Reasoning about Relational Granulation in Modal Logics

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***Abstract*— It is well known that the Kripke model for the modal logic system S5 can be interpreted as an approximation space in rough set theory. In this paper, we generalize the interpretation to relational granulation. We consider two multi-modal logics for reasoning about relational granulation in open world and closed world environments respectively. In an open world environment, two objects are granulated into the same equivalence class *only if* they have the same relationship with other objects, while in a closed world environment, two objects are granulated into the same equivalence class *if and only if* they have the same relationship with other objects. Such equivalence relations are represented by derived modalities from modal operators representing the relationships between objects.**

I. Introduction

Granular computing is a novel problem-solving methodology deeply rooted in human thinking. Many daily "things" have been granulated into "sub-things". For example, the human body can be granulated into the head, the neck, and so forth, and geographic features can be granulated into mountains, plains, etc. Although the notion is essentially fuzzy, vague, and imprecise, mathematicians have idealized it into partitions (equivalence relations) and developed a fundamental problem-solving methodology based on it. The notion has played a major role in solving many important problems throughout the history of mathematics. In recent years, rough set theory [1], [2] has introduced the idea to computer science, and it has been successfully applied to data analysis and uncertainty management. Nevertheless, the notion of partitions, which does not permit any overlapping among its granules (equivalence classes), is too restrictive for real world applications. Even in the natural sciences, classifications permit a small degree of overlapping. For example, there are creatures that are the proper subjects of both zoology and botany. A more general theory is thus needed. Granular computing is a new, rapidly emerging paradigm designed to meet this need [3]–[11].

In rough set theory, objects are partitioned into equivalence classes based on their attribute values, which are essentially functional information associated with the objects. A natural generalization is to consider granulation defined by the relational information between objects. Such information is defined by general binary relations, which are extensions of the functional attributes of the objects. Geometrically, such granulation is derived from the neighborhood system of topological spaces [12], where each point/object is assigned at most one neighborhood/granule. This kind of granulation is called *relational granulation*, whereas granulation based only on attribute values is called *functional granulation*.

It is well known that the Kripke model for the modal logic system S5 [13] can be interpreted as an approximation space in rough set theory [14]. In this paper, we generalize this interpretation to relational granulation. We consider two multi-modal logics for reasoning about relational granulation in open world and closed world environments respectively. In an open world environment (OWE), two objects are granulated into the same equivalence class *only if* they have the same relationship with other objects. In a closed world environment (CWE), on the other hand, two objects are granulated into the same equivalence class *if and only if* they have the same relationship with other objects. Such equivalence relations are represented by derived modalities from modal operators representing the relationships between objects.

The remainder of this paper is organized as follows. In Section II, we review rough set theory and relational granulation. In Sections III and IV, we present modal logics for reasoning about relational granulation in open world and closed world environments respectively. Finally, in Section V, we present our conclusions and indicate some future research directions.

II. Rough Set Theory and Relational Granulation

Rough set theory was originally defined with respect to data tables. A data table[1] is a pair $S = (U, A)$, where U is a nonempty finite set, called the universe of objects; A is a nonempty finite set of primitive attributes; and, for each $a \in A$, $a : U \to V_a$ is a total function, where V_a is the set of values for a, called the domain of a. Given any subset of attributes $B \subseteq A$, we can derive an equivalence relation over U , defined by $Ind(B) \subseteq U \times U$, as follows:

$$(x, y) \in Ind(B) \Leftrightarrow \forall a \in B,\ a(x) = a(y).$$

For any subset of objects $X \subseteq U$, the lower and upper approximations of X with respect to B are defined as

$$\underline{B}X = \{x \in U \mid \forall (x, y) \in Ind(B),\ y \in X\},$$

[1] Also called a knowledge representation system, information system, or attribute-value system.

and

$$\overline{B}X = \{x \in U \mid \exists(x,y) \in Ind(B),\ y \in X\}.$$

Since each attribute in A is considered as a total function from the set of objects to the domain of values, the equivalence relation is completely defined with respect to the functional information associated with the objects. Thus, in rough set theory, objects are granulated according to their functional attributes. Sometimes, the relationships between objects provide important information for data analysis. A notable example is social network analysis, in which the principal types of data are attribute data and relational data. According to [15],

> *Attribute data relates to the attitudes, opinions and behavior of agents, in so far as these are regarded as the properties, qualities or characteristics that belong to them as individuals or groups.* $\ldots, \ldots$
>
> *Relational data, on the other hand, are the contacts, ties and connections, the group attachments and meetings, which relate one agent to another and so cannot be reduced to the properties of the individual agents themselves.*

