An Optimization Method for Designing High Rate and High Performance SCTCM Systems with in-line Interleavers

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***Abstract*— We present a method for designing high-rate, high-performance SCTCM systems with in-line interleavers. Using in-line EXIT charts and ML performance analysis, we develop criteria for choosing constituent codes and optimization methods for selecting the best ones. To illustrate our methods, we show that an optimized SCTCM system with an in-line interleaver for rate $r = 5/6$ and 64QAM has better performance than other turbo-like TCMs with the same parameters.**

I. Introduction

Turbo codes [1], and their relatives, turbo-like codes, have revolutionized the field of error-correction. Turbo-like codes include not only the original parallel concatenated convolutional codes (PCCC), but also serially concatenated convolutional codes (SCCC) [2].

Applications of turbo-like codes to trellis-coded modulation include turbo trellis coded modulation (Turbo TCM) [3] and serially concatenated trellis coded modulation (SCTCM) [4].

The performance of turbo-like codes can be evaluated by the height of the error floor and convergence threshold characteristics. Benedetto et al. [5] showed that the average performance of the error floor of an ensemble of turbo-like codes under ML decoding can be analyzed using the uniform interleaver technique. In [6], ten Brink devised the EXIT chart method for computing ensemble thresholds.

The authors of this paper have previously designed and analyzed SCTCM systems for AWGN channel using in-line interleavers for rate $r = 2/3$, 8PSK, which enjoy both excellent convergence thresholds and low error floors [7][8]. In this paper, we extend our methods to general high-rate and high-performance SCTCM systems. As a detailed design example we construct an optimized rate $r = 5/6$ 64QAM system.

II. SCTCM system with an in-line interleaver

In this section, we briefly review the performance analysis methods for SCTCM systems with in-line interleavers developed in [7] and [8].

Fig. 1 represents the encoder structure of an SCTCM system with an in-line interleaver for rate $r = k/(k+1)$. The outer code and the inner code are convolutional codes of rate $R_o = k/(k+1)$ and $R_i = (k+1)/(k+1)$, respectively. There is an in-line interleaver between the outer and inner code, which independently interleaves each of the $(k+1)$ output streams. The $(k+1)$ bits of output of the inner code are mapped to a $2^{(k+1)}$-ary modulation constellation.

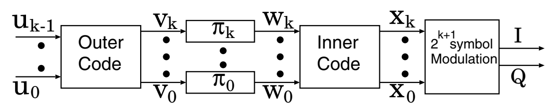

Fig. 1. The encoder structure of an SCTCM system with an in-line interleaver.

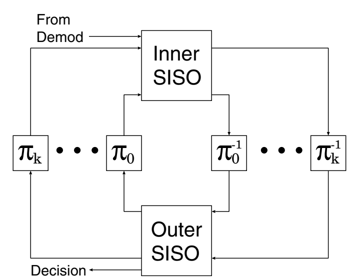

Fig. 2. The decoder structure of an SCTCM system with an in-line interleaver.

A convergence threshold analysis method for codes of this type based on a multi-dimension generalization of the EXIT chart was described in [7]. Fig. 2 represents the decoder structure of an SCTCM system with an in-line interleaver. Each SISO decoder can be viewed as a mutual information converter with multiple inputs and outputs, if we assume that the a priori probability inputs from the in-line interleaver are i.i.d. Gaussians. Equation (1) shows how the mutual information of the j-th output of the SISO decoder $x \in \{i, o\}$ is obtained from the transfer function of the mutual informations from the $(k+1)$ inputs $I^{(x)}_{in0}, \cdots, I^{(x)}_{ink}$.

$$I^{(x)}_{outj} = T^{(x)}_j \left(I^{(x)}_{in0}, \cdots, I^{(x)}_{ink} \right) \quad (1)$$

These $(k+1)$ functions, which are obtained by simulations for both the outer and inner codes, allow us to construct multidimensional EXIT charts and compute mutual information trajectories. If each of the mutual informations of the outer SISO decoder converges to 1, it means that the ensemble iterative decoder error probability approaches zero at this particular SNR. Fig. 3 shows an example of a mutual information trajectory of an SCTCM system with an in-line interleaver for rate $r = 2/3$ 8PSK.

