Scaling Deep Learning-based Decoding of Polar Codes via Partitioning

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Abstract—The training complexity of deep learning-based channel decoders scales exponentially with the codebook size and therefore with the number of information bits. Thus, neural network decoding (NND) is currently only feasible for very short block lengths. In this work, we show that the conventional iterative decoding algorithm for polar codes can be enhanced when sub-blocks of the decoder are replaced by neural network (NN) based components. Thus, we partition the encoding graph into smaller sub-blocks and train them individually, closely approaching maximum a posteriori (MAP) performance per sub-block. These blocks are then connected via the remaining conventional belief propagation decoding stage(s). The resulting decoding algorithm is non-iterative and inherently enables a highlevel of parallelization, while showing a competitive bit error rate (BER) performance. We examine the degradation through partitioning and compare the resulting decoder to state-of-theart polar decoders such as successive cancellation list and belief propagation decoding.

I. Introduction

Non-iterative and consequently low-latency decoding together with close to maximum a posteriori (MAP) decoding performance are two advantages of deep learning-based channel decoding. However, this concept is mainly restricted by its limited scalability in terms of the supported block lengths, known as *curse of dimensionality* [1]. For k information bits, the neural network (NN) needs to distinguish between 2^k different codewords, which results in an exponential training complexity in case that the full codebook needs to be learned. In this work we focus on NN decoding of polar codes and show that the scalability can be significantly improved towards practical lengths when the NN only replaces sub-components of the current decoding algorithm.

List decoding [2] with manageable list sizes and excellent bit error rate (BER) performance for short block lengths makes polar codes [3] a potential candidate for future communication standards such as the upcoming 5G standard or *internet of things (IoT)* applications. As polar codes are currently proposed for the 5G control channel [4], decoding algorithms for very short block lengths are of practical importance [5]. Besides that, the rate can be completely flexible adjusted with a single-bit granularity. However, the price to pay is an inherently serial decoding algorithm which is in general hard to accelerate, e.g., through parallel processing [6]. This leads to high decoding latency when compared to state-of-the-art low density parity check (LDPC) codes/decoders [7]. Thus,

there is a demand for alternative decoding strategies. Besides algorithmic optimizations [7] of the existing algorithms, modifying the code structure [8] can be considered to overcome this issue. However, once standardized, the encoder cannot be changed. In this work, we propose an alternative approach by applying machine learning techniques to find an alternative decoding algorithm instead of changing the code structure. Once trained, the final decoding algorithm (i.e., the weights of a deep neural network) itself is static and can be efficiently implemented and parallelized on a graphical processing unit (GPU), field programmable gate array (FPGA), or application-specific integrated circuit (ASIC).

A first investigation of the topic learning to decode was already done in [1]. The authors showed that the main difficulty lies within the curse of dimensionality meaning that for k information bits 2^k classes exist, leading to exponential complexity during the training phase. In other applications, such as computer vision, the number of possible output classes is typically limited, e.g., to the number of different objects. In contrast to many other machine learning fields, an unlimited amount of labeled training data is available, since the encoding function and the channel model are well known. Additionally, a clear benchmark with existing decoders is possible. Although very powerful machine learning libraries such as Theano [9] and Tensorflow [10] are available nowadays and the computation power increased by order of magnitudes, the exponential complexity still hinders straight-forward learning of practical code lengths as shown in [11]. It was observed in [11], that there is a certain generalization of NN decoding, meaning that the NN can infer from certain codewords to others it has never seen before. This is essential for learning longer codes and gives hope that neural network decoding (NND) can be scaled to longer codes. However, to the best of our knowledge, the naive approach of learning to decode only works for rather small block lengths.

The authors in [12] proposed the idea of using machine learning techniques to train the weights of a belief propagation factor graph in order to improve its decoding performance for high density parity check (HDPC) codes. As the Tanner graph is already given initially and only its weights are refined, their approach scales very well for larger block lengths and does not suffer from the curse of dimensionality. However, in this case, the use of machine learning refines an existing solution.

The decoding algorithm itself is not learned, since the iterative nature of the BP algorithm is kept.

