This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2011 proceedings.

On the Exchange Rate for Bi-Directional Relaying over Inter-Symbol Interference Channels

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***Abstract*—We propose two compute-and-forward coding schemes for the bi-directional relay channel with inter-symbol interference (ISI) based on lattice codes and study their achievable rates. The first coding scheme is similar in spirit to coded orthogonal frequency division multiplexing (OFDM) with independent coding across sub-carriers and uses nested-lattice code with a power allocation strategy that can exploit the group property of lattices. The second coding scheme is a time-domain coding scheme which uses a novel precoding scheme at the transmitter in combination with lattice precoding and a minimum mean squared error receiver to recover linear combinations of lattice codewords. The proposed compute-and-forward coding schemes substantially outperform decode-and-forward schemes. While it is well known that for the point-to-point communication case, both the coded OFDM approach and the time-domain coding scheme can approach the capacity limit, we show that for the bi-directional relaying case, the performance of the two coding schemes are different. Particularly, we show that independent coding across sub-channels is not optimal and joint coding across sub-channels can improve the exchange capacity for some channel realizations.**

***Index Terms*—Bi-directional relay channel, joint physical layer coding and network coding, and inter-symbol interference.**

I. Introduction and Problem Statement

In this paper, we consider the bi-directional relay channel in which two nodes A and B wish to exchange information through a relay R between them as shown in Fig. 1. In addition, there is no direct link between nodes A and B. Different from other works on this problem, we consider the case that the communication is taken place in the presence of inter-symbol interference (ISI). Specifically, the time-invariant ISI channel from node A to node R (and node R to node A) is denoted as $\mathbf{h_A} \in \mathbb{C}^{L_A}$ and that from node B to node R (and node R to node B) is denoted as $\mathbf{h_B} \in \mathbb{C}^{L_B}$. Each node is assumed to have global channel knowledge. Let $\mathbf{w_A} = (\mathbf{w_A^R}, \mathbf{w_A^I}) \in \mathbb{F}_2^k \times \mathbb{F}_2^k$ and $\mathbf{w_B} = (\mathbf{w_B^R}, \mathbf{w_B^I}) \in \mathbb{F}_2^k \times \mathbb{F}_2^k$ be the message vectors at nodes A and B, respectively. These messages are then mapped to length-n codewords $\mathbf{x_A} \in \mathbb{C}^n$ and $\mathbf{x_B} \in \mathbb{C}^n$. The nodes are subject to a total power constraint, namely $P_A + P_B + P_R \leq 3P$. The transmission protocol we consider is the two phase protocol consisting of a multiple access channel (MAC) phase and a broadcast channel (BC) phase. Each of phases occupies a half of channel uses and is assumed to be orthogonal to each other.

This work was supported by the National Science Foundation under Grant CCF 0729210.

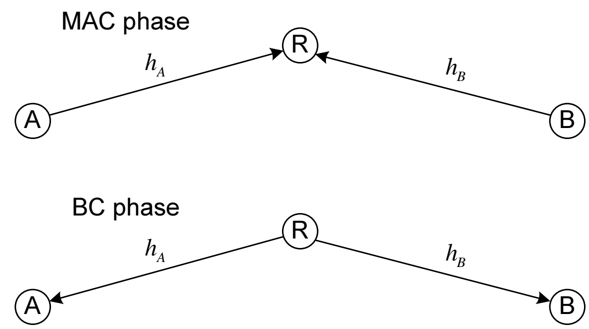

Fig. 1. Bi-directional relay network.

During the MAC phase, both nodes transmit their signals to the relay simultaneously and the relay keeps silent. The received signal at relay is given by

$$\mathbf{y_R} = \mathbf{x_A} * \mathbf{h_A} + \mathbf{x_B} * \mathbf{h_B} + \mathbf{z_R}, \tag{1}$$

where $*$ denotes the linear convolution operator and $\mathbf{z_R} \sim \mathcal{CN}(0, \sigma^2\mathbf{I})$ be i.i.d. Gaussian noise. Upon receiving $\mathbf{y_R}$, the relay maps it to the transmitted signal in the BC phase $\mathbf{x_R}$. This mapping depends on the forwarding strategy and will be discussed later.

