Benefits of Improper Signaling for Overlay Cognitive Radio

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***Abstract*—This paper considers improper Gaussian signaling (IGS) in an overlay cognitive radio scenario. We follow a protocol in which the secondary user (SU) uses part of its power to relay the message for the primary user (PU) and consider a simple yet illustrative 2-user scenario. We analyze two communication schemes depending on whether or not the PU cooperates with the SU and derive closed-form expressions for the optimal transmission parameters that maximize the SU rate while ensuring a specified minimum performance of the PU. Our numerical results show that IGS may significantly outperform proper signaling and that, interestingly, the cooperative approach provides negligible performance gains over its non-cooperative counterpart.**

***Index Terms*—Improper Gaussian signaling, interference channel, overlay cognitive radio, spectrum sharing.**

I. Introduction

In multiuser communications, the use of proper Gaussian signaling (PGS) is typically assumed, as this scheme achieves capacity in point-to-point communications [1]. Proper signals have uncorrelated real and imaginary parts with equal variance. On the contrary, improper Gaussian signals have real and imaginary parts that are correlated or have unequal variance [2]. Although improper Gaussian signaling (IGS) is suboptimal in point-to-point communications, it has been shown to be advantageous in interference-limited networks when interference is treated as noise [3]–[10].

One important example of interference-limited communications is cognitive radio (CR), in both underlay and overlay versions (UCR and OCR, respectively) [11]. CR is regarded as one of the key enabling technologies of future broadband communication systems, as it permits a more efficient utilization of the already scarce radio-frequency spectrum [12]. There are two sets of users in CR networks: primary users (PU) and secondary users (SU). The former are the licensed users of the spectrum and thus have strict quality-of-service (QoS) constraints. The SUs, which do not have a spectrum license, may use the PUs' channel provided that they do not disrupt their communications.

In UCR, where the SUs limit their transmit power so as to ensure the PU's QoS, IGS was shown to outperform PGS in [7]. In a simple yet illustrative 2-user scenario, it was shown that a SU can improve its data rate with IGS if the ratio between the gains of the interference and direct link was above a certain threshold. After that, the benefits of IGS for UCR have been proved for other scenarios [9], [10].

Contrarily to UCR, in OCR the SU has non-causal knowledge of the PU's message. By using a portion of the transmit power to relay the PU's message, the SU can compensate the additional interference created by the transmission of its own message. Additionally, dirty paper coding (DPC) can be used to eliminate the interference at the secondary receiver caused by the PU's communication [11]. The secondary transmitter can acquire the PU's message in a variety of ways. For example, the secondary transmitter may be connected to the primary transmitter through a backhaul link [13]. Alternatively, the PU's message can be shared through a wireless link prior to transmission if the secondary transmitter is close by [14]. Finally, the message can also be acquired on a transmission that is not correctly decoded at the primary receiver due to bad channel quality and then exploited during the retransmission of the packet [15].

IGS for OCR has been considered in [13], where a 2-user scenario is analyzed under different assumptions about the availability of the channel state information (CSI). This work considers that the message transmitted by the PU is different from the message relayed by the SU and intended for the PU. This assumption is made for the purpose of mathematical tractability, but it might be questionable in practice. Here we consider the more classical OCR scenario, in which the message intended for the PU is sent from both primary and secondary transmitters, assuming that the latter has already acquired this message using some of the aforementioned techniques. We consider two different schemes depending on the cooperation level between the two users. In the first, the SU is allowed to use IGS, but the PU is unaware of the SU and thus employs PGS. In the second scheme there is some cooperation between both networks. Specifically, the PU is aware of the SU and may use IGS as well. In both schemes, the PU is protected by means of a minimum rate constraint, and the SU splits its available power between its own message and the message intended for the primary receiver. We will show that, while IGS significantly outperforms PGS also for this scenario, the cooperative approach provides negligible gains over the non-cooperative approach, and thus the latter is more

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interesting from a practical perspective.

II. System model

A. Preliminaries about improper random variables

We first provide the necessary background on improper random variables. We refer the reader to [2] for a comprehensive treatment of the topic.

Definition 1 ([2]). *The complementary variance of a zero-mean complex random variable x is defined as $\tilde{\sigma}_x = \mathrm{E}[x^2]$, where $\mathrm{E}[\cdot]$ is the expectation operator. If $\tilde{\sigma}_x = 0$, then x is called proper, otherwise improper.*

Furthermore, σ_x^2 and $\tilde{\sigma}_x$ are a valid pair of variance and complementary variance if and only if $\sigma_x^2 \geq 0$ and $|\tilde{\sigma}_x| \leq \sigma_x^2$.

