

Broadcast Function Computation with Complementary Side Information

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Abstract—We consider the function computation problem in a three node network with one encoder and two decoders. The encoder has access to two correlated sources X and Y . The encoder encodes X^n and Y^n into a message which is given to two decoders. Decoder 1 and decoder 2 have access to X and Y respectively, and they want to compute two functions $f(X, Y)$ and $g(X, Y)$ respectively using the encoded message and their respective side information. We want to find the optimum (minimum) encoding rate under the zero error and ϵ -error (i.e. vanishing error) criteria. For the special case of this problem with $f(X, Y) = Y$ and $g(X, Y) = X$, we show that the ϵ -error optimum rate is also achievable with zero error. This result extends to a more general ‘complementary delivery index coding’ problem with arbitrary number of messages and decoders. For other functions, we show that the cut-set bound is achievable under ϵ -error if X and Y are binary, or if the functions are from a special class of ‘compatible’ functions which includes the case $f = g$.

I. INTRODUCTION

We consider the broadcast function network with complementary side information as shown in Fig. 1. Here, (X_i, Y_i) is an i.i.d. discrete random process with an underlying probability mass function $p_{XY}(x, y)$. An encoder encodes X^n and Y^n into a message, which is given to two decoders. Decoder 1 and decoder 2 have side information X and Y respectively, and want to compute $Z_1 = f(X, Y)$ and $Z_2 = g(X, Y)$ respectively. We study this problem under ϵ -error and zero error criteria. We are interested in finding the optimum broadcast rate in both cases.

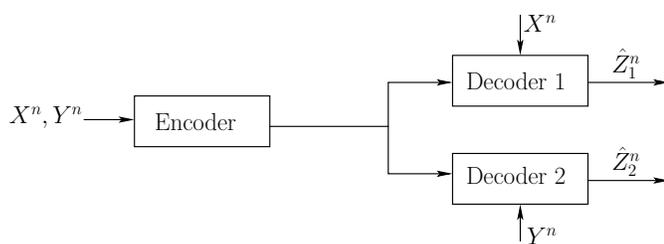


Fig. 1: Function computation in broadcast function network with complementary side information. Here $Z_1 = f(X, Y)$, $Z_2 = g(X, Y)$.

We first consider a special case of the problem with $Z_1 = Y$ and $Z_2 = X$, known as the complementary delivery problem. This special case is an instance of index coding problem

with two messages. This problem has been addressed under noisy broadcast channel in [1]–[3] for ϵ -error recovery of the messages. In contrast to their model of independent messages, we consider correlated messages over a noiseless broadcast channel. Lossy version of this problem was studied in [4], [5]. For the lossless case, the optimal ϵ -error rate can be shown to be $\max\{H(Y|X), H(X|Y)\}$ using the Slepian-Wolf result. We show that this rate is also achievable with zero error. We then extend this to n random variables which can also be considered as a special case of the index coding problem. Here, the server has messages X_1, \dots, X_K and there are m receivers. Each receiver has a subset of $\{X_1, \dots, X_K\}$ as side information, and all the receivers want to recover all the random variables that it does not have access to. We call this setup as *complementary delivery index coding* problem. Cut-set bound in this case can be shown to be achievable for ϵ -error using the Slepian-Wolf result. We show that this rate is also achievable with zero error.

Next we address the function computation problem shown in Fig. 1, where each decoder wants to recover a function of the messages. For ϵ -error criteria, we give a single letter characterization of the optimal broadcast rate when either (i) $Z_1 = Z_2$, (ii) X, Y are binary random variables, or (iii) Z_1, Z_2 belong to a special class of ‘compatible’ functions (defined in Section II). For zero error criteria with variable length coding, we give single letter upper and lower bounds for the optimal broadcast rate.

In contrast to correlated messages in our model, most work on index coding consider independent messages. On the other hand, in index coding problems in general, each receiver wants to recover an arbitrary subset of the messages. The goal is to minimize the broadcast rate of the message sent by the server (see [6]–[10] and references therein). For correlated sources, index coding problem has been studied for ϵ -error where the receivers demand their messages to be decoded with ϵ -error (see for example [11]). They gave an inner bound, and showed that it is tight for three receivers. To the best of our knowledge, index coding problem has not been considered for correlated sources with zero error. When the sources are independent and uniformly distributed, it was shown that the optimal rate for zero error and ϵ -error are the same [12]. Our result extends this to correlated sources with arbitrary distribution in the specific case of complementary delivery. The technique followed in [12] does not directly extend to correlated sources.