During the BC phase, the relay broadcasts $\mathbf{x_R}$ back to nodes and both nodes keep silent. Then the received signal at nodes are

$$\mathbf{y_A} = \mathbf{x_R} * \mathbf{h_A} + \mathbf{z_A}, \tag{2}$$
$$\mathbf{y_B} = \mathbf{x_R} * \mathbf{h_B} + \mathbf{z_B}, \tag{3}$$

where again $\mathbf{z_A}, \mathbf{z_B} \sim \mathcal{CN}(0, \sigma^2\mathbf{I})$. Nodes A and B then form estimates of $\mathbf{w_B}$ and $\mathbf{w_A}$, namely $\hat{\mathbf{w}}_\mathbf{B}$ and $\hat{\mathbf{w}}_\mathbf{A}$, respectively. The error probability is given by

$$P_e = \Pr\left(\{\mathbf{w_A} \neq \hat{\mathbf{w}}_\mathbf{A}\} \cup \{\mathbf{w_B} \neq \hat{\mathbf{w}}_\mathbf{B}\}\right). \tag{4}$$

The communication is said to be reliable if this P_e can be made arbitrarily small as the block length n goes to infinity. Moreover, we say an exchange rate is achievable if there exists a coding scheme with this rate and a corresponding decoding function such that the reliable communication is possible. Therefore, the exchange rate can be defined as the maximum possible rate for both nodes to exchange their information reliably with the same rate. i.e.,

$$R_{ex} = \max_{P_e \longrightarrow 0} \frac{2k}{n}. \tag{5}$$

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Our goal in this paper is to propose coding/decoding schemes for the setup mentioned above and to give the corresponding achievable exchange rate.

II. Related Works

The problem of communication over the bi-directional relaying channel has attracted a great deal of attention; however, most of works focused on the AWGN channel without memory (see [1] and the reference therein.) For this memoryless setup, it has been shown that classical relaying strategies such as amplify-and-forward (AF) and decode-and-forward (DF) can be strictly suboptimal. Recently, an important observation has been pointed out in several independent works by Wilson *et al.* [2] and Nazer and Gastpar [3] that instead of decoding to the individual messages of both nodes at relay, it is sufficient for the relay to decode just enough information such that both nodes can correctly recover the other's message from it. With this knowledge, the authors in [2] exploited the group property of lattice codes to show that for the case that $h_A = h_B = 1$ and each node is subject to a same power constraint P, the following exchange rate is achievable

$$R_{ex,AWGN} = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{\sigma^2}\right). \tag{6}$$

Note that the 1/2 outside the logarithm comes from the fact that each phase occupies a half of channel uses. The main idea here is that two users first map their messages to codewords from the same nested lattice code, and then the relay tries to decode to the modulo sum of two codewords which is again a codeword in the same nested lattice code. The nodes are able to decode the other's message correctly according to this modulo sum and their own message. In [4], this coding scheme was extended to the case when both $h_A \in \mathbb{C}$ and $h_b \in \mathbb{C}$ (i.e., flat fading channels) with an total power constraint on the inputs to the channel. This can be done by introducing a power allocation strategy among nodes such that after the channel attenuation, the nested lattice codes used by node A and node B are perfectly aligned.

