

# Degrees of Freedom Region of the Gaussian MIMO Broadcast Channel with Common and Private Messages\*

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## Abstract

We consider the Gaussian multiple-input multiple-output (MIMO) broadcast channel with common and private messages. We obtain the degrees of freedom (DoF) region of this channel. We first show that a parallel Gaussian broadcast channel with unmatched sub-channels can be constructed from any given Gaussian MIMO broadcast channel by using the generalized singular value decomposition (GSVD) and a relaxation on the power constraint for the channel input, in a way that the capacity region of the constructed parallel channel provides an outer bound for the capacity region of the original channel. The capacity region of the parallel Gaussian broadcast channel with unmatched sub-channels is known, using which we obtain an explicit outer bound for the DoF region of the Gaussian MIMO broadcast channel. We finally show that this outer bound for the DoF region can be attained both by the achievable scheme that uses a classical Gaussian coding for the common message and dirty-paper coding (DPC) for the private messages, as well as by a variation of the zero-forcing (ZF) scheme.

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# 1 Introduction

We study the two-user Gaussian multiple-input multiple-output (MIMO) broadcast channel, where each link between the transmitter and each receiver is a linear additive Gaussian channel. We consider the scenario where the transmitter sends a private message to each user in addition to a common message which is directed to both users. The capacity region for this scenario, i.e., the capacity region of the Gaussian MIMO broadcast channel with common and private messages, is unknown. However, when one of these three messages is absent, the corresponding capacity region is known. In particular, the capacity region is known when there is no common message, i.e., each user gets only a private message [1], and for the degraded message set case, i.e., there is a common message directed to both users, and only one of the users gets a private message [2, 3].

The first work that considers the Gaussian MIMO broadcast channel with common and private messages is [4]. Reference [4] proposes an achievable scheme which uses a classical Gaussian coding scheme for the common message, and dirty-paper coding (DPC) for the private messages. The corresponding achievable rate region is called the DPC region. In addition, [4] obtains the capacity region when the Gaussian MIMO broadcast channel is equivalent to a set of parallel independent Gaussian channels by using the results from [5]. The Gaussian MIMO broadcast channel with common and private messages is further studied in [2, 3], where the partial optimality of the DPC region [4] is shown. References [2, 3] first propose an outer bound for the capacity region of the Gaussian MIMO broadcast channel with common and private messages, and then prove that it is tight on certain sub-regions of the capacity region by showing that it matches the DPC region given in [4]. Moreover, [2, 3] show that for a given common message rate, the private message sum capacity is attained by the achievable scheme in [4]. Finally, [2, 3] show the optimality of the DPC region in [4] when the common message rate is beyond a certain threshold.

A more recent work on the Gaussian MIMO broadcast channel is reported in [6]<sup>1</sup>. In [6], we first obtain an outer bound for the capacity region of the two-user discrete memoryless broadcast channel with common and private messages. We next show that if jointly Gaussian random variables are sufficient to evaluate this outer bound for the Gaussian MIMO broadcast channel, the DPC region is the capacity region of the Gaussian MIMO broadcast channel with common and private messages. However, we can evaluate only a loosened version of this outer bound, which yields the result that extending the DPC region in the common message rate direction by a fixed amount is an outer bound for the capacity region of the Gaussian MIMO broadcast channel with common and private messages. However, this fixed amount, i.e., the gap, does not have suitable scaling with the available power at the transmitter to enable us to obtain the degrees of freedom (DoF) region of the Gaussian MIMO broadcast channel with common and private messages.

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<sup>1</sup>Some of the results in [6] are concurrently and independently obtained in [7].

In this work, we follow a different approach and establish the DoF region of the Gaussian MIMO broadcast channel with common and private messages. We first show that we can construct a parallel Gaussian broadcast channel with unmatched sub-channels [5] from any given Gaussian MIMO broadcast channel such that the capacity region of this parallel Gaussian broadcast channel with unmatched sub-channels includes the capacity region of the Gaussian MIMO broadcast channel. To construct such a parallel channel, we use the generalized singular value decomposition (GSVD) [8] on the channel gain matrices of the Gaussian MIMO broadcast channel and also relax the power constraint on the channel input. This relaxation on the power constraint enlarges the capacity region during the transformation of the Gaussian MIMO broadcast channel into a parallel Gaussian broadcast channel with unmatched sub-channels. Consequently, the capacity region of the constructed parallel channel provides an outer bound for the capacity region of the Gaussian MIMO channel. Since the capacity region of the parallel Gaussian broadcast channel with unmatched sub-channels is known due to [5], we are able to characterize the DoF region of the parallel Gaussian broadcast channel with unmatched sub-channels, which serves as an outer bound for the DoF region of the Gaussian MIMO broadcast channel. We next show that this outer bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages can be attained by a proper selection of the covariance matrices involved in the DPC region [4]. Moreover, we also show that, in addition to the DPC scheme, a variation of the zero-forcing (ZF) scheme [9, 10] can attain the DoF region of the Gaussian MIMO broadcast channel with common and private messages.

## 2 Channel Model and Definitions

The Gaussian MIMO broadcast channel is defined by

$$\mathbf{Y}_1 = \mathbf{H}_1 \mathbf{X} + \mathbf{N}_1 \quad (1)$$

$$\mathbf{Y}_2 = \mathbf{H}_2 \mathbf{X} + \mathbf{N}_2 \quad (2)$$

where the channel input  $\mathbf{X}$  is a  $t \times 1$  column vector,  $\mathbf{H}_j$  is the  $j$ th user's channel gain matrix of size  $r_j \times t$ ,  $\mathbf{Y}_j$  is the channel output of the  $j$ th user which is an  $r_j \times 1$  column vector, and the Gaussian random vector  $\mathbf{N}_j$  is of size  $r_j \times 1$  with an identity covariance matrix. The channel input is subject to an average power constraint as follows

$$E[\mathbf{X}^\top \mathbf{X}] = \text{tr}(E[\mathbf{X}\mathbf{X}^\top]) \leq P \quad (3)$$

We study the Gaussian MIMO broadcast channel for the scenario where the transmitter sends a common message to both users, and a private message to each user. We call the channel model arising from this scenario the *Gaussian MIMO broadcast channel with common and private messages*. An  $(n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})$  code for this channel consists of three

message sets  $\mathcal{W}_0 = \{1, \dots, 2^{nR_0}\}$ ,  $\mathcal{W}_1 = \{1, \dots, 2^{nR_1}\}$ ,  $\mathcal{W}_2 = \{1, \dots, 2^{nR_2}\}$ , one encoder  $f_n : \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \rightarrow \mathcal{X}^n$ , one decoder at each receiver  $g_n^j : \mathcal{Y}_j^n \rightarrow \mathcal{W}_0 \times \mathcal{W}_j$ ,  $j = 1, 2$ . The probability of error is defined as  $P_e^n = \max\{P_{e1}^n, P_{e2}^n\}$ , where  $P_{ej} = \Pr[g_n^j(f_n(W_0, W_1, W_2)) \neq (W_0, W_j)]$ ,  $j = 1, 2$ , and  $W_j$  denotes the message which is a uniformly distributed random variable in  $\mathcal{W}_j$ ,  $j = 0, 1, 2$ . A rate triple  $(R_0, R_1, R_2)$  is said to be achievable if there exists a code  $(n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})$  which has  $\lim_{n \rightarrow \infty} P_e^n = 0$ . The capacity region  $\mathcal{C}(P)$  is defined as the convex closure of all achievable rate triples  $(R_0, R_1, R_2)$ .

Our main concern is to investigate how the capacity region  $\mathcal{C}(P)$  behaves when the available power at the transmitter  $P$  is arbitrarily large, i.e.,  $P$  goes to infinity. This investigation can be carried out by characterizing the DoF region of the Gaussian MIMO broadcast channel with common and private messages. A DoF triple  $(d_0, d_1, d_2)$  is said to be achievable if there exists a rate triple  $(R_0, R_1, R_2) \in \mathcal{C}(P)$  such that

$$d_j = \lim_{P \rightarrow \infty} \frac{R_j}{\frac{1}{2} \log P}, \quad j = 0, 1, 2 \quad (4)$$

The DoF region  $\mathcal{D}$  is defined as the convex closure of all achievable DoF triples  $(d_0, d_1, d_2)$ .

