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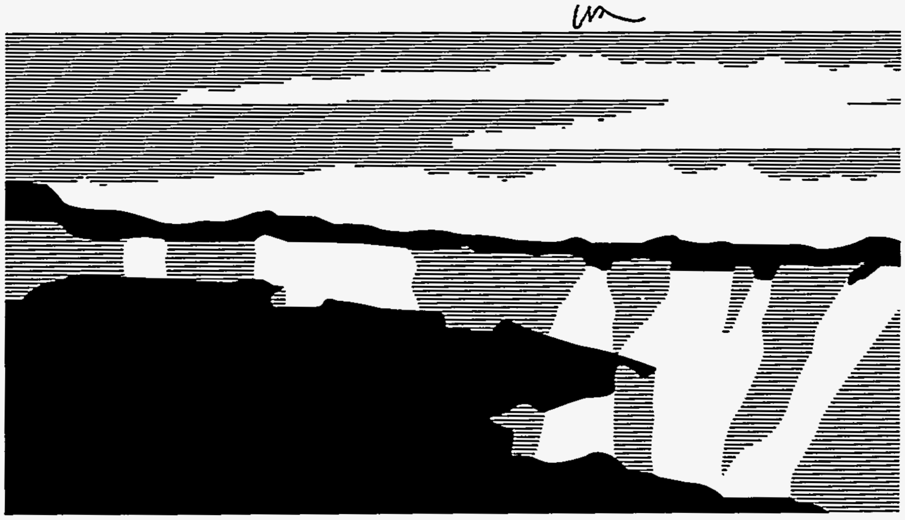

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# State Analysis of Nonlinear Systems Using Local Canonical Variate Analysis

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## 1.0 Introduction

There are many instances in which time series measurements are used to derive an empirical model of a dynamical system. State space reconstruction from time series measurement has applications in many scientific and engineerig disciplines including structural engineering, biology, chemistry, climatology, control theory, and physics. Prediction of future time series values from empirical models was attemped as early as 1927 by Yule, who applied linear prediction methods to the sunspot values (Yule, 1927). More recently work has been done by many investigtors, including Priestly (Priestly, 1980), Tong (1990) , Packard (Packard, 1980), Farmer (Farmer, 1988), Casdagli (Casdagli, 1991), and Larimore (Larimore, 1983). Efforts in this area have centered on two related aspects of time series analysis, namely prediction and modeling. In prediction future time series values are estimated from past values, in modeling, fundamental characteristics of the state model underlying the measurements are estimated, such as dimension and eigenvalues. In either approach a measured time series

$$\{\mathbf{y}(t_i)\}, i=1,\ldots,N \tag{1}$$

is assumed to derive from the action of a smooth dynamical system

$$\mathbf{s}(t+\tau)=\mathbf{a}(\mathbf{s}(t)) \tag{2}$$

where the bold notation indicates the (potentially ) multivariate nature of the time series. The time series is assumed to derive from the state evolution via a measurement function c.

$$\mathbf{y}(t)=\mathbf{c}(\mathbf{s}(t)) \tag{3}$$

In general the states $\mathbf{s}(t)$, the state evolution function a and the measurement function c are unknown, and must be inferred from the time series measurements.

We approach this problem from the standpoint of time series analysis. We review the principles of state space reconstruction in Section 2. Section 3 deals with the specific model formulation used in the local canonical variate analysis algorithm. A detailed description of the state space reconstruction algorithm is included in Section 3 and the references therein. Applications are illustrated in Section 4. Section 4.1 covers the application of the algorithm to a single-degree-of-freedom Duffing-like Oscillator. Section 4.2 illustrates the difficulties involved in reconstruction of an unmeasured degree of freedom in a four degree of freedom nonlinear oscillator, while illustrating a successful reconstruction. In the concluding section the advantages and current limitations of state space reconstruction are summarized. Improvements in neighborhood selection algorithms, noise elimination, and error estimation are suggested as further topics of research.

## 2.0 State Space Reconstruction.

We assume that the measured data $\mathbf{y}(t)$, derived from the action of the dynamical system, consists of samples separated by a sampling interval $\tau$, and that this sampling interval is chosen to incude frequencies of interest in the analysis.