To the best of our knowledge, the achievable rate for bi-directional relaying over ISI channels has not been studied. For point to point communications with ISI, the capacity can be achieved by two different approaches. The first one is by partitioning the whole spectrum (which is frequency-selective) into infinitely many infinitesimal sub-channels that are flat fading (frequency-nonselective) and then use good AWGN codes at each sub-channel separately. Moreover, the power allocated at each sub-channel is obtained by the conventional water-filling algorithm for maximizing the overall rate [5]. This naturally leads to the family of multi-carrier transmission systems which partition the whole spectrum into a large number of sub-channels that can be considered as experiencing flat fading. The second approach proposed by Cioffi *et al.* [6] adopts single-carrier transmissions with infinitely-length unbiased minimum mean-squared-error decision-feedback equalizer (MMSE-DFE).

In what follows, we will extend the above two approaches to the bi-directional relay channel with ISI and compute the corresponding achievable exchange rate. Interestingly, we will show that different from the AWGN channel with ISI, the performance of these two approaches are quite different. Moreover, we will provide an example to show that for the considered setup, coding (or processing) across sub-channels is required for achieving exchange capacity. Due to space limitations, the background material on lattices and lattice codes is not covered in this paper, the interested reader is referred to [7] and [8].

III. Frequency-Domain Approach

We first partition the whole spectrum into M sub-channels and the corresponding channel gains at m^{th} sub-channel are $\tilde{h}_A(m)$ and $\tilde{h}_B(m)$ with $m \in \{1, 2, \dots M\}$ such that every sub-channel can be regarded as experiencing flat fading. Then we can directly extend the scheme proposed in [4] to the setup we consider here. In the frequency domain, the corresponding power constraint can be expressed as

$$\frac{1}{M}\sum_{m=1}^{M} P_A(m) + \frac{1}{M}\sum_{m=1}^{M} P_B(m) + \frac{1}{M}\sum_{m=1}^{M} P_R(m) \leq 3P, \tag{7}$$

where $P_i(m)$ is the power allocated at the m^{th} sub-channel of node i.

A. Proposed Coding scheme

For the m-th sub-channel, we choose an appropriate (in the sense that it achieves the AWGN capacity in the limit of dimension) nested lattice code $(\Lambda_f^{(n)}(m), \Lambda^{(n)}(m))$ where $\Lambda_f^{(n)}(m)$ is the fine lattice and $\Lambda^{(n)}(m)$ is the coarse lattice (shaping lattice) where m represents the sub-channel index and n denotes the lattice dimension. The second moment per unit dimension of the coarse lattice is denoted as $P_\Lambda(m)/2$. Nodes A and B first respectively map their messages onto lattice codewords $\mathbf{t_A}(m)$ and $\mathbf{t_B}(m)$ and then transmit the dithered version of these codewords on the m^{th} sub-channel. The frequency domain transmitted signals $\tilde{\mathbf{x}}_\mathbf{A}(m)$ and $\tilde{\mathbf{x}}_\mathbf{B}(m)$ are given by

$$\tilde{\mathbf{x}}_\mathbf{i}(m) = \sqrt{\frac{P_i(m)}{P_\Lambda(m)}}\left([\mathbf{t_i}(m) - \mathbf{d_i}(m)] \mod \Lambda^{(n)}(m)\right), \tag{8}$$

where $i \in \{A, B\}$ and $\mathbf{d_i}(m)$'s are uniformly distributed over the Voronoi region of the coarse lattice and are known by the relay also. Both nodes then collect transmitted signals at all sub-channels and perform an inverse discrete Fourier transform operation to get the time domain transmitted signals.

Suppose a sufficient cyclic prefix is inserted at both nodes and is perfectly removed at the relay, after a discrete Fourier transform the equivalent frequency domain received signal is given by

$$\tilde{\mathbf{y}}_\mathbf{R}(m) = \tilde{h}_A(m)\tilde{\mathbf{x}}_\mathbf{A}(m) + \tilde{h}_B(m)\tilde{\mathbf{x}}_\mathbf{B}(m) + \tilde{\mathbf{z}}_\mathbf{R}(m), \tag{9}$$