Definition 2 ([2]). *The circularity coefficient of a complex random variable x, which measures the degree of impropriety, is defined as $\kappa_x = |\tilde{\sigma}_x|/\sigma_x^2$. The circularity coefficient satisfies $0 \leq \kappa_x \leq 1$. If $\kappa_x = 0$, then x is proper, otherwise improper. If $\kappa_x = 1$ we call x maximally improper (or rectilinear).*

B. System description

We consider an OCR scenario comprised of a PU and an SU, both single-antenna. Following the OCR paradigm, the secondary transmitter has access to the primary message. We consider a protocol in which the SU uses part of its power to transmit the PU's message and the remaining power to deliver its own message to a secondary receiver. By doing so, the impact of the interference at the PU due to the SU's message can be canceled. Furthermore, we consider that the PU has a minimum rate requirement $\bar{R}$. The signal at the primary receiver can be expressed as

$$y_p' = h_p\sqrt{p'}x_p + h_{ps}(\sqrt{\alpha q'}z_p + \sqrt{(1-\alpha)q'}x_s) + n_p' \ , \quad (1)$$

where x_p and x_s are the desired messages for the primary and the secondary receiver, respectively, with unit variance, n_p' is additive proper Gaussian noise with variance σ_p^2, h_p and h_{ps} are the channel coefficient of the direct and interference link, respectively, p' and q' are the power budgets of the primary and secondary transmitter, respectively, and α is the power splitting factor of the SU. The signal intended for the PU that is transmitted by the SU is denoted as z_p.

Since the secondary transmitter knows x_p, it can use DPC to achieve interference-free communication. Therefore, we can express the received signal at the secondary receiver as

$$y_s' = h_s\sqrt{(1-\alpha)q'}x_s + n_s' \ , \quad (2)$$

where n_s' is the additive proper Gaussian noise with variance σ_s^2, and h_s is the channel coefficient. For ease of illustration, we will use the standard or canonical model for the communication channel described by (1), (2). Thereby, these two expressions are rewritten such that the direct links and noise variance are set to 1. Specifically, (1) and (2) can be equivalently expressed as

$$y_p = \sqrt{p}x_p + \sqrt{a_{ps}}(\sqrt{\alpha q}z_p + \sqrt{(1-\alpha)q}x_s) + n_p \ , \quad (3)$$

$$y_s = \sqrt{(1-\alpha)q}x_s + n_s \ , \quad (4)$$

where n_p and n_s have now unit variance, $\sqrt{a_{ps}} = \frac{\sigma_s|h_{ps}|}{\sigma_p|h_s|}$, $\sqrt{p} = \frac{\sqrt{p'}|h_p|}{\sigma_p}$, and $\sqrt{q} = \frac{\sqrt{q'}|h_s|}{\sigma_s}$.

Since interference is treated as noise at the primary receiver, the SU can benefit from transmitting improper Gaussian signals. That is, if x_s is improper, the rate degradation of the PU is smaller than if it is proper, and the SU can thus use more power for the transmission of its own message. Hence, the achievable rate of the SU is [7]

$$R_s = \frac{1}{2}\log_2\left[1 + (1-\alpha)q\left((1-\alpha)q(1-\kappa^2) + 2\right)\right] \ , \quad (5)$$

where κ is the circularity coefficient of x_s. We will consider two approaches for the PU. In the first, the primary transmitter is unaware of the SU and thus transmits proper Gaussian signals, i.e., x_p is proper and Gaussian. In this case, the SU signal is $z_p = x_p$. In the second approach, we permit a higher level of cooperation. Specifically, the primary receiver feeds back the interference parameters to the primary transmitter, so that it can adapt its strategy. Therefore, the PU uses its optimal improper signaling strategy. In this case, x_p and z_p are both improper Gaussian, each one obtained through a different widely-linear transformation (WLT) of the PU's message. In both cases, the optimization problem that we address is

$$\underset{0\leq\{\alpha,\kappa\}\leq 1}{\text{maximize}} \quad R_s \ , \quad (6a)$$

$$\text{subject to} \quad R_p \geq \bar{R} \ , \quad (6b)$$

where $\bar{R}$ is the (feasible) PU rate constraint and R_p is the rate achieved by the PU, which will follow a different expression depending on the adopted approach.

III. Unaware primary transmitter

Let us first consider the non-cooperative approach, in which the primary transmitter, unaware of the SU, transmits a proper signal. Thus, we take also $z_p = x_p$, and the PU rate is given by [5]

$$R_p(\alpha,\kappa) = \frac{1}{2}\log_2\left[\frac{\left(p + qa_{ps} + 1 + 2\sqrt{\alpha pqa_{ps}}\right)^2}{\left((1-\alpha)qa_{ps}+1\right)^2 - \left((1-\alpha)qa_{ps}\kappa\right)^2} - \frac{\left((1-\alpha)qa_{ps}\kappa\right)^2}{\left((1-\alpha)qa_{ps}+1\right)^2 - \left((1-\alpha)qa_{ps}\kappa\right)^2}\right] . \quad (7)$$

The optimal solution of (6) for this case is presented in the next theorem.