To model relational data, we employ relation algebra [16], [17]. A *proper relation algebra* is a structure

$$\mathfrak{R} = (\mathcal{R}, \cup, ^-, \circ, ^\smile, \mathbf{i}),$$

where

- $\mathcal{R}$ is a nonempty family of binary relations over a set U such that $U \times U \in \mathcal{R}$,
- $(x,y) \in R \cup S$ iff $(x,y) \in R$ or $(x,y) \in S$,
- $(x,y) \in \overline{R}$ iff $(x,y) \notin R$,
- $(x,y) \in R \circ S$ iff there exists $z \in U$ such that $(x,z) \in R$ and $(z,y) \in S$
- $(x,y) \in R^\smile$ iff $(y,x) \in R$, and
- $\mathbf{i} = \{(x,x) \mid x \in U\}$,

for any $R, S \in \mathcal{R}$ and $x, y \in U$. The set U is called the *field* of the relation algebra. For any binary relation $R \subseteq U \times U$ and $x \in U$, we define $R(x) = \{y \in U \mid (x,y) \in R\}$.

A binary relation in $\mathcal{R}$ provides a kind of relational information between objects, just as a subset of attributes yields functional information about objects. Based on such relational information, objects are granulated into equivalence classes, as in rough set theory. Formally, for any binary relation $R, S \in \mathcal{R}$, S is said to be an indiscernibility relation based on R if S is an equivalence relation and $S \subseteq \{(x,y) \in U \times U \mid R(x) = R(y)\}$. We use $\simeq_R$ to denote an arbitrary indiscernibility relation based on R, and $\cong_R$ to denote the least specific indiscernibility relation based on R, i.e., $\cong_R = \{(x,y) \in U \times U \mid R(x) = R(y)\}$.

For reasoning about relational granulation, we consider open world environments (OWE) and closed world environments (CWE). In OWE, it is assumed that, in addition to the relational information, R, other information may be available for the granulation of objects. Thus, the indiscernibility relation for granulating objects may be finer than $\cong_R$, so that an arbitrary indiscernibility relation based on R can serve the purpose. On the other hand, in CWE, we assume that R is the *only* information available for the granulation of objects. Thus, the least specific indiscernibility relation $\cong_R$ is used to granulate objects.

Example 1: Assume U is a set of agents who can receive and provide information which may be confidential. For any agent $x, y \in U$, we define three relations R, S_1, and S_2 as follows:

1) $(x,y) \in R$ iff x would like to acquire information about y,
2) $(x,y) \in S_1$ iff there is a channel for sending information from x to y,
3) and $(x,y) \in S_2$ iff x and y have a conflict of interest.

Assume the agents with the same goal of information acquisition might form an alliance. If R is the only criterion for formation of the alliance, then $\cong_R$ denotes the alliance relation between the agents. If, in addition to R, other criteria, such as personal preferences, affect formation of the alliance, then we only know that the alliance relation is an indiscernibility relation $\simeq_R$ based on R. We can state a security requirement as

$$S_1 \circ \simeq_R \subseteq \overline{S_2}$$

or

$$S_1 \circ \cong_R \subseteq \overline{S_2},$$

which means that an agent, x, can send information to another agent, y, only when no agents in the same alliance as y have a conflict of interest with x. This is related to the well-known Chinese Wall security policy [18]. ■

III. Reasoning about Relational Granulation in Open World Environments

To reason about relational granulation in OWE, we propose a multi-modal logic, $\mathrm{K}_n^{\simeq}$. The alphabet of $\mathrm{K}_n^{\simeq}$ contains the following symbols:

1) a countable set $P_0 = \{p, q, r, \ldots\}$ of atomic propositions,
2) the propositional constants $\bot$ (falsum or falsity constant) and $\top$ (verum or truth constant),
3) the binary Boolean operator $\vee$ (or), and the unary Boolean operator $\neg$ (not),
4) a set $A_0 = \{a, b, \ldots\}$ of primitive modalities,
5) and the modal operator-forming symbols $\simeq$, [, and]

The set Φ of well-formed formulas (wffs) is defined as the smallest set containing $P_0 \cup \{\bot, \top\}$ and closed under Boolean and modal operators:

$$\Phi := p \mid \bot \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid [a]\varphi \mid [\simeq_a]\varphi,$$

where $p \in P_0$, $a \in A_0$, and $\varphi, \psi \in \Phi$.