Reconstruction of a state model from a time series relies on the fact that the past values of a time series contain information about unobserved state values at the present time. In a simiar manner, evolution of the time series contains information about the state evolution function $a(\mathbf{s}(t)$. A proof of this equivilance was demonstrated by Takens (Takens, 1981). Takens showed that, for a state space $\mathbf{s}(t)$ of dimension d and a scalar time series y(t) that 2d+1 values of the time series provide, in principle, all of the information neccessary to reconstruct the system state space at a given time t. Taken's theorem relies on the fact that 2d+1 nonlinear equations are sufficient to determine d variables. The "reconstructed" state space will contain information about all of the "observable" state variables whose influence effects the observed time series response. Taken's theorm forms the basis for the approach used in this paper. We emphasize the modeling of driven systems, where system response depends explictly on delayed values of the measured response and input time series. These systems are described by the state formulation in equation 4.

$$\begin{aligned}\mathbf{s}(t+\tau)&=a(\mathbf{s}(t))+b(\mathbf{u}(t))\\ \mathbf{y}(t)&=c(\mathbf{s}(t))+d(\mathbf{u}(t))\end{aligned} \tag{4}$$

Models of driven systems have been considered by Larimore( Larimore, 1983), Casdagli (Casdagli, 1991) and Hunter (Hunter 1990, 1991). Casdagli showed that, given 2d+1 lagged values of both the measured response y and known input u, a state model of the nonlinear sytem can be constructed. This is the "driven system" equivilant to Takens' theorem.

Figure 1 illustrates the general state reconstruction problem. The true state values $\mathbf{s}(t)$ and the state evolution a(t) are unknown. Measured values of the system response $\mathbf{y}(t)$ are related to the states $\mathbf{s}(t)$ by an unknown measurement function $c(\mathbf{s}(t))$. Inputs to the system are known, and the functions which relate inputs to states, $b(\mathbf{u}(t))$, and inputs to measurements, $d(\mathbf{u}(t))$ are unknown. To reconstruct the dynamics of the system, an "embedding" space is constructed from delayed time series values of the measured responses and inputs. A canonical transformation of the embedding space yields a reconstructed state space. Since the functions which evolve the reconstructed states are potentially highly nonlinear, a local linear model, which fits local hyperplanes to each region of the state and response spaces, is used to construct the unmeasured functions.

Reconstruction of the state space in these circumstances is a challenging problem and a number of procedures have been proposed, including local linear models (Farmer, 1988), NARMAX models (Billings and Tomlinson, 1988),local Principal component analysis (Casdagli,1991b, and nonlinear Canonical Variate Analysis (Larimore, 1983). Our local Canonical variate analysis model is related to these, and especially relies on the work of Farmer, Casdagli, and Larimore. The basic assumptions underlying the local CVA modeling procedure include:

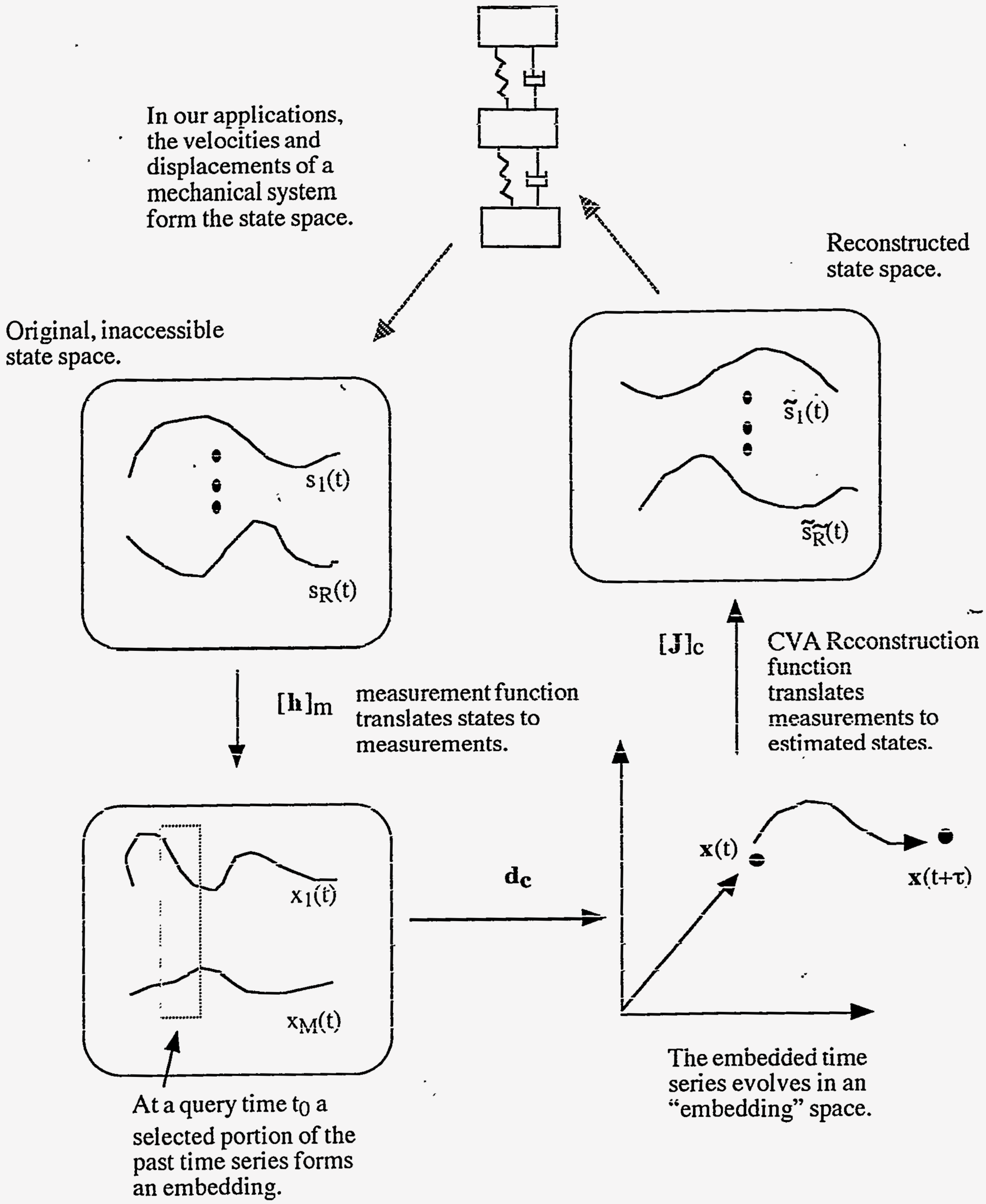
1. An emphasis on accurate prediction of future values of the (potentially multivariate ) time series from past values of the time series. The time series values are encoded as future and past "waveforms" whose evolution is basic to the system dynamics.