where $\tilde{\mathbf{z}}_{\mathbf{R}}(m) \sim \mathcal{CN}(0, \sigma^2 I)$. For enforcing the lattices $\tilde{h}_A(m)\sqrt{P_A(m)}\Lambda^{(n)}(m)$ and $\tilde{h}_B(m)\sqrt{P_B(m)}\Lambda^{(n)}(m)$ to be perfectly aligned at relay, we must have the following condition

$$|\tilde{h}_A(m)|^2 P_A(m) = |\tilde{h}_B(m)|^2 P_B(m), \tag{10}$$

for all $m = \{1, 2, \ldots M\}$. If this condition can be satisfied, from the main result in [2], we know that the relay can correctly decode $\mathbf{t}(m) = [\mathbf{t}_{\mathbf{A}}(m) + \mathbf{t}_{\mathbf{B}}(m)] \mod \Lambda^{(n)}(m)$ with high probability if the rate used by both node A and B at m-th sub-channel satisfy the following conditions

$$R_i(m) \leq \frac{1}{2M}\log(\frac{1}{2} + \frac{|\tilde{h}_i(m)|^2 P_i(m)}{\sigma^2}), \quad i \in \{A, B\} \tag{11}$$

Note that in the above rate expressions, the constant $1/2M$ comes from the fact that each sub-channel occupies $1/M$ fractions of channel bandwidth and both uplink and downlink occupy a half of the channel uses.

Upon decoding the $\mathbf{t}(m)$, the relay broadcasts it back by either a nested lattice code or a random Gaussian codebook. It is easy to observe that as long as the rate achieved in the BC phase is no less that achieved in the MAC phase, i.e., for $i, j \in \{A, B\}$ and $i \neq j$

$$1 + \frac{|\tilde{h}_i(m)|^2 P_R(m)}{\sigma^2} \geq \frac{1}{2} + \frac{|\tilde{h}_j(m)|^2 P_j(m)}{\sigma^2}, \tag{12}$$

the nodes can correctly decode $\mathbf{t}(m)$ (and hence $\mathbf{t}_{\mathbf{B}}(m)$ and $\mathbf{t}_{\mathbf{A}}(m)$, respectively) with high probability. Therefore, a rate of $R_A(m) = R_B(m)$ can be exchanged at the m-th sub-channel and the total exchange rate achieved by this coding scheme is $\sum_{m=1}^{M} R_A(m) = \sum_{m=1}^{M} R_B(m)$.

B. Power allocation strategy

We now try to maximize the total achievable exchange rate of this scheme through an optimal power allocation strategy among nodes. The corresponding optimization problem is

$$\max \frac{1}{M}\sum_{m=1}^{M}\frac{1}{2}\log(\frac{1}{2} + \frac{|\tilde{h}_A(m)|^2 P_A(m)}{\sigma^2}) \tag{13}$$

subject to (7), (10), (12), and the nonnegativity of $P_A(m)$, $P_B(m)$, and $P_R(m)$.

It is easy to verify that this problem is convex since its objective function is concave and all constraints are convex. Therefore, one can efficiently solve this optimization problem and obtain the optimal power allocation strategy.

IV. Time-Domain Approach

In this section, we will first introduce the coding scheme under the assumption that two links have the same l_2-norm ($\|\mathbf{h}_{\mathbf{A}}\|^2 = \|\mathbf{h}_{\mathbf{B}}\|^2$) and nodes are subject to individual power constraints $P_A = P_B = P_R = P$, respectively. Similar to the frequency domain approach, we will then extend the proposed scheme to general channels (which may have different l_2-norms) through a power allocation strategy.