We conclude this section by presenting the achievable rate region, hereafter called the DPC region, given in [4]. In the achievable scheme in [4], the common message is encoded by a standard Gaussian codebook, and the private messages are encoded by DPC. Each user decodes the common message by treating the signals carrying the private messages as noise. Next, users decode their private messages. Since a DPC scheme is used to encode the private messages, one of the users observes an interference-free link depending on the encoding order at the transmitter. We next define

$$R_{0j}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|\mathbf{H}_j(\mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2)\mathbf{H}_j^\top + \mathbf{I}|}{|\mathbf{H}_j(\mathbf{K}_1 + \mathbf{K}_2)\mathbf{H}_j^\top + \mathbf{I}|}, \quad j = 1, 2 \quad (5)$$

$$R_1(\mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|\mathbf{H}_1(\mathbf{K}_1 + \mathbf{K}_2)\mathbf{H}_1^\top + \mathbf{I}|}{|\mathbf{H}_1\mathbf{K}_2\mathbf{H}_1^\top + \mathbf{I}|} \quad (6)$$

$$R_2(\mathbf{K}_2) = \frac{1}{2} \log |\mathbf{H}_2\mathbf{K}_2\mathbf{H}_2^\top + \mathbf{I}| \quad (7)$$

where  $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$  denote the covariance matrices allotted for the common message, the first user's private message, and the second user's private message, respectively. The DPC region is stated in the following theorem.

**Theorem 1** ([4]) *The rate triples  $(R_0, R_1, R_2)$  lying in the region*

$$\mathcal{R}^{\text{DPC}}(P) = \text{conv}(\mathcal{R}_1^{\text{DPC}}(P) \cup \mathcal{R}_2^{\text{DPC}}(P)) \quad (8)$$

*are achievable, where conv is the convex hull operator,  $\mathcal{R}_1^{\text{DPC}}(P)$  consists of rate triples*

$(R_0, R_1, R_2)$  satisfying

$$R_0 \leq R_{0j}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2), \quad j = 1, 2 \quad (9)$$

$$R_1 \leq R_1(\mathbf{K}_1, \mathbf{K}_2) \quad (10)$$

$$R_2 \leq R_2(\mathbf{K}_2) \quad (11)$$

for some positive semi-definite matrices  $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$  such that  $\text{tr}(\mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2) \leq P$ , and  $\mathcal{R}_2^{\text{DPC}}(P)$  can be obtained from  $\mathcal{R}_1^{\text{DPC}}(P)$  by swapping the subscripts 1 and 2.

The DPC region is tight in several cases. The first one is the case where each receiver gets only a private message, i.e.,  $R_0 = 0$  [1]. The other case is the degraded message sets scenario in which we have either  $R_1 = 0$  or  $R_2 = 0$  [2]. In both of these cases, there are only two messages to be sent. The case when both private messages and a common message are present is investigated in [2, 3]. In [2, 3], outer bounds on the capacity region with private and common messages are given and these outer bounds are shown to match the DPC region in certain regions. Furthermore, [2, 3] show that for a given common message rate  $R_0$ , the DPC region achieves the private message sum rate capacity, i.e., the maximum of  $R_1 + R_2$ . Finally, [2, 3] show that if the common message rate is beyond a certain threshold, the DPC region matches the capacity region if the channel input is subject to a covariance constraint, i.e.,  $E[\mathbf{X}\mathbf{X}^\top] \preceq \mathbf{S}$  for some  $\mathbf{S} \succeq \mathbf{0}$ . In [6], we show that an outer bound for the capacity region of the Gaussian MIMO broadcast channel with common and private messages can be obtained by extending the DPC region in the common message rate direction by a fixed amount. This fixed amount, i.e., the gap, depends on the channel gain matrices  $\mathbf{H}_1, \mathbf{H}_2$ , and is not finite for all possible channel gain matrices  $\mathbf{H}_1, \mathbf{H}_2$ .

### 3 Main Result

We now present our main result which characterizes the DoF region of the Gaussian MIMO broadcast channel with common and private messages. Our result shows that this DoF region can be attained by using the achievable scheme in Theorem 1, i.e., the DPC region in Theorem 1 is asymptotically tight. Moreover, we also show that in addition to the achievable scheme in Theorem 1, a variation of the ZF scheme [9, 10] can achieve the DoF region as well. Before stating our main result, we introduce the GSVD [8, 11] which plays a crucial role in the proof of our main result, and provides the necessary notation to express this result.

**Definition 1 ([8], Theorem 1)** *Given two matrices  $\mathbf{H}_1 \in \mathbb{R}^{r_1 \times t}$  and  $\mathbf{H}_2 \in \mathbb{R}^{r_2 \times t}$ , there exist orthonormal matrices  $\Psi_1 \in \mathbb{R}^{r_1 \times r_1}, \Psi_2 \in \mathbb{R}^{r_2 \times r_2}, \Psi_0 \in \mathbb{R}^{t \times t}$ , a non-singular, lower*

triangular matrix  $\mathbf{\Omega} \in \mathbb{R}^{k \times k}$ , and two matrices  $\mathbf{\Sigma}_1 \in \mathbb{R}^{r_1 \times k}$ ,  $\mathbf{\Sigma}_2 \in \mathbb{R}^{r_2 \times k}$  such that

$$\mathbf{\Psi}_1^\top \mathbf{H}_1 \mathbf{\Psi}_0 = \mathbf{\Sigma}_1 \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \quad (12)$$

$$\mathbf{\Psi}_2^\top \mathbf{H}_2 \mathbf{\Psi}_0 = \mathbf{\Sigma}_2 \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \quad (13)$$

where  $\mathbf{\Sigma}_1$  and  $\mathbf{\Sigma}_2$  are given by

$$\mathbf{\Sigma}_1 = \begin{bmatrix} \mathbf{I}_{k-p-s \times k-p-s} & & \\ & \mathbf{D}_{1,s \times s} & \\ & & \mathbf{0}_{r_1+p-k \times p} \end{bmatrix} \quad (14)$$

$$\mathbf{\Sigma}_2 = \begin{bmatrix} \mathbf{0}_{r_2-p-s \times k-p-s} & & \\ & \mathbf{D}_{2,s \times s} & \\ & & \mathbf{I}_{p \times p} \end{bmatrix} \quad (15)$$

and the constants  $k, p$  are given as

$$k = \text{rank} \left( \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix} \right) \quad (16)$$

$$p = \dim (\text{Null}(\mathbf{H}_1) \cap \text{Null}(\mathbf{H}_2)^\perp) \quad (17)$$

and  $s$  depends on the matrices  $\mathbf{H}_1, \mathbf{H}_2$ . The matrices  $\mathbf{D}_1, \mathbf{D}_2$  are diagonal with the diagonal elements being strictly positive.

We define the sets  $\mathcal{S}_1, \mathcal{S}_c, \mathcal{S}_2$  as follows

$$\mathcal{S}_1 = \{1, \dots, k-p-s\} \quad (18)$$

$$\mathcal{S}_c = \{k-p-s+1, \dots, k-p\} \quad (19)$$

$$\mathcal{S}_2 = \{k-p+1, \dots, k\} \quad (20)$$

Our main result is stated in the following theorem.

**Theorem 2** *The DoF region of the Gaussian MIMO broadcast channel with common and private messages is given by the union of DoF triples  $(d_0, d_1, d_2)$  satisfying*

$$d_0 \leq |\mathcal{S}_c| - \alpha_1 - \alpha_2 + \beta \quad (21)$$

$$d_1 \leq \alpha_1 + |\mathcal{S}_1| - \beta \quad (22)$$

$$d_2 \leq \alpha_2 + |\mathcal{S}_2| - \beta \quad (23)$$

for some non-negative  $\alpha_1, \alpha_2, \beta$  such that  $\alpha_1 + \alpha_2 \leq |\mathcal{S}_c|$ ,  $\beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\}$ . The DoF region of the Gaussian MIMO broadcast channel with common and private messages can be attained by the DPC region given in Theorem 1 as well as by a variation of the ZF scheme.