The paper is organized as follows. In Section II, we present our problem formulation and some definitions. We provide the main results of the paper in Section III. Proof of the results are presented in Section IV.

II. PROBLEM FORMULATION AND DEFINITIONS

A. Problem formulation: function computation

There are one encoder and two decoders for the function computation problem shown in Fig 1. A $(2^{nR}, n)$ code for variable length coding consists of one encoder

$$\phi : \mathcal{X}^n \times \mathcal{Y}^n \longrightarrow \{0, 1\}^*$$

and two decoders

$$\psi_1 : \phi(\mathcal{X}^n \times \mathcal{Y}^n) \times \mathcal{X}^n \longrightarrow \mathcal{Z}_1^n, \quad (1)$$

$$\psi_2 : \phi(\mathcal{X}^n \times \mathcal{Y}^n) \times \mathcal{Y}^n \longrightarrow \mathcal{Z}_2^n. \quad (2)$$

Here $\{0, 1\}^*$ denotes the set of all finite length binary sequences and we assume that the encoding is prefix free. Let us define $\hat{Z}_1^n = \psi_1(\phi(X^n, Y^n), X^n)$ and $\hat{Z}_2^n = \psi_2(\phi(X^n, Y^n), Y^n)$. The probability of error for a n length code is defined as

$$P_e^{(n)} \triangleq Pr\{(\hat{Z}_1^n, \hat{Z}_2^n) \neq (Z_1^n, Z_2^n)\} \quad (3)$$

The rate of the code is defined as

$$R = \frac{1}{n} \sum_{(x^n, y^n)} Pr(x^n, y^n) |\phi(x^n, y^n)|,$$

where $|\phi(x^n, y^n)|$ denotes the length of the encoded sequence $\phi(x^n, y^n)$. A rate R is said to be achievable with zero error if there is a zero-error code of some length n with rate R and $P_e^{(n)} = 0$. Let R_0^n denote the optimal zero error rate for n length code. Then the optimal zero error rate R_0^* is defined as $R_0^* = \lim_{n \rightarrow \infty} R_0^n$.

A fixed length $(2^{nR}, n)$ code consists of one encoder map

$$\phi : \mathcal{X}^n \times \mathcal{Y}^n \longrightarrow \{1, 2, \dots, 2^{nR}\}$$

and the two decoder maps as defined in (1), (2).

A rate R is said to be achievable with ϵ -error if there exists a sequence of $(2^{nR}, n)$ codes for which $P_e^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. The optimal broadcast rate in this case is the infimum of the set of all achievable rates and it is denoted by R_ϵ^* .

B. Problem formulation: Index coding

Let $H(i)$ denote the indices of the messages that receiver i has and let $X_{H(i)}$ denote their corresponding values. Let us denote the complement of the set $H(i)$ by $H^c(i)$. The set of messages that receiver i has, is denoted by $X_{H(i)}$. The set of messages receiver i wants is $X_{W(i)}$. For the complementary delivery index coding problem, $W(i) = H^c(i)$. The encoder, decoders, probability of error, achievable rate, etc. are defined similarly as before.

C. Graph theoretic definitions

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $I \subseteq V(G)$ is called an independent set if no two vertices in I are adjacent in G . Let $\Gamma(G)$ denote the set of all independent sets of G . A clique of a graph G is a complete subgraph of G . A clique of the largest size is called a maximum clique. The number of vertices in a maximum clique is called clique number of G and is denoted by $\omega(G)$. The chromatic number of G , denoted by $\chi(G)$, is the minimum number of colors required to color the graph G . A graph G is said to be perfect if for any vertex induced subgraph G' of G , $\omega(G') = \chi(G')$. Note that the vertex disjoint union of perfect graphs is also perfect.

The n -fold OR product of G , denoted by $G^{\vee n}$, is defined by $V(G^{\vee n}) = (V(G))^n$ and $E(G^{\vee n}) = \{(v^n, v'^n) : (v_i, v'_i) \in E(G) \text{ for some } i\}$. The n -fold AND product of G , denoted by $G^{\wedge n}$, is defined by $V(G^{\wedge n}) = (V(G))^n$ and $E(G^{\wedge n}) = \{(v^n, v'^n) : \text{either } v_i = v'_i \text{ or } (v_i, v'_i) \in E(G) \text{ for all } i\}$.