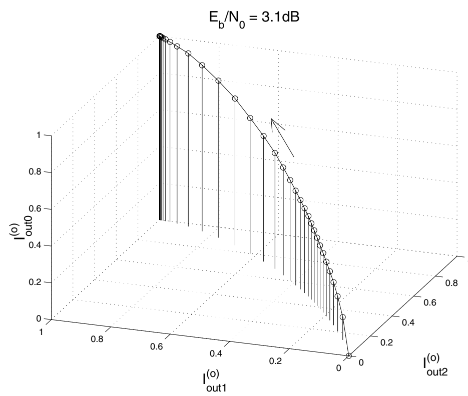

Fig. 3. Mutual information trajectory.

A ML performance analysis method for the in-line ensemble is described in [8], as an extension of the error floor analysis [5]. It estimates the error-floor by identifying dominant terms of the Input each-Output Weight Enumerator Function (IeOWEF), the each-Input Output Weight Enumerator Function (eIOWEF), using the concept of a uniform in-line interleaver. IeOWEF in (2) independently denotes the weight distributions of each of the multiple output streams in an error event of the outer code, and eIOWEF in (3) independently denotes the weight distribution of each of the multiple input streams in an error event of the inner code.

$$\begin{aligned} &T_{eO}(W, D_0, \cdots, D_{p-1}) \\ &= \sum_{w=1}^{\infty} \sum_{d_0=0}^{\infty} \cdots \sum_{d_{p-1}=0}^{\infty} B_{w,d_0,\cdots,d_{p-1}} W^w D_0^{d_0} \cdots D_{p-1}^{d_{p-1}} \end{aligned} \quad (2)$$

$$\begin{aligned} &T_{eI}(W_0, \cdots, W_{p-1}, D) \\ &= \sum_{w_0=0}^{\infty} \cdots \sum_{w_{p-1}=0}^{\infty} \sum_{d=1}^{\infty} B_{w_0,\cdots,w_{p-1},d} W_0^{w_0} \cdots W_{p-1}^{w_{p-1}} D^d \end{aligned} \quad (3)$$

The upper bounds of the serially concatenated codes with in-line interleavers can be described as (5), where B^C_{w,d_H} in (4) is the number of code words of input and output weight (w, d_H), which is calculated using IeOWEF, eIOWEF and the concept of uniform in-line interleaver.

$$B^C_{w,d_H} \approx \sum_{\ell_0=0}^{N/p} \cdots \sum_{\ell_{p-1}=0}^{N/p} \frac{B^{C_o}_{w,\ell_0,\cdots,\ell_{p-1}} \cdot B^{C_i}_{\ell_0,\cdots,\ell_{p-1},d_H}}{\binom{N/p}{\ell_0} \cdots \binom{N/p}{\ell_{p-1}}} \quad (4)$$

$$\begin{aligned} P_b(e) \;\le\; & \frac{1}{2} \sum_{w=1}^{NR_{C_o}} \sum_{d_H=d_{Hf}}^{N/R_{C_i}} \frac{w}{NR_{C_o}} \cdot B^C_{w,d_H} \\ & \times \mathrm{erfc}\left(\sqrt{\frac{d_H R_C E_b}{N_0}}\right) \end{aligned} \quad (5)$$

The dominant terms for error floor are the terms of maximum exponent of N ($\alpha_M = \max_{n_o,n_i,\ell_j}\{\alpha\}$). When the minimum output weight of outer codes which are fed into IIR input of the inner code (d^o_{qf}) is equal to or greater than 2, then the $\alpha_M < 0$ and this code has interleaver gain (6).

$$\begin{aligned} \alpha &= n_o + n_i - \sum_{j=0}^{p-1} \ell_j - 1 \\ &\le n_o - \left\lfloor \frac{\sum_{j=0}^{q-1} \ell_j + 1}{2} \right\rfloor - 1 \\ &\le -\left\lfloor \frac{n_o(d^o_{qf} - 2) + 3}{2} \right\rfloor, \end{aligned} \quad (6)$$

where n_o, n_i are the numbers of error events of outer and inner codes.