2.Orderly selection of the most critical features, here referred to as "fundamental waveforms", or "pseudo-states", which lead to effective prediction of the future from the past.

3. Analysis of model features emphasizing reconstructed states, state evolution, eigenvalues and local mode shapes. Where practical, model features are related to the physical system parameters.

### 3.0 **Local Canonical Variate Analysis**

The model used is based on two specific concepts, the Canonical Variate Analysis algorithm for constructing models from a time series (Larimore,1983) and the local modeling approach for dealing with systems whose properties change as a function of state (Farmer, 1988). In canonical variate analysis past waveforms are selected based on their utility in predicting future waveforms. this contrasts to approaches which emphasize prediction of individual future time series values.




The State Reconstruction Problem
Figure 1

To construct the model, we define the past p(t) and future f(t) of a multivariate time series as :

$$\mathbf{p}(t_0)=\{\mathbf{y}(t_0-\tau),\ldots,\mathbf{y}(t_0-l_o\tau),\mathbf{u}(t_0),\ldots\mathbf{u}(t_0-l_i\tau)\}$$
$$\mathbf{f}(t_0)=\{\mathbf{y}(t_0),\ldots,\mathbf{y}(t_0+l_o\tau)\} \tag{5}$$

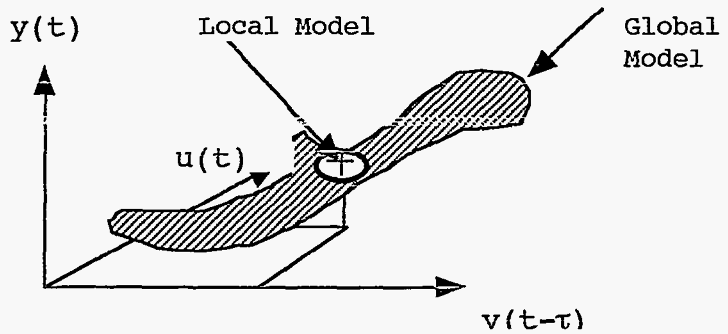
where the **y**'s refer to response time series values and the **u**'s refer to input time series values, $t_0$ is the time at index 0, $l_i$ is the number of input lags ans $l_o$ is the number of response lags. In a local region of the space of past and future vectors the matrices of past and future behavior are:

$$\mathbf{P}=\begin{bmatrix}\{\mathbf{y}(t_0-\tau),\ldots,\mathbf{y}(t_0-l_o\tau),\mathbf{u}(t_0),\ldots\mathbf{u}(t_0-l_i\tau)\}\\ \{\mathbf{y}(t_1-\tau),\ldots,\mathbf{y}(t_1-l_o\tau),\mathbf{u}(t_0),\ldots\mathbf{u}(t_1-l_i\tau)\}\\ \vdots \\ \{\mathbf{y}(t_j-\tau),\ldots,\mathbf{y}(t_j-l_o\tau),\mathbf{u}(t_j),\ldots\mathbf{u}(t_j-l_i\tau)\}\end{bmatrix} \tag{6a}$$