Other classical Boolean connectives $\wedge$ (and), $\supset$ (implication), and $\equiv$ (equivalence) are defined as abbreviations, i.e., $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi), \varphi \supset \psi = \neg\varphi \vee \psi$, and $\varphi \equiv \psi = (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Also, we write $\langle a \rangle\varphi$ (resp. $\langle \simeq_a \rangle\varphi$) as an abbreviation of $\neg[a]\neg\varphi$ (resp. $\neg[\simeq_a]\neg\varphi$). Furthermore, the auxiliary symbols "(" and ")" (i.e. left and right parentheses) are used to avoid ambiguity in wffs.

For the semantics, a possible world model for $K_n^{\simeq}$ is the structure

$$(U, (R_a, \simeq_{R_a})_{a\in A_0}, V),$$

where

- U is a set of possible worlds (the universe of objects),
- for each $a \in A_0$,
  - $R_a \subseteq U \times U$ is a binary relation over U,
  - and $\simeq_{R_a} \subseteq \{(x,y) \in U \times U \mid R_a(x) = R_a(y)\}$ is an equivalence relation over U,
- and $V : P_0 \to 2^U$ is a truth assignment that maps each atomic proposition to the set of worlds in which the proposition is true.

Let $\mathfrak{M} = (U, (R_a, \simeq_{R_a})_{a\in A_0}, V)$ be a model and Φ be the set of wffs for $K_n^{\simeq}$. The satisfaction relation $\models_{\mathfrak{M}} \subseteq U \times \Phi$ can then be defined by the following inductive rules (we use the infix notation for the relation and omit the subscript $\mathfrak{M}$ for brevity):

1) for each $p \in P_0$, $u \models p$ iff $u \in V(p)$,
2) $u \not\models \perp$ and $u \models \top$,
3) $u \models \neg\varphi$ iff $u \not\models \varphi$,
4) $u \models \varphi \vee \psi$ iff $u \models \varphi$ or $w \models \psi$,
5) $u \models [a]\varphi$ iff for all $(u, w) \in R_a$, $w \models \varphi$,
6) $u \models [\simeq_a]\varphi$ iff for all $(u, w) \in \simeq_{R_a}$, $w \models \varphi$.

A set of wffs, Σ, is satisfied in a world, w, written as $w \models \Sigma$, if $w \models \varphi$ for all $\varphi \in \Sigma$. We write $\Sigma \models_{\mathfrak{M}} \varphi$ if, for each possible world w in $\mathfrak{M}$, $w \models \Sigma$ implies $w \models \varphi$; and $\Sigma \models_{K_n^{\simeq}} \varphi$ if $\Sigma \models_{\mathfrak{M}} \varphi$ for each $K_n^{\simeq}$ model $\mathfrak{M}$. When $\Sigma = \emptyset$, it can be omitted. We say that a wff, φ, is valid in $\mathfrak{M}$ if $\models_{\mathfrak{M}} \varphi$, and φ is valid if $\models_{K_n^{\simeq}} \varphi$. For brevity, the subscript is usually omitted.

Given the language and semantics, the valid wffs of $K_n^{\simeq}$ are captured by the axiomatic system shown in Figure 1. In this presentation of the system, we use $\Box$ to denote modalities $[a]$ or $[\simeq_a]$.

The axiom K is the standard axiom for normal modal operators, whereas axioms T, 4, and 5 are characterizations of equivalence relations. The characteristic axioms Ch1 and Ch2 stipulate the connections between R_a and $\simeq_{R_a}$. According to the requirements of the semantics, $(x, y) \in \simeq_{R_a}$ implies $R_a(x) = R_a(y)$. Thus, every R_a successor of x must also be an R_a successor of any objects that are $\simeq_{R_a}$-equivalent to x.

A wff φ is derivable from the system $K_n^{\simeq}$, or simply, φ is a *theorem* of $K_n^{\simeq}$, if there is a finite sequence $\varphi_1, \ldots, \varphi_m$ such that $\varphi = \varphi_m$, and every φ_i is an instance of an axiomatic schema or obtained from earlier φ_j's by the application of an inference rule. We write $\vdash_{K_n^{\simeq}} \varphi$ if φ is a theorem of $K_n^{\simeq}$. Let $\Sigma \cup \{\varphi\}$ be a subset of wffs, then φ is derivable from Σ in the system $K_n^{\simeq}$, written as $\Sigma \vdash_{K_n^{\simeq}} \varphi$, if there is a finite subset Σ' of Σ such that $\vdash_{K_n^{\simeq}} \bigwedge \Sigma' \supset \varphi$. We usually drop the subscript if no confusion occurs. We then have the soundness and completeness theorem for $K_n^{\simeq}$.