For a graph G and a random variable X taking values in $V(G)$, (G, X) represents a *probabilistic graph*. Chromatic entropy [17] of (G, X) is defined as

$$H_\chi(G, X) = \min\{H(c(X)) : c \text{ is a coloring of } G\}.$$

Let W be distributed over the power set $2^{\mathcal{X}}$. The graph entropy of the probabilistic graph (G, X) is defined as

$$H_G(X) = \min_{X \in W \in \Gamma(G)} I(W; X), \quad (4)$$

where $\Gamma(G)$ is the set of all independent sets of G . Here the minimum is taken over all conditional distributions $p_{W|X}$ which are non-zero only for $X \in W$. The following result was shown in [17].

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_\chi(G^{\vee n}, X^n) = H_G(X). \quad (5)$$

The complementary graph entropy of (G, X) is defined as

$$\bar{H}_G(X) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 \{\chi(G^{\wedge n}(T_{P_X, \epsilon}^n))\},$$

where $T_{P_X, \epsilon}^n$ denotes the ϵ -typical set of length n under the distribution P_X . It was shown in [18] that

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_\chi(G^{\wedge n}, X^n) = \bar{H}_G(X). \quad (6)$$

To address the function computation problem, we define some suitable graphs. Let $S_{\mathcal{X}^n \times \mathcal{Y}^n}$ denote the support set of (X^n, Y^n) . A rook's graph defined over $\mathcal{X} \times \mathcal{Y}$ has its vertex set $\mathcal{X} \times \mathcal{Y}$ and edge set $\{(x, y), (x', y') : x = x' \text{ or } y = y', \text{ but } (x, y) \neq (x', y')\}$. For functions $Z_1 = f(X, Y)$, $Z_2 = g(X, Y)$ defined over $\mathcal{X} \times \mathcal{Y}$, we now define a graph called $Z_1 Z_2$ -modified rook's graph which is similar to the f -modified rook's graph defined in [14].

Definition 1 $Z_1 Z_2$ -modified rook's graph $G_{\mathcal{X} \times \mathcal{Y}}^{Z_1 Z_2}$ is a subgraph of the rook's graph on $\mathcal{X} \times \mathcal{Y}$, which has its vertex

set S_{XY} , and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if

- 1) $x_1 = x_2$ and $f(x_1, y_1) \neq f(x_2, y_2)$,
- or 2) $y_1 = y_2$ and $g(x_1, y_1) \neq g(x_2, y_2)$.

Example 1 Let us consider a doubly symmetric binary source (DSBS(p)) (X, Y) where $p_{X,Y}(0, 0) = p_{X,Y}(1, 1) = (1-p)/2$ and $p_{X,Y}(0, 1) = p_{X,Y}(1, 0) = p/2$, and functions Z_1, Z_2 given by

$$Z_1 = X \cdot Y \quad (7)$$

$$Z_2 = \begin{cases} Y & \text{if } Y = 0 \\ X & \text{if } Y = 1, \end{cases} \quad (8)$$

$Z_1 Z_2$ -modified rook's graph of these functions is shown in Fig. 2a.

Next we extend the definition of $G_{XY}^{Z_1 Z_2}$ to n instances:

Definition 2 $G_{XY}^{Z_1 Z_2}(n)$ has its vertex set $S_{X^n Y^n}$, and two vertices (x^n, y^n) and (x'^n, y'^n) are adjacent if and only if

- 1) $x^n = x'^n$ and $f(x_i, y_i) \neq f(x'_i, y'_i)$ for some i ,
- or 2) $y^n = y'^n$ and $g(x_i, y_i) \neq g(x'_i, y'_i)$ for some i .

Clearly, $G_{XY}^{Z_1 Z_2}(n)$ is the $Z_1^n Z_2^n$ -modified rook's graph on the vertex set $S_{X^n Y^n}$. We note here from the definitions that $G_{XY}^{Z_1 Z_2}(n)$ is a subgraph of $(G_{XY}^{Z_1 Z_2})^{\vee n}$.

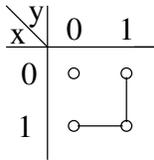
Definition 3 Functions Z_1, Z_2 are said to be compatible if there exists a function $Z = h(X, Y)$ such that $G_{XY}^{ZZ} = G_{XY}^{Z_1 Z_2}$. We call such a graph $G_{XY}^{Z_1 Z_2}$ compatible.

Example 2 Let us consider another pair Z_1, Z_2 which is also defined over a DSBS(p).