This theorem states that, if the available power  $P$  is sufficiently large, the Gaussian MIMO broadcast channel behaves as if it is a parallel Gaussian broadcast channel with  $|\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2|$  sub-channels.  $|\mathcal{S}_1|$  of these sub-channels can be accessed by only the first user,  $|\mathcal{S}_2|$  of these sub-channels can be accessed by only the second user, and  $|\mathcal{S}_c|$  of these sub-channels can be accessed by both users. For a fixed  $(\alpha_1, \alpha_2, \beta)$ ,  $|\mathcal{S}_c| - \alpha_1 - \alpha_2$  of the sub-channels that both users can access, need to be used for the transmission of the common message in addition to  $\beta$  of the  $|\mathcal{S}_1|$  sub-channels that only the first user can access and  $\beta$  of the  $|\mathcal{S}_2|$  sub-channels that only the second user can access. Thus, each user gets the common message over some common sub-channels, namely  $|\mathcal{S}_c| - \alpha_1 - \alpha_2$  sub-channels, which can be observed by both users, and some private sub-channels, namely  $\beta$  sub-channels, which can be observed by only one user. This leads to a  $\beta + |\mathcal{S}_c| - \alpha_1 - \alpha_2$  DoF for the common message. The first user's message needs to be transmitted over  $\alpha_1$  of the  $|\mathcal{S}_c|$  sub-channels that are observed by both users, and the remaining  $|\mathcal{S}_1| - \beta$  of the first user's private sub-channels (the rest of these sub-channels were dedicated to the transmission of the common message) that cannot be observed by the second user. This results in an  $\alpha_1 + |\mathcal{S}_1| - \beta$  DoF for the first user's private message. Similarly, the second user's message needs to be transmitted over  $\alpha_2$  of the  $|\mathcal{S}_c|$  sub-channels that are observed by both users, and the remaining  $|\mathcal{S}_2| - \beta$  of the second user's private sub-channels (the rest of these sub-channels were dedicated to the transmission of the common message) that cannot be observed by the first user. This results in an  $\alpha_2 + |\mathcal{S}_2| - \beta$  DoF for the second user's private message.

We provide the proof of Theorem 2 in the next three sections. In the next section, we obtain an outer bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages by using the GSVD, and also a relaxation on the power constraint for the channel input. In Section 5, we obtain an inner bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages. We obtain this inner bound by using two different achievable schemes. The first one directly uses Theorem 1, i.e., we make an explicit selection of the covariance matrices  $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$  involved in the DPC region to obtain this inner bound. The second one employs a variation of the ZF scheme [9, 10] to obtain the inner bound for the DoF region. The equivalence of these inner and outer bounds are shown in Section 6 to complete the proof of Theorem 2.

## 4 Outer Bound

We first obtain a new channel from the original one in (1)-(2)-(3) by using the GSVD, where the capacity region of the new channel includes the capacity region of the original one in (1)-(2)-(3). To this end, we note that

$$\mathbf{\Psi}_j^\top \mathbf{H}_j = \mathbf{\Sigma}_j [\mathbf{\Omega}^{-1} \quad \mathbf{0}_{k \times t-k}] \mathbf{\Psi}_0^\top, \quad j = 1, 2 \quad (24)$$

which is due to (12)-(13), and the fact that  $\mathbf{\Psi}_0$  is orthonormal. Since  $\mathbf{\Psi}_j$  is also orthonormal, i.e., non-singular, the capacity region of the following channel

$$\tilde{\mathbf{Y}}_j = \mathbf{\Psi}_j^\top \mathbf{Y}_j, \quad j = 1, 2 \quad (25)$$

is equal to the capacity region of the original one in (1)-(2)-(3). The channel defined in (25) can be explicitly expressed as

$$\tilde{\mathbf{Y}}_j = \mathbf{\Psi}_j^\top \mathbf{H}_j \mathbf{X} + \mathbf{\Psi}_j^\top \mathbf{N}_j \quad (26)$$

$$= \mathbf{\Sigma}_j \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \mathbf{\Psi}_0^\top \mathbf{X} + \mathbf{\Psi}_j^\top \mathbf{N}_j, \quad j = 1, 2 \quad (27)$$

where we used (24). We define  $\tilde{\mathbf{N}}_j = \mathbf{\Psi}_j^\top \mathbf{N}_j$  which is also a white Gaussian random vector, i.e.,

$$E \left[ \tilde{\mathbf{N}}_j \tilde{\mathbf{N}}_j^\top \right] = \mathbf{I} \quad (28)$$

due to the fact that  $\mathbf{\Psi}_j$  is orthonormal and  $\mathbf{N}_j$  is white. We also define

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \mathbf{\Psi}_0^\top \mathbf{X} \quad (29)$$

using which the channel in (27) can be written as

$$\tilde{\mathbf{Y}}_j = \mathbf{\Sigma}_j \tilde{\mathbf{X}} + \tilde{\mathbf{N}}_j, \quad j = 1, 2 \quad (30)$$

where the channel input  $\tilde{\mathbf{X}}$  should be chosen according to the trace constraint on  $\mathbf{X}$  stated in (3). We now relax the power constraint on  $\tilde{\mathbf{X}}$ , and consequently, obtain a new channel whose capacity region includes the capacity region of the original channel in (1)-(2)-(3). To this end, we note that

$$\text{tr} \left( E \left[ \tilde{\mathbf{X}} \tilde{\mathbf{X}}^\top \right] \right) = \text{tr} \left( \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \mathbf{\Psi}_0^\top E \left[ \mathbf{X} \mathbf{X}^\top \right] \mathbf{\Psi}_0 \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix}^\top \right) \quad (31)$$

$$= \text{tr} \left( E \left[ \mathbf{X} \mathbf{X}^\top \right] \mathbf{\Psi}_0 \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix}^\top \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \mathbf{\Psi}_0^\top \right) \quad (32)$$

where (31) comes from the definition of  $\tilde{\mathbf{X}}$  in (29), and (32) comes from the fact that  $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ . Since

$$\begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix}^\top \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \quad (33)$$

is a positive semi-definite matrix, there exists a  $\zeta > 0$  such that

$$\begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix}^\top \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \preceq \zeta \mathbf{I} \quad (34)$$



Since  $\text{tr}(\mathbf{AB}) \geq 0$  if  $\mathbf{A} \succeq \mathbf{0}, \mathbf{B} \succeq \mathbf{0}$ , using (34) in (32), we get

$$\text{tr} \left( E \left[ \tilde{\mathbf{X}} \tilde{\mathbf{X}}^\top \right] \right) \leq \zeta \text{tr} \left( E \left[ \mathbf{X} \mathbf{X}^\top \right] \mathbf{\Psi}_0 \mathbf{\Psi}_0^\top \right) \quad (35)$$

$$= \zeta \text{tr} \left( E \left[ \mathbf{X} \mathbf{X}^\top \right] \right) \quad (36)$$

$$\leq \zeta P \quad (37)$$

where (36) comes from the fact that  $\mathbf{\Psi}_0$  is orthonormal, and (37) is due to the total power constraint on  $\mathbf{X}$  given in (3). We now consider the following channel

$$\tilde{\mathbf{Y}}_j = \mathbf{\Sigma}_j \tilde{\mathbf{X}} + \tilde{\mathbf{N}}_j, \quad j = 1, 2 \quad (38)$$

where the channel input is subject to the following trace constraint

$$\text{tr} \left( E \left[ \tilde{\mathbf{X}} \tilde{\mathbf{X}}^\top \right] \right) \leq \zeta P \quad (39)$$

We note that this new channel in (38)-(39) is obtained from the original channel in (1)-(2)-(3) by two main operations: The first one is the multiplication of the channel outputs in the original channel, i.e., (1)-(2), with invertible matrices  $\mathbf{\Psi}_1, \mathbf{\Psi}_2$  which preserves the capacity region. The second operation is the relaxation of the power constraint in the new channel to get (39) which increases the capacity region by means of increasing the set of all feasible input distributions. Thus, due to this second operation, the capacity region of the new channel in (38)-(39) serves as an outer bound for the capacity region of the original channel in (1)-(2)-(3). Similarly, the DoF region of the new channel in (38)-(39) is an outer bound for the DoF region of the original channel in (1)-(2)-(3).