In both analysis methods, a partial infinite impulse response (IIR) inner code is employed. If a partial IIR code is used as an inner code of SCTCM system with an overall interleaver, the BER performance curve starts to decrease at low SNR, but yields a high error floor independent of the interleaver length. The reason for this undesirable error floor is easily explained: the EXIT chart trajectories have a cross point (see Fig. 4). But with an in-line interleaver, the cross point vanishes under certain conditions. Similarly, it is found by an ML performance analysis that the upper bound on $P_b(e)$ has the terms independent of the interleaver length N when an overall interleaver is employed, whereas these troublesome terms become dependent on the interleaver length N under some conditions (ex. $d^o_{qf} \ge 2$) when an in-line interleaver replaces the overall interleaver. Using these two analysis methods, we can construct SCTCM systems with low convergence threshold SNR and low error floor employing the partial IIR inner codes and the in-line interleavers.

In this paper, we propose a method for constructing and optimizing high performance SCTCM systems with in-line interleavers by applying these two analysis methods to a high rate codes.

III. Conditions for a High Performance SCTCM System with an In-Line Interleaver

In this section we explain the conditions to obtain a high performance SCTCM system when using an in-line interleaver for rate $r = k/(k+1)$. We assume that the outer and inner codes are convolutional codes with rate $R_o = k/(k+1)$ and $R_i = (k+1)/(k+1)$, respectively. We also assume that the $(k+1)$ outputs of the inner code are modulated to $2^{(k+1)}$ values and mapped to a transmission constellation symbol.

A. Conditions for low convergence threshold

We first explain the conditions derived from the EXIT chart analysis. If an SCTCM ensemble with an in-line interleaver converges at low SNR, the BER curve starts to decrease at low SNR even with an overall interleaver. This implies that the overall EXIT chart curves for the outer code and the set of the inner code and the signal mapping resemble each other except near the points with $I_{in}^{(i)} = 0$ or $I_{in}^{(i)} = 1$ (see Fig. 4). The shape of the overall EXIT chart curve for the outer code depends only on its own distance distribution and does not change significantly when the code rate is fixed, in contrast to the EXIT chart curves for the inner code and the signal mapping. Therefore we can construct an SCTCM ensemble with a low convergence threshold SNR by properly choosing the the inner code and the signal mapping.

Two important parameters have a large impact on the EXIT characteristics of the inner code and the signal mapping. One is the output Euclidean distance distribution with the input Hamming distance 1 $d_{E1}(Z)$, and the other is $b_{d_{free}}$. Both $d_{E1}(Z)$ and $b_{d_{free}}$ can be obtained from the Input Output Weight Enumerating Function (IOWEF) [5] $T_I(W, Z)$,

$$d_{E1}(Z) = \frac{\partial}{\partial W} T_I(W, Z)|_{W=0}, \quad (7)$$

$$b_{d_{free}} = \left. \frac{\frac{\partial}{\partial W} T_I(W, Z)}{Z^{d^2_{Emin}}} \right|_{W=1, Z=0}, \quad (8)$$

where d_{Emin} is the minimum output Euclidean distance of the inner code, and is equal to the minimum Euclidean distance between transmission signals because the rate of inner code is 1.

The input Hamming distance between transmission symbols d_{in} is defined as

$$d_{in}(\boldsymbol{x}, \boldsymbol{x}') \equiv d_H(\boldsymbol{W}, \boldsymbol{W}') \quad (\boldsymbol{x} \neq \boldsymbol{x}'), \quad (9)$$
$$(\boldsymbol{X}(\boldsymbol{W}) = (\boldsymbol{x}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}, \cdots),$$
$$\boldsymbol{X}(\boldsymbol{W}') = (\boldsymbol{x}', \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}, \cdots)),$$

where $\boldsymbol{x}, \boldsymbol{x}'$ are the sets of transmission values of $(k+1)$ bits and corresponds to a point in the signal constellation space, and $\boldsymbol{X}(\boldsymbol{W})$ is the output sequence of the inner code generated by the input sequence $\boldsymbol{W}$, and $\boldsymbol{x}^{(t)}$ represents a transmission value at time t on trellis, and d_H represents Hamming distance. That is, there exist two different input sequence of the inner code which have two different output symbols only at one time slot and the same ones at the other, because the rate of the inner code is 1. Thus the Hamming distance between these two sequences is the input Hamming distance between transmission symbols.