Theorem 1: For any wff of $K_n^{\simeq}$, $\models \varphi$ iff $\vdash \varphi$.

Example 2: Continuing with Example 1, let A_0 consist of three primitive modalities a, b_1, and b_2 such that the intended interpretation of R_a, R_{b_1}, and R_{b_2} corresponds respectively to R, S_1, and $\overline{S_2}$ in Example 1. Then, the security policy in Example 1 can be written as a proper axiom in $K_n^{\simeq}$ as

$$[b_2]\varphi \supset [b_1][\simeq_a]\varphi.$$

■

1) Axioms:
   - P: all tautologies of propositional calculus
   - K: $(\Box\varphi \wedge \Box(\varphi \supset \psi)) \supset \Box\psi$
   - T: $[\simeq_a]\varphi \supset \varphi$
   - 4: $[\simeq_a]\varphi \supset [\simeq_a][\simeq_a]\varphi$
   - 5: $\neg[\simeq_a]\varphi \supset [\simeq_a]\neg[\simeq_a]\varphi$
   - Ch1: $[a]\varphi \supset [\simeq_a][a]\varphi$
   - Ch2: $\neg[a]\varphi \supset [\simeq_a]\neg[a]\varphi$
2) Rules of Inference:
   - R1(Modus ponens, MP):
   $$\frac{\varphi \quad \varphi \supset \psi}{\psi}$$
   - R2(Generalization, Gen):
   $$\frac{\varphi}{\Box\varphi}$$

Fig. 1. The axiomatic system for $K_n^{\simeq}$

IV. Reasoning about Relational Granulation in Closed World Environments

To reason about relational granulation in CWE, we propose an extension of Boolean modal logic $B_n^{\cong}$, which is closely related to Boolean modal logic [19] and dynamic logic [20]. The alphabet of $B_n^{\cong}$ is obtained from that of $K_n^{\simeq}$ by adding the symbols $\sqcup$, $\sim$, ;, $^-$, and ι for compound modalities and replacing the modal operator-forming symbol $\simeq$ with $\cong$.

The set of modalities (Π) and the set of wffs (Φ) are defined inductively as follows:

$$\Pi := a \mid \iota \mid \ \sim\alpha \mid \alpha^- \mid \alpha \sqcup \beta \mid \alpha;\beta,$$

where $a \in A_0$ and $\alpha, \beta \in \Pi$; and

$$\Phi := p \mid \perp \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid [\alpha]\varphi \mid [\cong_\alpha]\varphi,$$

where $p \in P_0$, $\alpha \in \Pi$, and $\varphi, \psi \in \Phi$. The abbreviation of other logical connectives is defined as in the case of $K_n^{\simeq}$.

For the semantics, a possible world model for $B_n^{\cong}$ is a structure $(U, (R_a)_{a\in A_0}, V)$, where U, R_a, and V are defined as in $K_n^{\simeq}$ models. Given a $B_n^{\cong}$ model $\mathfrak{M} = (U, (R_a)_{a\in A_0}, V)$, we define $\mathfrak{R}_{\mathfrak{M}}$ as the least relation algebra containing $U \times U$ and $\{R_a \mid a \in A_0\}$ with field U. The domain of the mapping $a \mapsto R_a : A_0 \to \mathfrak{R}_{\mathfrak{M}}$ is extended from A_0 to Π by the following homomorphic constraints:

1) $R_\iota = \mathbf{i}$
2) $R_{\sim\alpha} = \overline{R_\alpha}$,
3) $R_{\alpha^-} = R_\alpha^{\smile}$,
4) $R_{\alpha\sqcup\beta} = R_\alpha \cup R_\beta$,

5) $R_{\alpha;\beta} = R_\alpha \circ R_\beta$.

Furthermore, for each $\alpha \in \Pi$, we define

$$\cong_{R_\alpha} = \{(x,y) \in U \times U \mid R_\alpha(x) = R_\alpha(y)\}.$$

The satisfaction condition for Boolean connectives is the same as in $\mathrm{K}_n^{\cong}$ models, but the clauses for the satisfaction of modal formulas are modified as follows:

1) $u \models [\alpha]\varphi$ iff for all $(u,w) \in R_\alpha$, $w \models \varphi$,
2) $u \models [\cong_\alpha]\varphi$ iff for all $(u,w) \in \cong_{R_\alpha}$, $w \models \varphi$.