In our work, we tackle the problem of completely replacing the polar code decoder by a machine learning approach. As it turns out, only small codeword lengths can be trained efficiently, and thus we divide the polar encoding graph into several sub-graphs (cf. [13]). We learn sub-block wise decoding and couple the components by conventional belief propagation (BP) stages. This scales the results from [11] towards practical block lengths.

II. POLAR CODES

An encoder for polar codes maps the k information bits onto the k most reliable bit positions of the vector \mathbf{u} of length N, denoted as information set \mathbb{A} , while the remaining N-k positions are treated as frozen positions. These frozen positions are denoted as $\overline{\mathbb{A}}$ and must be known at the decoder side. Now the input block \mathbf{u} is encoded according to $\mathbf{x} = \mathbf{u} \cdot \mathbf{G}_N$, where $\mathbf{G} = \mathbf{F}^{\otimes n}$ is the generator matrix and $\mathbf{F}^{\otimes n}$ denotes the n^{th} Kronecker power of the kernel $\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. The resulting encoding circuit is depicted in Fig. 1 for N = 8, which also defines the decoding graph. This factor graph consists of $n+1=\log_2(N)+1$ stages, each consisting of N nodes.

The BER performance of a polar code highly depends on the type of decoder used and has been one of the most exciting and active areas of research related to polar coding. There are two main algorithmic avenues to tackle the polar decoding problem:

- 1) successive cancellation-based decoding, following a serial "channel layer unwrapping" decoding strategy [3],
- 2) belief propagation-based decoding based on Gallager's BP iterative algorithm [14].

Throughout this work, we stick with the BP decoder as its structure is a better match to neural networks and enables parallel processing. For details about successive cancellation (SC) decoding and its list extension called successive cancellation list (SCL) decoding, we refer the interested reader to [2] and [3]. The BP decoder describes an iterative message passing algorithm with soft-values, i.e., log likelihood ratio (LLR) values over the encoding graph. For the sake of simplicity, we assume binary phase shift keying (BPSK) modulation and an additive white Gaussian noise (AWGN) channel. However, other channels can be implemented straightforwardly. For a received value y, it holds that

$$LLR(y) = \ln\left(\frac{P(x=0|y)}{P(x=1|y)}\right) = \frac{2}{\sigma^2}y$$

where σ^2 is the noise variance. There are two types of LLR messages: the right-to-left messages (L-messages) and the left-to-right messages (R-messages). One BP iteration consists of two update propagations:

1) Left-to-right propagation: The R-messages are updated starting from the leftmost stage (i.e., the stage of a priori information) until reaching the rightmost stage.

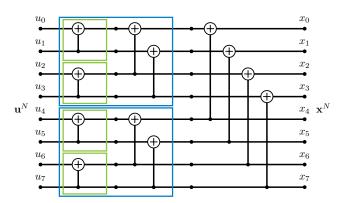


Fig. 1: Polar Encoding circuit for N=8; blue boxes indicate the independent partitions of the code for M=2 and green boxes for M=4.

 Right-to-left propagation: The L-messages are updated starting from the rightmost stage (i.e., the stage of channel information) until reaching the leftmost stage.

The output from two nodes becomes the input to a specific neighboring processing element (PE) (for more details we refer to [15]). One PE updates the **L**- and **R**-messages as follows [15]:

$$L_{out,1} = f(L_{in,1}, L_{in,2} + R_{in,2})$$

$$R_{out,1} = f(R_{in,1}, L_{in,2} + R_{in,2})$$

$$L_{out,2} = f(R_{in,1}, L_{in,1}) + L_{in,2}$$

$$R_{out,2} = f(R_{in,1}, L_{in,1}) + R_{in,2}$$
(1)

where

$$f(a,b) = \ln\left(\frac{1+e^{a+b}}{e^a+e^b}\right).$$

For initialization, all messages are set to zero, except for the first and last stage, where

$$L_{i,n+1} = L_{i,ch}$$
, and
$$R_{i,1} = \begin{cases} L_{max} & \forall i \in \bar{\mathbb{A}} \\ 0 & \text{else} \end{cases}$$
 (2)

with L_{max} denoting the clipping value of the decoder (in theory: $L_{max} \to \infty$), as all values within the simulation are clipped to be within $(-L_{max}, L_{max})$. This prevents from experiencing numerical instabilities.