$b_{d_{free}}$ represented in (8) is a number which is proportional to the sum of the input Hamming distance d_{in} of the all pairs of signal points with Euclidean distance d_{Emin}.

The segment at 0 of the EXIT characteristics (see Fig. 4) of the set of the inner code and the signal mapping can be increased by decreasing the value of $b_{d_{free}}$. However, in order to decrease $b_{d_{free}}$ beyond a certain value, we need to use an inner code with input Hamming distance 1. The segment at 1 can also be decreased if the output Euclidean distance distribution $d_{E1}(Z)$ is decreased when the inner code has input Hamming distance 1. These features can be derived from the analysis of the BER performance of TCM under conditions of perfectly known (segment at 1) or unknown (segment at 0) a priori probabilities. However, the inner code should be partial IIR to have the input Hamming distance equal to 1. Here the partial IIR code is a code whose inputs can be separated to finite impulse response (FIR) and infinite impulse response (IIR) (see Fig. 7). Such a code has a feature that the output mutual information value of the overall EXIT characteristics does not reach to the value 1 when the input value is 1. However, we have already mentioned that this problem can be solved with an in-line interleaver in some conditions.

B. Conditions for low error floor

Next, we explain the design rules derived from ML performance analysis. As noted above, an SCTCM system has a high error floor, independent of the interleaver length, when we use an overall interleaver and a partial IIR inner code. However, an SCTCM system with an in-line interleaver will have an error floor which depends on the interleaver length if $d^o_{qf} \geq 2$. Here d^o_{qf} is the total minimum weight of outputs $v_0 \sim v_{q-1}$ of the outer code when inputs $w_0 \sim w_{q-1}$ $(q < k+1)$ of the inner code are IIR (see Fig. 6 and 7).

C. Conditions for high performance SCTCM with an in-line interleaver

The design rules for the components of SCTCM are shown below, based on the considerations described above.

- The outer code is a convolutional code with rate $k/(k+1)$ and with $d^o_{qf} \geq 2$.
- An in-line interleaver should be used.
- The inner code is a convolutional code with rate $(k+1)/(k+1)$ with one FIR input and k IIR inputs.
- $2^{(k+1)}$-ary modulation.

The conditions for the inner code are explained as follows. The number of IIR inputs of the inner code must be greater than or equal to k, in order to fulfill the condition of $d^o_{qf} \geq 2$ of the rate $k/(k+1)$ outer code. Furthermore, at least 1 FIR input is necessary so that error events with input Hamming distance 1 exist.

IV. Optimization method

In this section, we explain the optimization method which we used to obtain a high rate and high performance SCTCM system with an in-line interleaver.

The optimization is done by simulations of BER performance and EXIT charts. EXIT chart analyses predict performance in the limit of arbitrarily long information blocks and arbitrarily many iterations. However, because of computational limitations, we restricted the information block length to 10000 and the number of iterations to 20.The overall EXIT charts and the BER performance curves are used to select the constituent codes, and the performance of the optimized

SCTCM system with an in-line interleaver with the selected constituent codes is evaluated using the BER performance curves and the convergence threshold SNR using the in-line EXIT charts.

The optimization method follows.

(a) List the candidate constituent codes and signal mappings.
(b) Select good pairs from the list using overall EXIT charts and BER performance curves.
(c) Optimize using the in-line EXIT charts and the BER performance curves.

In (a) the constituent codes should fulfill the conditions and the required decoding complexity. The signal mappings are selected among various $d_{E1}(Z)$ and $b_{d_{free}}$.

In (b) the good pairs are selected using overall EXIT charts which have only one cross point at low SNR. The BER performance curves are also measured in order to verify that the selected pairs really converge at low SNR.