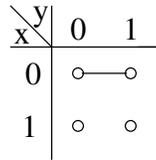
$$Z_1 = \begin{cases} Y & \text{if } X = 0 \\ X & \text{if } X = 1, \end{cases} \quad (9)$$

$$Z_2 = Y. \quad (10)$$

$Z_1 Z_2$ -modified rook's graph of the above functions is shown in Fig. 2b. $G_{XY}^{Z_1 Z_2}$ in Fig. 2b is not a compatible graph. Whereas $G_{XY}^{Z_1 Z_2}$ in Fig. 2a is a compatible graph because it is the same as G_{XY}^{ZZ} for $Z = X \cdot Y$.



(a) $G_{XY}^{Z_1 Z_2}$ for Z_1, Z_2 defined in (7),(8)



(b) $G_{XY}^{Z_1 Z_2}$ for Z_1, Z_2 defined in (9),(10)

Fig. 2: $Z_1 Z_2$ -modified rook's graphs

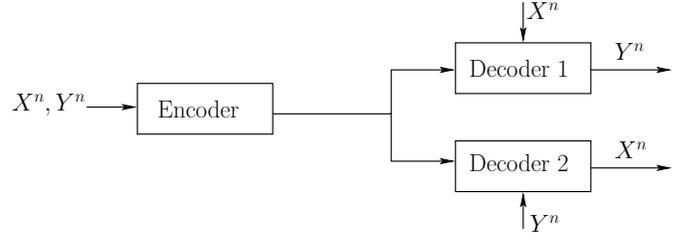


Fig. 3: Complementary delivery

III. MAIN RESULTS

Our first result shows that the optimal rate for zero error and ϵ -error are the same for the complementary delivery problem* shown in Fig 3.

Theorem 1 For the complementary delivery problem shown in Fig. 3, the optimal zero error broadcast rate $R_0^* = \max\{H(Y|X), H(X|Y)\}$.

We now extend Theorem 1 to a more general complementary delivery index coding problem with arbitrary number of messages/decoders.

Theorem 2 For the complementary delivery index coding problem, where each receiver demands the complement of its side information, the optimal zero error broadcast rate $R_0^* = \max_i H(X_{H^c(i)} | X_{H(i)})$.

We now consider broadcast function computation with complementary side information, and characterize the optimal rate under ϵ -error in two special cases, and also give single letter bounds for the optimal rate under ϵ -error and zero error.

Theorem 3 For the broadcast function computation with complementary delivery problem shown in Fig. 1

(i) The optimal rate R_ϵ^* is given by

$$R_\epsilon^* = \max(H(Z_1|X), H(Z_2|Y))$$

if either of the following conditions hold

- a) Z_1, Z_2 are compatible. In particular, this condition is satisfied when $Z_1 = Z_2$.
- b) X, Y are binary random variables.

(ii) Let

$$R_I = \min_{p(u|x,y)} \max(I(X; U|Y), I(Y; U|X)),$$

where $(X, Y) \in U \in \Gamma(G_{XY}^{Z_1 Z_2})$.

$$R_O = \max_{p(v|x,y)} \max(I(X; V|Y), I(Y; V|X))$$

with $|\mathcal{V}| \leq |\mathcal{X}| \cdot |\mathcal{Y}| + 2$.

Then $R_O \leq R_\epsilon^* \leq R_I$.

(iii) The optimal zero error rate R_0^* satisfies $\max\{H(Z_1|X), H(Z_2|Y)\} \leq R_0^* \leq H_{G_{XY}^{Z_1 Z_2}}(X, Y)$.

* In Section IV before proving Theorem 1, we argue that the scheme of binning which achieves the optimal ϵ -error rate does not work with zero-error.

IV. PROOFS OF THE RESULTS

A. Proof of Theorem 1

Remark 1 To achieve rates R close to $\max\{H(X|Y), H(Y|X)\}$, let us first consider the obvious scheme of random binning $X^n \oplus Y^n$ into 2^{Rn} bins. The decoders can do joint typicality decoding of $X^n \oplus Y^n$ similar to Slepian-Wolf scheme. However, there are two sources of errors. The decoding errors for non-typical sequences (x^n, y^n) can be avoided by transmitting those $x^n \oplus y^n$ unencoded, with an additional vanishing rate. However, for the same y^n , there is a non-zero probability of two different $x^n \oplus y^n, x'^n \oplus y'^n$, both of which are jointly typical with y^n , being in the same bin; leading to an error in decoding for at least one of them. It is not clear how to avoid this type of error with the help of an additional vanishing rate.