A. Partitionable Codes

As opposed to other random-like channel codes with close-to-capacity performance, polar codes exhibit a very regular (algebraic) structure. It is instructive to realize that the encoding graph, as visualized in Fig. 1 for N=8, can be partitioned into independent sub-graphs [13], [16], i.e., there is no interconnection in the first $\log_2(N_p)$ stages, where N_p denotes the number of bits per sub-block. We define a partitionable code in a sense that each sub-block can be decoded independently (i.e., no interconnections within the same stage exist; leading to a tree like factor graph). This algorithm can be adopted to all partitionable codes and is

not necessarily limited to polar codes. Each sub-graph (in the following called sub-block) is now coupled with the other sub-blocks only via the remaining polar stages as depicted in Fig. 1. In order to simplify polar decoding, several sub-blocks B_i can now be decoded on a per-sub-block basis [13]. The set of frozen bit positions $\bar{\mathbb{A}}$ need to be split into sub-sets $\bar{\mathbb{A}}_i$ corresponding to the sub-blocks B_i (with information vector \mathbf{u}_i) and, thus, each sub-block might show a different code rate.

In a more abstract view, we follow the spirit of [17]. Their simplified successive cancellation (SSC) algorithm partitions the decoding tree into single-parity checks (SPC) and repetition codes (RC). This turns out to be a very efficient way of improving the overall throughput, as the SPC and RC sub-decoders can be efficiently implemented. Our approach replaces these partitions by NNs.

B. Deep-Learning for Channel Coding

For the fundamentals of deep learning, we refer the reader to [18]. However, for the sake of terminology, we provide a brief overview on the topic of machine learning. For a more detailed explanation how to train the sub-block NND we refer to [11]. Feedforward NNs consist of many connected neurons which are arranged in L layers without feedback connections. The output y of each neuron depends on its weighted inputs $\theta_i x_i$ and its activation function g, given by

$$y = g\left(\sum_{i} \theta_{i} x_{i} + \theta_{0}\right). \tag{3}$$

The whole network composes together many different functions $\mathbf{f}^{(l)}$ of each layer l and describes an input-output mapping

$$\mathbf{w} = \mathbf{f}(\mathbf{v}; \mathbf{\Theta}) = \mathbf{f}^{(L-1)} \left(\mathbf{f}^{(L-2)} \left(\dots \left(\mathbf{f}^{(0)} (\mathbf{v}) \right) \right) \right)$$
 (4)

where ${\bf v}$, ${\bf w}$ and ${\bf \Theta}$ denote the input vector, output vector and the weights of the NN, respectively. It was shown in [19] that such a multi-layer NN with L=2 and nonlinear activation functions can theoretically approximate any continuous function on a bounded region arbitrarily closely—if the number of neurons is large enough. A training set of known input-output mappings is required in order to find the weights ${\bf \Theta}$ of the NN with gradient descent optimization methods and the backpropagation algorithm [20]. After training, the NN is able to find the right output even for unknown inputs which is called generalization.

As described in [11], we use a feed-forward deep NN that can learn to map a noisy version of the codeword to its corresponding information bits. Each hidden layer employs rectified linear unit (ReLU) activation functions and the final stage is realized with a sigmoid activation function [18] in order to obtain output values in between zero and one giving the probability that the output bit is "1". In order to keep the training set small, we extend the decoder NN with additional layers which model an abstract channel [11], i.e., a training set containing every codeword is sufficient to train with as much training samples as desired.

It was shown in [11], that it is possible to decode polar codes with MAP performance for small block lengths. The

BER performance gap between a NND and MAP decoding is shown in Fig. 2. It illustrates that learning to decode is limited through exponential complexity as the number of information bits in the codewords increases.