Now, we explain how to select the signal mappings. As mentioned above, the parameters $d_{E1}(Z)$ and $b_{d_{free}}$ have a great impact on the shape of the EXIT characteristics of the inner code and signal mapping. The parameter $d_{E1}(Z)$ affects the value $I_{out}^{(i)}$ when $I_{in}^{(i)} = 1$. That is, the output mutual information of FIR input of inner codes is the mutual information of received value itself. When the distribution $d_{E1}(Z)$ has low weight terms, segment at 1 decreases. If two signal mappings have same distribution $d_{E1}(Z)$, the segment at 1 is the same value.

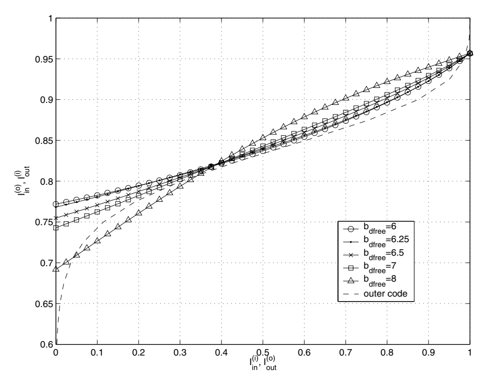

Fig. 4. Effect of $b_{d_{free}}$ on EXIT chart ($d_{E1}(Z) = Z^{d^2_{Emin}}$).

On the other hand, the parameter $b_{d_{free}}$ affects the value $I_{out}^{(i)}$ at $I_{in}^{(i)} = 0$ (segment at 0). That is, the output mutual information of an inner code with no a priori probability depends on $b_{d_{free}}$ of the code. The smaller $b_{d_{free}}$, the larger the segment at 0 for fixed $d_{E1}(Z)$.

Fig. 4 shows the overall EXIT charts of some SCTCM systems for rate $r = 5/6$ and 64QAM, for which all signal mapping have the same distribution $d_{E1}(Z)$. We can see that the segments at 1 meet at same point and the segments at 0 depend on the values of $b_{d_{free}}$.

In this way, the shape of EXIT characteristics of the sets of inner codes and signal mappings can be controlled by adjusting $d_{E1}(Z)$ and $b_{d_{free}}$ for each outer code.

In (c) the BER performance curves are measured using the in-line interleaver with the constituent codes selected in (b). There are $(k+1)!$ ways to connect the $(k+1)$ output streams of the outer codes to the interleaver. Although the good codes are selected using the BER performance curves, it is difficult to select the best one by lengthy simulations. Instead, we use the convergence threshold of the selected good codes to select the best one.

V. Optimization results

We applied our optimization methods to design a rate $r = 5/6$ and 64QAM SCTCM system using an in-line interleaver. The number of candidates considered for each part is more than 10 for the outer code, 1 for the inner code, and about 30 for the signal mapping. All simulations used information block length = 10000, and the number of iterations 20.

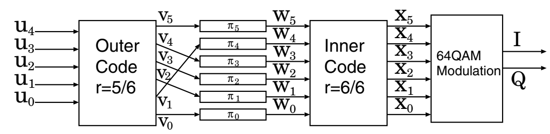

Fig. 5. The optimized SCTCM system with an in-line interleaver for $r = 5/6$ 64QAM.

The encoder structure of optimized code is shown in Fig. 5, the outer code in Fig. 6, the inner code in Fig. 7, and the signal mapping is shown in Fig. 8.

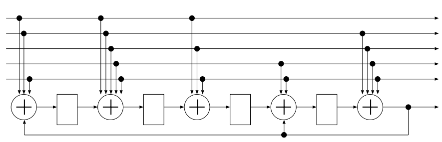

Fig. 6. The outer code of the optimized SCTCM system with an in-line interleaver.

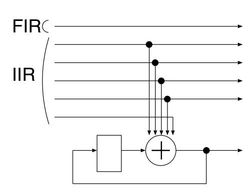

Fig. 7. The inner code of the optimized SCTCM system with an in-line interleaver.