To prove Theorem 1, we first consider the problem for single receiver case as shown in Fig. 4. Witsenhausen [16] studied this problem under fixed length coding, and gave a single letter characterization of the optimal rate. For variable length coding, optimal rate R_0^* can be argued to be $R_0^* = H(Y|X)$ by using one codebook for each x . Here, we give a graph theoretic proof for this, and later extend this technique to prove Theorem 1.

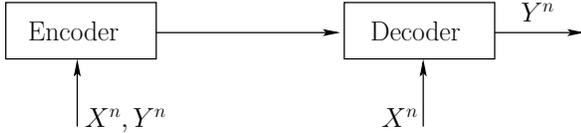


Fig. 4: One receiver with side information

Lemma 1 For the problem depicted in Fig. 4, $R_0^* = H(Y|X)$.

To prove Lemma 1, we first prove some claims. The graph that we use to prove Lemma 1, is a special case of the graph $G_{XY}^{Z_1 Z_2}$ defined in Section II-C, obtained by setting $Z_1 = Y$ and $Z_2 = \emptyset$. For simplicity, let us denote this graph by G . Graph G has its vertex set S_{XY} , and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $x_1 = x_2$ and $y_1 \neq y_2$. Similarly, we can obtain the n -instance graph for this problem from Definition 2. For simplicity, this graph is denoted by $G(n)$.

It is easy to observe that G is the disjoint union of complete row graphs G_i for $i = 1, 2, \dots, |\mathcal{X}|$, where each G_i has vertex set $\{(x_i, y) : (x_i, y) \in S_{XY}\}$.

Claim 1 For any n , the decoder can recover Y^n with zero error if and only if ϕ is a coloring of $G(n)$.

Proof: The decoder can recover Y^n with zero error \Leftrightarrow for any $(x^n, y^n), (x^n, y'^n) \in S_{X^n Y^n}$ with $y^n \neq y'^n$, $\phi(x^n, y^n) \neq \phi(x^n, y'^n) \Leftrightarrow$ for any $((x^n, y^n), (x^n, y'^n)) \in E(G(n))$, $\phi(x^n, y^n) \neq \phi(x^n, y'^n) \Leftrightarrow \phi$ is a coloring of $G(n)$. ■

In the following claim, we identify the vertices of $G(n)$ with the vertices of $G^{\wedge n}$ by identifying (x^n, y^n) with $((x_1, y_1), \dots, (x_n, y_n))$.

Claim 2 $G(n) = G^{\wedge n}$.

Proof: For both the graphs, (x^n, y^n) is a vertex if and only if $p(x_i, y_i) > 0$ for all i . Thus both the graphs have the same vertex set.

Next we show that both the graphs have the same edge set. Suppose $(x^n, y^n), (x'^n, y'^n) \in S_{X^n Y^n}$ are two distinct pairs. $((x^n, y^n), (x'^n, y'^n)) \in E(G(n)) \Leftrightarrow x^n = x'^n$ and $y^n \neq y'^n \Leftrightarrow x_i = x'_i$ for all i , and $y_j \neq y'_j$ for some $j \Leftrightarrow$ for each i , either $(x_i, y_i) = (x'_i, y'_i)$ or $((x_i, y_i), (x'_i, y'_i)) \in E(G) \Leftrightarrow (((x_1, y_1), \dots, (x_n, y_n)), ((x'_1, y'_1), \dots, (x'_n, y'_n))) \in E(G^{\wedge n})$. This shows that $G(n) = G^{\wedge n}$. ■

Claim 3 $R_0^* = \bar{H}_G(X, Y)$.

Proof: Claim 1 and the definition of chromatic entropy imply that $\frac{1}{n} H_\chi(G(n), (X^n, Y^n)) \leq R_0^n \leq \frac{1}{n} H_\chi(G(n), (X^n, Y^n)) + \frac{1}{n}$. Using Claim 2, and taking limit, we get $R_0^* = \lim_{n \rightarrow \infty} \frac{1}{n} H_\chi(G^{\wedge n}, (X^n, Y^n))$. Using (6), this implies $R_0^* = \bar{H}_G(X, Y)$. ■

Claim 4 G is a perfect graph.