$$\mathbf{F}=\begin{bmatrix}\{\mathbf{y}(t_0),\ldots,\mathbf{y}(t_0+l_o\tau)\}\\ \{\mathbf{y}(t_1),\ldots,\mathbf{y}(t_1+l_o\tau)\}\\ \vdots \\ \{\mathbf{y}(t_j),\ldots,\mathbf{y}(t_j+l_o\tau)\}\end{bmatrix} \tag{6b}$$

The time indices 0,1,...,j refer to response and input values selected as "neighbors" in a given region of the space of past and future vectors. A Euclidian metric is used as the distance measure. Figure 2 illustrates the local modeling concept. Direct formulation of the system a $F=\alpha P$ usually leads to serious over fitting and the resultant estimation of a large number of parameters. The dimension of the past vector is $N_o*l_o+N_i*l_i+1$, where the number of input channels is $N_i$ and the number of measured response channels is $N_o$. The measured time series are derived from a d dimensional dynamical system. The number of parameters necessary to describe the d dimensional system are (Larimore, 1992):

$$N_p=d\left(2N_o+N_i+N_oN_i+\frac{N_o(N_o+1)}{2}\right) \tag{7}$$




Local Modeling
Figure 2
A global model would fit a single function to the entire response surface relating past and future. A local model, in contrast, fits a function the the relationship in a local neighborhood. Use of local modeling allows simple functional forms like linear models to be used in the context of nonlinear systems.

Suppose, for example, we measure one input and four responses from an 8 state system, and use 12 lagged values of each time series to represent the past. Then 60*12=720 parameters are needed to formulate the $P=\alpha F$ relationship, while the actual number of parameters needed to fully describe the system is, from equation (7), 216.

Canonical Variate Analysis accomplishes the necessary dimensional reduction by diagonaization of three fundamental relationships between the past and future matrices P and F such that:

$$JP^TPJ^T=I_p$$
$$LF^TFL^T=I_F$$
$$JP^TFL^T=E \tag{8}$$

$I_p$ and $I_f$ are identity matrices with rank of the past and future respectively. E is a diagonal matrix of singular values. The singular values are arranged in order of decreasing magnitude. The magnitude of each singular value is proportional to the importance of the corresponding row of J in predicting the future. The matrix J, which is computed from a generalized singular value decomposition of P and F as above, converts the past vectors to estimated states as:

$$\hat{s}(t)=\mathbf{J}\mathbf{p}(t) \tag{9}$$

J is computed by from:

$$J=U^T\left[P^TP\right]^{\frac{1}{2}}$$
$$SVD\left[\left[P^TP\right]^{\frac{1}{2}}\left[P^TF\right]\left[F^TF\right]^{\frac{1}{2}}\right]=UWV^T \tag{10}$$

This generalized SVD is neccessary to produce the relationships defined in equation 8. The algorithm is implemented in matlab™. Once the estimated states have been obtained from equation (9) functions a(s(t),b(u(t)),c(s(t)), and d(u(t) are approximated using a least squares solution of equations 12.

$$\hat{s}(t+\tau)=\hat{a}(\hat{s}(t))+\hat{b}(\mathbf{u}(t))+\mathbf{e}(\mathbf{t})$$
$$\mathbf{y}(t)=\hat{c}(\hat{s}(t))+\hat{d}(\mathbf{u}(t))+\mathbf{G}\mathbf{e}(\mathbf{t})+\mathbf{h}(\mathbf{t}) \tag{11}$$

The terms e(t) and h(t) are explicit estimates of the errors in the estimation of future states from past states. In the least squares formulation the estimated states and measurements are known.

Equations 11 govern the behavior in the reconstructed state space of Figure 1.

Future responses are estimated for short term predictions (one step predictions) and long term predictions (iterated predictions). The form of the estimated state transition matrix a(s(t)) is reviewed, state evolution of the states s are computed, and local estimates of eigenvalues and mode shapes are calculated. If chaotic behavior is of interest the largest Lyapunov exponent is estimated.

Local CVA has been applied to measured data from numerous systems. Some, like a Duffing oscillator, were synthesized to test the method. Others, like a climatic time series, were analyzed to obtain insight into unknown dynamics. Hardening Oscillators, bilinear oscillators, and a beam vibrating chaotically between two potential wells have been analyzed, as has a global climate time series(Hunter,1991 ). To illustrate the results of nonlinear state space modeling, we study data from two examples of nonlinear systems.