We next rewrite the channel in (38)-(39) in an alternative form. To this end, we note that the last  $(r_1 + p - k)$  entries of  $\tilde{\mathbf{Y}}_1$  come from only the noise. Since the noise is white, see (28), we can omit these last  $r_1 + p - k$  entries of  $\tilde{\mathbf{Y}}_1$  without loss of generality. Furthermore, we define

$$\tilde{h}_{1\ell} = \Sigma_{1,\ell\ell}, \quad 1 \leq \ell \leq k - p \quad (40)$$

Similarly, the first  $r_2 - p - s$  entries of  $\tilde{\mathbf{Y}}_2$  come from only the noise. Since the noise is white, see (28), we can again omit these first  $r_2 - p - s$  entries of  $\tilde{\mathbf{Y}}_2$  without loss of generality. Similarly, we also define

$$\tilde{h}_{2\ell} = \Sigma_{2,(r_2-k+\ell)\ell}, \quad k - p - s + 1 \leq \ell \leq k \quad (41)$$

Using the definitions in (40)-(41) and omitting the entries of  $\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2$  which contain only

noise, the channel in (38) can be expressed as

$$\tilde{Y}_{1\ell} = \tilde{h}_{1\ell}\tilde{X}_\ell + \tilde{N}_{1\ell}, \quad \ell = 1, \dots, |\mathcal{S}_1| + |\mathcal{S}_c| \quad (42)$$

$$\tilde{Y}_{2\ell} = \tilde{h}_{2\ell}\tilde{X}_\ell + \tilde{N}_{2\ell}, \quad \ell = |\mathcal{S}_1| + 1, \dots, |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| \quad (43)$$

where we used the definitions of  $\mathcal{S}_1, \mathcal{S}_c, \mathcal{S}_2$  given in (18)-(20) in conjunction with (40)-(41), and  $\{\tilde{N}_{1,\ell}\}_{\ell=1}^{\ell=|\mathcal{S}_1|+|\mathcal{S}_c|}, \{\tilde{N}_{2,\ell}\}_{\ell=|\mathcal{S}_1|+1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|}$  are i.i.d. Gaussian random variables with unit variance. The power constraint on the channel input in (39) can be rewritten as

$$\sum_{\ell=1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|} E[\tilde{X}_\ell^2] \leq \zeta P \quad (44)$$

We note that the channel defined by (42)-(43) is a parallel Gaussian broadcast channel with unmatched sub-channels, whose capacity region is obtained in [5]. In particular, the capacity region of this channel can be obtained by evaluating the following region

$$R_0 \leq \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_c} I(U_\ell; \tilde{Y}_{1\ell}) \quad (45)$$

$$R_0 \leq \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_c} I(U_\ell; \tilde{Y}_{2\ell}) \quad (46)$$

$$R_0 + R_1 \leq \sum_{\ell \in \mathcal{S}_{c2}} I(U_\ell; \tilde{Y}_{1\ell}) + \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_{c1}} I(X_\ell; \tilde{Y}_{1\ell}) \quad (47)$$

$$R_0 + R_2 \leq \sum_{\ell \in \mathcal{S}_{c1}} I(U_\ell; \tilde{Y}_{2\ell}) + \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} I(X_\ell; \tilde{Y}_{2\ell}) \quad (48)$$

$$R_0 + R_1 + R_2 \leq \sum_{\ell \in \mathcal{S}_{c2}} I(U_\ell; \tilde{Y}_{1\ell}) + \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} I(X_\ell; \tilde{Y}_{2\ell}|U_\ell) + \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_{c1}} I(X_\ell; \tilde{Y}_{1\ell}) \quad (49)$$

$$R_0 + R_1 + R_2 \leq \sum_{\ell \in \mathcal{S}_{c1}} I(U_\ell; \tilde{Y}_{2\ell}) + \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_{c1}} I(X_\ell; \tilde{Y}_{1\ell}|U_\ell) + \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} I(X_\ell; \tilde{Y}_{2\ell}) \quad (50)$$

by jointly Gaussian  $\{(U_\ell, X_\ell)\}_{\ell=1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|}$  [5], where  $\mathcal{S}_{c1}$  and  $\mathcal{S}_{c2}$  are given by

$$\mathcal{S}_{c1} = \left\{ \ell \in \mathcal{S}_c : \tilde{h}_{1\ell}^2 \geq \tilde{h}_{2\ell}^2 \right\} \quad (51)$$

$$\mathcal{S}_{c2} = \left\{ \ell \in \mathcal{S}_c : \tilde{h}_{2\ell}^2 \geq \tilde{h}_{1\ell}^2 \right\} \quad (52)$$

Hence, the capacity region of the channel in (42)-(43) is given by the union of the rate triples

$(R_0, R_1, R_2)$  satisfying

$$R_0 \leq \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_c} C \left( \frac{\tilde{h}_{1\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{1\ell}^2 \bar{\gamma}_\ell P_\ell} \right) \quad (53)$$

$$R_0 \leq \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_c} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right) \quad (54)$$

$$R_0 + R_1 \leq \sum_{\ell \in \mathcal{S}_{c2}} C \left( \frac{\tilde{h}_{1\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{1\ell}^2 \bar{\gamma}_\ell P_\ell} \right) + \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_{c1}} C \left( \tilde{h}_{1\ell}^2 P_\ell \right) \quad (55)$$

$$R_0 + R_2 \leq \sum_{\ell \in \mathcal{S}_{c1}} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right) + \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \tilde{h}_{2\ell}^2 P_\ell \right) \quad (56)$$

$$R_0 + R_1 + R_2 \leq \sum_{\ell \in \mathcal{S}_{c2}} C \left( \frac{\tilde{h}_{1\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{1\ell}^2 \bar{\gamma}_\ell P_\ell} \right) + \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell \right) + \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_{c1}} C \left( \tilde{h}_{1\ell}^2 P_\ell \right) \quad (57)$$

$$R_0 + R_1 + R_2 \leq \sum_{\ell \in \mathcal{S}_{c1}} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right) + \sum_{\ell \in \mathcal{S}_1 \cup \mathcal{S}_{c1}} C \left( \tilde{h}_{1\ell}^2 \bar{\gamma}_\ell P_\ell \right) + \sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \tilde{h}_{2\ell}^2 P_\ell \right) \quad (58)$$

for some  $\gamma_\ell = 1 - \bar{\gamma}_\ell \in [0, 1]$ ,  $\ell = 1, \dots, |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2|$ , and  $\{P_\ell\}_{\ell=1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|}$  such that  $\sum_{\ell=1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|} P_\ell = \zeta P$ , where  $C(x) = (1/2) \log(1 + x)$ .

We now obtain the DoF region of the channel in (42)-(43) by using (53)-(58), which will serve as an outer bound for the DoF region of the original channel in (1)-(2). To this end, we define

$$\eta_j = \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_j} C \left( \frac{\tilde{h}_{j\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{j\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P}, \quad j = 1, 2 \quad (59)$$

We define  $\delta_{c1}$  as follows

$$\delta_{c1} = \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_{c1}} C \left( \frac{\tilde{h}_{1\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{1\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P} \quad (60)$$

$$\geq \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_{c1}} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P} \quad (61)$$

where the inequality comes from the fact that  $\tilde{h}_{1\ell}^2 \geq \tilde{h}_{2\ell}^2$ ,  $\ell \in \mathcal{S}_{c1}$ . Similarly, we define  $\delta_{c2}$  as

follows

$$\delta_{c2} = \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_{c2}} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P} \quad (62)$$

$$\geq \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_{c2}} C \left( \frac{\tilde{h}_{1\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{1\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P} \quad (63)$$

where the inequality comes from the fact that  $\tilde{h}_{2\ell}^2 \geq \tilde{h}_{1\ell}^2$ ,  $\ell \in \mathcal{S}_{c2}$ . Using (59), (60),(63) and the bound in (53), we get

$$d_0 \leq \eta_1 + \delta_{c1} + \delta_{c2} \quad (64)$$

Similarly, using (59), (61),(62) and the bound in (54), we get

$$d_0 \leq \eta_2 + \delta_{c1} + \delta_{c2} \quad (65)$$

Using (63) and the rate bound in (55), we get

$$d_0 + d_1 \leq \delta_{c2} + |\mathcal{S}_{c1}| + |\mathcal{S}_1| \quad (66)$$

Similarly, using (61) and the rate bound in (56), we get

$$d_0 + d_2 \leq \delta_{c1} + |\mathcal{S}_{c2}| + |\mathcal{S}_2| \quad (67)$$

We next consider the rate bounds in (57) and (58) to obtain bounds for  $d_0 + d_1 + d_2$ . To this end, we note that

$$C \left( \tilde{h}_{j\ell}^2 \bar{\gamma}_\ell P_\ell \right) = C \left( \tilde{h}_{j\ell}^2 P_\ell \right) - C \left( \frac{\tilde{h}_{j\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{j\ell}^2 \bar{\gamma}_\ell P_\ell} \right) \quad (68)$$