A. Proposed Coding scheme

In the MAC phase, since we wish to again make use of the addition property of lattice codes, we need to somehow enforce the lattices chosen at node A and node B to be perfectly aligned at the relay. To this end, we introduce linear filters $\mathbf{f}_{\mathbf{A}}$ and $\mathbf{f}_{\mathbf{B}}$ respectively at two nodes so that the transmitted signals become

$$\mathbf{x}_{\mathbf{i}} = \bar{\mathbf{x}}_{\mathbf{i}} * \mathbf{f}_{\mathbf{i}}, \tag{14}$$

where $i \in \{A, B\}$ and $\bar{\mathbf{x}}_{\mathbf{i}}$'s with power P_{Λ_c} will be determined later. The key idea of our time domain coding scheme is to choose

$$\mathbf{f}_{\mathbf{A}} * \mathbf{h}_{\mathbf{A}} = \mathbf{f}_{\mathbf{B}} * \mathbf{h}_{\mathbf{B}} = \mathbf{g}, \tag{15}$$

so that two lattices are perfectly aligned at the relay.

Upon receiving, the relay first performs unbiased MMSE equalization so that at each time index t we have

$$v(t) = \bar{x}_A(t) + \bar{x}_B(t) + s_A(t) + s_B(t) + e(t) \tag{16}$$

where $s_A(t)$ and $s_B(t)$ denote the post-cursor ISI induced respectively by node A and node B and $e(t)$ is the sum of pre-cursor ISI and the filtered AWGN noise. Note that $e(t)$ can be regarded as the error obtained by the unbiased MMSE-DFE equalizer. Thus, the variance of this error can be computed as [6]

$$\sigma_e^2 = \frac{2P_{\Lambda_c}N_0}{S_0 - N_0}, \tag{17}$$

with

$$\log\frac{S_0}{N_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[\frac{2P_{\Lambda_c}}{\sigma^2}|S_{\mathbf{g}}(\theta)|^2 + 1\right]d\theta, \tag{18}$$

where $S_{\mathbf{g}}(\theta)$ is the power spectral density of g. Note that $e(t)$ is in general not Gaussian since the pre-cursor ISI is in general not Gaussian. It should also be noted that we do not literally perform decision-feedback equalization at the relay. Instead we adopt the lattice precoding at both nodes to get rid of the post-cursor ISI. The reason of this substitution will be discussed later.

We now have the collection of $v(t)$ as

$$\mathbf{v} = \bar{\mathbf{x}}_{\mathbf{A}} + \bar{\mathbf{x}}_{\mathbf{B}} + \mathbf{s}_{\mathbf{A}} + \mathbf{s}_{\mathbf{B}} + \mathbf{e}. \tag{19}$$

Similar to the frequency domain approach, we use a sequence of identical nested lattice $(\Lambda_f^{(n)}, \Lambda_c^{(n)})$ code for node A and node B. The coarse lattice has second moment per unit dimension $P_{\Lambda_c} = P/\|\mathbf{f}_{\mathbf{A}}\|^2$. Each node first maps its message to a codeword $\mathbf{t}_{\mathbf{A}}$ (and $\mathbf{t}_{\mathbf{B}}$) of this nested lattice code in an injective manner. Since all the post-cursor ISI are the linear combinations of previously transmitted signals weighted by effective channel gains, it is known at the transmitter. We then perform the lattice precoding [8] to cancel the known interference. The transmitted signals are given by

$$\bar{\mathbf{x}}_{\mathbf{i}} = [\mathbf{t}_{\mathbf{i}} - \alpha\mathbf{s}_{\mathbf{i}} - \mathbf{d}_{\mathbf{i}}] \mod \Lambda_c^{(n)}, \tag{20}$$

where $i \in \{A, B\}$ and $\mathbf{d}_{\mathbf{i}}$'s are again random dithers. Note that if we choose the coarse lattice $\Lambda_c^{(n)}$ to be good for quantization [7], then the $\bar{\mathbf{x}}_{\mathbf{A}}$ and $\bar{\mathbf{x}}_{\mathbf{B}}$ will tend to Gaussian distribution when $n \to \infty$. i.e., we can now assume $e(t)$ to have a Gaussian distribution.