Proof: As mentioned before, G is disjoint union of complete graphs. Since a complete graph is a perfect graph, it follows that G is also a perfect graph. ■

We now state a lemma from [13].

Lemma 2 [13] Let the connected components of the graph A be subgraphs A_i . Let $Pr(A_i) = \sum Pr(x)$, $x \in V(A_i)$. Further, set

$$Pr_i(x) = Pr(x)[Pr(A_i)]^{-1}, \quad x \in V(A_i).$$

Then $H_A(X) = \sum_i Pr(A_i) H_{A_i}(X_i)$.

We now prove Lemma 1.

Proof of Lemma 1: For any perfect graph A , it is known that $\bar{H}_A(X) = H_A(X)$ [20], [21]. So Claims 3 and 4 imply that $R_0^* = H_G(X, Y)$. We now use Lemma 2 to compute $H_G(X, Y)$. Recall that each connected component of graph G is a complete graph, and the connected component G_i , for each i , has vertex set $\{(x_i, y) : (x_i, y) \in S_{XY}\}$ and $Pr(G_i) = Pr(x_i)$. So we can set the probability of each vertex $(x_i, y) \in G_i$ as $Pr(x_i, y)/Pr(x_i)$. Since all the vertices in G_i are connected, we get $H_{G_i}(x_i, Y) = H(Y|X = x_i)$. Then by using Lemma 2, we get $H_G(X, Y) = H(Y|X)$. This completes the proof of Lemma 1. ■

Now let us consider the complementary delivery problem shown in Fig 3. This is a special case of the problem shown in Fig. 1 with $Z_1 = Y$ and $Z_2 = X$. In this case, the $Z_1 Z_2$ -modified rook's graph G_{XY}^{YX} has its vertex set S_{XY} , and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1 \neq y_2$, or $y_1 = y_2$ and $x_1 \neq x_2$. Now onwards,

we denote G_{XY}^{YX} and the n -instance graph $G_{XY}^{YX}(n)$ by G and $G(n)$ respectively.

We now state a Theorem from [19] which is used to prove Theorem 1.

Theorem 4 [19] *Let $\mathcal{G} = (G_1, \dots, G_k)$ be a family of graphs on the same vertex set. If $R_{\min}(\mathcal{G}, P_X) := \lim_{n \rightarrow \infty} \frac{1}{n} (H_X(\bigcup_i G_i^{\wedge n}, P_X^n))$, then $R_{\min}(G, P_X) = \max_i R_{\min}(G_i, P_X)$ where $R_{\min}(G_i, P_X) = \bar{H}_{G_i}(X)$.*

We are now ready to prove Theorem 1.

Proof of Theorem 1: For $i = 1, 2$, let G_i be the modified rook's graphs corresponding to decoding with side information at decoder i . So the modified rook's graph for the problem with two decoders is given by $G = G_1 \cup G_2$. Two vertices (x^n, y^n) and (x'^n, y'^n) are connected in the corresponding n instance graph $G(n)$ if and only if they are connected either in $G_1(n)$ or in $G_2(n)$. This implies that $G(n) = G_1(n) \cup G_2(n)$. This shows that both the decoders can decode with zero error if and only if ϕ is a coloring of $G(n)$. This fact and the definition of chromatic entropy imply that $R_0^* = \lim_{n \rightarrow \infty} \frac{1}{n} H_X(G(n), (X^n, Y^n))$. From Claim 2, it follows that $G(n) = G_1^{\wedge n} \cup G_2^{\wedge n}$. Then by using Theorem 4, we get $R_0^* = \max\{\bar{H}_{G_1}(X, Y), \bar{H}_{G_2}(X, Y)\}$. As argued in the proof of Lemma 1, $\bar{H}_{G_1}(X, Y) = H(Y|X)$ and $\bar{H}_{G_2}(X, Y) = H(X|Y)$. Thus $R_0^* = \max\{H(Y|X), H(X|Y)\}$. ■

B. Proof of Theorem 2

The proof of Theorem 2 follows by the same arguments as that of Theorem 1, and is thus omitted.

C. Proof of Theorem 3

Lemma 3 below is used in the achievability proof of part (i).