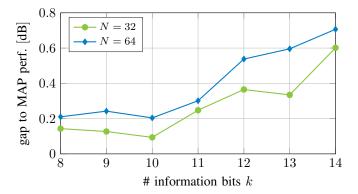


Fig. 2: Scalability shown by the gap to MAP performance in dB at BER = 0.01 for 32/64 bit-length polar codes with different code rates for fixed learning epoches.

III. PARTITIONED POLAR NEURAL NETWORK DECODING

Instead of using sequential SCL sub-block decoders as in [13], we replace them by NN decoders. Every sub-block NN_i covers k_i information bits. As the number of possible encoder output states (i.e., different codewords) is 2^{k_i} , efficient training is only possible for small sub-blocks, containing a small amount of information bits k_i [11]. The advantage of this concept is that each sub-block NN decoder can be trained independently. However, each NN has its own corresponding frozen bit positions \mathbb{A}_i and a specific block lengths N_i . As we deal with man made signals, the training and validation dataset can be created by random input data, a conventional polar encoder and a channel. Thus, an infinite amount of labeled training data is available.

Now, the NN decoder can be efficiently trained offline to MAP performance [11], since the effective block-size per subblock N_i reduces by the number of sub-blocks M. Each NN decoder outputs the decoded codeword (and/or the extracted information bits) either as soft-values, i.e., probabilities, or after a hard-decision which might require re-encoding. These bits are now treated as known values.

Our proposed decoder consists of two stages (see Fig. 1 and Fig. 3):

- 1) M deep learning blocks, trained to decode the corresponding sub-codeword with length N_i and frozen bit position vector \mathbb{A}_i
- 2) A *conventional* BP part, when the already decoded subblocks are propagated via the *coupling* stages of the remaining polar encoding graph.

After initializing the rightmost stages with the received channel LLR-values, messages are propagated from stage to stage according to the BP update rules in (1) until the first NN

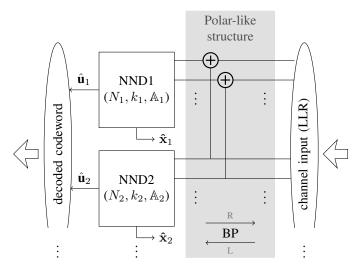


Fig. 3: Partitioned neural network polar decoding structure.

decoder stage $n_{\rm NN}$ (see Fig. 1). Then, the first NN decoder estimates the received sub-block ${\rm NN_1}$. After having decoded the first sub-block, the results are propagated via conventional BP decoding algorithms through the remaining *coupling* stages (see Fig. 1 and 3). Thus, this algorithm is block-wise sequential where M sub-blocks ${\rm NN}_i$ are sequentially decoded from top to bottom. The detailed decoding process works as follows:

- 1) Initialize stage n+1 with received channel LLR-values
- 2) Update stages n_{NN} to n+1 according to BP algorithm rules (propagate LLRs)
- 3) Decode next sub-block (top to bottom)
- 4) Re-encode results and treat results as perfectly known, i.e., as frozen bits
- 5) If not all sub-blocks are decoded go to step 2

The interface between the NND and the BP stages depends on the trained input format of the NN. Typically, the NN prefers normalized input values between 0 and 1. Fortunately, it was observed in [11] that the NN can handle both input formats and effective training is possible.

In summary, the system itself can be modeled as one large NN as well. Each BP update iterations defines additional layers, which are deterministic and thus do not effect the training complexity, similar to regularization layers [18]. This finally leads to a pipelined structure as depicted in Fig. 4. As each NN is only passed *once* and to emphasize the difference compared to iterative decoding, we term this kind of decoding as *one-shot-decoding*.

A. Further Optimizations

As it can be seen in Fig. 2, the limiting parameter is the number of information bits per sub-block k_i , since it defines the number of possible estimates of the NND. One further improvement in terms of sub-block size can be done by merging multiple equally-sized sub-blocks such that $\sum_i k_i < k_{\text{max}}$,

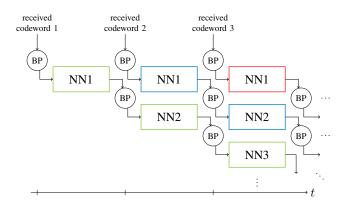


Fig. 4: Pipelined implementation of the proposed PNN decoder.

leading to an unequal sub-block size. This helps to obtain as few as possible sub-blocks. The results for M=8 partitions are shown in Fig. 5.