Theorem 1. *The optimal solution to* (6)*, when the PU rate follows* (7)*, is*

$$\kappa^\star = \sqrt{\max(\kappa', 0)} \ ,$$

$$\kappa' = \frac{2^{2\bar{R}}[(1-\alpha^\star)qa_{ps}+1]^2 - (p + qa_{ps} + 1 + 2\sqrt{\alpha^\star qa_{ps}p})^2}{(2^{2\bar{R}}-1)(1-\alpha^\star)^2q^2a_{ps}^2} \ , \quad (8)$$

where $\alpha^\star = \max(\alpha_0, \min(\alpha_1, \alpha_c))$, with

$$\alpha_0 = \inf\{\alpha \in [0,1] \mid R_p(\alpha, 0) \geq \bar{R}\} \ , \quad (9)$$

$$\alpha_1 = \inf\{\alpha \in [0,1] \mid R_p(\alpha, 1) \geq \bar{R}\} \ , \quad (10)$$

and $\sqrt{\alpha_c}$ *is the unique positive root of*

$$f_1(x) = -q^2 a_{ps}^2 x^3 + q a_{ps}[2p + 2^{2\bar{R}} - a_{ps}(2^{2\bar{R}} - 1 - q)]x + \sqrt{q a_{ps}} p (p + q a_{ps} + 1) = 0 \ . \quad (11)$$

Proof: Please refer to Appendix A. ■

IV. Aware primary transmitter

We now consider the cooperative approach, where the primary transmitter is aware of the SU and thus transmits improper Gaussian signals with the optimal IGS scheme. As the PU signal x_p is now improper, the PU signal transmitted by the SU, z_p, is also improper. The improper signals x_p and z_p are generated at the PU and SU transmitters, respectively, by means of two different WLT of a common proper, unit power, signal s_p that conveys the message intended for the PU. Therefore, they can be expressed as

$$x_p = g_1 e^{j\phi_1} s_p + g_2 e^{j\phi_2} s_p^* \ , \quad (12)$$

$$z_p = g_3 e^{j\phi_3} s_p + g_4 e^{j\phi_4} s_p^* \ , \quad (13)$$

where s_p is proper Gaussian with unit variance and $g_i \geq 0$, $i = 1, \ldots, 4$, with $g_1^2 + g_2^2 = g_3^2 + g_4^2 = 1$. The optimal WLTs are stated in the next lemma.

Lemma 1. *The optimal WLTs* (12), (13), *which maximize the rate of the PU, are*

$$\phi_i = \frac{1}{2}(\phi + \pi) \ , \ i = 1, \ldots, 4 \ , \quad (14)$$

$$g_1^2 = g_3^2 = \frac{1 + \sqrt{1 - \kappa_p^2}}{2} \ , \quad (15)$$

$$g_2^2 = g_4^2 = 1 - g_1^2 \ , \quad (16)$$

where ϕ *is the phase of the complementary variance of* x_s *and*

$$\kappa_p = \min\left(\frac{(1-\alpha) q a_{ps} \kappa}{(\sqrt{p} + \sqrt{\alpha q a_{ps}})^2}, 1\right) \quad (17)$$

is the circularity coefficient of the aggregate PU signal $\sqrt{p} x_p + \sqrt{a_{ps} \alpha q} z_p$, *with* κ *being the circularity coefficient of* x_s.

Proof: Please refer to Appendix B. ■

Using the optimal WLTs, the rate achieved by the PU is [8]

$$R_p(\alpha, \kappa) = \begin{cases} R_{p,1}(\alpha, \kappa) \ , & \text{if } \kappa_p < 1 \ , \\ R_{p,2}(\alpha, \kappa) \ , & \text{if } \kappa_p = 1 \ , \end{cases} \quad (18)$$

where

$$R_{p,1}(\alpha, \kappa) = \frac{1}{2} \log_2 \left\{ \frac{\left[(1-\alpha) q a_{ps} + (\sqrt{p} + \sqrt{\alpha q a_{ps}})^2 + 1\right]^2}{1 + (1-\alpha) q a_{ps} \left[(1-\alpha) q a_{ps} (1 - \kappa^2) + 2\right]} \right\} ,$$

$$R_{p,2}(\alpha, \kappa) = \frac{1}{2} \log_2 \left\{ 1 + \frac{2(\sqrt{p} + \sqrt{\alpha q a_{ps}})^2 \left[(1-\alpha) q a_{ps} (1 + \kappa) + 1\right]}{1 + (1-\alpha) q a_{ps} \left[(1-\alpha) q a_{ps} (1 - \kappa^2) + 2\right]} \right\} . \quad (19)$$