Using this identity, we get

$$\lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell \right)}{\frac{1}{2} \log P} = \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \tilde{h}_{2\ell}^2 P_\ell \right)}{\frac{1}{2} \log P} - \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P} \quad (69)$$

$$\leq |\mathcal{S}_2| + |\mathcal{S}_{c2}| - \lim_{P \rightarrow \infty} \frac{\sum_{\ell \in \mathcal{S}_2 \cup \mathcal{S}_{c2}} C \left( \frac{\tilde{h}_{2\ell}^2 \gamma_\ell P_\ell}{1 + \tilde{h}_{2\ell}^2 \bar{\gamma}_\ell P_\ell} \right)}{\frac{1}{2} \log P} \quad (70)$$

$$= |\mathcal{S}_2| + |\mathcal{S}_{c2}| - \eta_2 - \delta_{c2} \quad (71)$$

where in (71), we used the definitions of  $\eta_2, \delta_{c2}$  given in (59), (62), respectively. Using (57)

and (71), we get

$$d_0 + d_1 + d_2 \leq \delta_{c2} + |\mathcal{S}_2| + |\mathcal{S}_{c2}| - \eta_2 - \delta_{c2} + |\mathcal{S}_{c1}| + |\mathcal{S}_1| \quad (72)$$

$$= |\mathcal{S}_2| + |\mathcal{S}_{c2}| + |\mathcal{S}_{c1}| + |\mathcal{S}_1| - \eta_2 \quad (73)$$

Similarly, using the rate bound in (58), we can get the following

$$d_0 + d_1 + d_2 \leq |\mathcal{S}_2| + |\mathcal{S}_{c2}| + |\mathcal{S}_{c1}| + |\mathcal{S}_1| - \eta_1 \quad (74)$$

Thus, we have obtained the DoF region of the channel in (42)-(43), which, by combining (64)-(67), (73)-(74), can be expressed as the union of the triples  $(d_0, d_1, d_2)$  satisfying

$$d_0 \leq \min\{\eta_1, \eta_2\} + \delta_{c1} + \delta_{c2} \quad (75)$$

$$d_0 + d_1 \leq \delta_{c2} + |\mathcal{S}_{c1}| + |\mathcal{S}_1| \quad (76)$$

$$d_0 + d_2 \leq \delta_{c1} + |\mathcal{S}_{c2}| + |\mathcal{S}_2| \quad (77)$$

$$d_0 + d_1 + d_2 \leq |\mathcal{S}_1| + |\mathcal{S}_2| + |\mathcal{S}_{c1}| + |\mathcal{S}_{c2}| - \max\{\eta_1, \eta_2\} \quad (78)$$

for some non-negative  $\eta_1, \eta_2, \delta_{c1}, \delta_{c2}$  such that  $\eta_j \leq |\mathcal{S}_j|$ ,  $\delta_{cj} \leq |\mathcal{S}_{cj}|$ . We define  $\eta = \min\{\eta_1, \eta_2\}$ . Using this definition, we can enlarge the region in (75)-(78) as follows

$$d_0 \leq \eta + \delta_{c1} + \delta_{c2} \quad (79)$$

$$d_0 + d_1 \leq \delta_{c2} + |\mathcal{S}_{c1}| + |\mathcal{S}_1| \quad (80)$$

$$d_0 + d_2 \leq \delta_{c1} + |\mathcal{S}_{c2}| + |\mathcal{S}_2| \quad (81)$$

$$d_0 + d_1 + d_2 \leq |\mathcal{S}_1| + |\mathcal{S}_2| + |\mathcal{S}_{c1}| + |\mathcal{S}_{c2}| - \eta \quad (82)$$

where  $0 \leq \eta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\}$ ,  $\delta_{cj} \leq |\mathcal{S}_{cj}|$ . Furthermore, we let  $\delta = \delta_{c1} + \delta_{c2}$  and define the region  $\mathcal{D}^{\text{out}}$  as the union of the DoF triples  $(d_0, d_1, d_2)$  satisfying

$$d_0 \leq \eta + \delta \quad (83)$$

$$d_0 + d_1 \leq |\mathcal{S}_c| + |\mathcal{S}_1| \quad (84)$$

$$d_0 + d_2 \leq |\mathcal{S}_c| + |\mathcal{S}_2| \quad (85)$$

$$d_0 + d_1 + d_2 \leq |\mathcal{S}_1| + |\mathcal{S}_2| + |\mathcal{S}_c| - \eta \quad (86)$$

for some non-negative  $\eta, \delta$  such that  $\eta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\}$ , and  $\delta \leq |\mathcal{S}_c|$ . It is clear that  $\mathcal{D}^{\text{out}}$  contains the region in (79)-(82), and hence, is an outer bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages given in (1)-(2)-(3).

## 5 Inner Bound

In this section, we provide an inner bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages, i.e., we show the achievability of the DoF region given in Theorem 2. In particular, we provide two different achievable schemes for the DoF region in Theorem 2. The first one, presented in Section 5.1, uses the DPC region in Theorem 1 directly. The second one, presented in Section 5.2, can be viewed as a variation of the ZF scheme that eliminates the inter-user interference and inter-transmit antenna interference by means of linear pre-processing at the transmitter and linear post-processing at the receivers.

### 5.1 DPC-based Achievable Scheme

We now obtain an inner bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages in (1)-(2)-(3) by using the achievable scheme given in Theorem 1. In particular, we make explicit selections for the covariance matrices  $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$  involved in the achievable scheme of Theorem 1, and show that the corresponding DoF region is equal to the one given in Theorem 2. To this end, we define the covariance matrices  $\mathbf{K}_u$  as follows

$$\mathbf{K}_u = (\xi P) \Psi_0 \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{0}_{t-k \times k} \end{bmatrix} \mathbf{\Lambda}_u \begin{bmatrix} \mathbf{\Omega}^\top & \mathbf{0}_{k \times t-k} \end{bmatrix} \Psi_0^\top, \quad u = 0, 1, 2 \quad (87)$$

where  $\mathbf{\Lambda}_u$  is a diagonal matrix of size  $k \times k$ .  $\xi$  in (87) is selected to ensure that  $\text{tr}(\mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2) \leq P$ . We next note the following identity

$$\frac{1}{\xi P} \mathbf{H}_j \mathbf{K}_u \mathbf{H}_j^\top = \Psi_j \Sigma_j \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \Psi_0^\top \Psi_0 \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{0}_{t-k \times k} \end{bmatrix} \mathbf{\Lambda}_u \begin{bmatrix} \mathbf{\Omega}^\top & \mathbf{0}_{k \times t-k} \end{bmatrix} \Psi_0^\top \mathbf{H}_j^\top \quad (88)$$

$$= \Psi_j \Sigma_j \mathbf{\Lambda}_u \begin{bmatrix} \mathbf{\Omega}^\top & \mathbf{0}_{k \times t-k} \end{bmatrix} \Psi_0^\top \mathbf{H}_j^\top \quad (89)$$

$$= \Psi_j \Sigma_j \mathbf{\Lambda}_u \begin{bmatrix} \mathbf{\Omega}^\top & \mathbf{0}_{k \times t-k} \end{bmatrix} \Psi_0^\top \Psi_0 \begin{bmatrix} \mathbf{\Omega}^{-\top} \\ \mathbf{0}_{t-k \times k} \end{bmatrix} \Sigma_j^\top \Psi_j^\top \quad (90)$$

$$= \Psi_j \Sigma_j \mathbf{\Lambda}_u \Sigma_j^\top \Psi_j^\top, \quad j = 1, 2, \quad u = 0, 1, 2 \quad (91)$$

where (88) and (90) come from the following identity

$$\mathbf{H}_j = \Psi_j \Sigma_j \begin{bmatrix} \mathbf{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \Psi_0^\top \quad (92)$$

which is a consequence of the GSVD in (12)-(13). Thus, using the covariance matrices  $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2$  defined by (87) for the achievable scheme in Theorem 1, we can get the following

achievable rates

$$R_0(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2) = \min_{j=1,2} \frac{1}{2} \log \frac{|(\xi P) \mathbf{\Psi}_j \mathbf{\Sigma}_j (\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{\Sigma}_j^\top \mathbf{\Psi}_j^\top + \mathbf{I}|}{|(\xi P) \mathbf{\Psi}_j \mathbf{\Sigma}_j (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{\Sigma}_j^\top \mathbf{\Psi}_j^\top + \mathbf{I}|} \quad (93)$$

$$R_1(\mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|(\xi P) \mathbf{\Psi}_1 \mathbf{\Sigma}_1 (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{\Sigma}_1^\top \mathbf{\Psi}_1^\top + \mathbf{I}|}{|(\xi P) \mathbf{\Psi}_1 \mathbf{\Sigma}_1 \mathbf{\Lambda}_2 \mathbf{\Sigma}_1^\top \mathbf{\Psi}_1^\top + \mathbf{I}|} \quad (94)$$

$$R_2(\mathbf{K}_2) = \frac{1}{2} \log |(\xi P) \mathbf{\Psi}_2 \mathbf{\Sigma}_2 \mathbf{\Lambda}_2 \mathbf{\Sigma}_2^\top \mathbf{\Psi}_2^\top + \mathbf{I}| \quad (95)$$