We evaluated the performance of this code by applying the ML performance analysis method and convergence threshold analysis method. The dominant terms of error floor can be calculated as (10) using the equation described in [8].

$$P_b(e) \approx \frac{27}{20} N^{-1} \cdot \mathrm{erfc}\left(\sqrt{\frac{5}{14} \cdot \frac{E_b}{N_0}}\right). \tag{10}$$

Fig. 9 shows this term and simulation results for information block length 2040, 10000 and 50000. The convergence threshold SNR using the in-line EXIT chart [7] is $E_b/N_0 = 9.40$[dB] (Shannon Limit = 9.21[dB] for 5 bps/Hz).

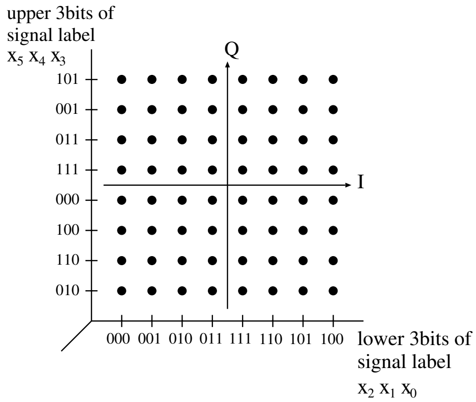

Fig. 8. The signal mapping of the optimized SCTCM system with an in-line interleaver, $(d_{E1}(Z) = Z^{d^2_{Emin}}, b_{d_{free}} = 6.25)$.

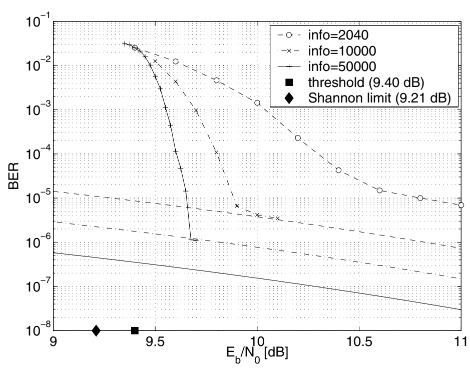

Fig. 9. BER performance of SCTCM with an in-line interleaver and dominant terms. (AWGN, information length : 2040, 10000, 50000, iterations: 20).

We compare the performance of our optimized SCTCM system with an in-line interleaver to Divsalar's SCTCM[4] and Robertson's Turbo TCM[3] in Fig. 10 (a) and (b). Divsalar's SCTCM is designed according to [4] by us. For both short and long block length, our proposed SCTCM system with an in-line interleaver exhibits better or equal performance than these turbo-like TCMs.

VI. Conclusion

In this paper we proposed an optimization method for constructing a high rate and high performance SCTCM system with an in-line interleaver. Simulation results show that our optimized SCTCM system with an in-line interleaver for rate $r = 5/6$ and 64QAM has convergence threshold which is near the Shannon limit, low error floor, and better performance than comparable turbo-like TCMs.

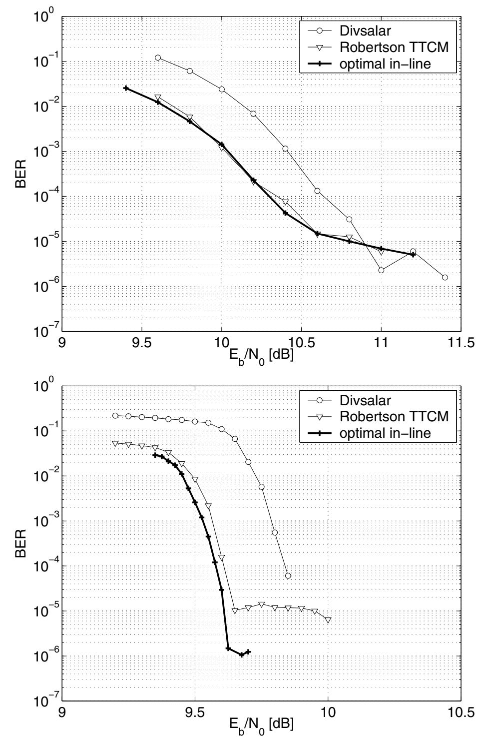

Fig. 10. Performance comparison of the optimized SCTCM system with an in-line interleaver v.s. other turbo-like TCM codes (AWGN, (upper) information length: 2040, iterations: 20, (lower) information length: 50000, iterations: 50).

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