Theorem 2. *The optimal solution to* (6), *when the PU rate is* (18), *can be characterized as follows.*

- *PGS is optimal, i.e.,* $\kappa^\star = 0$, *if and only if*
  - $a_{ps} \leq 2p 2^{-2\bar{R}} + 1$, *or*
  - $a_{ps} > 2p 2^{-2\bar{R}} + 1$ *and* $\alpha_c \geq \alpha_0$,

  where

$$\alpha_0 = \inf\{\alpha \in [0,1] \mid R_p(\alpha, 0) \geq \bar{R}\} \ , \quad (20)$$

$$\alpha_c = \frac{p(p + q a_{ps} + 1)^2}{2^{4\bar{R}} q a_{ps} (a_{ps} - 1 - 2p 2^{-2\bar{R}})^2} \ , \quad (21)$$

  in which case $\alpha^\star = \alpha_0$.
- *Otherwise, IGS is optimal with following parameters.*
  - *If* $\max(\alpha_c, \alpha_1) \geq \alpha_t$ *then*

$$\alpha^\star = \max(\alpha_c, \alpha_1) \ , \quad (22)$$

$$\kappa^{\star 2} = \frac{\left[(1-\alpha) q a_{ps} + 1\right]^2}{(1-\alpha)^2 (q a_{ps})^2} - \frac{2^{-2\bar{R}} \left[(1-\alpha) q a_{ps} + (\sqrt{p} + \sqrt{\alpha q a_{ps}})^2 + 1\right]^2}{(1-\alpha)^2 (q a_{ps})^2} \ , \quad (23)$$

    where

$$\alpha_1 = \inf\{\alpha \in [0,1] \mid R_{p,1}(\alpha, 1) \geq \bar{R}\} \ , \quad (24)$$

$$\sqrt{\alpha_t} = \frac{(2^{2\bar{R}} - 1)^{1/2} \left[2^{2\bar{R}} 2(q a_{ps} + 1) - p(2^{2\bar{R}} + 1)\right]^{1/2}}{2^{2\bar{R}} 2 \sqrt{q a_{ps}}} - \frac{\sqrt{p}(2^{2\bar{R}} + 1)}{2^{2\bar{R}} 2 \sqrt{q a_{ps}}} \ . \quad (25)$$

  - *Otherwise,*

$$\alpha^\star = \max(\alpha_c', \alpha_1') \ , \quad (26)$$

$$\kappa^\star = 1 - \frac{2(\sqrt{p} + \sqrt{\alpha q a_{ps}})^2 - (2^{2\bar{R}} - 1)}{(2^{2\bar{R}} - 1)(1-\alpha) q a_{ps}} \ , \quad (27)$$

    where

$$\alpha_1' = \inf\{\alpha \in [0,1] \mid R_{p,2}(\alpha, 1) \geq \bar{R}\} \ , \quad (28)$$

    and $\sqrt{\alpha_c'}$ *is the unique positive root of*

$$\begin{aligned} f_2(x) = & -x^3 2^{2\bar{R}} 4 q^2 a_{ps}^2 - x^2 6 (2^{2\bar{R}} + 1) q a_{ps} \sqrt{p q a_{ps}} \\ & + x \Big\{ (2^{2\bar{R}} - 1) q a_{ps} \big[2(q a_{ps} + 1 - p) \\ & - (2^{2\bar{R}} - 1)(a_{ps} - 1) \big] - 12 p q a_{ps} \Big\} \\ & + 2(2^{2\bar{R}} - 1) \sqrt{p q a_{ps}} (q a_{ps} + 1) - 4p \sqrt{p q a_{ps}} = 0 \ . \end{aligned} \quad (29)$$

Proof: Please refer to Appendix C. ■

From Theorem 2, we observe that a necessary condition for the optimality of IGS is that the gain of the interference channel is above $2p2^{-2\bar{R}} + 1$, which is equal to or greater than 1. Notice as well that this is also a necessary condition for the case of unaware primary transmitter. A similar condition was obtained for underlay CR in [7] and for the Z-IC in [8], which is also sufficient for those scenarios. However, the threshold in a_{ps} for those scenarios is equal to or smaller than 1. This means that the IGS optimality conditions for OCR are stricter, and hence less benefits are to be expected. The IGS optimality condition for underlay CR obtained in [7] is