Additionally, *finetuning*-learning can be applied to the overall network in order to adjust the independently trained components to the overall system. This means that the whole decoding setup is used to re-train the system such that the conventional stages are taken into consideration for decoding. This prevents from performance degradation due to potentially non-Gaussian NN-input distributions, which was assumed during the training. Such an effect can be observed whenever clipping of the LLRs is involved. However, the basic structure is already fixed and thus only a small amount of the 2^k possible codewords is sufficient for good training results. The required training set is created with the free-running decoder.

The coupling could be also done by an SC stage (as originally proposed in [13]) without having additional iterations. However, the BP structure suits better to the NN structure as both algorithms can be efficiently described by a graph and their corresponding edge weights. Thus, the BP algorithm is preferred.

For cyclic redundancy check (CRC) aided decoding (a CRC check over the whole codeword) the CRC check can be split into smaller parts as in [13], where each CRC only protects one sub-block, i.e., the CRC can be considered by the NN decoders and thus straightforwardly learned. However, this requires at least some larger NN-sub-blocks, otherwise the rate-loss due to the CRC checks becomes prohibitive and is thus not considered at the moment.

IV. COMPARISON WITH SCL/BP

In general, a fair comparison with existing solutions is hard, as many possible optimizations need to be considered. We see this idea as an alternative approach, for instance in cases whenever low-latency is required. The BER results for N=128 and different decoding algorithms are shown in Fig. 5. As shown in Tab I, the size of the partitions is chosen such each partition does not contain more than $k_{\rm max}=12$ information bits, which facilitates learning the sub-blocks. If

TABLE I: Number of information bits k_i for each sub-block

sub-block i	1	2	3	4	5	6	7	8	
sub-block size N_i	32	16	16	16	16	8	8	16	
information bits k_i	1	3	11	5	13	7	8	16	

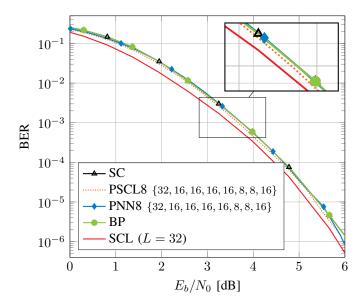


Fig. 5: BER performance of the proposed partitioned NN decoder for N=128 and M=8 partitions with variable size in comparison to state-of-the-art SC, BP and SCL (L=32) decoding.

the sub-block does not contain any frozen bits, a simple hard-decision decoder is used. Basically, the concept of partitioning polar codes helps to scale NND to longer codes.

We denote a partitioned neural network (PNN) with M partitions as PNNM. Although each NN in the PNN can be learned to approximately MAP performance, there is a loss λ between PNN decoding and conventional SCL decoding with list size L. We limit ourselves in this work to L=32. The loss λ can be explained by two main reasons:

- 1) Loss through partitioning λ_{part} as the concept only applies sub-block MAP decoding.
- 2) Loss due to sub-optimality of the NND λ_{NN} , due to insufficient training or non-Gaussian input distributions.¹

For further analysis of this problem, we replace the NNDs in each partition by SCL decoders with list size L=32 as in [13] and obtain a partitioned successive cancellation list (PSCL) decoder. As before, PSCLM terms a PSCL decoder with M partitions. This enables the investigation of larger partition sizes and thus the effect of partitioning, which is currently not possible with NNs due to the limited training length. Since SCL decoding approximates MAP performance

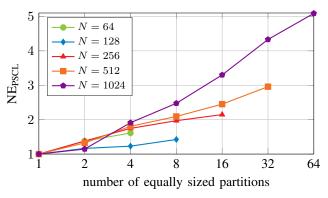


Fig. 6: Effect of the partitioning on the BER performance.