Using Sylvester's determinant theorem, i.e.,  $|\mathbf{A}_{m \times n} \mathbf{B}_{n \times m} + \mathbf{I}_{m \times m}| = |\mathbf{B}_{n \times m} \mathbf{A}_{m \times n} + \mathbf{I}_{n \times n}|$ , these rates can be expressed as follows

$$R_0(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2) = \min_{j=1,2} \frac{1}{2} \log \frac{|(\xi P) (\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{\Sigma}_j^\top \mathbf{\Sigma}_j + \mathbf{I}|}{|(\xi P) (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{\Sigma}_j^\top \mathbf{\Sigma}_j + \mathbf{I}|} \quad (96)$$

$$R_1(\mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|(\xi P) (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1 + \mathbf{I}|}{|(\xi P) \mathbf{\Lambda}_2 \mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1 + \mathbf{I}|} \quad (97)$$

$$R_2(\mathbf{K}_2) = \frac{1}{2} \log |(\xi P) \mathbf{\Lambda}_2 \mathbf{\Sigma}_2^\top \mathbf{\Sigma}_2 + \mathbf{I}| \quad (98)$$

We note that  $\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1, \mathbf{\Sigma}_2^\top \mathbf{\Sigma}_2$  are  $k \times k$  ( $k = |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2|$ ) diagonal matrices with the following structures

$$(\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1)_{\ell\ell} \begin{cases} > 0 & \text{if } 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| \\ = 0 & \text{if } |\mathcal{S}_1| + |\mathcal{S}_c| + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| \end{cases} \quad (99)$$

$$(\mathbf{\Sigma}_2^\top \mathbf{\Sigma}_2)_{\ell\ell} \begin{cases} = 0 & \text{if } 1 \leq \ell \leq |\mathcal{S}_1| \\ > 0 & \text{if } |\mathcal{S}_1| + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| \end{cases} \quad (100)$$

We next specify the diagonal matrices  $\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2$  as follows

$$\Lambda_{0,\ell\ell} = \begin{cases} 1 & \text{if } 1 \leq \ell \leq \beta \\ 0 & \text{if } \beta + 1 \leq \ell \leq |\mathcal{S}_1| + \alpha_1 \\ 1 & \text{if } |\mathcal{S}_1| + \alpha_1 + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| - \alpha_2 \\ 0 & \text{if } |\mathcal{S}_1| + |\mathcal{S}_c| - \alpha_2 + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| - \beta \\ 1 & \text{if } |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| - \beta + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| \end{cases} \quad (101)$$

$$\Lambda_{1,\ell\ell} = \begin{cases} 0 & \text{if } 1 \leq \ell \leq \beta \\ 1 & \text{if } \beta + 1 \leq \ell \leq |\mathcal{S}_1| + \alpha_1 \\ 0 & \text{if } |\mathcal{S}_1| + \alpha_1 + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| \end{cases} \quad (102)$$

$$\Lambda_{2,\ell\ell} = \begin{cases} 0 & \text{if } 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| - \alpha_2 \\ 1 & \text{if } |\mathcal{S}_1| + |\mathcal{S}_c| - \alpha_2 + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| - \beta \\ 0 & \text{if } |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| - \beta + 1 \leq \ell \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| \end{cases} \quad (103)$$

where  $0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\}$ ,  $0 \leq \alpha_j$ ,  $\alpha_1 + \alpha_2 \leq |\mathcal{S}_c|$ . These selections of  $\Lambda_0, \Lambda_1, \Lambda_2$  yield

$$R_{01}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|(\xi P)(\Lambda_0 + \Lambda_1 + \Lambda_2)\Sigma_1^\top \Sigma_1 + \mathbf{I}|}{|(\xi P)(\Lambda_1 + \Lambda_2)\Sigma_1^\top \Sigma_1 + \mathbf{I}|} \quad (104)$$

$$= \frac{1}{2} \sum_{\ell=1}^{|\mathcal{S}_1|+|\mathcal{S}_c|} \log \frac{(\xi P)(\Lambda_{0,\ell} + \Lambda_{1,\ell\ell} + \Lambda_{2,\ell\ell})(\Sigma_1^\top \Sigma_1)_{\ell\ell} + 1}{(\xi P)(\Lambda_{1,\ell\ell} + \Lambda_{2,\ell\ell})(\Sigma_1^\top \Sigma_1)_{\ell\ell} + 1} \quad (105)$$

$$= \frac{1}{2} \sum_{\ell=1}^{\beta} \log ((\xi P)(\Sigma_1^\top \Sigma_1)_{\ell\ell} + 1) + \frac{1}{2} \sum_{\ell=|\mathcal{S}_1|+\alpha_1+1}^{|\mathcal{S}_1|+|\mathcal{S}_c|-\alpha_2} \log ((\xi P)(\Sigma_1^\top \Sigma_1)_{\ell\ell} + 1) \quad (106)$$

where (105) comes from the fact that  $\Lambda_0, \Lambda_1, \Lambda_2, \Sigma_1^\top \Sigma_1$  are diagonal by noting the structure of  $\Sigma_1^\top \Sigma_1$  stated in (99), and (106) is a consequence of our  $\Lambda_0, \Lambda_1, \Lambda_2$  choices given in (101)-(103), respectively. Equation (106) implies that

$$\lim_{P \rightarrow \infty} \frac{R_{01}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2)}{\frac{1}{2} \log P} = \beta + |\mathcal{S}_c| - \alpha_1 - \alpha_2 \quad (107)$$

Similarly, we have

$$R_{02}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|(\xi P)(\Lambda_0 + \Lambda_1 + \Lambda_2)\Sigma_2^\top \Sigma_2 + \mathbf{I}|}{|(\xi P)(\Lambda_1 + \Lambda_2)\Sigma_2^\top \Sigma_2 + \mathbf{I}|} \quad (108)$$

$$= \frac{1}{2} \sum_{\ell=|\mathcal{S}_1|+1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|} \log \frac{(\xi P)(\Lambda_{0,\ell} + \Lambda_{1,\ell\ell} + \Lambda_{2,\ell\ell})(\Sigma_2^\top \Sigma_2)_{\ell\ell} + 1}{(\xi P)(\Lambda_{1,\ell\ell} + \Lambda_{2,\ell\ell})(\Sigma_2^\top \Sigma_2)_{\ell\ell} + 1} \quad (109)$$

$$= \frac{1}{2} \sum_{\ell=|\mathcal{S}_1|+\alpha_1+1}^{|\mathcal{S}_1|+|\mathcal{S}_c|-\alpha_2} \log ((\xi P)(\Sigma_2^\top \Sigma_2)_{\ell\ell} + 1) + \frac{1}{2} \sum_{\ell=|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|-\beta+1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|} \log ((\xi P)(\Sigma_2^\top \Sigma_2)_{\ell\ell} + 1) \quad (110)$$

where (109) comes from the fact that  $\Lambda_0, \Lambda_1, \Lambda_2, \Sigma_2^\top \Sigma_2$  are diagonal by noting the structure of  $\Sigma_2^\top \Sigma_2$  stated in (100), and (110) is a consequence of our  $\Lambda_0, \Lambda_1, \Lambda_2$  choices given in (101)-(103), respectively. Equation (110) implies that

$$\lim_{P \rightarrow \infty} \frac{R_{02}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2)}{\frac{1}{2} \log P} = \beta + |\mathcal{S}_c| - \alpha_1 - \alpha_2 \quad (111)$$