The relay first forms the estimate of $\mathbf{t_R} = (\mathbf{t_A} + \mathbf{t_B}) \mod \Lambda_c^{(n)}$ from (16) as

$$\begin{aligned}\hat{\mathbf{t}}_\mathbf{R} &= [\alpha \mathbf{v} + \mathbf{d_A} + \mathbf{d_B}] \mod \Lambda_c^{(n)} \\ &= [\mathbf{t_R} - (1-\alpha)(\bar{\mathbf{x}}_\mathbf{A} + \bar{\mathbf{x}}_\mathbf{B}) + \alpha \mathbf{e}] \mod \Lambda_c^{(n)} \\ &= \mathbf{t_R} + \mathbf{n_{eq}} \mod \Lambda_c^{(n)}, \end{aligned} \tag{21}$$

where $\mathbf{n_{eq}} = \alpha \mathbf{e} - (1-\alpha)(\bar{\mathbf{x}}_\mathbf{A} + \bar{\mathbf{x}}_\mathbf{B})$.

In (21) we use the distributive property of modulo operation several times. By applying the group property of lattices and the *crypto lemma* [7], we can see that $\mathbf{t_R}$ is uniformly distributed over $\{\Lambda_f^{(n)} \cap \mathcal{V}(\Lambda_c^{(n)})\}$. Moreover, it can be seen that $\mathbf{e}$ is independent from $\bar{\mathbf{x}}_\mathbf{A}$ and $\bar{\mathbf{x}}_\mathbf{B}$. We then minimize the variance of the random variable $\mathbf{N_{eq}}$ (whose particular realizations are $\mathbf{n_{eq}}$) by choosing α as

$$\alpha = \frac{2P_{\Lambda_c}}{2P_{\Lambda_c} + \sigma_e^2}, \tag{22}$$

and the corresponding variance is given by

$$\sigma_{eq}^2 = \frac{2P_{\Lambda_c}\sigma_e^2}{2P_{\Lambda_c} + \sigma_e^2}. \tag{23}$$

The relay next performs lattice decoding to decode to the nearest lattice point of $\hat{\mathbf{t}}_\mathbf{R}$. Then, applying the *Theorem* 1 in [2] to here, we have that during the MAC phase, each node can transmit their message reliably as long as

$$R_{MAC}(\mathbf{g}) = \frac{1}{2}\log\left(\frac{P_{\Lambda_c}}{\sigma_{eq}^2}\right) = \frac{1}{2}\log\left(\frac{1}{2} + SNR_{MAC}\right), \tag{24}$$

where we define

$$SNR_{MAC} = \frac{P_{\Lambda_c}}{\sigma_e^2} = \frac{1}{2}\left(\frac{S_0}{N_0} - 1\right), \tag{25}$$

with S_0/N_0 defined previously in (18).

Remark 1: If we directly apply the lattice-based encoding and decoding scheme in [2] to here and use an unbiased MMSE-DFE at the relay, we still cannot get rid of the post-cursor ISI $\mathbf{s_A} + \mathbf{s_B}$. It is due to the fact that in lattice decoding, instead of $\mathbf{t_A}$ and $\mathbf{t_B}$, the relay tries to decode to $\mathbf{t_R} = (\mathbf{t_A} + \mathbf{t_B}) \mod \Lambda_c^{(n)}$, making it very difficult for the relay to reconstruct $\mathbf{s_A} + \mathbf{s_B}$.