## 4.0 Applications

Our applications are taken from the context of mechanical vibrations. The application of local modeling to other fields is covered in the references. In our first example a nonlinear Duffing-like oscillator, driven by a known realization of band limited random noise, is simulated using an analog computer. The states and eigenvalues are computed and iterated prediction accuracy demonstrated for this nonlinear, non chaotic single-degree-of-freedom system. In the second example a nonlinear four-degree-of-freedom system is simulated and a "hidden" state detected. Neither of these systems is chaotic, though both are significantly nonlinear.

### 4.1 Application to a Nonlinear Hardening Oscillator.

Consider the nonlinear oscillator described by equation 12. In this two state system the stiffness $k_{01}$ varies as a function of the relative displacement $x_1$-$x_0$. The system is similar to the Duffing Oscillator (Moon, 1994) but the stiffness increases as the absolute value of a quadratic, rather than a cubic, function of the displacement. The quadratic formulation is more convenient to implement on the analog computer used to simulate the system behavior. The system is driven by an input $x_0''(t)$, a realization of band-limited random noise. The response acceleration $x_1''$ and the drive acceleration $x_0''$ are digitized at 150 samples/second. Embedding is accomplished using the measured input accelerations and the estimated velocities and displacements obtained from integration of the accelerations. A local CVA model computes the functions and states in equations (11), and estimates responses and eigenvalues.

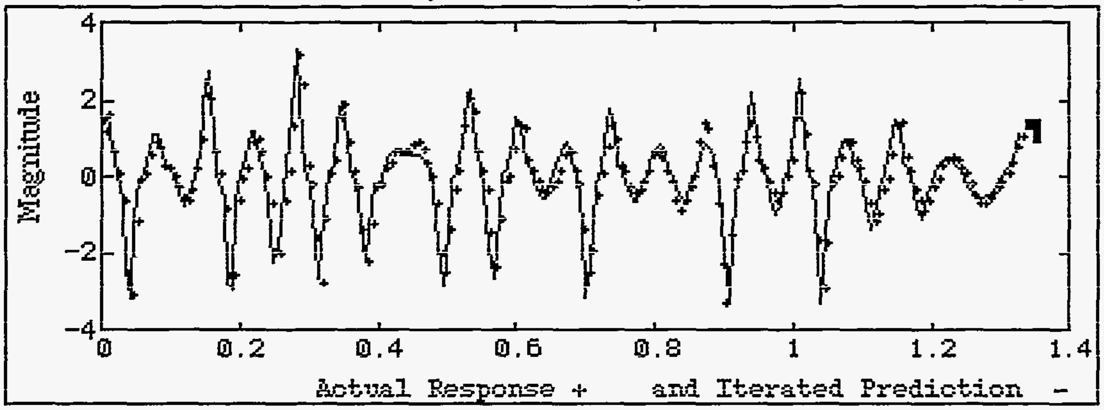
The nonlinear model is approximately described by :

$$x_1'' + 2\zeta\omega_n\left(x_1' - x_0'\right) + \omega_n^2\,(x_1 - x_0) + \alpha\omega_n^2 (x_1 - x_0)^2 + \beta\omega_n^2\,(x_1 - x_0)\,|x_1 - x_0| = 0 \qquad (12)$$

$$\alpha = 3000$$
$$\beta = 3500$$
$$\omega_n = 2\pi(11.5)$$
$$\zeta = 0.04$$

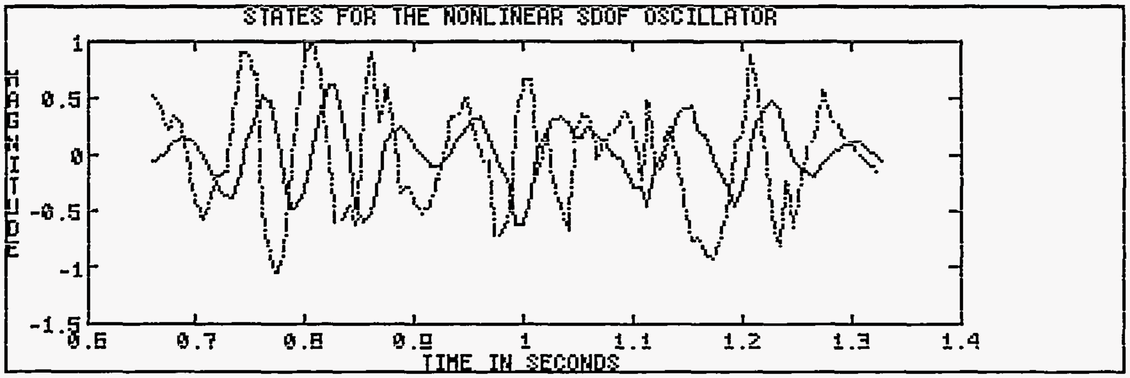
The nonlinear oscillator is simulated using an analog computer. Band limited random noise is applied to the input $x_0''$. The acceleration response $x1''$ is digitized at 150 samples/second. At low level we expect a resonant frequency of 11.5 Hz. The input level is adjusted to provide approximately equal rms levels from the linear and nonlinear stiffness terms. For positive excursions of $x1$-$x0$ the quadratic and absolute value terms add. For negative excursions they subtract. Since $\beta > \alpha$ we expect increasing stiffness in either direction, but less stiffness increase occurs in the direction of negative $x_1$-$x_0$.

