{|\mathcal{K}|n} \sum_{\mathbf{L} \in \mathcal{K}} \sum_{i=1}^{n} \frac{\mathbf{L}\mathbf{s}_{i}\mathbf{s}_{i}^{T}\mathbf{L}^{T}}{\mathbf{s}_{i}^{T}\mathbf{L}^{T}\mathbf{Q}_{k}^{-1}L\mathbf{s}_{i}}$$
(14)

$$= \frac{p}{|\mathcal{K}|n} \sum_{\mathbf{L}\in\mathcal{K}} \sum_{i=1}^{n} \frac{(\mathbf{L}\mathbf{s}_i)(\mathbf{L}\mathbf{s}_i)^T}{(\mathbf{L}\mathbf{s}_i)^T \mathbf{Q}_k^{-1}(\mathbf{L}\mathbf{s}_i)}.$$

We repeat the computations for 100 times for the four estimators with 150 - 600 samples. In order to make the results consistent we divide all the matrices by their traces.

VII. ACKNOWLEDGEMENT

This work was partially supported by Israel Science Foundation Grant No. 786/11 and Kaete Klausner Scholarship. The authors would like to thank Alba Sloin for numerous and helpful discussions.

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