For the BC phase, the relay broadcasts the decoded $\mathbf{t_R}$ back to the nodes by a nested lattice code or a random Gaussian codebook. Since each node knows their own transmitted signals, the decodability of the desired message is guaranteed once each node can correctly decode $\mathbf{t_R}$. Therefore, the achievable rates for node A and node B are given by

$$R_{BC,i} = \frac{1}{2}\log\left(1 + SNR_{BC,i}\right), \tag{26}$$

where

$$SNR_{BC,i} = \frac{S_{0,i}}{N_{0,i}} - 1, \tag{27}$$

with

$$\log\frac{S_{0,i}}{N_{0,i}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[\frac{P_R}{\sigma^2}|S_{\mathbf{h_i}}(\theta)|^2 + 1\right]d\theta, \tag{28}$$

and $i \in \{A, B\}$. Therefore, the achievable exchange rate is

$$R_{ex} = \min(R_{MAC}(\mathbf{g}), R_{BC,A}, R_{BC,B}). \tag{29}$$

Remark 2: Since all the pre-cursor ISI are also known at its own transmitter, it is natural to argue that one can further improve the rate using the lattice precoding to get rid of them also. However, since the pre-cursor ISI are dependent on the current symbol, any attempts of pre-processing the pre-cursor ISI may change the pre-cursor ISI themselves. This leads to a logical fallacy [8].

Now, the problem becomes to choose a valid pair of transmitted filters $\mathbf{f_A}$ and $\mathbf{f_B}$. We will give a feasible choice for the case $\|\mathbf{h_A}\|^2 = \|\mathbf{h_B}\|^2$ here and leave the optimal choice as the potential future work. Without loss of generality, we assume $\mathbf{h_A} \neq \mathbf{h_B}$. Otherwise, (15) is automatically satisfied. One feasible choice of filters is making $\mathbf{f_A} = \mathbf{h_B}$ and $\mathbf{f_B} = \mathbf{h_A}$ so that we have

$$\mathbf{g} = \mathbf{h_A} * \mathbf{h_B}, \tag{30}$$

and the corresponding P_{Λ_c} becomes

$$P_{\Lambda_c} = \frac{P}{\|\mathbf{h_A}\|^2} = \frac{P}{\|\mathbf{h_B}\|^2}. \tag{31}$$

Remark 3: This choice of $\mathbf{f_A}$ and $\mathbf{f_B}$ is by no means optimal. Of course one can try to optimize the choice of transmitted filters $\mathbf{f_A}$ and $\mathbf{f_B}$. However, since the corresponding optimization problem seems to be non-convex, we will leave the optimal filter design problem for the future work and will stick to the choice of transmitted filters to be $\mathbf{f_A} = \mathbf{h_B}$ and $\mathbf{f_B} = \mathbf{h_A}$ from now on. Therefore, in the following discussion, we define the achievable rate during the MAC phase $R_{MAC} = R_{MAC}(\mathbf{g})$ with $\mathbf{g}$ defined in (30).

Remark 4: Another possibility to improve the rate is to optimize the input power spectral density for $\bar{\mathbf{x}}_\mathbf{A}$ and $\bar{\mathbf{x}}_\mathbf{B}$ as in [6]. However, as discussed in [6], this helps only in low SNR regime since in the high SNR regime the optimal input power spectral density tends to be uniform.

B. Extension to general channels

For general channels that two links may have different l_2-norms, we can try to allocate power among nodes to maximize the exchange rate. Since we choose $\mathbf{f_A} = \mathbf{h_B}$ and $\mathbf{f_B} = \mathbf{h_A}$, for perfectly aligning lattices, we shall choose the power P_A and P_B such that

$$P_{\Lambda_c} = \frac{P_A}{\|\mathbf{h_B}\|^2} = \frac{P_B}{\|\mathbf{h_A}\|^2}. \tag{32}$$

For the same reason as that in constraint (12), P_R is chosen such that

$$1 + SNR_{BC,i} \geq \frac{1}{2} + SNR_{MAC}, \quad i \in \{A, B\}. \tag{33}$$

Therefore, the achievable exchange rate is given by (24). We then maximize the exchange rate through the power allocation strategy that maximizes R_{ex} subject to (32), (33), and the total power constraint.