Figure 3 illustrates the measured and predicted responses. Predictions are based on data outside of the sample range used for the model. The complete 1.4 seconds of response waveform is estimated using the known input and the initial conditions at t=0 seconds. This is a much more demanding task than estimating a series of one step predictions based on the known data from the past measured time step.




Measured and Predicted Responses for the Nonlinear Oscillator
Figure 3

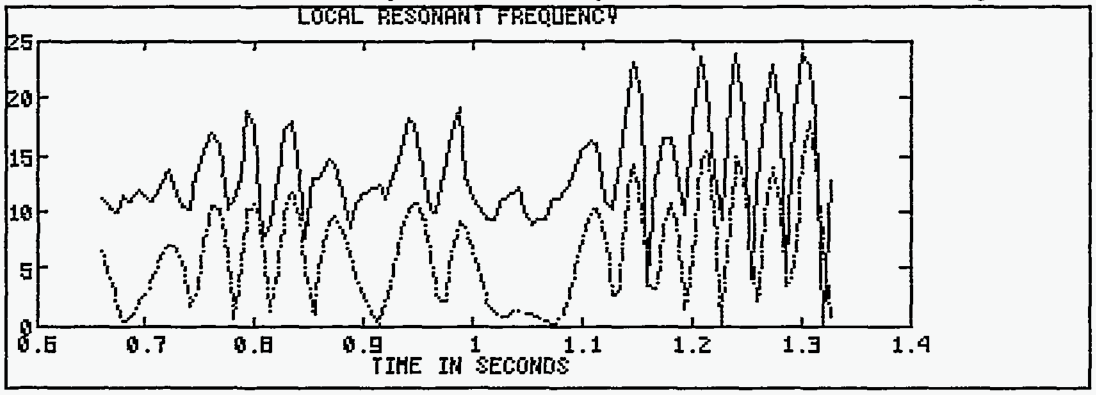
Estimated States for the nonlinear oscillator are illustrated in Figure 4. Note the approximate sine and cosine nature of the waveforms, combined with the occurrence of sharp peaks associated with the nonlinear stiffening behavior.




States for the Nonlinear Oscillator
Figure 4

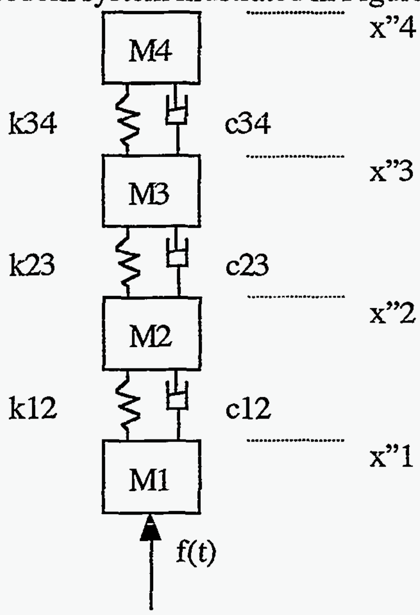
With the local linear formulation, the model results may be illustrated in several ways. At each time step the state transition matrix A[1] is estimated. For a linear system the elements of A are constant. For a nonlinear system, the elements of a change with time (or more fundamentally, with state). Both A and the states s(t) associated with a represent one example out of the set of {A,s} which can be used to formulate the state model. All A's in the set are related by similarity transforms and possess identical eigenvalues. Figure 4 illustrates the eigenvalues of a from t=0 to t=1.4 seconds. The absolute value of the relative displacement is plotted just below the resonant frequency to show the correlation between response level and stiffness. The system resonant frequency varies from approximately 10.0 Hz. at low relative displacements to nearly 25 Hz. at large relative displacements.