$$a_{ps} > 1 - \frac{p}{2^{2\bar{R}} - 1} \ . \quad (30)$$

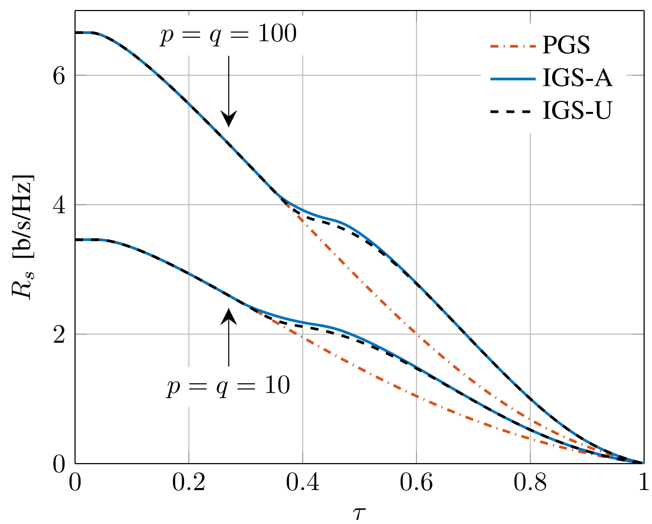

Fig. 1. Rate achieved by the SU for the different signaling schemes. The cross-channel gain is $a_{ps} = 5$.

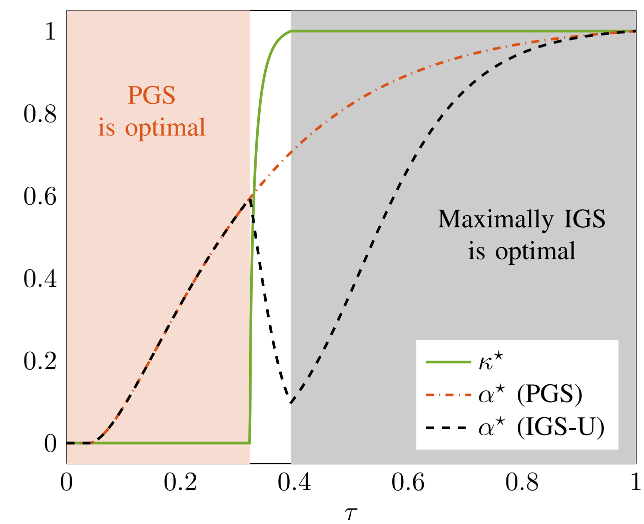

Fig. 3. Optimal power splitting factor and circularity coefficient for $p = q = 10$ and $a_{ps} = 5$.

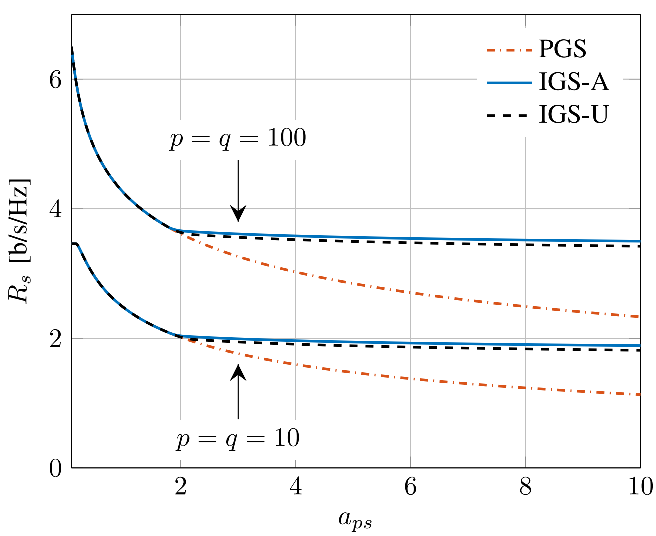

Fig. 2. Rate achieved by the SU for the different signaling schemes. The loading factor is $\tau = 0.5$.

As can be observed, the right-hand side of (30) decreases when $\bar{R}$ decreases and when p increases. However, the threshold on a_{ps} presented in Theorem 2 behaves in the opposite way. Thus, while in underlay CR (and also in the Z-IC) the payoffs of IGS are more noticeable for lower values of $\bar{R}$, the benefits of IGS in OCR are more prominent for higher values of $\bar{R}$, as we will show in the next section with some numerical examples.

V. Numerical examples

We now provide some numerical examples to illustrate our results. The PU rate constraint is expressed as $\bar{R} = \tau R_{\max}$, where $\tau \in [0, 1]$ is the loading factor and $R_{\max} = R_p(1, 0)$ is the maximum PU rate (achieved when the SU uses all of its power to deliver the PU's message). Figures 1 and 2 show the achievable rate of the SU as a function of τ and a_{ps}, respectively, for the different schemes. IGS-A and IGS-U denote the aware and unaware schemes, respectively. In both figures we observe that IGS provides a significant gain in achievable rate with respect to PGS. In Fig. 1, in which $a_{ps} = 5$, this is observed for intermediate and high values of τ. In Fig. 2, where $\tau = 0.5$, the benefits of IGS are observed for approximately $a_{ps} > 2$. Indeed, the rate achieved by IGS is almost flat with respect to a_{ps} in the considered example, whereas that of PGS decreases much faster.