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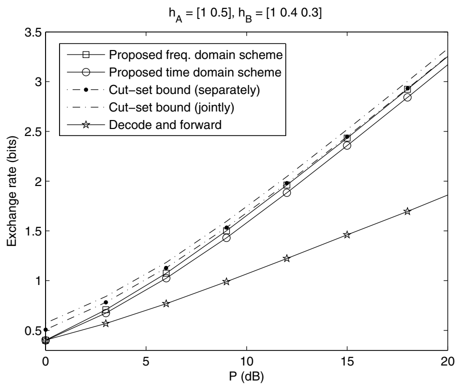

Fig. 2. Performance comparison for $\mathbf{h_A} = [1\ 0.5]^t$ and $\mathbf{h_B} = [1\ 0.3\ 0.4]^t$.

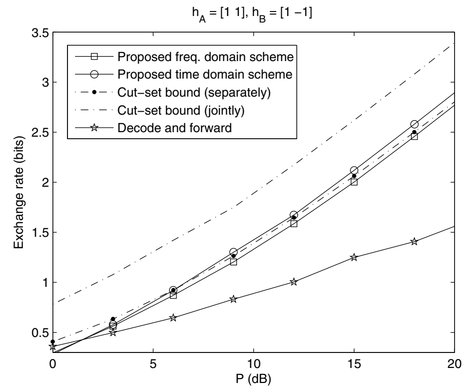

Fig. 3. Performance comparison for $\mathbf{h_A} = [1\ 1]^t$ and $\mathbf{h_B} = [1\ -1]^t$.

V. Numerical results

We present some numerical results here. In each figure, we plot the exchange rate achieved by both the frequency domain scheme and time domain scheme with optimal input spectrum as mentioned in *Remark* 4. Moreover, we provide the cut-set bound [9] for the case that joint encoding across sub-channels is allowed and the case that it is not allowed. For the cut-set bound that joint encoding is allowed, since we wish to have an outer bound for the time domain scheme, we restrict the power constraint to the individual power constraint P as we did in the time domain scheme. For comparison, we also provide the achievable exchange rate for the scheme performing the two-phase DF [10] at each sub-channel with the optimal power allocation strategy and each phase uses a half of channel uses.

The ISI channel parameters are set to be $\mathbf{h_A} = [1\ 0.5]^t$ and $\mathbf{h_B} = [1\ 0.3\ 0.4]^t$ for Fig. 2 and $\mathbf{h_A} = [1\ 1]^t$ and $\mathbf{h_B} = [1\ -1]^t$ for Fig. 3. It can be observed that in both figures, both proposed schemes outperform the DF substantially. We also observe that in Fig. 2 the frequency domain scheme performs better than the time domain scheme and approaches the cut-set bound very well. However, in Fig. 3, the time domain scheme outperforms the frequency domain scheme and even beats the cut-set bound of separate encoding. This result points out that, similar to the parallel Gaussian interference channel [11], joint encoding across sub-channels is required for achieving the capacity of the considered setup.

The superiority of the time domain scheme over the frequency domain scheme in Fig. 3 can be explained by the corresponding channel transfer functions which are *more mismatched* than that in Fig. 2. Thus, the frequency domain scheme may end up in wasting more power to the sub-channels that are mediocre for both directions and discarding some channels which are very good for one direction but very poor for the other. On the other hand, instead of channel gains at individual sub-channels, what really matters in the time domain scheme are the effective SNR obtained by the unbiased MMSE-DFE (25) and (27). Therefore, this severely mismatched channel condition provides an opportunity for the time domain scheme to outperform the frequency domain scheme.

VI. Conclusions and Future Works

In this work, we studied the achievable exchange rate for bi-directional relaying over ISI channels from both frequency and time domain viewpoints. Interestingly, different from its point to point communication counterpart, for the bi-directional relaying over ISI channels, there exist examples for which the time domain scheme performs better than the frequency domain scheme and even beats the cut-set bound of separate encoding schemes despite the fact that our choice of linear filters is by no means optimal. One potential future work is to further close the gap between the achievable exchange rate and the cut-set bound through optimizing the choice of linear filters.

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