[1] The function a(s) is nonlinear. Here A, a square matrix of constants represents the local linear approximation to the unknown function a(s).




Instantaneous Resonant Frequency for the Nonlinear Oscillator
Figure 5

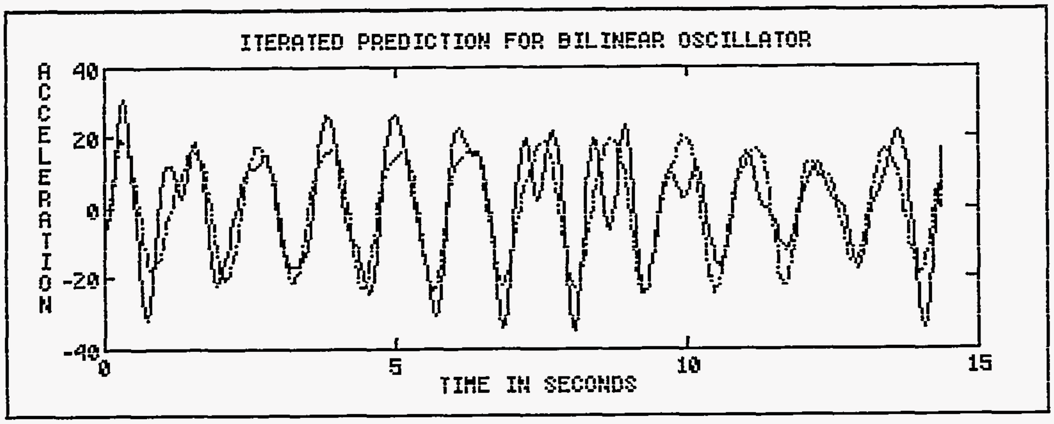
**4.2 Application to a Four-Degree-of-Freedom Bilinear Oscillator.**
Consider the multi-degree-of-freedom system illustrated in Figure 6.



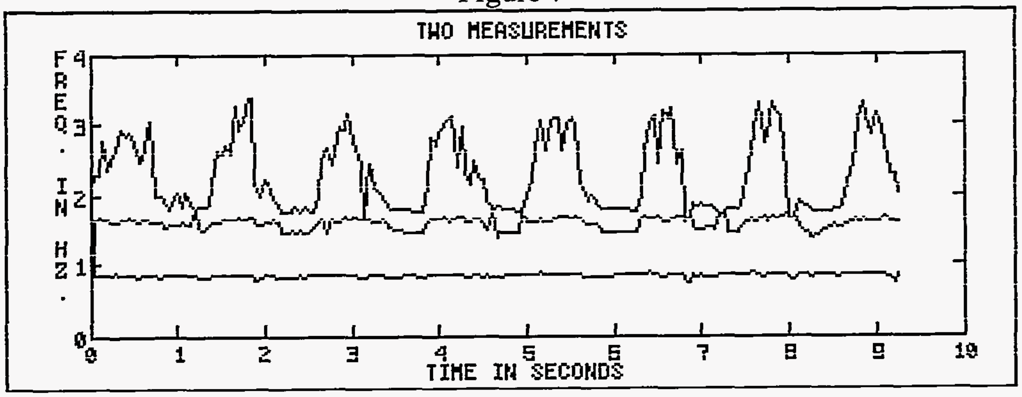

$$M1=M2=M3=M4=1.0$$
$$k12=k23=4\pi^2$$
$$x_4-x_3>0 \Rightarrow k34=4\pi^2$$
$$x_4-x_3<0 \Rightarrow k34=48\pi^2$$
$$c12=c23=c34=2(\zeta\omega)$$
$$\zeta=0.01$$
$$\omega=2\pi(1.0)$$

Four Degree of Freedom System With A Single Nonlinear Stiffness
Figure 6

Four masses are connected in chain fashion. Band limited random excitation is applied to the base mass M1. The rigid body mode is eliminated by connecting Mass 1 to ground through a soft spring. The stiffness k34 is high in compression and low in tension. In the first case an eigenvalue analysis shows frequencies of 0.1094 Hz., 0.7793 Hz., 1.4187 Hz., and 1.8488 Hz. In the second case, with the stiffer spring, the eigenvalues are 0.1096 Hz., 0.8580 Hz., 1.6638 Hz., and 4.9531 Hz. Local canonical variate analysis is applied to the system with 4000 response points used to train the model. Two formulations are made: in the first the acceleration response of each of the four masses are measured. In the second case the acceleration responses of masses one and four are measured. The iterated prediction, based on the initial condition at t=0 seconds, for a region of the response time series not used in the model formulation, is shown in Figure 7. The local frequencies, based on four measurements and two lags, are shown in Figure 8. Comparative local frequencies based on two measurements and four lags are shown in Figure 9. In both cases the state rank of the system is eight, corresponding to four eigenvalues. The indicated eigenfrequencies correspond to the theoretical values for the three higher system modes. Note the major change in frequency for the highest mode as predicted by the analysis of the system eigenvalues.