Lemma 3 *If Z_1, Z_2 are compatible such that $G_{XY}^{ZZ} = G_{XY}^{Z_1 Z_2}$ for $Z = h(X, Y)$, then $H(Z_1|Z, X) = 0$ and $H(Z_2|Z, Y) = 0$. As a consequence, $H(Z|X) = H(Z_1|X)$ and $H(Z|Y) = H(Z_2|Y)$.*

Proof: For any (x, y) and (x, y') , observe that

$$h(x, y) = h(x, y') \iff f(x, y) = f(x, y'). \quad (11)$$

Similarly, for any (x, y) and (x', y) ,

$$h(x, y) = h(x', y) \iff g(x, y) = g(x', y) \quad (12)$$

For a given $X = x$ and $Z = h(X, Y) = z$, let us consider the set of possible y , $A_{x,z} = \{y' : h(x, y') = z\}$.

By (11), $f(x, y') = f(x, y'') \forall y', y'' \in A_{x,z}$. Thus, denoting this unique value by $z_1 := f(x, y')$, we have $\Pr\{Z_1 = z_1 | X = x, Z = z\} = 1$. So we have $H(Z_1|Z, X) = 0$ and similarly $H(Z_2|Z, Y) = 0$. Using similar lines of arguments,

we get $H(Z|Z_1, X) = 0$ and $H(Z|Z_2, Y) = 0$. Then we get the following.

$$\begin{aligned} H(Z|X) &= H(Z|X) + H(Z_1|Z, X) \\ &= H(Z_1, Z|X) \\ &= H(Z_1|X) + H(Z|Z_1, X) \\ &= H(Z_1|X), \end{aligned}$$

Similarly, we get $H(Z|Y) = H(Z_2|Y)$. ■

Proof of part (i): We first prove part (i) a). Converse for R^* follows from the cut-set bound. Now let us consider the achievability of R_ϵ^* . The encoder first computes $h(x^n, y^n)$ and then uses Slepian-Wolf binning to compress it at a rate $\max(H(Z|X), H(Z|Y))$. Then decoder 1 and 2 can compute Z^n with negligible probability of error. From Lemma 3, it follows that encoder 1 can recover Z_1^n from Z^n and X^n . Similarly, encoder 2 computes Z_2^n from Z^n and Y^n . From Lemma 3, we have $\max(H(Z|X), H(Z|Y)) = \max(H(Z_1|X), H(Z_2|Y))$. When $Z_1 = Z_2 = Z$, from the above arguments it is easy to see that $\max(H(Z|X), H(Z|Y))$ is achievable.

Now let us consider part (i) b). Here also converse for R_ϵ^* follows from the cut-set bound. For achievability, let us consider $G_{XY}^{Z_1 Z_2}$. When X, Y are binary random variables, any $G_{XY}^{Z_1 Z_2}$ is a subgraph of the ‘‘square’’ graph with four edges. When $S_{XY} = \mathcal{X} \times \mathcal{Y}$, if graph $G_{XY}^{Z_1 Z_2}$ has one edge then Z_1, Z_2 are not compatible. It can be checked that any other possible graph $G_{XY}^{Z_1 Z_2}$ is compatible. For those compatible graphs, the proof follows from part (i) a). For a graph with only one edge, w.l.o.g., let us consider the graph shown in Fig. 2b. It is clear that $H(Z_2|Y) = 0$ and so decoder 2 can recover Z_2 only from Y . For decoder 1, we need an encoding rate $R = H(Z_1|X)$. Thus the rate $\max(H(Z_1|X), H(Z_2|Y)) = H(Z_1|X)$ is achievable. ■

Before proving part (ii) of Theorem 3, we present a useful lemma.

Lemma 4 *Let $W \in \Gamma(G_{XY}^{Z_1 Z_2})$ be a random variable such that $(X, Y) \in W$. Then $H(Z_1|W, X) = 0$ and $H(Z_2|W, Y) = 0$.*

Proof: Since w is an independent set of $G_{XY}^{Z_1 Z_2}$, for each $x \in \mathcal{X}$, $f(x, y') = f(x, y'')$ for all $(x, y'), (x, y'') \in w$. So decoder 1 can compute $f(x, y)$ from (w, x) whenever $p(w, x, y) > 0$. Similarly, decoder 2 can compute $g(x, y)$ from (w, y) whenever $p(w, x, y) > 0$. This implies that $H(Z_1|W, X) = 0$ and $H(Z_2|W, Y) = 0$. ■

Given x and independent set w , since the value of z_1 is unique, this unique value is denoted by $z_1(w, x)$ with abuse of notation.

Proof of part (ii): First we prove $R_\epsilon^* \leq R_I$. Let U be a random variable such that it satisfies the conditions of R_I in part (ii).