In Fig. 3 we depict the optimal power splitting factor and the optimal circularity coefficient for the scenario $p = q = 10$ and $a_{ps} = 5$. For the sake of illustration, we only plot the values for PGS and IGS-U. The shaded area on the left-hand side indicates the interval where PGS is optimal, while the shaded area on the right-hand side indicates the interval where maximally IGS is optimal. We observe that there is an interval where the power splitting factor of IGS-U decreases. That is, even though the PU rate increases, more power is allocated to the message intended for the SU. Obviously, this is only possible if the circularity coefficient increases as well, as observed in the figure. Indeed, when the circularity coefficient equals 1, $\alpha^\star$ increases again with τ.

Comparing IGS-A and IGS-U we observe that the improvement of IGS-A with respect to IGS-U is negligible, while the latter entails higher cooperation between the primary and secondary systems. This is a surprising result and indicates that a non-cooperative IGS approach exhibits little performance degradation compared to the fully-cooperative approach. In light of these results, we conclude that IGS is a promising approach for OCR scenarios, where a one-sided approach, in which only the SU uses IGS and thus no cooperation is required, permits obtaining most of the performance gains that are possible with IGS.

VI. Conclusion

We have analyzed two IGS schemes for an OCR scenario comprised of a single-antenna SU and a single-antenna PU, namely, a cooperative and a non-cooperative scheme. Considering that the SU splits its transmit power between the primary and secondary messages, and that the PU is protected by a minimum rate constraint, we have derived closed-form expressions for the transmission parameters that maximize the SU rate. Our results have shown that, while IGS significantly outperforms PGS, the non-cooperative approach provides similar performance gains than the cooperative approach but with less signaling overhead.

Appendix A
Proof of Theorem 1

After some manipulations, the rate constraint (6b) can equivalently be written as $\kappa^2 \geq \kappa'$, with κ' as in (8). Since R_s is increasing in κ (see (5)), R_s is maximized when $\kappa^2 = \kappa'$ as long as $0 \leq \kappa' \leq 1$. The values of α for which $\kappa' > 1$ are then not feasible, as the PU rate constraint is violated in this case. Since κ' is decreasing in α (smaller α implies higher interference power, so the circularity coefficient of this interference has to be also higher to ensure the PU rate), $\kappa' \leq 1$ holds as long as $\alpha \geq \alpha_1$, where α_1 is given by (10) and is the required value of α (to ensure (6b)) when $\kappa = 1$. Similarly, the required value of α for $\kappa = 0$ is given by (9). Hence, the optimal value of α lies in the interval $[\alpha_1, \alpha_0]$. To find this value, we first assume that $\kappa' \geq 0$ holds for the optimal α. We then take $\kappa^2 = \kappa'$ and plug it into (5), which yields an expression for R_s that depends only on α. Let us denote this expression as $R_s(\alpha)$. Since the rate constraint has already been accounted for with κ', we can analyze the derivative of $R_s(\alpha)$ with respect to α to obtain its optimal value. By doing so, we obtain that $R_s(\alpha)$ is increasing in α when $f_1(x) > 0$, with $x = \sqrt{\alpha}$ and $f_1(x)$ given by (11), i.e.,

$$\begin{aligned} f_1(x) = &-q^2 a_{ps}^2 x^3 + q a_{ps}[2p + 2^{2\bar{R}} - a_{ps}(2^{2\bar{R}} - 1 - q)]x \\ &+ \sqrt{q a_{ps} p}(p + q a_{ps} + 1) \ . \end{aligned} \tag{31}$$

Now we will use the Descartes' rule of signs to show that $f_1(x)$ crosses zero at most at one valid point, $\sqrt{\alpha_c}$. This rule states that the number of positive roots of a polynomial is at most equal to the change of signs between consecutive coefficients of the polynomial. Following these lines, we observe that $f_1(x)$ has only one sign change between consecutive terms, which implies that the number of positive roots is not greater than one. Furthermore, $f_1(x)$ is positive at $x = 0$. Therefore, $R_s(\alpha)$ is increasing in the interval $[0, \alpha_c]$ and decreasing elsewhere. Since $\alpha \in [\alpha_1, \alpha_0]$, we then obtain $\alpha^\star = \max(\alpha_0, \min(\alpha_1, \alpha_c))$. Since this value may yield $\kappa' < 0$, we finally obtain $\kappa^\star = \sqrt{\max(\kappa', 0)}$.