The definition of validity and logical consequence in $\mathrm{B}_n^{\cong}$ is the same as above.

Unfortunately, we can not find a complete axiomatization for $\mathrm{B}_n^{\cong}$. Instead, a sound axiomatic system can be obtained by combining the axioms for Boolean modal logic [19] and dynamic logic [20] as shown in Figure 2.

1) Axioms:

   P: all tautologies of propositional calculus
   K: $([\alpha]\varphi \wedge [\alpha](\varphi \supset \psi)) \supset [\alpha]\psi$
   D1: $[\alpha \sqcup \beta]\varphi \equiv [\alpha]\varphi \wedge [\beta]\varphi$
   D2: $[\alpha;\beta]\varphi \equiv [\alpha][\beta]\varphi$
   D3 $\varphi \supset [\alpha]\langle\alpha^-\rangle\varphi$
   D4 $\varphi \supset [\alpha^-]\langle\alpha\rangle\varphi$
   B1: $[\iota]\varphi \equiv \varphi$
   B2 $[\sim\sim \alpha]\varphi \equiv [\alpha]\varphi$
   Ch: $[\cong_\alpha]\varphi \equiv [\sim (\alpha; \sim \alpha^- \sqcup \sim \alpha; \alpha^-)]\varphi$

2) Rules of Inference:

   R1(Modus ponens, MP):

$$\frac{\varphi \quad \varphi \supset \psi}{\psi}$$

   R2(Generalization, Gen):

$$\frac{\varphi}{[\alpha]\varphi}$$

Fig. 2. An axiomatic system

D1-D4 are the axioms of propositional dynamic logic and B1-B2 are the axioms for Boolean modal logic. The axiom Ch characterizes the connection between R_α and $\cong_{R_\alpha}$. The soundness of the axiom is justified by the corresponding equation in relation algebra:

$$\cong_{R_\alpha} = \overline{R_\alpha \circ \overline{R_\alpha^{\smile}} \cup \overline{R_\alpha} \circ R_\alpha^{\smile}}.$$

This equation follows easily from the definition of $\cong_{R_\alpha}$. The definition of derivability and theoremhood is the same as above. We then have the soundness theorem.

Theorem 2: For any wff of $\mathrm{B}_n^{\cong}$, $\vdash \varphi$ implies $\models \varphi$.

Example 3: Continuing with Example 1, let A_0 consist of three primitive modalities a, b_1, and b_2 such that the intended interpretation of R_a, R_{b_1}, and R_{b_2} corresponds respectively to R, S_1, and S_2 in Example 1. Then, the security policy in Example 1 can be written as a proper axiom in $\mathrm{B}_n^{\cong}$ as

$$[\sim b_2]\varphi \supset [b_1; \cong_a]\varphi.$$

■

V. Conclusion

In this paper, we present two modal logics for reasoning about relational granulation. The first, called $\mathrm{K}_n^{\cong}$, is for reasoning in OWE, while the second, called $\mathrm{B}_n^{\cong}$, is for reasoning in CWE. To represent the indiscernibility relation in OWE, we do not need full relation algebra on the modalities, so we propose $\mathrm{K}_n^{\cong}$ as a moderate extension of normal multi-modal logic. On the other hand, to represent the indiscernibility relation in CWE, we have to employ compound modalities based on the operations of relation algebra. Consequently, we propose $\mathrm{B}_n^{\cong}$ as an extended combination of dynamic logic and Boolean modal logic. Obviously, the syntax of $\mathrm{B}_n^{\cong}$ is more expressive than that of $\mathrm{K}_n^{\cong}$. Nevertheless, we can not find a complete axiomatization for the former, while the latter has one. Thus, the next research problem is to investigate the possibility of axiomatizing the validity in $\mathrm{B}_n^{\cong}$.

In the definition above, we use the condition $R(x) = R(y)$ to determine the indiscernibility of x and y, based on the relational information R. This kind of flat definition is in fact an approximation. More precisely, we should define $\equiv_R$ recursively as

- $(x,y) \in \equiv_R$ iff (or only if) there exists a bijection $\sigma : R(x) \to R(y)$ such that for all $u \in R(x)$, $(u, \sigma(u)) \in \equiv_R$.

How to axiomatize a modality corresponding to $\equiv_R$ is another problem that deserves further investigation.

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