Generation of codebooks: Let $\{U^n(l)\}, l \in [1 : 2^{n\bar{R}}]$, be a set of sequences, each chosen i.i.d. according to $\prod_{i=1}^n p_U(u_i)$.

Partition the set of sequences $U^n(l)$, $l \in [1 : 2^{n\tilde{R}}]$, into equal-size bins, $B(m) = [(m-1)2^{n(\tilde{R}-R)} + 1 : m2^{n(\tilde{R}-R)}]$, where $m \in [1 : 2^{nR}]$.

Encoding: Given (x^n, y^n) , the encoder finds an index l such that $(x^n, y^n, u^n(l)) \in T_\epsilon^n(X, Y, U)$. If there is more than one such index, it selects one of them uniformly at random. If there is no such index, it selects an index uniformly at random from $[1 : 2^{n\tilde{R}}]$. The encoder sends the bin index m such that $l \in B(m)$.

Decoding: Once decoder 1 receives the message from the encoder, it finds the unique index $\hat{l} \in B(m)$ such that $(x^n, u^n(\hat{l})) \in T_\epsilon^n(X, U)$. If there is no unique $\hat{l} \in B(m)$, it sets $\hat{l} = 1$. It then computes the function values z_{1i} as $\hat{z}_{1i} = z_{1i}(u_i(\hat{l}), x_i)$ for $i \in [1; n]$. Decoder 2 operates similarly.

Analysis of error: Let (L, M) denote the chosen codeword and bin indices at encoder and let \hat{L} be the index estimate given by decoder 1. Decoder 1 makes an error if and only if the following event \mathcal{E}_1 happens.

$$\mathcal{E}_1 = \{(U^n(\hat{L}), X^n, Y^n) \notin T_\epsilon^n\}$$

Event \mathcal{E}_1 happens only if one of the following events happens.

$$\mathcal{E}_{11} = \{(U^n(l), X^n, Y^n) \notin T_\epsilon^n \text{ for all } l \in [1 : 2^{n\tilde{R}}]\}$$

$$\mathcal{E}_{12} = \{\exists \tilde{l} \neq L \text{ such that } \tilde{l} \in B(M), (U^n(\tilde{l}), X^n) \in T_\epsilon^n\}$$

Under \mathcal{E}_{11}^c , if $\hat{L} = L$, then decoder 1 can compute Z_1^n with no error. The probability of error for decoder 1 is upper bounded as $P(\mathcal{E}_1) \leq P(\mathcal{E}_{11}) + P(\mathcal{E}_{12})$.

By covering lemma [15], $P(\mathcal{E}_{11}) \rightarrow 0$ as $n \rightarrow \infty$ if $\tilde{R} > I(X, Y; U) + \delta(\epsilon')$. $P(\mathcal{E}_{12})$ is the same as the probability of error $P(\mathcal{E}_3)$ in [15, Lemma 11.3] if we replace Y^n with X^n . By packing lemma, $P(\mathcal{E}_{12}) \rightarrow 0$ if $\tilde{R} - R < I(X; U) - \delta(\epsilon)$. Combining these two bounds, we get $P(\mathcal{E}_1) \rightarrow 0$ as $n \rightarrow \infty$ if $R > I(X, Y; U) - I(X; U) + \delta(\epsilon) + \delta(\epsilon')$. This shows that any rate $R > I(U; Y|X)$ is achievable for decoder 1.

Similarly for decoder 2, any rate $R > I(U; X|Y)$ is achievable under the same encoding. So we get that $R > \max(I(X; U|Y), I(Y; U|X))$ is an achievable rate. Now we show that $R_O \leq R_\epsilon^*$.

$$\begin{aligned} nR &\geq H(M) \\ &\geq H(M|X^n) \\ &= I(M; Y^n|X^n) \text{ (} M \text{ is a function of } (X^n, Y^n)\text{)} \\ &\stackrel{(a)}{\geq} \sum_{i=1}^n H(Y_i|X_i) - \sum_{i=1}^n H(Y_i|Y^{i-1}, X_i, X^{i-1}, M) \\ &= \sum_{i=1}^n I(Y_i; V_i|X_i) \text{ (where } V_i = (M, X^{i-1}, Y^{i-1})\text{)}, \end{aligned}$$

where (a) follows from the fact that conditioning reduces entropy. Now defining a timesharing random variable $Q, V = (V_Q, Q), X_Q = X$ and $Y_Q = Y$; and using support lemma, the result follows. ■

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