Appendix B
Proof of Lemma 1

The received signal at the PU can be expressed as

$$\begin{aligned} y_p = &\left(\sqrt{p} g_1 e^{j\phi_1} + \sqrt{\alpha q a_{ps}} g_3 e^{j\phi_3}\right) x_p \\ &+ \left(\sqrt{p} g_2 e^{j\phi_2} + \sqrt{\alpha q a_{ps}} g_4 e^{j\phi_4}\right) x_p^* + \sqrt{(1-\alpha) q a_{ps}} x_s \\ &+ n_p = x_p' + \sqrt{(1-\alpha) q a_{ps}} x_s + n_p \ . \end{aligned} \tag{32}$$

Signal x_p' contains the PU's message and is improper Gaussian with variance and complementary variance given as

$$\begin{aligned} \sigma_p^2 = &\left|\sqrt{p} g_1 e^{j\phi_1} + \sqrt{\alpha q a_{ps}} g_3 e^{j\phi_3}\right|^2 \\ &+ \left|\sqrt{p} g_2 e^{j\phi_2} + \sqrt{\alpha q a_{ps}} g_4 e^{j\phi_4}\right|^2 \ , \end{aligned} \tag{33}$$

$$\begin{aligned} \tilde{\sigma}_p = &\, 2\left(\sqrt{p} g_1 e^{j\phi_1} + \sqrt{\alpha q a_{ps}} g_3 e^{j\phi_3}\right) \\ &\times \left(\sqrt{p} g_2 e^{j\phi_2} + \sqrt{\alpha q a_{ps}} g_4 e^{j\phi_4}\right) \ . \end{aligned} \tag{34}$$

The received signals (4) and (32) follow the expressions of the Z-interference channel (Z-IC). Thus, using the results from [8], the optimal complementary variance is

$$\tilde{\sigma}_p^\star = \sigma_p^2 \min\left(\frac{(1-\alpha) q a_{ps} \kappa}{\sigma_p^2}, 1\right) e^{j(\phi+\pi)} \ . \tag{35}$$

At the same time, the received signal power, σ_p^2, has to be maximized in order to maximize as well the PU achievable rate. We now show in the following that this is possible with the WLT described by (14)–(17). To this end, let us first analyze the received signal power, which yields

$$\begin{aligned} \sigma_p^2 = &\, p\left(g_1^2 + g_2^2\right) + \alpha q a_{ps}\left(g_3^2 + g_4^2\right) \\ &+ 2\sqrt{\alpha p q a_{ps}}\left(\Re\left\{g_1 g_3 e^{j(\phi_1 - \phi_3)}\right\} + \Re\left\{g_2 g_4 e^{j(\phi_2 - \phi_4)}\right\}\right) \\ \leq &\, p + \alpha q a_{ps} + 2\sqrt{\alpha p q a_{ps}}\left(g_1\sqrt{1 - g_4^2} + g_4\sqrt{1 - g_1^2}\right) \ , \end{aligned} \tag{36}$$

where $\Re\{\cdot\}$ returns the real part, and we have used $g_1^2 + g_2^2 = g_3^2 + g_4^2 = 1$. The above expression is achieved with equality for $\phi_1 = \phi_3$ and $\phi_2 = \phi_4$. Taking these values, it is easy to see through the derivative of σ_p^2 that the maximum is achieved for $g_1 = g_3$ and $g_2 = g_4$. Using these values, the complementary variance of the received signal is

$$\tilde{\sigma}_p = \sigma_p^2 2 g_1 \sqrt{1 - g_1^2} e^{j(\phi_1 + \phi_2)} \ . \tag{37}$$

Finally, equating (37) to (35) yields (14)–(17).

Appendix C
Proof of Theorem 2

Let us first assume that $\kappa_p < 1$ holds for the optimal solution. The rate constraint (6b) holds with equality for κ^2 equal to (23). Since (23) is decreasing in α (the circularity coefficient has to be higher to maintain the PU rate if more power is allocated to the SU's message), there exist α_1 and α_0, respectively given by (24) and (20), such that $\alpha_1 \leq \alpha \leq \alpha_0 \Leftrightarrow 0 \leq \kappa \leq 1$, with $\kappa = 0$ for $\alpha = \alpha_0$ and $\kappa = 1$ for $\alpha = \alpha_1$. To find the value of $\alpha \in [\alpha_1, \alpha_0]$

that maximizes R_s, assuming $\kappa_p < 1$, we plug (23) into (5), which yields an expression for R_s that only depends on α. After some manipulations, the derivative of this expression with respect to α can be shown to be non-negative if

$$\sqrt{p}(p + qa_{ps} + 1) + \sqrt{\alpha}2^{2\bar{R}}\sqrt{qa_{ps}}(2p2^{-2\bar{R}} + 1 - a_{ps}) \geq 0 \,. \tag{38}$$