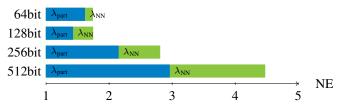


Fig. 7: Normalized error NE_{PSCL} due to λ_{part} (blue) and NE_{PNN} due to λ_{NN} (green) for fixed size of partitions $N_i = 16$.

for properly chosen parameters [2], the PSCL gives a lower bound on the expected BER according to the actual number of partitions. This enables to observe λ_{part} and λ_{NN} separately.

In order to quantify the loss, we introduce a similar concept as in [11]: the normalized error (NE) for a decoding concept, e.g., PNN, PSCL, which is given by

$$NE = \frac{1}{D} \sum_{t=1}^{D} \frac{BER(\rho_t)}{BER_{SCL}(\rho_t)}$$
 (5)

where ρ_t is a certain signal to noise ratio (SNR) value, BER (ρ_t) denotes the decoders BER for this specific SNR and BER_{SCL} (ρ_t) is the corresponding BER of the SCL decoder, respectively. Thus, NE compares the decoding performance over a range of D different SNR values ρ_1, \ldots, ρ_T .

It can be observed in Fig. 6, that NE_{PSCL} increases with the number of partitions, i.e., decreases with the partition sizes. Fig. 7 relates the effect of λ_{part} and λ_{NN} . It can be observed that the main error originates from partitioning and only a small part from suboptimal NNs. The amount of sub-blocks with larger k_i increases with larger codelength and at the same time it is more difficult to achieve MAP performance for these blocks [11]. Therefore, the loss λ_{NN} becomes more important for longer codes. The training complexity limits the feasible sub-block size of PNN decoding and thus long codes require a lot of partitions. The larger the number of partitions, the larger λ_{part} . However, the fast progress in the machine learning domain might enable larger sub-blocks, which should improve λ_{part} and therefore the overall performance of the PNN concept.

¹We did not focus on how to find the best NN structure for each NN because we want to introduce the concept of partitioned polar codes in order to scale NND. We expect that for sufficient training and hyperparameter tuning [21] this loss vanishes.

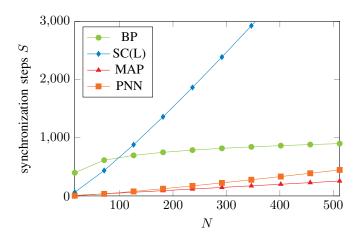


Fig. 8: Comparison of required synchronization steps for different polar decoding approaches.

In order to approximate the decoding latency in Fig. 8, we count the number of synchronization steps as operations can be done in parallel, but need to be synchronized after each synchronization step. The latency of the SCL algorithm can be described by $\mathcal{O}(N\log N)$ because it estimates each bit sequentially. Thus, the number of required synchronization steps is

$$S_{\text{SCL}} = N \log N.$$

The BP algorithm scales much better with $\mathcal{O}(\log N)$ because synchronization is required after each BP stage but depends on the number of iterations I, namely,

$$S_{\rm BP} = 2I \log N$$
.

The PNN decoder enforces less BP updates and the NNDs itself only synchronize after each layer

$$S_{\text{PNN}} = \frac{N}{N_P} N_H + \frac{N}{N_P} 2 \log \frac{N}{N_P}$$

where N_P and N_H respectively denote the size of each partition and the number of hidden layers in the NN. Fig. 8 is given for $N_P=16$ and $N_H=3$. To sum up, our approach enables latency reduction of BP decoding, while being competitive with the SC and BP BER performance.

V. CONCLUSION

In this work, we have shown that one way to reach scalability of deep-learning based channel decoding can be described by replacing sub-components of an existing decoder by NN-based components. This enables scalability in terms of block length and number of information bits towards practical lengths. Meanwhile, the length is still limited to short codes as the degradation through partitioning limits the overall performance of this concept. The BER performance of our decoder turns out to be similar to the SC and BP performance. However, the latency reduces a lot when the inherent parallel structure of this algorithm is exploited, since *one-shot-decoding* (i.e., non-iterative decoding) becomes possible.

We have shown that the performance degradation is mainly a result of the small partitions, as the sub-block size is currently strictly limited by the training. Nonetheless, the proposed setup would scale very well for larger sub-blocks, therefore future work needs to be done on potential improvements of the NN structure such that larger components become available.

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