Hence, combining (107) and (111) yields that

$$d_0 = \beta + |\mathcal{S}_c| - \alpha_1 - \alpha_2 \quad (112)$$

is an achievable DoF for the common message. We now consider the first user's rate as follows

$$R_1(\mathbf{K}_1, \mathbf{K}_2) = \frac{1}{2} \log \frac{|(\xi P)(\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2)(\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1) + \mathbf{I}|}{|(\xi P)\mathbf{\Lambda}_2(\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1) + \mathbf{I}|} \quad (113)$$

$$= \frac{1}{2} \sum_{\ell=1}^{|\mathcal{S}_1|+|\mathcal{S}_c|} \log \frac{|(\xi P)(\Lambda_{1,\ell\ell} + \Lambda_{2,\ell\ell})(\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1)_{\ell\ell} + 1|}{|(\xi P)\Lambda_{2,\ell\ell}(\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1)_{\ell\ell} + 1|} \quad (114)$$

$$= \frac{1}{2} \sum_{\ell=\beta+1}^{|\mathcal{S}_1|+\alpha_1} \log ((\xi P)(\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1)_{\ell\ell} + 1) \quad (115)$$

where (114) comes from the fact that  $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1$  are diagonal by noting the structure of  $\mathbf{\Sigma}_1^\top \mathbf{\Sigma}_1$  stated in (99), and (115) is a consequence of our  $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2$  choices given in (102)-(103), respectively. Equation (115) implies that

$$d_1 = \alpha_1 + |\mathcal{S}_1| - \beta \quad (116)$$

is an achievable DoF for the first user's private message. We finally consider the second user's rate as follows

$$R_2(\mathbf{K}_2) = \frac{1}{2} \log |(\xi P)\mathbf{\Lambda}_2(\mathbf{\Sigma}_2^\top \mathbf{\Sigma}_2) + \mathbf{I}| \quad (117)$$

$$= \frac{1}{2} \sum_{\ell=|\mathcal{S}_1|+|\mathcal{S}_c|-\alpha_2+1}^{|\mathcal{S}_1|+|\mathcal{S}_c|+|\mathcal{S}_2|-\beta} \log ((\xi P)(\mathbf{\Sigma}_2^\top \mathbf{\Sigma}_2)_{\ell\ell} + 1) \quad (118)$$

where (118) comes from (100) and (103). Equation (118) implies that

$$d_2 = \alpha_2 + |\mathcal{S}_2| - \beta \quad (119)$$

is an achievable DoF for the second user's private message. Thus, we have obtained an inner bound  $\mathcal{D}^{\text{in}}$  for the DoF region of the Gaussian MIMO broadcast channel with common and private messages, where  $\mathcal{D}^{\text{in}}$  consists of DoF triples  $(d_0, d_1, d_2)$  satisfying

$$d_0 \leq |\mathcal{S}_c| - \alpha_1 - \alpha_2 + \beta \quad (120)$$

$$d_1 \leq \alpha_1 + |\mathcal{S}_1| - \beta \quad (121)$$

$$d_2 \leq \alpha_2 + |\mathcal{S}_2| - \beta \quad (122)$$

for some non-negative  $\alpha_1, \alpha_2, \beta$  such that  $\alpha_1 + \alpha_2 \leq |\mathcal{S}_c|, \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\}$ .

As a final remark, we note that here we obtain an inner bound for the DoF region of the Gaussian MIMO broadcast channel with common and private messages without any recourse to the alternative parallel channel-like representation of the Gaussian MIMO channel given in (30). Indeed, this inner bound can also be obtained by using this alternative representation, and in the next section, the ZF-based achievable scheme implicitly uses this alternative scheme.

## 5.2 ZF-based Achievable Scheme

In this section, we provide an alternative achievable scheme to show that the DoF region given in Theorem 2 is achievable. This alternative achievable scheme can be viewed as a variation of the ZF scheme [9, 10], where the ZF scheme is originally proposed for the Gaussian MIMO broadcast channel with only private messages, i.e., without a common message. In this ZF scheme, the transmitter eliminates the inter-user interference via a linear pre-processing of its transmitted signals. In particular, the transmitter sends each user's message in the null space of the other user's channel gain matrix such that each user sees an interference-free link between itself and the transmitter. However, this complete elimination of the inter-user interference can be accomplished only under certain conditions on the ranks of the channel gain matrices  $\mathbf{H}_1, \mathbf{H}_2$ , i.e., under certain conditions on the number of transmit and receive antennas  $t, r_1, r_2$ . In particular, [9, 10] show that the ZF scheme can attain the DoF for the private message sum rate<sup>2</sup> when  $r_1 + r_2 \leq t$ . This restriction comes from the fact that in the ZF scheme, each user's message is sent through the null-space of the other user's channel gain matrix. Alternatively, this restriction can be explained by examining the methodology of the ZF scheme, which uses individual singular value decompositions of the channel gain matrices  $\mathbf{H}_1, \mathbf{H}_2$  to obtain the pre-coding matrix of each user [9, 10]. However, by using the GSVD of the two channel gain matrices simultaneously to obtain the precoding matrices of the two users, this restriction can be removed as we do here. In the variation of the ZF scheme we propose here, the transmitter sends

$$\mathbf{X} = \Psi_0 \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{0}_{t-k \times k} \end{bmatrix} (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_c + \hat{\mathbf{X}}_2) \quad (123)$$

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<sup>2</sup>The DoF for the private message sum rate is given by

$$d^{\text{sum}} = \lim_{P \rightarrow \infty} \frac{R_1 + R_2}{\frac{1}{2} \log P}$$

where  $\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_c, \hat{\mathbf{X}}_2$  are given by

$$\hat{\mathbf{X}}_1 = \begin{bmatrix} \hat{X}_{11} \dots \hat{X}_{1|\mathcal{S}_1|} & \mathbf{0}_{1 \times (|\mathcal{S}_c| + |\mathcal{S}_2|)} \end{bmatrix}^\top \quad (124)$$

$$\hat{\mathbf{X}}_c = \begin{bmatrix} \mathbf{0}_{1 \times |\mathcal{S}_1|} & \hat{X}_{c1} \dots \hat{X}_{c|\mathcal{S}_c|} & \mathbf{0}_{1 \times |\mathcal{S}_2|} \end{bmatrix}^\top \quad (125)$$

$$\hat{\mathbf{X}}_2 = \begin{bmatrix} \mathbf{0}_{1 \times (|\mathcal{S}_1| + |\mathcal{S}_c|)} & \hat{X}_{21} \dots \hat{X}_{2|\mathcal{S}_2|} \end{bmatrix}^\top \quad (126)$$

Consequently, the received signal at the first user can be written as

$$\mathbf{Y}_1 = \mathbf{H}_1 \mathbf{X} + \mathbf{N}_1 \quad (127)$$

$$= \boldsymbol{\Psi}_1 \boldsymbol{\Sigma}_1 \begin{bmatrix} \boldsymbol{\Omega}^{-1} & \mathbf{0}_{k \times t-k} \end{bmatrix} \boldsymbol{\Psi}_0^\top \boldsymbol{\Psi}_0 \begin{bmatrix} \boldsymbol{\Omega} \\ \mathbf{0}_{t-k \times k} \end{bmatrix} (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_c + \hat{\mathbf{X}}_2) + \mathbf{N}_1 \quad (128)$$

$$= \boldsymbol{\Psi}_1 \boldsymbol{\Sigma}_1 (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_c + \hat{\mathbf{X}}_2) + \mathbf{N}_1 \quad (129)$$

$$= \boldsymbol{\Psi}_1 \boldsymbol{\Sigma}_1 (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_c) + \mathbf{N}_1 \quad (130)$$

where (128) is a consequence of the GSVD and (123), and (130) comes from the fact that  $\boldsymbol{\Sigma}_1 \hat{\mathbf{X}}_2 = \mathbf{0}$ . After multiplying  $\mathbf{Y}_1$  by the orthonormal matrix  $\boldsymbol{\Psi}_1^\top$ , we get

$$\hat{\mathbf{Y}}_1 = \boldsymbol{\Psi}_1^\top \mathbf{Y}_1 \quad (131)$$

$$= \boldsymbol{\Sigma}_1 (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_c) + \hat{\mathbf{N}}_1 \quad (132)$$

where  $\hat{\mathbf{N}}_1 = \boldsymbol{\Psi}_1^\top \mathbf{N}_1$  is additive white Gaussian noise with unit covariance matrix. Thus, the channel outputs resulting from the use of the channel input defined by (123)-(126) are given by

$$\hat{\mathbf{Y}}_1 = \boldsymbol{\Sigma}_1 (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_c) + \hat{\mathbf{N}}_1 \quad (133)$$

$$\hat{\mathbf{Y}}_2 = \boldsymbol{\Sigma}_2 (\hat{\mathbf{X}}_c + \hat{\mathbf{X}}_2) + \hat{\mathbf{N}}_2 \quad (134)$$