Since the above function is linear, it crosses zero at most at one point. Furthermore, it has a positive slope for $a_{ps} \leq 2p2^{-2\bar{R}} + 1$. In such a case, the rate of the SU increases monotonically with α. Since the maximum value of α is α_0, we then have that the maximum is achieved for $\alpha = \alpha_0$ and $\kappa = 0$. In this case $\kappa_p = 0$ (see (17)), so that the assumption $\kappa_p < 1$ is satisfied and this is therefore the optimal solution for this case. If this condition does not hold, we observe that (38) is non-negative for $\alpha \leq \alpha_c$, where α_c is the root of (38) and is given by (21). Since $\alpha \in [\alpha_1, \alpha_0]$, the optimal value of α equals α_0 if $\alpha_c \geq \alpha_0$, which yields $\kappa = 0$. This characterizes the optimality of PGS.

IGS is then optimal if and only if the slope of (38) is negative and $\alpha_c < \alpha_0$. In this case, the optimal value of $\alpha \in [\alpha_1, \alpha_0]$ is then $\max(\alpha_c, \alpha_1)$ if the resulting value of κ_p is smaller than 1 (as (38) is obtained assuming this). Let us therefore obtain the range of α for which $\kappa_p < 1$. Plugging (23) into (17) we obtain

$$\begin{aligned}\kappa_p < 1 \;\Leftrightarrow\; & -\alpha 2^{2\bar{R}}2qa_{ps} - \sqrt{\alpha}2(2^{2\bar{R}}+1)\sqrt{pqa_{ps}} \\ & + \left(2^{2\bar{R}} - 1\right)(qa_{ps}+1) - \left(2^{2\bar{R}}+1\right)p < 0 \,.\end{aligned} \tag{39}$$

The above function is concave and decreasing in $\sqrt{\alpha}$. Hence we obtain that $\kappa_p < 1$ if and only if $\alpha > \alpha_t$, where α_t is the root of the right-hand side of (39) and is given by (25). This yields (22)–(25). If this condition does not hold, we have that $\kappa_p = 1$ is satisfied for the optimal solution and the analysis has to be repeated using $R_{p,2}(\alpha, \kappa)$.

When $\kappa_p = 1$, the rate constraint (6b) holds with equality for κ equal to (27). Again, (27) is decreasing in α, and thus there exists α_1', given by (28), such that $\alpha \geq \alpha_1' \Leftrightarrow \kappa \leq 1$, with $\kappa = 1$ for $\alpha = \alpha_1'$. Plugging (27) into (5), we obtain again an expression for R_s that depends only on α. After some manipulations, the derivative of this function with respect to α can be shown to be non-negative if $f_2(x)$, with $x = \sqrt{\alpha}$ and

$$\begin{aligned}f_2(x) = & -x^3 2^{2\bar{R}}4q^2a_{ps}^2 - x^2 6(2^{2\bar{R}}+1)qa_{ps}\sqrt{pqa_{ps}} \\ & + x\Big\{(2^{2\bar{R}}-1)qa_{ps}\big[2(qa_{ps}+1-p) \\ & -(2^{2\bar{R}}-1)(a_{ps}-1)\big] - 12pqa_{ps}\Big\} \\ & + 2(2^{2\bar{R}}-1)\sqrt{pqa_{ps}}(qa_{ps}+1) - 4p\sqrt{pqa_{ps}} \,,\end{aligned} \tag{40}$$

is equal to or greater than zero. Now we will use the Descartes' rule of signs to show that (40) crosses zero at most at one valid point. We first observe that the coefficients corresponding to the third- and second-order terms are negative. However, the sign of the remaining terms is not clear. Let us consider the zero-order term. This term is non-positive if

$$p \geq \frac{1}{2}(2^{2\bar{R}} - 1)(qa_{ps} + 1) \,. \tag{41}$$

The number of positive roots could be greater than one if the first-order term is positive while the above condition holds, since in this case the number of sign changes would be two. Since the first-order term decreases with p, its maximum under condition (41) is achieved when (41) holds with equality. In this case the first-order term is given by

$$-4(2^{2\bar{R}} - 2)(qa_{ps}+1)qa_{ps} - (2^{2\bar{R}}-1)^2 qa_{ps}^2(q+1) \,, \tag{42}$$

which is always negative. This means that the number of sign changes in the polynomial is not greater than one, and thus the number of positive roots is at most one. Therefore, the optimal solution for $\kappa_p = 1$ is $\max(\alpha_c', \alpha_1')$, with $\sqrt{\alpha_c'}$ being the root of (40), which yields (26)–(29).

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