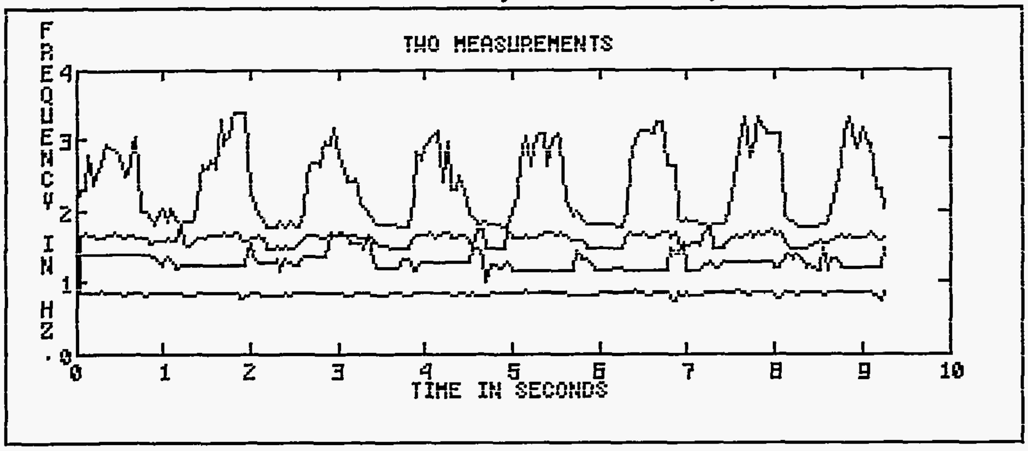

Iterated Prediction for the Four Degree of Freedom System
Figure 7




Local Frequency for the Four Degree of Freedom System- Four Measurement Locations.
Figure 8

The instantaneous frequency agrees quite well with the theoretical values when $k34=4\pi^2$. A significant increase in frequency occurs when $k34=48\pi^2$, though the frequency does not increase to 4.95 Hz.

This example clearly illustrates three features of the local CVA algorithm: reasonably accurate iterated predictions, approximate measurements of the nature of the nonlinearity, and detection of two hidden states when direct measurements of the state variables are unavailable.




Local Frequency for The Four Degree of Freedom System-Two Measurement Locations.
Figure 9

**Conclusion**

The examples shown, in concert with numerous other examples illustrated in the references, demonstrate the utility of local linear time series models for characterization and prediction. In this paper the the diagnostic capabilities of local CVA are illustrated by detecting hidden states, quantifying state rank, and showing, through the change in eigenvalues, the nature of a nonlinearity.

Significant problems remain. With real data, as opposed to analytically generated time series, the state rank determined from the separation of significant and trivial singular values is more difficult. More important is the estimation error inherent in the selection of nearest neighbors in the measurement space. Casdagli (Casdagli, 1991b) has pointed out the fundamental problem in selection of neighbors, namely that neighbors in the space of measured variables do not in general correspond to optimal neighbors in the space of the true state variables. Selection of neighbors in the measurement space gives some "false" neighbors whose inclusion in the local model leads to increased estimation errors. These estimation errors can in principle be reduced through a transformation of coordinates prior to the selection of nearest neighbors. Several algorithms have been suggested (Casdagli, 1991b). We intend to investigate some of these transformations in a future paper.

Characterization of nonlinear systems from measured response data is a difficult and challenging problem. In general, as the dimension of the system increases, exponentially increasing numbers of data points are required for accurate characterization. We have shown two approaches which mitigate this problem for systems of moderate dimension. Local Canonical variate analysis makes the most of the available data by emphasizing directions in the variable space critical for predicitons of system response. In principle, the states of a dynamic system can be constructed from delayed values of the time series from a single measurement. In practice, measurements at a number of points in a multidimensional system drastically reduce estimation error. Local Canonical Variate Analysis provides a means of effectively combining measurements at a number of locations into a single model.

Many interesting topics of research remain. Neighborhood selection can be improved through use of nonlinear coordinate transformations. The effects of noise on estimation errors requires further investigation. Finally, the problem of constructing a viable global model from the piecewise linear system derived from Local Analysis methods needs further attention.

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