This equivalent form of the channel in (133)-(134), which results from the use of the ZF scheme, imply that, since  $\boldsymbol{\Sigma}_1$  and  $\boldsymbol{\Sigma}_2$  are diagonal, the ZF transforms the channel into a parallel Gaussian broadcast channel with unmatched sub-channels, where both users have access to  $|\mathcal{S}_c|$  sub-channels through which they observe a noisy version of  $\hat{\mathbf{X}}_c$ . In addition to these common sub-channels, the  $j$ th user has access to  $|\mathcal{S}_j|$  sub-channels through which it observes a noisy version of  $\hat{\mathbf{X}}_j$ , and the other user cannot observe these sub-channels. Now, we consider independent Gaussian coding across all sub-channels to obtain the DoF region given in Theorem 2. In particular, we send the common message through  $|\mathcal{S}_c| - \alpha_1 - \alpha_2$  sub-channels of the  $|\mathcal{S}_c|$  common sub-channels that both users can access and  $\beta$  sub-channels of each user's private sub-channels which cannot be observed by the other user. The  $j$ th user's private message is transmitted through  $\alpha_j$  sub-channels of the  $|\mathcal{S}_c|$  common sub-channels in

addition to the  $|\mathcal{S}_j| - \beta$  sub-channels of the  $j$ th user's private sub-channels that cannot be observed by the other user. Consequently, this coding scheme yields the following achievable rate triples

$$R_0 \approx \frac{|\mathcal{S}_c| - \alpha_1 - \alpha_2 + \beta}{2} \log P \quad (135)$$

$$R_1 \approx \frac{\alpha_1 + |\mathcal{S}_1| - \beta}{2} \log P \quad (136)$$

$$R_2 \approx \frac{\alpha_2 + |\mathcal{S}_2| - \beta}{2} \log P \quad (137)$$

for any non-negative  $\alpha_1, \alpha_2, \beta$  satisfying  $\alpha_1 + \alpha_2 \leq |\mathcal{S}_c|, \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\}$ . The achievable rate triples given by (135)-(137) imply that the DoF region that ZF scheme can attain is equal to the one that DPC attains, i.e.,  $\mathcal{D}^{\text{in}}$ , where  $\mathcal{D}^{\text{in}}$  is the DoF region given in Theorem 2.

## 6 Equivalence of the Inner and Outer Bounds

We now show that the inner bound  $\mathcal{D}^{\text{in}}$  for the DoF region of the Gaussian MIMO broadcast channel with common and private messages given in (120)-(122) is equal to the outer bound  $\mathcal{D}^{\text{out}}$  for the DoF region of the Gaussian MIMO broadcast channel with common and private messages given in (83)-(86).  $\mathcal{D}^{\text{in}}$  is defined by the following equations

$$d_0 \leq |\mathcal{S}_c| - \alpha_1 - \alpha_2 + \beta \quad (138)$$

$$d_1 \leq \alpha_1 + |\mathcal{S}_1| - \beta \quad (139)$$

$$d_2 \leq \alpha_2 + |\mathcal{S}_2| - \beta \quad (140)$$

$$0 \leq \alpha_1 \quad (141)$$

$$0 \leq \alpha_2 \quad (142)$$

$$\alpha_1 + \alpha_2 \leq |\mathcal{S}_c| \quad (143)$$

$$0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\} \quad (144)$$

We define  $\alpha = \alpha_1 + \alpha_2$ , using which in (138)-(144), we get

$$d_0 \leq |\mathcal{S}_c| - \alpha + \beta \quad (145)$$

$$d_1 \leq \alpha - \alpha_2 + |\mathcal{S}_1| - \beta \quad (146)$$

$$d_2 \leq \alpha_2 + |\mathcal{S}_2| - \beta \quad (147)$$

$$0 \leq \alpha - \alpha_2 \quad (148)$$

$$0 \leq \alpha_2 \quad (149)$$

$$\alpha \leq |\mathcal{S}_c| \quad (150)$$

$$0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\} \quad (151)$$

We can eliminate  $\alpha_2$  from (145)-(151) by using Fourier-Motzkin elimination, which yields

$$d_0 \leq |\mathcal{S}_c| - \alpha + \beta \quad (152)$$

$$d_1 + d_2 \leq \alpha + |\mathcal{S}_1| + |\mathcal{S}_2| - 2\beta \quad (153)$$

$$d_1 \leq \alpha + |\mathcal{S}_1| - \beta \quad (154)$$

$$d_2 \leq \alpha + |\mathcal{S}_2| - \beta \quad (155)$$

$$0 \leq \alpha \leq |\mathcal{S}_c| \quad (156)$$

$$0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\} \quad (157)$$

We next note that if  $(d_0, d_1, d_2)$  is an achievable DoF triple, so is  $(d_0 - t_1 - t_2, d_1 + t_1, d_2 + t_2)$  for any  $(t_1, t_2)$  such that  $0 \leq t_1, 0 \leq t_2, t_1 + t_2 \leq d_0$ . We define

$$d'_0 = d_0 - t_1 - t_2 \quad (158)$$

$$d'_1 = d_1 + t_1 \quad (159)$$

$$d'_2 = d_2 + t_2 \quad (160)$$

using which, (152)-(157) can be expressed as

$$d'_0 + t_1 + t_2 \leq |\mathcal{S}_c| - \alpha + \beta \quad (161)$$

$$d'_1 + d'_2 - t_1 - t_2 \leq \alpha + |\mathcal{S}_1| + |\mathcal{S}_2| - 2\beta \quad (162)$$

$$d'_1 - t_1 \leq \alpha + |\mathcal{S}_1| - \beta \quad (163)$$

$$d'_2 - t_2 \leq \alpha + |\mathcal{S}_2| - \beta \quad (164)$$

$$0 \leq t_1 \quad (165)$$

$$0 \leq t_2 \quad (166)$$

$$0 \leq \alpha \leq |\mathcal{S}_c| \quad (167)$$

$$0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\} \quad (168)$$

We can eliminate  $t_1$  from (161)-(168) by using Fourier-Motzkin elimination, which yields

$$d'_0 + d'_1 + d'_2 \leq |\mathcal{S}_c| + |\mathcal{S}_1| + |\mathcal{S}_2| - \beta \quad (169)$$

$$d'_0 + d'_1 + t_2 \leq |\mathcal{S}_1| + |\mathcal{S}_c| \quad (170)$$

$$d'_2 - t_2 \leq \alpha + |\mathcal{S}_2| - \beta \quad (171)$$

$$d'_0 + t_2 \leq |\mathcal{S}_c| - \alpha + \beta \quad (172)$$

$$0 \leq t_2 \quad (173)$$

$$0 \leq \alpha \leq |\mathcal{S}_c| \quad (174)$$

$$0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\} \quad (175)$$

After eliminating  $t_2$  from (169)-(175), we get

$$d'_0 + d'_1 + d'_2 \leq |\mathcal{S}_c| + |\mathcal{S}_1| + |\mathcal{S}_2| - \beta \quad (176)$$

$$d'_0 + d'_1 + d'_2 \leq |\mathcal{S}_1| + |\mathcal{S}_c| + |\mathcal{S}_2| - \beta + \alpha \quad (177)$$

$$d'_0 + d'_1 \leq |\mathcal{S}_1| + |\mathcal{S}_c| \quad (178)$$

$$d'_0 + d'_2 \leq |\mathcal{S}_2| + |\mathcal{S}_c| \quad (179)$$

$$d'_0 \leq |\mathcal{S}_c| - \alpha + \beta \quad (180)$$

$$0 \leq \alpha \leq |\mathcal{S}_c| \quad (181)$$

$$0 \leq \beta \leq \min\{|\mathcal{S}_1|, |\mathcal{S}_2|\} \quad (182)$$

We note that the bound in (177) is redundant. Since the region described by (176)-(182) is equal to the region  $\mathcal{D}^{\text{out}}$  in (83)-(86), this completes the proof.

## 7 Conclusions

In this work, we consider the Gaussian MIMO broadcast channel with common and private messages and obtain the DoF region of this channel. The crucial step in obtaining this result is to construct a parallel Gaussian broadcast channel with unmatched sub-channels from the Gaussian MIMO broadcast channel by using the GSVD. The capacity region of the constructed parallel channel provides an outer bound for the capacity region of the Gaussian MIMO broadcast channel. Using the capacity result for the parallel channel, we obtain an outer bound for the DoF region of the Gaussian MIMO broadcast channel. We show that this outer bound can be attained by the achievable scheme that combines a classical Gaussian coding for the common message and DPC for the private messages. In addition to the DPC scheme, we also show that a variation of the ZF scheme can attain the DoF region of the Gaussian MIMO broadcast channel with common and private messages.

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